One-loop Functional and Heat-Kernel Expansion

SE - Current topics in theoretical particle physics

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Outline

- Generating Functional of φ^4-theory
- Expansion of generating functional in powers of $\hbar$
- Quasiclassical approximation: W[J] up to order of $\hbar$
- Dimensional Regularization and Renormalization of W[J]
- Renormalization-Group equations
- Heat-Kernel expansion and Seeley-coefficients
- <u>Sources</u>:

  J. Gasser und H. Leutwyler: „Skript zur Vorlesung: Quantenfeldtheorie 1“, Sommersemester 1997, Bern

  Sidney Coleman and Erick Weinberg: „Radiative Corrections as the Origin of SSB“, PRD 7(1973) page 1888, 15 March 1973

QFT in Minkowski space $\to$ Euclidean space

- Transformation ... not necessary but sometimes convenient
- Transformation-rules for space-time coordinates:

  $x^0 \to -i\,\hat{x}^0$ and $x^i \to \hat{x}^i$

  $\Rightarrow$ derivatives: $\partial^0 \to i\,\hat{\partial}_0$ and $\partial^i \to -\,\hat{\partial}_i$
- Transformation for a four-vector: $A^0 \to i\hat{A}^0$ and $A^i \to -A^i$.
- $i\,\mathsf{S}_M \;\to\; -\,\mathsf{S}_E$.
- Example ... Φ^4- theory:

$$
\begin{aligned}
i\,\mathsf{S}_M \;&=\; i\int \mathrm{d}^d x\,\left(\frac{1}{2}(\partial_\mu\Phi\,\partial^\mu\Phi \,-\, \mathrm{m}^2\Phi^2) \,-\, \frac{\lambda}{4!}\Phi^4\right)\\
\to -\mathsf{S}_E \;&=\; -\int \mathrm{d}^d \hat{x}\,\left(\frac{1}{2}(\hat{\partial}_\mu\Phi\,\hat{\partial}_\mu\Phi \,+\, \mathrm{m}^2\Phi^2) \,+\, \frac{\lambda}{4!}\Phi^4\right)
\end{aligned}
$$

Generating Functional

- from now on: x means $\hat{x}$ ... always in Euclidean space
- $Z[J] = e^{-W[J]} = \langle 0|0\rangle_J = \langle 0| T\left(e^{\int d^d x\, J(x)\Phi(x)}\right)|0\rangle$

  with J(x) being an external source
- Path integral representation:

$$e^{-W[J]} = \frac{1}{\mathcal{N}}\int [d\varphi]\, e^{-S[\varphi]}\, e^{\int d^d x\, J(x)\varphi(x)}$$

$$\text{with} \qquad \mathcal{N} = \int [d\varphi]\, e^{-S[\varphi]}$$

Generating Functional

- Z[J] ... generating functional of **all** Green-functions!
- $\sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^d x_1 \cdots \mathrm{d}^d x_n J(x_1) \cdots J(x_n) \, \langle 0| \, T \left(\Phi(x_1) \cdots \Phi(x_n) \right) |0\rangle$
- $\langle 0| \, T \left(\Phi(x_1) \cdots \Phi(x_n) \right) |0\rangle = \frac{1}{\mathcal{N}} \int [\mathrm{d}\varphi] \, e^{-\mathrm{S}[\varphi]} \, \varphi(x_1) \cdots \varphi(x_n)$

- W[J] ... generating functional of **connected** Green-functions!
- $\text{-W[J]} = \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathrm{d}^d x_1 \cdots \mathrm{d}^d x_n J(x_1) \cdots J(x_n) \, \mathrm{G}_{\mathrm{n},c}(\Phi)$

  with $\quad \mathrm{G}_{\mathrm{n},c}(\Phi) = \langle 0| \, T \left(\Phi(x_1) \cdots \Phi(x_n) \right) |0\rangle_c$

Example: $\varphi^4 -$ Theory

- $\varphi^4 -$ theory: $\mathrm{e}^{-\mathrm{W}[J]} = \frac{1}{\mathcal{N}} \int [d\varphi]\ \mathrm{e}^{-\mathrm{S}[\varphi,\mathrm{J}]}$ with

$$\mathrm{S}[\varphi,\mathrm{J}] = \int \mathrm{d}^d x \left(\frac{1}{2}(\partial_\mu\varphi\partial_\mu\varphi + \mathrm{m}^2\varphi^2) + \frac{\lambda}{4!}\varphi^4 - \mathrm{J}\varphi \right)$$

and $\quad \mathcal{N} = \int [d\varphi]\, \mathrm{e}^{-\mathrm{S}[\varphi,0]}$

Legendre transformation of W[J]

- Effective Action Γ is defined by

$$\Gamma[\bar{\varphi}] = \mathrm{W}[\mathrm{J}] + \int \mathrm{d}^d x \; \mathrm{J}(x)\, \bar{\varphi}(x), \text{ where } \bar{\varphi}(x) = -\frac{\delta\, \mathrm{W}[\mathrm{J}]}{\delta\, \mathrm{J}(x)}$$

