Renormalization

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Contents

Origin of UV-Divergences

Perturbation Theory

- Quantum-Field-theoretic correlation functions and S-matrix elements calculated in Perturbation theory
- E.g.: with $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$, and $\mathcal{L}_{\text{int}} = O(\lambda)$:

$$\left\langle \tilde{0} | T\phi(x_1)...\phi(x_n) | \tilde{0} \right\rangle =$$

$$= \frac{\left\langle 0 | T\phi_I(x_1)...\phi_I(x_n) \, \exp\{i \int d^4x \, \mathcal{H}^I_{\text{int}}(x)\} | 0 \right\rangle}{\left\langle 0 | \int d^4x \exp\{i \int d^4x \, \mathcal{H}^I_{\text{int}}(x)\} | 0 \right\rangle} \sim$$

$$\sim \langle 0 | T\phi_I(x_1)...\phi_I(x_n) | 0 \rangle +$$

$$+ i \int d^4x \, \langle 0 | T\phi_I(x_1)...\phi_I(x_n) \, \mathcal{H}^I_{\text{int}}(x)\} | 0 \rangle + O(\lambda^2)$$

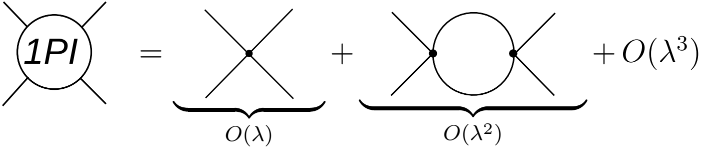
Locality and UV-Divergences

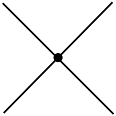
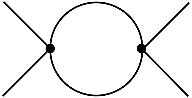
- QFT is based on concept of locality,
- $\Longrightarrow \mathcal{L}_0$ and $\mathcal{L}_{\text{int}}$ do only contain local operators , i.e. at same space-time point
- Examples:

$$\mathcal{L}_{\text{int}}^{\phi^4} = -\frac{\lambda}{4!}\,\phi^4(x), \qquad \mathcal{L}_{\text{int}}^{\text{QED}} = e\,\bar{\psi}(x)\not{A}(x)\psi(x)$$

- Locality leads to divergent integrals for higher order corrections to tree-level results (in configuration + Fourier space)

Example: 4-Point Vertex Function of ϕ^4-theory



- Tree-level:  $\sim -i\lambda$
- One-loop:  $\sim$

$$\sim \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{[k^2 - m_0^2 + i\varepsilon]} \frac{i}{[(p_1 + p_2 - k)^2 - m_0^2 + i\varepsilon]}$$

Locality and UV-Divergences

Observations:

1. High-energy processes are encoded in value of λ (not resolved, pointlike interaction)
   $\longrightarrow$ locality
2. Accuracy of λ is obviously different at various orders in perturbation theory
3. Loop-integral is UV-divergent, i.e. at large k-values
4. Divergent integrals will also appear in other loop-diagrams

Looks like an unsurmountable problem...

Locality and UV-Divergences

BUT: this is to be expected!

- We are trying to use a theory for all energies which is only valid up to energies where a certain amount of detail of the local interaction can be resolved
- Characteristic quantity: cutoff Λ
- Processes at $E > \Lambda$ effectively encoded in λ
- For $E > \Lambda$ new theory might take over (e.g. new particles)

Important conclusion (*running coupling*)

Coupling λ will be energy-dependend, $\lambda = \lambda(\Lambda)$, where Λ is an energy scale (*cutoff*), up to which the theory is valid. Similiar for the mass, $m = m(\Lambda)$, and the field(s), $\phi = \phi(\Lambda)$.

To-Do-List

A few points have to be tackled now:

- Categorize the UV-divergences in the various theories
- Find a convenient way to regularize the divergent integrals
- Extract the finite parts, get rid of the divergent parts
- Rewrite the infinite (*bare*) parameters of the theory in terms of new, finite (*renormalized*) ones
- Extract information about the running of the parameters

Classification of UV-Divergences

Superficial Degree of Divergence

As a first step define the naive degree of divergence of a diagram:

Superficial degree of divergence D:

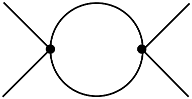
$$D = \text{powers of loop-momenta in numerator} \\ - \text{powers of loop-momenta in denominator}$$

when **all** loop-momenta of that diagram become large.

