The Coleman-Weinberg Potential

SE Current topics in theoretical particle physics

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Outline

- Discussion of V_{eff} of massless ϕ^4 - theory
- Massless scalar electrodynamics
- Computation of CW potential V_{eff} up to one-loop
- SSB generated by one-loop corrections
- Summary
- Outlook

Classical scale invariance

- Example: massless ϕ^4-theory

$$S = \int d^4x \left(\partial_\mu \phi \partial_\mu \phi + \frac{\lambda}{4!} \phi^4 \right)$$

- Scale transformation

$$x \to \rho x \,, \quad d^4x \to \rho^4 d^4x$$
$$\phi \to \frac{1}{\rho} \phi \,, \quad \partial_\mu \to \frac{1}{\rho} \partial_\mu$$

- Mass term $\frac{m^2}{2} \phi^2$ breaks scale invariance

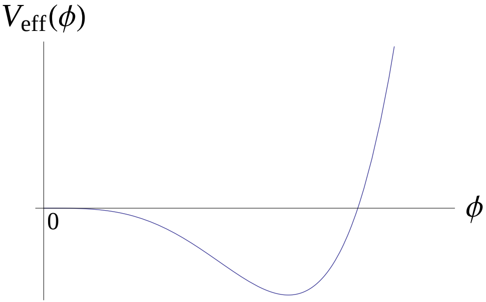
V_{eff} of the massless ϕ^4-theory

- V_{eff} in the $\overline{MS}$ scheme

$$V_{eff} = \frac{\lambda}{4!}\phi^4 + \frac{\lambda^2\phi^4}{256\pi^2}\left(\ln\frac{\frac{\lambda}{2}\phi^2}{\mu^2} - \frac{3}{2}\right)$$

- One-loop correction breaks scale invariance:

$$\int d^4x\,\phi^4 \ln\phi^2 \to \int d^4x\,\phi^4\left(\ln\phi^2 - \ln\rho^2\right)$$



- The value of ϕ at which the „new" minimum occurs is determined by

$$\lambda \ln \frac{\langle\phi\rangle^2}{\mu^2} = -\frac{32}{3}\pi^2 + \mathcal{O}(\lambda)$$

- Perturbation theory is only valid for small $\lambda \rightarrow$ new minimum lies outside the range of validity of the one-loop approximation

Massless Scalar Electrodynamics

- Calculations in dimensional regularization $\rightarrow$ gauge invariance is preserved
- Lagrangian $\mathcal{L}$ in Euclidean-space

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{\xi}{2} (\partial_\mu A_\mu)^2 + (D_\mu \phi)^* D_\mu \phi + \\ + V(\phi) - J_\mu A_\mu - f^* \phi - f \phi^*$$

with

$$V(\phi) = \mu^{4-d} \frac{\lambda}{6} (\phi^* \phi)^2$$

- Introduction of μ, $[\mu] = 1$, to keep λ and q dimensionless
- $\frac{\xi}{2}(\partial_\mu A_\mu)^2$.... gauge fixing term, R_ξ gauge

Introduction of real scalar fields φ_1,φ_2

- $\phi = \frac{\varphi_1 + i\varphi_2}{\sqrt{2}}$

$$\phi^*\phi = \frac{\varphi_1^2 + \varphi_2^2}{2} = \frac{1}{2}\varphi^T\varphi, \quad \text{with } \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

$$V(\varphi) = \mu^{4-d}\frac{\lambda}{4!}(\varphi^T\varphi)^2$$

$$(D_\mu\phi)^* D_\mu\phi = \frac{1}{2}(D_\mu\varphi)^T D_\mu\varphi$$

- with $D_\mu\varphi = (\partial_\mu - \mu^{\frac{4-d}{2}} q A_\mu \varepsilon)\varphi$ and $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- $f = \frac{f_1 + i f_2}{\sqrt{2}}$

$$f^*\phi + f\phi^* = f^T\varphi, \quad \text{with } f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

- Manipulation of terms to get an expression for the action $S = \int d^d x \, \mathcal{L}$

$$\int d^d x \left(\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{\xi}{2} (\partial_\mu A_\mu)^2 \right) = \\ = - \int d^d x \, \frac{1}{2} A_\mu \left(\delta_{\mu\nu} \partial^2 + (1 - \xi) \partial_\mu \partial_\nu \right) A_\nu$$

$$\int d^d x \, \frac{1}{2} \left(D_\mu \varphi \right)^T \left(D_\mu \varphi \right) = - \int d^d x \, \frac{1}{2} \varphi^T D_\mu D_\mu \varphi$$

$$\Rightarrow S = \int d^d x \left\{ - \frac{1}{2} A_\mu \left(\delta_{\mu\nu} \partial^2 + (1 - \xi) \partial_\mu \partial_\nu \right) A_\nu \right. \\ \left. - \frac{1}{2} \varphi^T D_\mu D_\mu \varphi + \mu^{4-d} \frac{\lambda}{4!} \left(\varphi^T \varphi \right)^2 - J_\mu A_\mu - f^T \varphi \right\}$$