- Γ ... generating functional of 1-particle-irreduzible graphs
- $\Gamma[\bar{\varphi}] = \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathrm{d}^d x_1 \cdots \mathrm{d}^d x_n \, \bar{\varphi}(x_1) \cdots \bar{\varphi}(x_n)\, \Gamma^{(n)}(x_1, \ldots, x_n)$
- with $\Gamma^{(n)}(x_1, \ldots, x_n)$ being the 1PI Green-functions
- It holds: $\frac{\delta\, \Gamma[\bar{\varphi}]}{\delta\, \bar{\varphi}} = \mathrm{J}(x)$
- Expanding in powers of momentum instead of $\bar{\varphi}$:
$\Gamma[\bar{\varphi}] = \int \mathrm{d}^d x \,(+V_{eff}(\bar{\varphi}) + \frac{1}{2} Z(\bar{\varphi})\, \partial_\mu \bar{\varphi} \partial_\mu \bar{\varphi} + \ldots)$

Expansion of W[J] in powers of $\hbar$

- $e^{-\frac{1}{\hbar} W[J]} = \frac{1}{\mathcal{N}} \int [d\varphi] \, e^{-\frac{1}{\hbar} S[\varphi, J]}$
- counting Lagrangian as independent of $\hbar$!
- parameter m in S[φ,J]: inverse of Compton-wavelength $m = \frac{m_1 c}{\hbar}$, (m_1 ... mass of the particle)
- expansion in powers in $\hbar$: $W[J] = \sum_{\ell=0}^{\infty} \hbar^{\ell} W_{\ell}[J]$
- $W_0[J]$ ... generating functional of connected tree diagrams
- $W_1[J]$ ... generating functional of the one-loop-graphs
- ....
- $W_{\ell}[J]$ ... generating functional of the ℓ-loop-graphs

Saddle-point approximation

$$S[\varphi, J] = \int d^d x \left(\frac{1}{2}(\partial_\mu \varphi \partial_\mu \varphi + m^2 \varphi^2) + \frac{\lambda}{4!}\varphi^4 - J\varphi \right)$$

- expanding around solution of classical equation $\varphi_{cl}(x)$:
  $\varphi(x) = \varphi_{cl}(x) + \sqrt{\hbar}\,\varphi'(x)$
- $\varphi_{cl}(x)$ obeys $-\partial^2 \varphi_{cl}(x) + m^2 \varphi_{cl}(x) + \frac{\lambda}{3!}\varphi_{cl}^3(x) = J(x)$
  $\Rightarrow$ terms linear in φ' vanish in $S[\varphi,J]$

$$S[\varphi, J] = S[\varphi_{cl}, J] + \hbar\, S^{(2)}[\varphi, J]$$

$$\text{with } S^{(2)}[\varphi, J] = \frac{1}{2} \int d^d x \underbrace{\left\{\partial_\mu \varphi' \partial_\mu \varphi' + m^2 (\varphi')^2 + \frac{\lambda}{2}\varphi_{cl}^2 (\varphi')^2\right\}}_{= \varphi' D \varphi' \ \text{ with } \ D = -\partial^2 + m^2 + \underbrace{\frac{\lambda}{2}\varphi_{cl}^2(x)}_{= v(x)}}$$

- $\varphi_{cl}(x)$ independent of $\varphi'(x)$; pull out $S[\varphi_{cl},J]$ of path integral

Saddle-point approximation

- remaining integral: $\mathcal{F}\{\mathrm{v}\} = \int [\mathrm{d}\varphi'] \, \mathrm{e}^{-\frac{1}{2} \int \mathrm{d}^d x \, \varphi'(x) D \varphi'(x)}$
- Namely, it holds: $\mathrm{e}^{-\mathrm{W}_1\{\mathrm{J}\}} = \frac{\mathcal{F}\{\mathrm{v}\}}{\mathcal{F}\{0\}}$
- expanding $\varphi'(x)$ as $\varphi'(x) = u_n(x)\xi^n$
- taking orthonormal set of functions $\mathrm{u}_n(x)$:

$$\Rightarrow \quad \mathcal{F}\{\mathrm{v}\} = \int [\mathrm{d}\xi] \, \mathrm{e}^{-\frac{1}{2} \mathrm{D}_{mn}\xi^m \xi^n} \text{ with } \mathrm{D}_{mn} = (u_m, \mathrm{D}\, u_n)$$

- It follows: $\frac{\mathcal{F}\{\mathrm{v}\}}{\mathcal{F}\{0\}} = \lim_{N\to\infty} \left(\frac{\det D_0}{\det D}\right)^{\frac{1}{2}}$

$$\Rightarrow \mathrm{W}[\mathrm{J}] = \mathrm{S}[\varphi_{cl}, \mathrm{J}] + \underbrace{\frac{\hbar}{2} \ln\left(\frac{\det \mathrm{D}}{\det \mathrm{D}_0}\right)}_{= \hbar \, \mathrm{W}_1[\varphi, \mathrm{J}]} + \ldots$$

