Exercises "Particle Physics II"

23. The one-dimensional harmonic oscillator is described by the Hamilton operator
$$H = \frac{P(t)^2}{2m} + \frac{m\omega^2 Q(t)^2}{2} \ .$$
The position operator $Q(t)$ and the momentum operator $P(t)$ fulfil the canonical commutation relation
$$[Q(t), P(t)] = i\hbar\mathbb{1} \ .$$
   (a) Verify that Heisenberg's equation of motion for $Q(t)$,
   $$\dot{Q}(t) = \frac{i}{\hbar}\left[H, Q(t)\right] \ ,$$
   implies the classical equation of motion $\ddot{Q}(t) + \omega^2 Q(t) = 0$.

   (b) Express $Q(t)$ in terms of the ladder operators a and $a^\dagger$ which satisfy the commutation relation $\left[a, a^\dagger\right] = \mathbb{1}$.

   (c) Calculate the two-point function $\langle 0|TQ(t_1)Q(t_2)|0\rangle$.

24. Determine the generating functional of the one-dimensional harmonic oscillator,
$$Z[f] = \langle 0|T e^{\frac{i}{\hbar}\int_{-\infty}^{+\infty} dt\, f(t)Q(t)}|0\rangle \ ,$$
using the path integral representation
$$Z[f] = \frac{1}{\mathcal{N}}\int [dq] \exp\left\{\frac{i}{\hbar}\int_{-\infty}^{+\infty} dt\, \left[\frac{m}{2}\dot{q}(t)^2 - \frac{m(\omega^2 - i\varepsilon)}{2} q(t)^2 + f(t)q(t)\right]\right\} \ .$$
Give a physical interpretation of the external field $f(t)$. Verify the result for the two-point function obtained with the operator method.

Hint: The path integral calculation is completely analogous to the one for a free field, discussed in detail in the lecture. The position variable $q(t)$ can be interpreted as a scalar field living in $0+1$-dimensional spacetime.

25. The generating functional of a free **non-hermitean** scalar field $\phi(x)$,
$$Z[f] = \langle 0|T e^{i\int d^4x\left(f(x)^*\phi(x) + f(x)\phi(x)^\dagger\right)}|0\rangle \ ,$$
can be deduced from the generating functionals of two hermitean scalar fields $\phi_{1,2}(x)$ with equal masses. Use this relation to derive the explicit form of $Z[f]$.

Hint: $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, $f = (f_1 + if_2)/\sqrt{2}$, $f_i^* = f_i$.