Saddle point method

- Expansion of the action S around φ^{cl} and A^{cl},
$$\left.\frac{\delta S[\varphi,A_\mu]}{\delta\varphi_a}\right|_{\varphi=\varphi^{cl},A=A^{cl}}=0\ ,\ \left.\frac{\delta S[\varphi,A_\mu]}{\delta A_\mu}\right|_{\varphi=\varphi^{cl},A=A^{cl}}=0$$

- $A_\mu = A^{cl}_\mu + B_\mu\ ,\ \varphi = \varphi^{cl} + \alpha$

$$S\left[\varphi^{cl}+\alpha, A^{cl}_\mu + B_\mu\right] \overset{\text{w/o cl}}{=} S\left[\varphi, A_\mu\right] +$$

$$+ \text{terms linear in } \alpha \text{ and } B_\mu +$$

$$+\int d^d x\left\{-\frac{1}{2}B_\mu\left(\delta_{\mu\nu}\partial^2-(1-\xi)\,\partial_\mu\partial_\nu\right)B_\nu - \right.$$
$$-\frac{1}{2}\alpha^T D^{cl}_\mu D^{cl}_\mu \alpha + \mu^{\frac{4-d}{2}}\frac{q}{2}\alpha^T B_\mu \varepsilon D^{cl}_\mu \varphi +$$
$$+\mu^{\frac{4-d}{2}}\frac{q}{2}\alpha^T D^{cl}_\mu\left(B_\mu\varepsilon\varphi\right) + \mu^{\frac{4-d}{2}}\frac{q}{2}\varphi^T D^{cl}_\mu\left(B_\mu\varepsilon\alpha\right)$$
$$+\mu^{\frac{4-d}{2}}\frac{q}{2}\varphi^T B_\mu\varepsilon D^{cl}_\mu\alpha - \mu^{4-d}\frac{q^2}{2}\varphi^T B_\mu\varepsilon B_\mu\varepsilon\varphi$$
$$\left.+\mu^{4-d}\frac{\lambda}{6}\left(\alpha^T\varphi\right)^2 + \mu^{4-d}\frac{\lambda}{12}\varphi^T\varphi\alpha^T\alpha\right\}$$

$$+ \text{terms cubic and quartic in } \alpha \text{ and } B_\mu$$

The differential operator D

- Terms quadratic in α and B_μ represent the one-loop correction to $W[f]$

$$Z[f] = \frac{1}{\mathcal{N}} \int [dB_\mu d\alpha]\, \mathrm{e}^{-\frac{S}{\hbar}} = \mathrm{e}^{-\frac{W}{\hbar}} = \mathrm{e}^{-\frac{W^{L=0}}{\hbar}} \mathrm{e}^{-W^{L=1}} \ldots$$

$$= \mathrm{e}^{-\frac{S^{cl}}{\hbar}} \frac{1}{\mathcal{N}} \mathrm{e}^{-\frac{1}{2}\int d^d x \Phi^T D \Phi} \ldots, \quad \text{with } \Phi = \begin{pmatrix} \alpha \\ B_\mu \end{pmatrix}$$

- with $A_\mu = 0$, D is determined by

$$D = \begin{pmatrix} -\partial^2 + \mu^{4-d} \frac{\lambda}{6} \varphi^T \varphi + \mu^{4-d} \frac{\lambda}{3} \varphi \varphi^T & \mu^{\frac{4-d}{2}} q \varepsilon \varphi \partial_\nu \\ \mu^{\frac{4-d}{2}} q \varphi^T \varepsilon \partial_\mu & \delta_{\mu\nu}(-\partial^2 + \mu^{4-d} q^2 \varphi^T \varphi) + (1-\xi) \partial_\mu \partial_\nu \end{pmatrix}$$