Calculation of W[J] of φ^4- theory: 1

- from now on: calculation in dimensional regularization !
- beginning: $S[\varphi_0] = \int d^d x \left(\frac{1}{2}(\partial_\mu \varphi_0 \, \partial_\mu \varphi_0 + m_0^2 \varphi_0^2) + \frac{\lambda_0}{4!} \varphi_0^4 \right)$
- dimensions: $[\varphi_0] = \frac{d-2}{2}$, $[m_0] = 1$ and $[\lambda_0] = 4 - d$
- renormalization: $\varphi_0 = \sqrt{Z_\varphi}\,\varphi$, $m_0^2 = m^2 - \delta m^2$ and

  $\lambda_0 = Z_\lambda \, \lambda \, \mu^{4-d}$
- introducing dimensions: $[Z_\varphi] = 0$, $[Z_\lambda] = 0$ and $[\mu] = 1$;
- introducing μ so that λ is dimensionless !
- λ and m^2 finite with $d \to 4$!
- $S[\varphi_0, J_0] = S[\varphi_0]$ - $\int d^d x \, \varphi_0(x) J_0(x)$ with $J_0 = \frac{1}{\sqrt{Z_\varphi}} J$

Calculation of W[J] of φ^4- theory: 2

$$S_r[\varphi, J] = \int d^d x \underbrace{\left(\frac{1}{2}(\partial_\mu\varphi\,\partial_\mu\varphi + m^2\varphi^2) + \frac{\lambda\mu^{4-d}}{4!}\varphi^4 - J\varphi\right)}_{O(\hbar^0)}$$

$$+\hbar \int d^d x \underbrace{\left(\frac{A}{2}\,\partial_\mu\varphi\,\partial_\mu\varphi + \frac{B}{2}\varphi^2 + \frac{C\,\mu^{4-d}}{4!}\varphi^4\right)}_{counterterms}$$

- $Z_r[J] = e^{-\frac{1}{\hbar}W_r[J]} = \frac{1}{\mathcal{N}} \int [d\varphi]\, e^{-\frac{1}{\hbar}S_r[\varphi,J]}$, $\varphi = \varphi_{cl} + \sqrt{\hbar}\varphi'$

$$Z_r[J] = Z[J_0] = e^{-\frac{1}{\hbar}S_r[\varphi_{cl},J]}\, e^{-\int d^d x\left(\frac{A}{2}\partial_\mu\varphi_{cl}\,\partial_\mu\varphi_{cl} + \frac{B}{2}\varphi_{cl}^2 + \frac{C\mu^{4-d}}{4!}\varphi_{cl}^4\right)}$$

$$\underbrace{\frac{1}{\mathcal{N}}\int [d\varphi']\, e^{-\frac{1}{2}\int d^d x\left(\partial_\mu\varphi'\,\partial_\mu\varphi' + m^2(\varphi')^2 + \frac{\lambda\mu^{4-d}}{4!}\varphi_{cl}^2(\varphi')^2\right)}}_{=\left(\frac{det\ D_0}{det\ D}\right)^{\frac{1}{2}}}$$

Calculation of W[J] of $\varphi^4 -$ theory: 3

- det A $= e^{Tr(\ln A)}$

- $W_r[J] =$

  $= S_r[\varphi_{cl}, J] + \hbar \int d^d x \left(\frac{A}{2} \partial_\mu \varphi_{cl} \, \partial_\mu \varphi_{cl} + \frac{B}{2} \varphi_{cl}^2 + \frac{C\mu^{4-d}}{4!} \varphi_{cl}^4 \right)$

  $+ \frac{\hbar}{2} \mathrm{Tr}(\ln D - \ln D_0) + O(\hbar^2)$

- $D = -\partial^2 + m^2 + \underbrace{\frac{\lambda \, \mu^{4-d}}{2} \varphi_{cl}^2(x)}_{= v(x)}$ and $D_0 = -\partial^2 + m^2$

Calculating Tr(lnD)

- consider D in QM in d dimensions (with $[X_\mu, P_\nu] = i\delta_{\mu\nu}$):

$$D = -\partial^2 + m^2 + v(x) \rightarrow P^2 + m^2 + v(X)$$

- only interested in V_{eff} $\rightarrow$ neglect derivate terms in effective action
- setting $\varphi_{cl}(x) = \text{const}$, $\int d^d x \, |x\rangle\langle x| = \mathbb{1} = \int d^d p \, |p\rangle\langle p|$:
- $\mathrm{Tr}(\ln D) = \int d^d x \, \langle x| \ln D |x\rangle = \int d^d x \int d^d p \, \langle x| \ln D \, |p\rangle\langle p|x\rangle$
- In that way: D is diagonalized
- $\mathrm{Tr}(\ \ln D\ ) = \int d^d x \int \frac{d^d p}{(2\pi)^d} \ln(p^2 + m^2 + \underbrace{\frac{\lambda\,\mu^{4-d}}{2} \varphi_{cl}^2}_{= v = const})$

$$\Rightarrow \ \mathrm{Tr}\,(\ln D - \ln D_0) = \int d^d x \int \frac{d^d p}{(2\pi)^d} \ln\left(\frac{p^2 + m^2 + v}{p^2 + m^2}\right)$$

