Exercises "Particle Physics II"

1. The Fourier decomposition of the vector potential $\vec{A}(t,\vec{x})$ describing a free electromagnetic field in a box (volume V) with periodic boundary conditions is given by

$$\vec{A}(t,\vec{x}) = \sum_{\vec{k},\lambda} \frac{1}{\sqrt{V}} \left(e^{i\vec{k}\cdot\vec{x}} e^{-i\omega_{\vec{k}} t} b_{\vec{k}}^{(\lambda)} \vec{\varepsilon}_{\vec{k}}^{(\lambda)} + \text{c.c.} \right) .$$

Show that the field energy is given by

$$H = \frac{1}{2} \int_{\Omega} d^3x \, (\vec{E}^2 + \vec{B}^2) = \sum_{\vec{k},\lambda} \underbrace{2\vec{k}^2 |b_{\vec{k}}^{(\lambda)}|^2}_{H_{\vec{k}}^{(\lambda)}} .$$

2. Show that the momentum of the electromagnetic field is given by

$$\vec{P} = \int_{\Omega} d^3x \, \vec{E} \times \vec{B} = \sum_{\vec{k},\lambda} \frac{\vec{k}}{|\vec{k}|} H_{\vec{k}}^{(\lambda)} .$$

3. The real variables $q_{\vec{k}}^{(\lambda)}(t)$ and $p_{\vec{k}}^{(\lambda)}(t)$ are defined by

$$q_{\vec{k}}^{(\lambda)}(t) = b_{\vec{k}}^{(\lambda)}(t) + b_{\vec{k}}^{(\lambda)}(t)^* , \quad p_{\vec{k}}^{(\lambda)}(t) = -i\omega_{\vec{k}} \left(b_{\vec{k}}^{(\lambda)}(t) - b_{\vec{k}}^{(\lambda)}(t)^* \right) ,$$

where

$$b_{\vec{k}}^{(\lambda)}(t) = e^{-i\omega_{\vec{k}} t} b_{\vec{k}}^{(\lambda)} .$$

   (a) Verify the equations of motion $\dot{p}_{\vec{k}}^{(\lambda)}(t) = \ddot{q}_{\vec{k}}^{(\lambda)}(t) = -\omega_{\vec{k}}^2 \, q_{\vec{k}}^{(\lambda)}(t)$.

   (b) Express the Hamiltonian in terms of $q_{\vec{k}}^{(\lambda)}(t)$ and $p_{\vec{k}}^{(\lambda)}(t)$.

4. In the quantized theory, it is convenient to introduce the annihilation and creation operators $a_{\vec{k}}^{(\lambda)}$, $a_{\vec{k}}^{(\lambda)\dagger}$ by

$$b_{\vec{k}}^{(\lambda)} = a_{\vec{k}}^{(\lambda)} / \sqrt{2\omega_{\vec{k}}} .$$

They fulfill the commution relations

$$\left[a_{\vec{k}}^{(\lambda)}, a_{\vec{p}}^{(\sigma)\dagger} \right] = \delta_{\vec{k}\vec{p}} \delta_{\lambda\sigma} , \quad \left[a_{\vec{k}}^{(\lambda)}, a_{\vec{p}}^{(\sigma)} \right] = 0 .$$

Verify that Heisenberg's equation of motion

$$\dot{\vec{A}}(t,\vec{x}) = i \left[H, \vec{A}(t,\vec{x}) \right]$$

describes indeed the correct time behaviour of the field operator.

5. The ladder operators of a one-dimensional harmonic oscillator $a, a^\dagger$ satisfy the commutation relation

$$\left[a, a^\dagger\right] = \mathbb{1} \ .$$

Show:

(a) $\left[a, (a^\dagger)^n\right] = n(a^\dagger)^{n-1}$

(b) $\left[a, f(a^\dagger)\right] = f'(a^\dagger)$ Hint: Consider the power series expansion of the function f.

6. The coherent state $|z\rangle$ ($z \in \mathbb{C}$) of a harmonic oscillator is defined by the property $a|z\rangle = z|z\rangle$. Show that the solution of this equation is given by

$$|z\rangle = Ce^{za^\dagger}|0\rangle \ .$$

Hint: Use the formula of the previous problem.

7. Write the coherent state $|z\rangle$ as a linear combination of the normalized energy eigenstates $|n\rangle$. $\langle n|z\rangle =?$ Determine the normalization factor C (up to a phase factor) from the state normalization $\langle z|z\rangle = 1$.

8. The Poisson distribution is defined by

$$p(n; \lambda) = e^{-\lambda}\frac{\lambda^n}{n!} \ .$$

(a) Check the normalization $\sum_n p(n; \lambda) = 1$.

(b) Determine the characteristic function $\phi(t) = \langle e^{itn}\rangle$.

(c) $\langle n^k\rangle$ can be obtained from the characteristic function.

(d) $\langle n\rangle =?$, $\langle n^2\rangle =?$, $\langle n^2\rangle - \langle n\rangle^2 =?$.

9. Convince yourself that the probabilities $|\langle n|z\rangle|^2$ obey a Poisson distribution. What does this imply for $\langle z|N|z\rangle$, $\langle z|N^2|z\rangle$ and ΔN? ($N = a^\dagger a$)

10. Express the operator of the electric field $\vec{E}(t, \vec{x})$ in terms of creation and annihilation operators. Show that the expectation value of $\vec{E}$ vanishes for a state with sharp photon number n.

11. Consider the coherent state $|\psi\rangle$ of the electromagnetic field defined by

$$a_{\vec{k}}^{(\lambda)}|\psi\rangle = z_{\vec{k}}^{(\lambda)}|\psi\rangle \ , \quad z_{\vec{k}}^{(\lambda)} \in \mathbb{C} \ .$$

What is the expectation value of the electric field in this state?