- $D < 0$: not superficially divergent
- $D = 0$: logarithmic divergence
- $D = 1$: linear divergence
- ...

Example

- Assume for the moment that loop-integral is regularized with a hard cutoff Λ:

 $\sim$

$$\sim \frac{(-i\lambda)^2}{2} \int^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{i}{[k^2 - m_0^2 + i\varepsilon]} \frac{i}{[(p_1+p_2-k)^2 - m_0^2 + i\varepsilon]}$$

$$\xrightarrow{k\to\infty} c \int^{\Lambda} dk \, \frac{k^3}{k^4} = c \int^{\Lambda} dk \, \frac{1}{k} = c \, \log(\Lambda)$$

- $D = 0$ as expected

Superficial Degree of Divergence

- Divergent part of a diagram $\approx \Lambda^D$
- For every diagram D can be obtained in a sytematic way by virtue of the Feynman rules (in every theory)
- Momentum dependence of propagators is fixed
- Ingredients to quantify D for a diagram:
    - Number of external lines
    - Number of vertices (loops)
    - Space-time dimension
    - Number of lines meeting at each vertex

Example: ϕ^n-theory

- Example: ϕ^n-theory in d space-time dimensions

$$D_{\phi^n} = d + \left[n \left(\frac{d-2}{2} \right) - d \right] V - \left(\frac{d-2}{2} \right) N$$

where

- N ... number of external lines
- n ... degree of self-interaction polynomial
- V ... number of vertices
- d ... space-time dimension

Example: ϕ^n-theory

$$D_{\phi^n} = d + \underbrace{\left[n \left(\frac{d-2}{2} \right) - d \right]}_{\eta} V - \left(\frac{d-2}{2} \right) N$$

Observation: the prefactor η of V plays a crucial role:

- $\eta > 0$: D grows with number of vertices
  $\longrightarrow$ **Non-renormalizable theory**
- $\eta = 0$: D independend of number of vertices
  $\longrightarrow$ **Renormalizable theory**
- $\eta < 0$: D decreases with number of vertices
  $\longrightarrow$ **Super-renormalizable theory**

Dimensional Analysis of ϕ^n-theory

- Some dimensional analysis (in d dimensions) reveals an interesting feature ($[\partial_\mu] = 1$):

$$[S] = \left[\int d^4x \mathcal{L}(x)\right] \stackrel{!}{=} 0 \quad \Rightarrow \quad [\mathcal{L}] \stackrel{!}{=} d$$

- Concretely: $\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{n!}\phi^n$:

$$[\phi] = \frac{d}{2} - 1, \qquad [m^2] = 2, \qquad [\lambda] = -n\frac{d-2}{2} + d$$

- Surprisingly: $\boxed{[\lambda] = -\eta}$

Operator Dimension and Renormalizability

- This is valid for the couplings of every theory
- Allows to classify the renormalizability of a theory by:
  a) Mass-dimension of the coupling: $[\lambda] = -\eta$
  b) Operator-dimension of $\mathcal{L}_{\text{int}}$, since

$$[\mathcal{L}_{\text{int}}] = [\lambda \hat{O}] \stackrel{!}{=} d \quad \Rightarrow \quad [\hat{O}] = d - [\lambda]$$

Renormalizability of a theory

$$[\hat{O}] = \begin{cases} > d, & \text{non-renormalizable} \\ = d, & \text{renormalizable} \\ < d, & \text{super-renormalizable} \end{cases}$$

Renormalizable Physical Theories in $d = 4$

Theory	$\mathcal{L}$
ϕ^4	$\frac{1}{2}(\partial^\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$
Yukawa	$\frac{1}{2}(\partial^\mu \phi)^2 - \frac{m^2}{2}\phi^2 + i\bar{\psi}(\not{\partial} - m)\psi - g\bar{\psi}\phi\psi$
Scalar QED	$(D^\mu \phi^\dagger)(D_\mu \phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \lambda(\phi^\dagger \phi)^2$
QED	$\bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$
GWS-Electro-Weak	$(D^\mu \phi^\dagger)(D_\mu \phi) + \mu\phi^\dagger\phi - \lambda(\phi^\dagger \phi)^2$
QCD	$\bar{Q}\left(i\not{D} - m\right)Q - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a$
Standard Model	