Calculating Tr $(\ln D - \ln D_0)$

- $\int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2+m^2} = \int\limits_0^\infty dt\, e^{-t\,m^2} \underbrace{\int \frac{d^d p}{(2\pi)^d} e^{-t\,p^2}}_{=\left(\frac{\pi}{t}\right)^{\frac{d}{2}} \frac{1}{(2\pi)^d}} = \ldots$

  $\ldots = \frac{m^{d-2}}{(4\pi)^{\frac{d}{2}}} \Gamma(1 - \frac{d}{2})$

- poles at $1 - \frac{d}{2} = 0, -1, -2, \ldots \Rightarrow$ thus a pole at $d = 4$!

- $\frac{\partial}{\partial x} \left(\int \frac{d^d p}{(2\pi)^d} \ln\left(\frac{p^2+x}{p^2+y}\right)\right) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2+x}$

- $\int \frac{d^d p}{(2\pi)^d} \ln\left(\frac{p^2+x}{p^2+y}\right) = \int\limits_y^x du \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2+u} =$

  $\frac{2}{d} \frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}} (x^{\frac{d}{2}} - y^{\frac{d}{2}})$

- $\Gamma(1 - \frac{d}{2}) \overset{d=4-2\varepsilon}{=} \Gamma(-1+\varepsilon) = -(\frac{1}{\varepsilon} + \Gamma'(1) + 1 + O(\varepsilon))$

Calculating Tr (ln D − ln D_0)

- thus with $x = m^2 + v$, $y = m^2$ and $d = 4 - 2\varepsilon$:

$$\int \frac{d^d p}{(2\pi)^d} \ln\left(\frac{p^2 + x}{p^2 + y}\right) =$$

$$= -\frac{1}{2}\frac{1}{1-\frac{\varepsilon}{2}}\frac{(4\pi)^\varepsilon}{(4\pi)^2}\Gamma(\varepsilon - 1)\left((m^2 + \frac{\lambda}{2}\mu^{2\varepsilon}\varphi_{cl}^2)^{2-\varepsilon} - m^{4-2\varepsilon}\right)$$

- multiplying with $\mu^{-2\varepsilon}\mu^{2\varepsilon}$ and using $c^\varepsilon = e^{\varepsilon \ln c} = 1 + \varepsilon \ln c$:

$$= -\frac{1}{32\pi^2}\mu^{-2\varepsilon}\left(\frac{1}{\varepsilon} + \frac{3}{2} + \ln 4\pi + \Gamma'(1) + O(\varepsilon)\right)$$

$$\Big[(m^2 + \frac{\lambda}{2}\mu^{2\varepsilon}\varphi_{cl}^2)^2(1 - \varepsilon \ln\left(\frac{m^2 + \frac{\lambda}{2}\mu^{2\varepsilon}\varphi_{cl}^2}{\mu^2}\right))$$

$$-m^4(1 - \varepsilon \ln \frac{m^2}{\mu^2})\Big]$$

Calculating Tr $(\ln D - \ln D_0)$

- Further calculation gives:

$$\int \frac{d^d p}{(2\pi)^d} \ln\left(\frac{p^2 + m^2 + \frac{\lambda\mu^{4-d}}{2}\varphi_{cl}^2}{p^2 + m^2}\right) =$$

$$\stackrel{d=4-2\varepsilon}{=} -\frac{1}{32\pi^2}\left(\frac{1}{\varepsilon} + \frac{3}{2} + \ln 4\pi + \Gamma'(1)\right)\mu^{-2\varepsilon}$$

$$\Big[\lambda m^2 \mu^{2\varepsilon}\varphi_{cl}^2 + \frac{\lambda^2}{4}\mu^{4\varepsilon}\varphi_{cl}^4 -$$

$$-\varepsilon\left(\lambda m^2 \varphi_{cl}^2 + \frac{\lambda^2}{4}\varphi_{cl}^4\right)\ln\left(\frac{m^2 + \frac{\lambda}{2}\mu^{2\varepsilon}\varphi_{cl}^2}{\mu^2}\right) -$$

$$-\varepsilon\, m^4 \ln\left(\frac{m^2 + \frac{\lambda}{2}\mu^{2\varepsilon}\varphi_{cl}^2}{m^2}\right)\Big]$$

