Exercises "Particle Physics II"

32. Use Noether's theorem to derive the form of the conserved current j^μ associated with the global $U(1)$ symmetry

$$\psi(x) \to e^{i\alpha}\psi(x) \ , \quad \bar{\psi}(x) \to e^{-i\alpha}\bar{\psi}(x)$$

of the Lagrangean $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$.

33. Use the Dirac equation to show that $\partial_\mu j^\mu = 0$ is indeed fulfilled for

$$j^\mu = q\bar{\psi}\gamma^\mu\psi \ .$$

34. The Fourier decomposition of the field operator of a free Dirac field is given by

$$\psi(x) = \sum_s \int d\mu(p) \ \left[b(p,s)u(p,s)e^{-ip\cdot x} + d(p,s)^\dagger v(p,s)e^{ip\cdot x}\right] \ .$$

The creation and annihilation operators fulfil the anticommutation relations

$$\begin{aligned} \left\{b(p,s),\, b(p',s')^\dagger\right\} &= \left\{d(p,s),\, d(p',s')^\dagger\right\} = \delta(p,p')\delta_{ss'} \ , \\ \left\{b(p,s),\, b(p',s')\right\} &= \left\{d(p,s),\, d(p',s')\right\} = 0 \ , \\ \left\{b(p,s),\, d(p',s')\right\} &= \left\{b(p,s),\, d(p',s')^\dagger\right\} = 0 \ . \end{aligned}$$

Using these relations, verify the canonical anticommutaion relations for the field operator:

$$\left\{\psi_a(t,\vec{x}),\, \psi_b(t,\vec{y})^\dagger\right\} = \delta_{ab}\delta^{(3)}(\vec{x}-\vec{y}) \ , \quad \left\{\psi_a(x),\, \psi_b(y)\right\}\Big|_{x^0=y^0} = 0 \ .$$

35. Express the energy-momentum operator

$$P^\mu = \int d^3x \ :\psi^\dagger i\partial^\mu\psi:$$

and the charge operator

$$Q = q\int d^3x \ :\psi^\dagger\psi:$$

in terms of creation and annihilation operators.

36. Verify the commutation relations of P^μ and Q with the creation and annihilation operators listed in the lecture notes.

37. Show:

$$i\,[P^\mu, \psi(x)] = \partial^\mu \psi(x)\ .$$

Discuss the space-time shift

$$\exp(ia\cdot P)\,\psi(x)\,\exp(-ia\cdot P)$$

following from this relation.

38. The γ matrices are 4×4 matrices satisfying the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4\ .$$

Show:

(a) $\not{a}\not{a} = a\cdot a\,\mathbb{1}_4$ where $\not{a} := a_\mu\gamma^\mu$

(b) $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$

(c) $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\,(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma})$

(d) The trace of the product of an odd number of γ matrices vanishes. Hint: $\gamma_5^2 = \mathbb{1}_4$, $\{\gamma^\mu, \gamma_5\} = 0$

39. Show the following relations:

$$\sum_s u(p,s)\bar{u}(p,s) = \not{p} + m\ , \quad \sum_s v(p,s)\bar{v}(p,s) = \not{p} - m\ .$$

Hint: Consult the lecture notes of “Particle Physics I”.

40. The generating functional of the free Dirac field was found to be

$$\begin{aligned} Z[\eta,\bar{\eta}] &\equiv \left\langle 0 \left| T\exp\left\{ i\int d^4x\ \left[\bar{\eta}(x)\Psi(x) + \overline{\Psi}(x)\eta(x)\right]\right\}\right| 0\right\rangle \\ &= \exp\left\{ i\int d^4x\,d^4y\,\bar{\eta}(x)S(x-y)\eta(y)\right\}\ . \end{aligned}$$

Using this formula, compute

$$\langle 0|T\{\Psi_{a_1}(x_1)\overline{\Psi}_{b_1}(y_1)\ldots\Psi_{a_n}(x_n)\overline{\Psi}_{b_n}(y_n)\}|0\rangle\ .$$