Explicit Renormalization

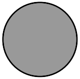
Explicit Renormalization - An Example

- As an illustrative example: ϕ^4-theory in 4 dimensions
- Refering to superficial degree of divergence for $d = 4, n = 4$:

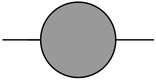
$$D = d - [\lambda]V - \left(\frac{d-2}{2}\right) N = 4 - N$$

- Only diagrams with up to 4 external legs should superficially diverge

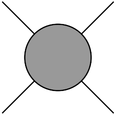
Divergent amplitudes in ϕ^4-theory



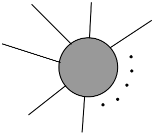
$D = 4$: (unobservable vacuum shift)



$D = 2$: $\sim \Lambda^2 + p^2 \log(\Lambda) +$ (finite terms)



$D = 0$: $\sim \log(\Lambda)$



$D < 0$: (finite)

Multiplicative Renormalization

- ϕ^4-Lagrangian in d space-time dimensions (setting $4-d=2\varepsilon$):

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi_0)^2 - \frac{m_0^2}{2}\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4$$

- Unobservable *bare* quantities ϕ_0, m_0, λ_0 are infinite
- Rewrite in terms of new, finite quantities ϕ, m, λ:

Renormalized parameters

$$\phi_0 \equiv \sqrt{Z_\phi}\,\phi, \qquad Z_\phi = 1 + \delta Z_\phi^{(1)} + \mathcal{O}(\lambda^2)$$

$$m_0^2 \equiv Z_m\, m^2, \qquad Z_m = 1 + \delta Z_m^{(1)} + \mathcal{O}(\lambda^2)$$

$$\lambda_0 \equiv \tilde{\mu}^{2\varepsilon} Z_\lambda\, \lambda, \qquad Z_\lambda = 1 + \delta Z_\lambda^{(1)} + \mathcal{O}(\lambda^2)$$

Expansion of the Lagrangian

$$\mathcal{L} = Z_\phi \, \frac{1}{2}(\partial^\mu \phi)^2 - Z_m \, Z_\phi \, \frac{m^2}{2}\phi^2 - Z_\lambda \, Z_\phi^2 \, \tilde{\mu}^{2\varepsilon} \frac{\lambda}{4!}\phi^4 =$$

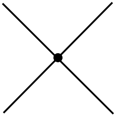
$$= \underbrace{\frac{1}{2}(\partial^\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \tilde{\mu}^{2\varepsilon}\frac{\lambda}{4!}\phi^4}_{\mathcal{L}_{\text{renormalized}}} +$$

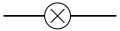
$$+ \underbrace{\frac{\delta Z_\phi^{(1)}}{2}(\partial^\mu \phi)^2 - \frac{\delta Z_\phi^{(1)} + \delta Z_m^{(1)}}{2} m^2 \phi^2 - \tilde{\mu}^{2\varepsilon}\frac{2\delta Z_\phi^{(1)} + \delta Z_\lambda^{(1)}}{4!}\lambda \phi^4}_{\mathcal{L}_{\text{counter}}} +$$

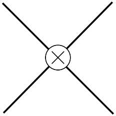
$$+ \underbrace{\mathcal{O}(\delta Z_i^{(1)} \delta Z_j^{(1)}, \delta Z_i^{(2)})}_{\mathcal{O}(\lambda^2)}$$

Renormalized Feynman Rules

 $$= \frac{i}{p^2 - m^2 + i\varepsilon}$$

 $$= -i\tilde{\mu}^{2\varepsilon}\lambda$$

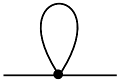
 $$= i(p^2 - m^2)\,\delta Z_\phi^{(1)} - im^2\,\delta Z_m^{(1)}$$

 $$= -i\tilde{\mu}^{2\varepsilon}\left(2\delta Z_\phi^{(1)} + \delta Z_\lambda^{(1)}\right)\lambda$$