Effective action $\Gamma[\varphi]$

- Writing whole expression in a function of $\overline{\varphi} \equiv \varphi = \varphi_{cl} + O(\hbar)$ (Legendre transformation):

$$\Gamma[\varphi] = \int d^{4-2\varepsilon}x \{ \tfrac{1}{2}(\mathrm{Z}(\varphi)\,\partial_\mu\varphi\,\partial_\mu\varphi + \mathrm{m}^2\varphi^2) + \tfrac{\lambda\mu^{4-d}}{4!}\,\varphi^4 \}$$

$$+ \int d^{4-2\varepsilon}x\,\mu^{-2\varepsilon}\Big\{ - \left(\tfrac{1}{\varepsilon} + \tfrac{3}{2} + \ln 4\pi + \Gamma'(1) \right) [\lambda \mathrm{m}^2\mu^{2\varepsilon}\,\varphi^2$$

$$+ \tfrac{\lambda^2}{4}\mu^{4\varepsilon}\varphi^4] + (\mathrm{m}^2 + \tfrac{\lambda}{2}\,\mu^{2\varepsilon}\varphi^2)^2 \ln\left(\frac{\mathrm{m}^2 + \frac{\lambda}{2}\,\mu^{2\varepsilon}\varphi^2}{\mu^2} \right)$$

$$- \mathrm{m}^4 \ln\left(\tfrac{\mathrm{m}^2}{\mu^2}\right) \Big\} \frac{1}{4\,(4\pi)^2}$$

$$+ \int d^{4-2\varepsilon}x \left(\tfrac{A}{2}\,\partial_\mu\varphi\,\partial_\mu\varphi + \tfrac{B}{2}\varphi^2 + \tfrac{C\,\mu^{4-d}}{4!}\,\varphi^4 \right) + \ldots$$

$$\text{with } \mathrm{Z}(\varphi) = 1 + \frac{\lambda}{(4\pi)^2}\,\frac{1}{6}\,\frac{\frac{\lambda}{2}\,\varphi^2}{\mathrm{m}^2 + \frac{\lambda}{2}\,\varphi^2} + \ldots$$

$\Gamma[\varphi]$ in $\overline{\mathrm{MS}}$: Renormalization

- Renormalization, adjusting the divergences in A, B and C:

$$A = 0$$

$$\frac{B}{2} = \left(\frac{1}{\varepsilon} + \ln 4\pi + \Gamma'(1)\right) \mathrm{m}^2 \lambda \frac{1}{4\,(4\pi)^2}$$

$$\frac{C}{4!} = \left(\frac{1}{\varepsilon} + \ln 4\pi + \Gamma'(1)\right) \frac{\lambda^2}{4} \frac{1}{4\,(4\pi)^2}$$

- $\mathrm{d} \to 4$:

$$\Gamma[\varphi] = \int \mathrm{d}^4x \,\{\tfrac{1}{2}(\mathrm{Z}(\varphi)\,\partial_\mu\varphi\,\partial_\mu\varphi + \mathrm{m}^2\varphi^2) + \tfrac{\lambda}{4!}\,\varphi^4\}$$

$$+ \int \mathrm{d}^4x \left\{ - \tfrac{3}{2}\,[\lambda\,\mathrm{m}^2\,\varphi^2 + \tfrac{\lambda^2}{4}\,\varphi^4]\right.$$

$$\left. + (\mathrm{m}^2 + \tfrac{\lambda}{2}\,\varphi^2)^2 \ln\left(\frac{\mathrm{m}^2 + \frac{\lambda}{2}\,\varphi^2}{\mu^2}\right) - \mathrm{m}^4 \ln\left(\frac{\mathrm{m}^2}{\mu^2}\right) \right\} \frac{1}{4\,(4\pi)^2}$$

$\Gamma[\varphi]$ in $\overline{\text{MS}}$

- interested in the physical mass !

$$\Gamma[\varphi] = \int d^4x \, \{ \tfrac{1}{2}(Z(\varphi)\,\partial_\mu\varphi\,\partial_\mu\varphi + m^2\varphi^2) + \tfrac{\lambda}{4!}\,\varphi^4 \}$$

$$+ \int d^4x \left\{ -\, \tfrac{3}{2}\,[\lambda\, m^2\,\varphi^2 + \tfrac{\lambda^2}{4}\,\varphi^4] \right.$$

$$\left. + (m^2 + \tfrac{\lambda}{2}\,\varphi^2)^2 \ln\left(\frac{m^2 + \frac{\lambda}{2}\,\varphi^2}{\mu^2}\right) - m^4 \ln\left(\frac{m^2}{\mu^2}\right) \right\} \frac{1}{4\,(4\pi)^2}$$

- expanding logarithm in a series:

$$\ln\left(\frac{m^2 + \frac{\lambda}{2}\,\varphi^2}{\mu^2}\right) = \ln\left(\frac{m^2}{\mu^2}\right) + \frac{\lambda}{2m^2}\,\varphi^2 - \frac{\lambda^2}{8m^4}\,\varphi^4 \pm \ldots$$