D in momentum-space, with $\varphi(x) = const$

$$\partial_\mu \to ik_\mu\,, \quad -\partial^2 \to k^2$$

$$\tilde{D} = \begin{pmatrix} k^2 + \mu^{4-d}\frac{\lambda}{6}\varphi^T\varphi + \mu^{4-d}\frac{\lambda}{3}\varphi\varphi^T & i\mu^{\frac{4-d}{2}} q\varepsilon\varphi k_\nu \\ & \\ i\mu^{\frac{4-d}{2}} q\varphi^T\varepsilon k_\mu & \delta_{\mu\nu}(k^2 + \mu^{4-d}q^2\varphi^T\varphi) - (1-\xi)k_\mu k_\nu \end{pmatrix}$$

$$\tilde{D} = \begin{pmatrix} \left(k^2 + \mu^{4-d}\frac{\lambda}{6}\varphi^T\varphi\right)\mathbb{1}_2 + 2\mu^{4-d}\frac{\lambda}{3}\varphi\varphi^T & i\mu^{\frac{4-d}{2}} q\varepsilon\varphi k^T \\ & \\ i\mu^{\frac{4-d}{2}} qk\varphi^T\varepsilon & \delta_{\mu\nu}(k^2 + \mu^{4-d}q^2\varphi^T\varphi)\mathbb{1}_d - (1-\xi)kk^T \end{pmatrix}$$

Eigenvalues of $\tilde{D}$

$$EW_1 = k^2 + \mu^{4-d}\frac{\lambda}{2}\varphi^T\varphi$$

$$EW_2 = k^2 + \mu^{4-d}q^2\varphi^T\varphi\,, \quad (d-1)\text{-fold degenerate}$$

$$EW_3 \cdot EW_4 = \xi\left[k^2 + \frac{1}{2}\mu^{4-d}\frac{\lambda}{6}\varphi^T\varphi\left(1 - \sqrt{1 - \frac{4!q^2\varphi^T\varphi}{\xi\lambda\varphi^T\varphi}}\right)\right] \times$$

$$\times\left[k^2 + \frac{1}{2}\mu^{4-d}\frac{\lambda}{6}\varphi^T\varphi\left(1 + \sqrt{1 - \frac{4!q^2\varphi^T\varphi}{\xi\lambda\varphi^T\varphi}}\right)\right]$$

- one-loop correction written as det of differential operators

$$
\begin{aligned}
\mathrm{e}^{-\frac{W}{\hbar}} =& \mathrm{e}^{-\frac{W^{L=0}}{\hbar}} \mathrm{e}^{-W^{L=1}} \dots \\
=& \mathrm{e}^{-\frac{S^{cl}}{\hbar}} \frac{1}{\mathcal{N}} \mathrm{e}^{-\frac{1}{2}\int d^d x \Phi^T D \Phi} \dots \\
=& \mathrm{e}^{-\frac{S^{cl}}{\hbar}} \left(\frac{\det D_0}{\det D} \right)^{\frac{1}{2}} \dots
\end{aligned}
$$

- with D_0 characterized by $f = 0 \Leftrightarrow \varphi = 0$

$$
\begin{aligned}
W =& S^{cl} + \frac{1}{2}\hbar \ln \frac{\det D}{\det D_0} + \dots \\
=& \underbrace{S^{cl}}_{=W^{L=0}} + \underbrace{\hbar \frac{1}{2} \mathrm{Tr} \left(\ln \frac{D}{D_0} \right)}_{=W^{L=1}} + \dots
\end{aligned}
$$

Calculation of $W^{L=1}$

$$
\begin{aligned}
W^{L=1} =& \frac{1}{2}\int d^d x \int \frac{d^d k}{(2\pi)^d}\left[\ln \frac{k^2 + \mu^{4-d}\frac{\lambda}{2}\varphi^T\varphi}{k^2} + \right. \\
&+ (d-1)\ln\frac{k^2+\mu^{4-d}q^2\varphi^T\varphi}{k^2} + \\
&+ \ln\frac{k^2 + \frac{1}{2}\mu^{4-d}\frac{\lambda}{6}\varphi^T\varphi\left(1-\sqrt{1-\frac{4!q^2}{\xi\lambda}}\right)}{k^2} + \\
&\left. + \ln\frac{k^2 + \frac{1}{2}\mu^{4-d}\frac{\lambda}{6}\varphi^T\varphi\left(1+\sqrt{1-\frac{4!q^2}{\xi\lambda}}\right)}{k^2}\right]
\end{aligned}
$$