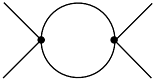
Example: Scalar-ϕ^4 at 1-loop Level

Using **dimensional regularization**: $$\int \frac{d^4k}{(2\pi)^4} \longrightarrow \int \frac{d^dk}{(2\pi)^d}$$

2 divergent graphs at one-loop:



$$= \frac{-i\lambda\tilde{\mu}^{2\varepsilon}}{2} \int \frac{d^dk}{(2\pi)^d} \frac{i}{[k^2 - m^2 + i\varepsilon]}$$



$$=$$

$$= \frac{(-i\lambda\tilde{\mu}^{2\varepsilon})^2}{2} \int \frac{d^dk}{(2\pi)^d} \frac{i}{[(p_1 + p_2 + k)^2 - m^2 + i\varepsilon]} \frac{i}{[k^2 - m^2 + i\varepsilon]}$$

Strategy for Computation

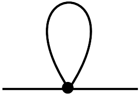
- Combine denominators with Feynman-parameter x
- Shift the integration variable: $k \longrightarrow l = k + xp$
- Wick-rotate: $l^0 = il_E^0, \quad l^2 = -l_E^2$
- Separate radial part from the d-dimensional angular part:

$$\int \frac{d^d l}{(2\pi)^d} = \int \frac{d\Omega_d}{(2\pi)^d} \int_0^\infty dl l^{d-1} = \frac{2}{(4\pi)^{\frac{d}{2}} \Gamma(\frac{d}{2})} \int_0^\infty dl l^{d-1}$$

- Map $l \in [0, \infty)$ to $z \in [0, 1]$, and read of the Beta-Function

1-Loop Contribution of 2-Pt-Fct.

Tadpole value:



$$= -\frac{i\lambda}{2}\frac{\tilde{\mu}^{2\varepsilon}}{(4\pi)^{2-\varepsilon}}\frac{\Gamma(\varepsilon-1)}{(m^2)^{\varepsilon-1}}$$

Expand around $\varepsilon \sim 0$: (note : $\mu^2 = 4\pi e^{-\gamma_E}\tilde{\mu}^2$)

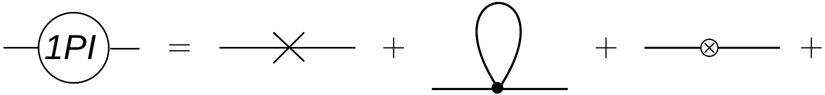
$$... = \frac{i\lambda}{32\pi^2} m^2 \left(\frac{1}{\varepsilon} + \log\left[\frac{\mu^2}{m^2}\right] + 1 + \mathcal{O}(\varepsilon)\right)$$

Observation:

- Divergence has been isolated $(\frac{1}{\varepsilon})$
- Tadpole is independend of momentum

1-Loop Contribution of 2-Pt-Fct.

Including the counterterm should give a finite result:



$$+ \quad \mathcal{O}(\lambda^2) \quad \overset{!}{=} \quad \textbf{finite to } \mathcal{O}(\lambda)$$

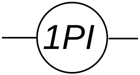
Counterterm value still undefined

$$= i(p^2 - m^2)\,\delta Z_\phi^{(1)} - im^2\,\delta Z_m^{(1)}$$

$\longrightarrow$ must be used to cancel the divergence of $\mathcal{O}(\lambda)$

1-Loop Contribution of 2-Pt-Fct.

Quantitatively:



$$= i(p^2 - m^2) + \frac{i\lambda}{32\pi^2} m^2 \left(\frac{1}{\varepsilon} + \log\left[\frac{\mu^2}{m^2}\right] + 1 \right) +$$

$$+ i(p^2 - m^2)\, \delta Z_\phi^{(1)} - i m^2\, \delta Z_m^{(1)} + \mathcal{O}(\lambda^2)$$

Since the tadpole is independend of p^2, the first counterterm can be fixed:

Field strength 1-loop correction

$$\delta Z_\phi^{(1)} = 0$$

Renormalization Schemes

Choice for $\delta Z_m^{(1)}$ is ambiguous however!
$\longrightarrow$ have to pick a **renormalization scheme**

a) $\overline{\text{MS}}$-**scheme**: only divergent terms are sucked in by counterterms

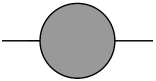
Mass 1-loop correction in $\overline{\text{MS}}$

$$\delta Z_m^{(1)} = \frac{\lambda}{32\pi^2\varepsilon}$$

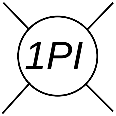
- All parameters **indirectly** fixed by experiment (e.g. mass parameter $\neq$ physical mass!!)
- Easy-to-handle counterterms (suitable for RG-analyis)

Renormalization Schemes

b) **On-shell-scheme**: renormalization conditions



$$\Big|_{p^2=m^2} \stackrel{!}{=} \frac{i}{p^2 - m^2 + i\varepsilon},$$



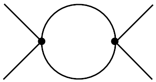
$$\Bigg|_{\substack{s=4m^2\\ t=u=0}} \stackrel{!}{=} -i\lambda$$

Mass 1-loop correction in On-shell-scheme

$$\delta Z_m^{(1)} = \frac{\lambda}{32\pi^2}\left(\frac{1}{\varepsilon} + \log\left[\frac{\mu^2}{m^2}\right] + 1\right)$$

- Mass parameter is physical
- Simplifies application of LSZ-formalism (no Z-factors)

1-Loop Contribution of 4-Pt-Fct.



$$= \frac{i\tilde{\mu}^{4\varepsilon}\lambda^2}{2(4\pi)^{2-\varepsilon}} \int_0^1 dx \frac{\Gamma(\varepsilon)}{[m^2 - x(1-x)\underbrace{(p_1+p_2)^2}_{=s}]^{\varepsilon}}$$

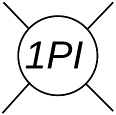
Expand around $\varepsilon \sim 0$: (note : $\mu^2 = 4\pi e^{-\gamma_E}\tilde{\mu}^2$)

$$\ldots = \frac{i\tilde{\mu}^{2\varepsilon}\lambda^2}{32\pi^2}\left(\frac{1}{\varepsilon} + \int_0^1 dx \, \log\left[\frac{\mu^2}{m^2 - x(1-x)s}\right] + \mathcal{O}(\varepsilon)\right)$$

Similiar contributions from t- and u-channel

1-Loop Contribution of 4-Pt-Fct.

Including the vertex counter term should give finite results:



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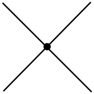
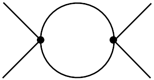
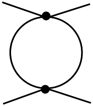
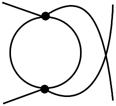
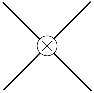
 $+$  $+$  $+$  $+$

 $+ \quad \mathcal{O}(\lambda^3) \quad \overset{!}{=}$ **finite to** $\mathcal{O}(\lambda^2)$

1-Loop Contribution of 4-Pt-Fct.

Quantitatively:

$$\ldots = \underbrace{-i\lambda}_{\text{tree level}} + \underbrace{\frac{3i\tilde{\mu}^{2\varepsilon}\lambda^2}{32\pi^2\varepsilon}}_{\text{divergence same for s,t,u}} + \frac{i\tilde{\mu}^{2\varepsilon}\lambda^2}{32\pi^2}\int_0^1 dx \left(\log\left[\frac{\mu^2}{m^2 - x(1-x)s}\right] + \right.$$

$$\left. + \log\left[\frac{\mu^2}{m^2 - x(1-x)t}\right] + \log\left[\frac{\mu^2}{m^2 - x(1-x)u}\right]\right) +$$

$$- i\tilde{\mu}^{2\varepsilon}\delta Z_\lambda^{(1)}\lambda - i\tilde{\mu}^{2\varepsilon} 2 \underbrace{\delta Z_\phi^{(1)}}_{=0} \lambda$$