- Finally: $\Gamma[\varphi] = \int d^4x \, \{ \frac{1}{2}(\partial_\mu\varphi\partial\mu\varphi + m_{ph}^2\varphi^2) + \frac{\lambda'}{4!}\varphi^4 \} + \ldots$
  with $m_{ph}^2 = m^2 + \frac{\lambda m^2}{2\,(4\pi)^2}\left(\ln\left(\frac{m^2}{\mu^2}\right) - 1\right)$
  and $\lambda' = \lambda + \frac{3\,\lambda^2}{2\,(4\pi)^2}\,\ln\left(\frac{m^2}{\mu^2}\right)$

Renormalization-Group Equations

- differentiate equations of m_{ph}^2 and λ' with respect to μ:

- $\Rightarrow \quad \mu \frac{\partial m^2}{\partial \mu} = \frac{\lambda m^2}{(4\pi)^2}$ ... γ_m−function
- $\Rightarrow \quad \mu \frac{\partial \lambda}{\partial \mu} = \frac{3\lambda^2}{(4\pi)^2}$ ... β−function

- Relations between $\lambda(\mu) \leftrightarrow \lambda(\mu_0)$ and $m^2(\mu) \leftrightarrow m^2(\mu_0)$:

- $m^2(\mu) = m^2(\mu_0) \left(1 + \frac{\lambda(\mu_0)}{(4\pi)^2} \ln\left(\frac{m^2}{\mu^2}\right) + ...\right)$
- $\lambda(\mu) = \lambda(\mu_0) + \frac{3\lambda(\mu_0)^2}{(4\pi)^2} \ln\left(\frac{m^2}{\mu^2}\right) + ...$

Heat Kernel

- $\mathrm{Tr}(\ln D - \ln D_0) = -\int_0^\infty \frac{dt}{t}\, \mathrm{Tr}(e^{-tD} - e^{-tD_0})$ ... Schwinger parametrization
- $\mathrm{Tr}(\ln D - \ln D_0)$ singular for d $\to$ 4, corresponds to small values of t
- $\Rightarrow$ Heat-Kernel expansion: procedure for analyzing the one-loop divergences completely
- Heat Kernel: $K(x,y,t) = \langle x | e^{-tD} | y \rangle = \sum_{n=1}^{\infty} u_n(x) u_n(y)\, e^{-tE_n}$
- K fulfills $\partial_t K(x,y,t) + D K(x,y,t) = 0$ with $K(x,y,0) = \delta^d(x - y)$
- for $D \to -\Delta$ above equation: Heat-Equation
- corresponding kernel:
  $k(x,y,t) = \int \frac{d^d p}{(2\pi)^d}\, e^{-tp^2}\, e^{ip(x-y)} = (4\pi t)^{-\frac{d}{2}}\, e^{-\frac{(x-y)^2}{4t}}$

Heat Kernel

- kernel for $D_0 = -\Delta + m^2$:
  $K_0(x,y,t) = k(x,y,t)e^{-tm^2} = (4\pi t)^{-\frac{d}{2}} e^{-\frac{(x-y)^2}{4t}} e^{-tm^2}$
- for D: kernel smooth function for $t>0$; for $t\to 0$ peak at $x=y$
- ansatz: $K(x,y,t) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{(x-y)^2}{4t}} H(x,y,t)$
  with $H(x,y,t) = H_0(x,y) + t\, H_1(x,y) + t^2\, H_2(x,y) + \ldots$
- H fulfills:
  $t\,\partial_t H(x,y,t) + (x-y)^\mu \partial_\mu H(x,y,t) + t\, DH(x,y,t) = 0$
- $(x-y)^\mu\, \partial_\mu H_0(x,y,t) = 0$ and
  $((x-y)^\mu \partial_\mu + n)\, H_n(x,y) + DH_{n-1}(x,y) = 0$ for $n = 1,2,\ldots$
- $\Rightarrow$ $H_0(x,x) = H_0(x,y) = 1$
  $H_1(x,x) = -(m^2 + v(x))$
  $H_2(x,x) = -\frac{1}{6}\,\Delta\, v(x) + \frac{1}{2}\,(m^2 + v(x))^2$

Seeley-Coefficients

- $\mathrm{Tr}(\mathrm{e}^{-t\mathrm{D}}) = \int \mathrm{d}^d x\, \mathrm{K}(x,x,t)$
- $$\begin{aligned}\mathrm{Tr}(\mathrm{e}^{-t\mathrm{D}} - \mathrm{e}^{-t\mathrm{D}_0}) &= \int \mathrm{d}^d x\, \{\mathrm{H}(x,x,t) - \mathrm{H}_0(x,x,t)\} \\ &= t^{-\frac{d}{2}}\, \{h_0 + t h_1 + t^2 h_2 + \ldots\}\end{aligned}$$
- $\Rightarrow$ Seeley-Coefficients:

$$h_n = (4\pi)^{-\frac{d}{2}} \int \mathrm{d}^d x\, \{\mathrm{H}_n(x,x,t) - \mathrm{H}_{0,n}(x,x,t)\}$$