- Landau gauge: $\xi \to \infty$

Calculation of $W^{L=1}$ in Landau gauge

$$W^{L=1} = \frac{1}{2}\int d^d x \int \frac{d^d k}{(2\pi)^d}\left[\ln\frac{k^2+\mu^{4-d}\frac{\lambda}{2}\varphi^T\varphi}{k^2} + \right.$$
$$+ (d-1)\ln\frac{k^2+\mu^{4-d}q^2\varphi^T\varphi}{k^2} +$$
$$\left. + \ln\frac{k^2+\mu^{4-d}\frac{\lambda}{6}\varphi^T\varphi}{k^2}\right]$$

$$f(x,y) = \int \frac{d^d k}{(2\pi)^d}\ln\frac{k^2+x}{k^2+y}$$
$$= \int_y^x du \int \frac{d^d k}{(2\pi)^d}\frac{1}{k^2+u} = \frac{2}{d}\frac{1}{(4\pi)^{\frac{d}{2}}}\Gamma\left(1-\frac{d}{2}\right) u^{\frac{d}{2}}\Big|_y^x$$

Calculation of $W^{L=1}$ in Landau gauge

$$W^{L=1} = \frac{1}{2}\int d^d x\, \frac{2}{d}\frac{1}{(4\pi)^{\frac{d}{2}}}\Gamma\left(1-\frac{d}{2}\right)\left(\left(\mu^{4-d}\frac{\lambda}{2}\varphi^T\varphi\right)^{\frac{d}{2}} + \right.$$

$$\left. + (d-1)\left(\mu^{4-d}q^2\varphi^T\varphi\right)^{\frac{d}{2}} + \left(\mu^{4-d}\frac{\lambda}{6}\varphi^T\varphi\right)^{\frac{d}{2}}\right)$$

Calculation of $W^{L=1}$ in Landau gauge

- $d \to 4 - 2\varepsilon$

$$
\begin{aligned}
W^{L=1} =& \frac{1}{4(4\pi)^2} \int d^{4-2\varepsilon} x \mu^{-2\varepsilon} \Bigg[-10 \left(\frac{1}{\varepsilon} + \Gamma'(1) + \frac{3}{2} + \ln(4\pi) \right) \left(\mu^{2\varepsilon} \frac{\lambda}{6} \varphi^T \varphi \right)^2 \\
& - 3 \left(\frac{1}{\varepsilon} + \Gamma'(1) + \frac{5}{6} + \ln(4\pi) \right) \left(\mu^{2\varepsilon} q^2 \varphi^T \varphi \right)^2 + \\
& + \left(\mu^{2\varepsilon} \frac{\lambda}{2} \varphi^T \varphi \right)^2 \ln \frac{\frac{\lambda}{2} \varphi^T \varphi}{\mu^2} + \left(\mu^{2\varepsilon} \frac{\lambda}{6} \varphi^T \varphi \right)^2 \ln \frac{\frac{\lambda}{6} \varphi^T \varphi}{\mu^2} + \\
& + \left(\mu^{2\varepsilon} q^2 \varphi^T \varphi \right)^2 \ln \frac{q^2 \varphi^T \varphi}{\mu^2} \Bigg]
\end{aligned}
$$

- Divergent terms $\sim \varphi^4 \to$ because of renormalizability of the theory

Effective action $\Gamma[\varphi]$, effective potential $V_{eff}(\varphi)$

$$\Gamma\left[\varphi^{cl}\right] = S\left[\varphi^{cl}\right] + \hbar W^{L=1} + \text{counter terms} + \mathcal{O}(\hbar^2)$$

- Terms $\sim \left(\frac{1}{\varepsilon} + \Gamma'(1) + \ln(4\pi)\right)$ absorbed by counter terms $\rightarrow$ $\overline{MS}$ scheme
- $\varepsilon \rightarrow 0$
- CW Potential V_{eff}:

$$\begin{aligned} V_{eff}(\varphi) = & \frac{\lambda}{4!}(\varphi^T\varphi)^2 + \\ & + \frac{1}{4(4\pi)^2}\left[\left(\frac{\lambda}{2}\varphi^T\varphi\right)^2\left(\ln\frac{\frac{\lambda}{2}\varphi^T\varphi}{\mu^2} - \frac{3}{2}\right) + \right. \\ & + 3(q^2\varphi^T\varphi)^2\left(\ln\frac{q^2\varphi^T\varphi}{\mu^2} - \frac{5}{6}\right) + \\ & \left. + \left(\frac{\lambda}{6}\varphi^T\varphi\right)^2\left(\ln\frac{\frac{\lambda}{6}\varphi^T\varphi}{\mu^2} - \frac{3}{2}\right)\right] \end{aligned}$$

