(33)

Bereich I: $x \in [-(b+\frac{a}{2}), -\frac{a}{2}]$

Bereich II: $x \in [-\frac{a}{2}, +\frac{a}{2}]$

Bereich III: $x \in [\frac{a}{2}, b+\frac{a}{2}]$

Schrödingergleichung:

$$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x) \quad \boxed{\begin{matrix} E > 0 \\ E < V_0 \end{matrix}}$$

$$\Rightarrow \psi''(x) - \frac{2m}{\hbar^2}(V(x)-E)\psi(x) = 0$$

I u. III : $V(x) = 0$

SGL: $\psi''(x) + \frac{2m}{\hbar^2}E\psi(x) = 0 \quad \boxed{\frac{2mE}{\hbar^2} := k^2} > 0$

$$\Rightarrow \boxed{\psi''(x) = -k^2\psi(x)}$$

allg. Lösungsansatz:

$$\psi(x) = A\sin k(x+\alpha) + B\cos k(x+\alpha)$$

$\rightarrow \psi''(x) = -k^2\psi(x)$ ✓ da Bereich nicht symm. um 0

I) Randbedingung: $\psi(-b-\frac{a}{2}) = 0$

$$\psi_I(-b-\tfrac{a}{2}) = A_I\sin k(-b-\tfrac{a}{2}+\alpha_I) + B_I\cos k(-b-\tfrac{a}{2}+\alpha_I)$$

wähle $\alpha_I = b+\frac{a}{2}$

$$\Rightarrow \psi_I(-b-\tfrac{a}{2}) = \underbrace{A\sin 0}_{0} + B_I\underbrace{\cos 0}_{1} = \underline{\underline{B_I = 0}}$$

$$\Rightarrow \underline{\underline{\psi_I(x) = A_I\sin k(x+b+\tfrac{a}{2})}}$$

III: RB: $\psi(b + \frac{a}{2}) = 0$

$$\psi_{III}(b + a/2) = A_{III} \sin k(b + \tfrac{a}{2} + \alpha_{III}) + B_{III} \cos k(b + \tfrac{a}{2} + \alpha_{III})$$

$$\alpha_{III} = -b - \tfrac{a}{2} \quad \Rightarrow \quad B_{III} = 0$$

$$\Rightarrow \psi_{III}(x) = A_{III} \sin k(x - b - \tfrac{a}{2}) = A'_{III} \sin k(b + \tfrac{a}{2} - x)$$

II $V(x) = V_0$

SGL: $\psi''_{II}(x) - \frac{2m}{\hbar^2}(V_0 - E)\psi(x) = 0$

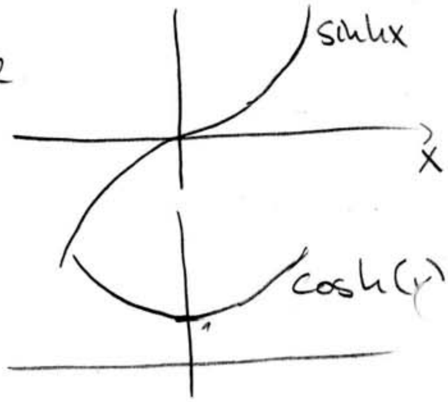
$$\Rightarrow \psi''_{II}(x) = q^2 \psi_{II}(x) \qquad q^2 := \frac{2m}{\hbar^2}(V_0 - E) > 0$$

allg. Ansatz:

$$\psi_{II}(x) = C \sinh(x) + D \cosh(x)$$

$\sinh(x) = \frac{e^x - e^{-x}}{2}$ ungerade

$\cosh(x) = \frac{e^x + e^{-x}}{2}$ gerade



$$\psi''_{II}(x) = q^2 \psi_{II}(x) \checkmark$$

I, II u. III: symmetr. Potential, betrachte nur gerade und ungerade Lösungen.

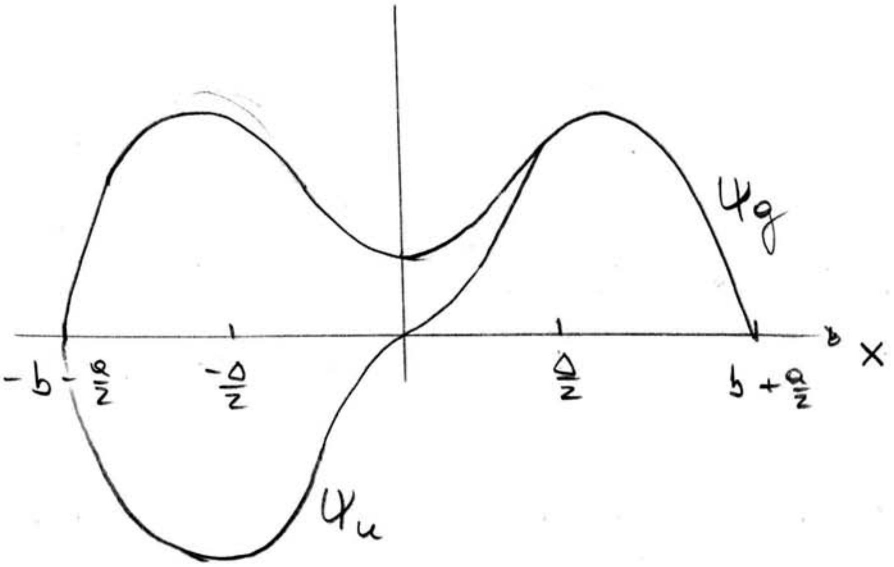
$\Rightarrow$ gerade: $A_I = A'_{III} = A$ da $\psi_{III}(-x) = \psi_I(x)$