- $$\Rightarrow \quad \begin{aligned} h_0(\mathrm{x,x}) &= 0 \\ h_1(\mathrm{x,x}) &= -(4\pi)^{-\frac{d}{2}} \int \mathrm{d}^d x\, v(x) \\ h_2(\mathrm{x,x}) &= (4\pi)^{-\frac{d}{2}} \int \mathrm{d}^d x\, \{\mathrm{m}^2\, v(x) + \tfrac{1}{2}\, v^2(x)\}\end{aligned}$$
- $\mathrm{Tr}(\ln\mathrm{D} - \ln\mathrm{D}_0) = \frac{2h_1}{d-2} + \frac{2h_2}{d-4} + \ldots$ poles at d = 2,4, ...
- d $\to$ 4:
  $\mathrm{Tr}(\ln\mathrm{D} - \ln\mathrm{D}_0) = \frac{1}{16\,\pi^2\,(d-4)} \int \mathrm{d}^d x\, \{2\mathrm{m}^2\, v(x) + v^2(x)\} + \ldots$

Generalization: Heat-Kernel expansion

Source: R. D. Ball: „Chiral Gauge Theory“, Phys. Rep. 182(1989)1

- quadratic term of euclidean action:

$$\mathrm{S}^{(2)} = \frac{1}{2}\int \mathrm{d}^4x\ \Phi^T \underbrace{\left(-\nabla\nabla + Y\right)}_{:=\mathrm{D}} \Phi \qquad \text{with } \Phi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix}$$

and $\nabla_\rho = \partial_\rho + X_\rho$ ... $X_\rho^T = -X_\rho$ and $Y^T = Y$

- $\mathrm{Z} = \mathrm{e}^{-\mathrm{W}} = \mathrm{e}^{-\left(\overbrace{\mathrm{S}^{cl}}^{\mathrm{W}^{L=0}} + \overbrace{\frac{1}{2}\,\mathrm{Tr}\,\ln\left(\frac{\mathrm{D}}{\mathrm{D}_0}\right)}^{\mathrm{W}^{L=1}}\right)}$
- $\mathrm{Z} = \mathrm{e}^{-\mathrm{W}} = \mathrm{e}^{-\mathrm{S}^{cl}} \int [\mathrm{d}\Phi]\, \mathrm{e}^{-\mathrm{S}^{(2)}[\Phi]}$

Generalization: Heat-Kernel expansion

- $W^{L=1} = \frac{1}{2}\,\mathrm{Tr}\,\ln\left(\frac{D}{D_0}\right) =$
  $$= -\frac{1}{2}\lim_{m^2\to 0}\int_0^\infty \frac{d\tau}{\tau}\, e^{-\tau m^2}\,\mathrm{Tr}(e^{-\tau D} - e^{-\tau D_0})$$

- important observation: $S^{(2)}$ invariant under local gauge-transformation:
  $\Phi(x) \to R(x)\Phi(x)$ $\qquad R(x)^T R(x) = \mathbb{1}$
  $X_\mu \to R\,\partial_\mu R^{-1} + R\,X_\mu R^{-1}$ $\qquad Y \to R\,Y\,R^{-1}$

- terms needed for a gauge-invariant action $W^{L=1}$:
  $Y,\qquad \nabla_\mu Y = \partial_\mu Y + [X_\mu, Y]$
  $X_{\mu\nu} = \partial_\mu X_\nu - \partial_\nu X_\mu + [X_\mu, X_\nu]$
  $D = -\nabla\cdot\nabla + Y = -\partial^2 - 2X_\mu\,\partial_\mu + Y - X^2$

- calculation for constant fields X_μ and $Y = X^2 \Rightarrow$
  $D = -\partial^2 - 2X_\mu\partial_\mu$

- reconstruction of result: $X^2 \to Y$ and $[X_\mu, X_\nu] \to X_{\mu\nu}$

Generalization: Heat-Kernel expansion

- $\mathrm{Tr}(\mathrm{e}^{-\tau \mathrm{D}}) = \int \mathrm{d}^d x \;\; \mathrm{tr} \langle x | \mathrm{e}^{-\tau \mathrm{D}} | x \rangle$
- calculation: $\mathrm{tr} \langle x | \mathrm{e}^{-\tau \mathrm{D}} | x \rangle \; = \; ... \; =$
$\frac{1}{(4\pi)^{\frac{d}{2}}} \, \tau^{-\frac{d}{2}} \, \mathrm{tr}(\mathbb{1} \, - \, \tau Y \, + \, \frac{1}{2} \tau^2 Y^2 \, + \, \frac{1}{12} \tau^2 X_{\mu\nu} X_{\mu\nu} \, + \, ...)$
- $\mathrm{W}^{L=1} \; = \; -\frac{1}{2} \lim_{\mathrm{m}^2 \to 0} \int_0^\infty \frac{\mathrm{d}\tau}{\tau} \, \mathrm{e}^{-\tau \mathrm{m}^2} \, \frac{1}{(4\pi)^{\frac{d}{2}}} \, \tau^{-\frac{d}{2}}$
$\int \mathrm{d}^d x \, \mathrm{tr}(\mathbb{1} \, - \, \tau Y \, + \, \frac{1}{2} \tau^2 Y^2 \, + \, \frac{1}{12} \tau^2 X_{\mu\nu} X_{\mu\nu} \, + \, ... \, - \, \mathbb{1})$
- $\mathrm{W}^{L=1} \; = \; \frac{1}{2} \, \frac{1}{(4\pi)^{\frac{d}{2}}} \, \lim_{\mathrm{m}^2 \to 0} \int \mathrm{d}^d x \, \Big\{ \mathrm{m}^{d-2} \Gamma(1 - \frac{d}{2}) \, \mathrm{tr}(\mathrm{Y})$
$- \, \mathrm{m}^{d-4} \Gamma(2 - \frac{d}{2}) \, \mathrm{tr}(\frac{1}{2} \, \mathrm{Y}^2 \, + \, \frac{1}{12} X_{\mu\nu} X_{\mu\nu}) \, + \, ... \Big\}$
- for $\mathrm{d} \, = \, 4 \, -2\varepsilon$:

$$- \, \frac{1}{32\pi^2} \, (4\pi)^{\varepsilon} \, \lim_{\mathrm{m}^2 \to 0} \int \mathrm{d}^{4-2\varepsilon} x \, \Big\{ (\mathrm{m}^2)^{-\varepsilon} \, \Gamma(\varepsilon) \, \mathrm{tr}(\frac{1}{2} \, \mathrm{Y}^2 \, + \, \frac{1}{12} X_{\mu\nu} X_{\mu\nu}) \Big\}$$

Generalization: Heat-Kernel expansion

$$-\frac{1}{32\pi^2}(4\pi)^\varepsilon \lim_{m^2\to 0}\int d^{4-2\varepsilon}x\left\{(m^2)^{-\varepsilon}\underbrace{\Gamma(\varepsilon)}_{=\frac{1}{\varepsilon}+\ldots}\ \mathrm{tr}(\frac{1}{2}Y^2+\frac{1}{12}X_{\mu\nu}X_{\mu\nu})\right\}$$

- for φ^4−theory: $X = 0$ and $Y = m^2 + v(x)$:

$$W^{L=1} = -\frac{1}{64\pi^2}\int d^4x\left\{\frac{1}{\varepsilon}\left[(m^2+v(x))^2\right]\right\}$$

- $\frac{1}{2}\,\mathrm{Tr}\,(\ln D - \ln D_0) = -\frac{1}{64\pi^2}\int d^4x\left\{\frac{1}{\varepsilon}\left(2m^2v(x)+v^2(x)\right)\right\}$

  equal to

  $\frac{1}{2}\,\mathrm{Tr}(\ln D - \ln D_0) = \frac{1}{2}\,\frac{1}{16\,\pi^2\,(d-4)}\int d^dx\,\{2m^2\,v(x)+v^2(x)\}$

- with $\mathrm{Tr}(\ln D - \ln D_0) = \frac{2\,h_2}{d-4}$
  $\Rightarrow h_2(x,x) = \frac{1}{32\pi^2}\int d^4x\,\{2m^2\,v(x)+v^2(x)\}$

Summary

- Generating functional of connected loop-diagrams can be written in powers of $\hbar$: $$W[J] = \sum_{\ell=0}^{\infty} \hbar^{\ell}\, W_{\ell}[J]$$
- Generating functional of connected one-loop graphs has a logarithmic term: $W[J] = S[\varphi_{cl}, J] + \frac{\hbar}{2} \ln\left(\frac{\det D}{\det D_0}\right) + \ldots$
- Renormalization-Group equations:
  $$\mu\,\frac{\partial\, m^2}{\partial\,\mu} = \frac{\lambda\, m^2}{(4\pi)^2} + \ldots$$
  $$\mu\,\frac{\partial\,\lambda}{\partial\,\mu} = \frac{3\lambda^2}{(4\pi)^2} + \ldots$$
- For analyzing the divergence-structure of φ^4-theory completely $\Rightarrow$ Heat-Kernel expansion
- pole for $d \to 4$: $h_2(x,x) = \frac{1}{32\pi^2} \int d^4x \,\{2m^2\, v(x) + v^2(x)\}$

Outlook and Applications

- effective potential V_{eff} useful for analyzing the vacuum-stability of the SM (here ϕ is Higgs field):

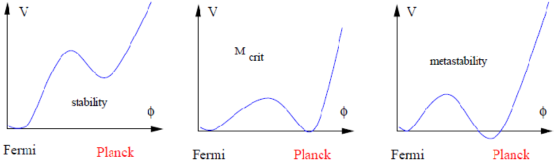

Figure: effective potential for the Higgs field ϕ [Mikhail Shaposhnikov: „Cosmology: theory“, arXiv:1311.4979v1, [hep-ph] 20 Nov 2013]

- Applications for Heat-Kernel expansion:

1) quantum gravity

2) one−loops divergences in effective field theories (e.g. chiral perturbation theory)