The CW Potential

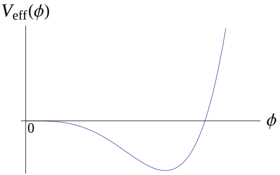
S. Coleman, E. Weinberg, Phys. Rev. Vol. 7, Nr. 6, 1888 (1973)

- Instead of $\overline{MS}$, renormalization condition employed by Coleman and Weinberg:

$$\left.\frac{d^4 V_{eff}}{d\varphi^4}\right|_{\varphi=M} = \lambda_{cw}$$

$$V_{eff}(\varphi) = \frac{\lambda_{cw}}{4!}(\varphi^T\varphi)^2 + \\ + \left(\frac{5\lambda_{cw}^2}{1152\pi^2} + \frac{3q^4}{64\pi^2}\right)\left(\varphi^T\varphi\right)^2\left(\ln\frac{\varphi^T\varphi}{M^2} - \frac{25}{6}\right)$$

- Like in the massless ϕ^4-theory: V_{eff} has a minimum away from the origin



SSB generated by one-loop corrections 1

$$V_{eff}(\varphi) = \frac{\lambda_{cw}}{4!}(\varphi^T\varphi)^2 + \\ + \left(\frac{5\lambda_{cw}^2}{1152\pi^2} + \frac{3q^4}{64\pi^2}\right)\left(\varphi^T\varphi\right)^2\left(\ln\frac{\varphi^T\varphi}{M^2} - \frac{25}{6}\right)$$

- Second coupling constant q : minimum obtained by balancing a term of $\mathcal{O}(\lambda)$ against a term of order $q^4 \ln \frac{\varphi^2}{M^2}$
- Choosing $M = \langle\varphi\rangle$, with $V'(\langle\varphi\rangle) = 0 \to \lambda$ is of order q^4

$$V_{eff} = \frac{\lambda_{cw}}{4!}(\varphi^T\varphi)^2 + \frac{3q^4}{64\pi^2}\left(\varphi^T\varphi\right)^2\left(\ln\frac{\varphi^T\varphi}{\langle\varphi\rangle^2} - \frac{25}{6}\right)$$

SSB generated by one-loop corrections 2

- From $V'(\langle\varphi\rangle) = 0$

$$\lambda_{cw} = \frac{33}{8\pi^2} q^4$$

- Redefinition of coupling leads to an expression of λ_{cw} in terms of q
- 2 dimensionless free parameters λ_{cw}, q $\rightarrow$ 2 free parameters q (dimensionless), $\langle\varphi\rangle$ (dimensional)
- $\rightarrow$ Dimensional transmutation

$$V_{eff} = \frac{3q^4}{64\pi^2}\left(\varphi^T\varphi\right)^2\left(\ln\frac{\varphi^T\varphi}{\langle\varphi\rangle^2} - \frac{1}{2}\right)$$

- Further analysis like for the Abelian Higgs model: scalar electrodynamics with a negative mass term
  - Determination of mass of the scalar meson $m(S)$

$$m^2(S) = V''(\langle\varphi\rangle) = \frac{3q^4}{8\pi^2}\langle\varphi\rangle^2$$

  - Photon mass $m(V)$ is given by

$$m^2(V) = q^2\langle\varphi\rangle^2$$

  - Scalar-to-vector mass ratio to lowest order

$$\frac{m^2(S)}{m^2(V)} = \frac{3}{2\pi}\frac{q^2}{4\pi}$$

Summary

- Massless scalar electrodynamics $\rightarrow$ Abelian Higgs model: massive real scalar meson, massive vector meson

$$V_{eff} = \frac{3q^4}{64\pi^2}\left(\varphi^T\varphi\right)^2\left(\ln\frac{\varphi^T\varphi}{\langle\varphi\rangle^2} - \frac{1}{2}\right)$$

- Scalar-to-vector mass ratio to lowest order

$$\frac{m^2(S)}{m^2(V)} = \frac{3}{2\pi}\frac{q^2}{4\pi}$$

- Remark: renormalization group analysis $\rightarrow$ restriction $\lambda_{cw} \sim q^4$ not necessary; $\lambda \sim q^2 \ll 1$ sufficient

Recent Applications

- One can see: all terms in the action of SM scale invariant, except Higgs mass term in the Higgs potential

$$rH^\dagger H$$

- Attempts of scale invariant extensions of SM:
  - Introduction of an additional field ϕ: singlet under $\mathrm{SU}(3)_{\mathrm{color}} \times \mathrm{SU}(2) \times \mathrm{U}(1)$

$$r \to \lambda \phi^* \phi$$

  - scale invariant term

$$\lambda \phi^* \phi H^\dagger H$$