$$\psi_g(x) = \begin{cases} \text{I} & A \sin k(b + \frac{a}{2} + x) \\ \text{II} & D \cosh(qx) \leftarrow C = 0 \text{ da } \psi_{II}(-x) = \psi_{II}(x) \\ \text{III} & A \sin k(b + \frac{a}{2} - x) \end{cases}$$

<u>ungerade</u>: $A_I = -A_{III}$ da $\psi_{III}(-x) = -\psi_I(x)$

$$\psi_u(x) = \begin{cases} -A \sin k\cdot(b+\frac{a}{2}+x) \\ C \sinh(q x) \\ A \sin k(b+\frac{a}{2}-x) \end{cases}$$

z. B.



b) <u>symmetrisch</u> (gerade):

$\psi^g_{III}(b-\frac{a}{2}) = \psi^g_{II}(b-\frac{a}{2})$:

$\Rightarrow$ (A) $A \sin(ka) = D\cosh(q(b-\frac{a}{2}))$ $\qquad \underline{b-\frac{a}{2} = \frac{\Delta}{2}}$

$\psi'_{III}(b-\frac{a}{2}) = \psi'_{II}(b-\frac{a}{2})$: (Ableitung)

$\Rightarrow$ (B) $-A k \cos(ka) = D\, q \sinh(q(b-\frac{a}{2}))$

↳ innere Abl.

(A) u. (B) homogenes Gleichungssystem nach A u. D, nur nichttriviale Lösung wenn Determinante = 0.

$\Rightarrow -q \sin(ka)\sinh(q\frac{\Delta}{2}) = k\cos(ka)\cosh(q\frac{\Delta}{2})$

$\Rightarrow \underline{\tan(ka) = -\frac{k}{q}\coth(q\frac{\Delta}{2})}$

asymmetrisch (ungerade):

(wieder bei $b-\frac{a}{2}$)

$$A \sin ka = C \sinh\left(q\frac{\Delta}{2}\right)$$
$$-Ak\cos ka = C\, q \cosh\left(q\frac{\Delta}{2}\right)$$

$\det = 0$

$$\Rightarrow -q\sin(ka)\cosh\left(q\frac{\Delta}{2}\right) = k\cos(ka)\sinh\left(q\frac{\Delta}{2}\right)$$
$$\Rightarrow \underline{\tan(ka) = -\frac{k}{q}\tanh\left(q\frac{\Delta}{2}\right)}$$

34 a)

$$E \ll V_0 : q^2 \simeq \frac{2mV_0}{\hbar^2} \gg k^2 = \frac{2mE}{\hbar^2}$$

$$q\cdot\Delta \gg 1$$

$$\coth\frac{q\Delta}{2} = \frac{e^{\frac{q\Delta}{2}} + e^{-\frac{q\Delta}{2}}}{e^{q\Delta/2} - e^{-q\Delta/2}} = \frac{1+e^{-q\Delta}}{1-e^{-q\Delta}}$$

$$\simeq \left(1+e^{-q\Delta}\right)^2 = 1 + 2e^{-q\Delta} + \underbrace{\left(e^{-q\Delta}\right)^2}_{\simeq 0}$$

$$\Rightarrow \coth\frac{q\Delta}{2} \simeq 1 + 2e^{-q\Delta}$$

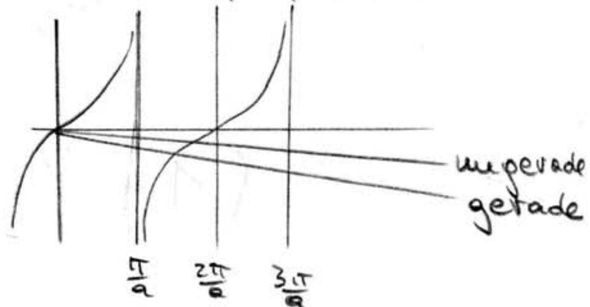
$$\tanh\frac{q\Delta}{2} \simeq 1 - 2e^{-q\Delta}$$

gerade: $\tan ka = -\frac{k}{q}\left(1+2e^{-q\Delta}\right)$

ungerade: $\tan ka = \underbrace{-\frac{k}{q}\left(1-2e^{-q\Delta}\right)}_{\text{klein, Tan} \simeq \text{linear}}$

$$ka + n\pi = -\frac{k}{q}(1+2e^{-q\Delta}) \qquad n = 1, 2, 3 \ldots$$

$$k_n^g = \frac{-n\pi}{a + \frac{1}{q}(1+2e^{-q\Delta})}$$



$$E_n^g = \frac{\hbar^2 (k_n^g)^2}{2m} = \frac{\hbar^2 \pi^2 n^2}{2m} \cdot \frac{1}{\left[a + \frac{1}{q}(1+2e^{-q\Delta})\right]^2}$$

ungerade:

$$E_n^u = \frac{\hbar^2 \pi^2 n^2}{2m} \cdot \frac{1}{\left[a + \frac{1}{q}(1-2e^{-q\Delta})\right]^2}$$

$$\Rightarrow E_n^g < E_n^u$$

mittlere Energie:

$$\tilde{\varepsilon}_g := \frac{1}{q}(1+2e^{-q\Delta}) \ll 1 \qquad \boxed{C_n = \hbar^2 \pi^2 n^2 / 2m}$$

$$\Rightarrow E_n^g = C_n \frac{1}{(a+\tilde{\varepsilon}_g)^2} \simeq \frac{C_n \cdot 1}{a^2 + 2a\tilde{\varepsilon}_g} = \frac{C_n \cdot 1}{a^2\left(1 + \frac{2\tilde{\varepsilon}_g}{a}\right)}$$

$$= \frac{C_n \cdot 1}{a^2} \cdot \frac{1}{1+\varepsilon_g}$$

$$\text{mit } \varepsilon_g := \frac{2\tilde{\varepsilon}_g}{a} = \frac{2}{aq}(1+2e^{-q\Delta})$$

$$\Rightarrow E_n^u \simeq \frac{C_n \cdot 1}{a^2} \cdot \frac{1}{1+\varepsilon_u}$$

$$\text{mit } \varepsilon_u := \frac{2}{aq}(1-2e^{-q\Delta})$$

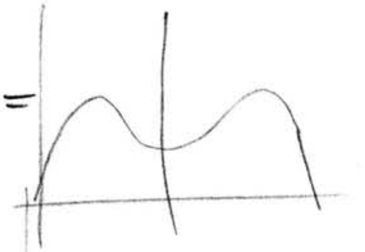
$$\overline{E} = \frac{E_n^g + E_n^u}{2} = \frac{C_n}{2} \cdot \frac{1}{a^2} \cdot \left(\frac{1}{1+\varepsilon_g} + \frac{1}{1+\varepsilon_u}\right)$$

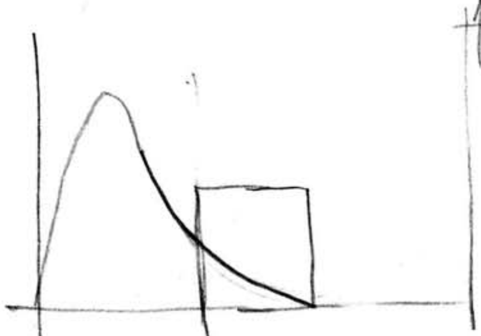
$$\simeq \frac{C_n}{2a^2} \cdot (1 - \varepsilon_g + 1 + \varepsilon_u) = \frac{C_n}{2a^2} \cdot (2 + \varepsilon_g + \varepsilon_u) =$$

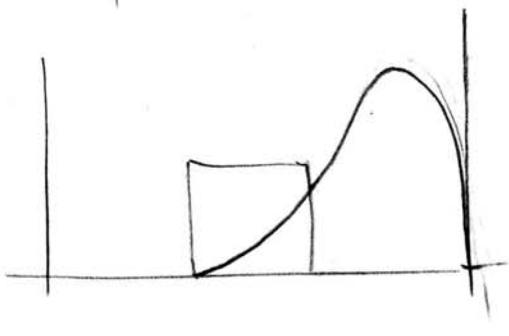
$$= \frac{C_n}{2a^2}\left(2 + \frac{2}{aq} + \cancel{\frac{4e^{-q\Delta}}{aq}} + \frac{2}{aq} - \cancel{\frac{4e^{-q\Delta}}{aq}}\right)$$

$$= \frac{C_n \cdot \cancel{2}}{\cancel{2}a^2}\left(1 + \frac{2}{aq}\right) = \frac{\hbar^2\pi^2 n^2}{2ma^2}\left(1 + \frac{2}{aq}\right)$$

b) Bild aus 33 a) für $n = 1$:

$\psi_L = \frac{1}{\sqrt{2}}(\psi_S - \psi_A) =$  $-$ 

$=$ 

$\psi_R =$ 

ψ_L, ψ_R **nicht** Lösungen zur zeitunabh. Schrödingergl. aus 33) sondern inkl. Zeitentw. zur zeitabh. SGL

$$\psi^n(x,t) = \frac{1}{\sqrt{2}}\left(\psi_S^n(x)e^{-i/\hbar E_S^n t} + \psi_A^n(x)e^{-i/\hbar E_A^n t}\right)$$

$\Rightarrow$ entspricht genau dem Term in Bsp. 24, oszillierender Interferenzterm mit $\delta E = E^n - E^S$