Coupling 1-loop correction in $\overline{\text{MS}}$

$$\delta Z_\lambda^{(1)} = \frac{3\lambda}{32\pi^2\varepsilon}$$

Counterterm Values in $\overline{\text{MS}}$

$$\delta Z_\phi^{(1)} = 0 \quad \Rightarrow \quad \phi_0 = \left(1 + \mathcal{O}(\lambda^2)\right)\phi$$

$$\delta Z_m^{(1)} = \frac{\lambda}{32\pi^2\varepsilon} \quad \Rightarrow \quad m_0^2 = \left(1 + \frac{\lambda}{32\pi^2\varepsilon} + \mathcal{O}(\lambda^2)\right) m^2$$

$$\delta Z_\lambda^{(1)} = \frac{3\lambda}{32\pi^2\varepsilon} \quad \Rightarrow \quad \lambda_0 = \left(1 + \frac{3\lambda}{32\pi^2\varepsilon} + \mathcal{O}(\lambda^2)\right) \tilde{\mu}^{2\varepsilon}\lambda$$

Renormalization Group

Renormalization-Group Equations

Possible to extract futher information by following observations:

- Mass scale μ introduced just to keep $[\lambda] = 0 \longrightarrow$ exact value arbitrary
- Observables/ unrenormalized quantities should be independend of μ:

$$\frac{d\lambda_0}{d\log[\mu^2]} = \frac{d\phi_0}{d\log[\mu^2]} = \frac{dm_0^2}{d\log[\mu^2]} \stackrel{!}{=} 0$$

- Above conditions will enforce μ-dependences of λ, m, ϕ (*RG-dependence*)

Renormalization-Group Equations

Renormalized parameters

$$\phi_0 \equiv \sqrt{Z_\phi}\,\phi, \qquad Z_\phi = 1 + \delta Z_\phi^{(1)} + \mathcal{O}(\lambda^2)$$

$$m_0^2 \equiv Z_m\, m^2, \qquad Z_m = 1 + \delta Z_m^{(1)} + \mathcal{O}(\lambda^2)$$

$$\lambda_0 \equiv \tilde{\mu}^{2\varepsilon} Z_\lambda\, \lambda, \qquad Z_\lambda = 1 + \delta Z_\lambda^{(1)} + \mathcal{O}(\lambda^2)$$

$\Longrightarrow$ ODEs for $\lambda(\mu), m^2(\mu), \phi(\mu)$

RG-Equation for λ at 1-loop

$$\frac{d}{d\log[\mu^2]}\,\lambda_0 \overset{!}{=} 0 =$$

$$= \frac{d}{d\log[\mu^2]}\left[\tilde{\mu}^{2\varepsilon} Z_\lambda \lambda\right] = \frac{d}{d\log[\mu^2]}\left[\tilde{\mu}^{2\varepsilon}\left(1+\frac{3\lambda}{32\pi^2\varepsilon}\right)\lambda\right] =$$

$$= \varepsilon\tilde{\mu}^{2\varepsilon}\left(1+\frac{3\lambda}{32\pi^2\varepsilon}\right)\lambda + \tilde{\mu}^{2\varepsilon}\left(1+\frac{6\lambda}{32\pi^2\varepsilon}\right)\underbrace{\frac{d\lambda}{d\log[\mu^2]}}_{\equiv\beta_\lambda^\varepsilon}$$

$$\beta_\lambda^\varepsilon = \frac{d\lambda}{d\log[\mu^2]} = -\varepsilon\lambda + \frac{3\lambda^2}{32\pi^2} + \mathcal{O}(\lambda^3)$$

RG-Equation for m^2 at 1-loop

$$\frac{d}{d\log[\mu^2]}\, m_0^2 \overset{!}{=} 0 =$$

$$= \frac{d}{d\log[\mu^2]}\left[Z_m m^2\right] = \frac{d}{d\log[\mu^2]}\left[\left(1+\frac{\lambda}{32\pi^2\varepsilon}\right)m^2\right] =$$

$$= \left(1+\frac{\lambda}{32\pi^2\varepsilon}\right)\underbrace{\frac{dm^2}{d\log[\mu^2]}}_{\equiv m^2\,\gamma_m} + \frac{1}{32\pi^2\varepsilon}\,\frac{d\lambda}{d\log[\mu^2]}\,m^2$$

$$\gamma_m = \frac{1}{m^2}\frac{dm^2}{d\log[\mu^2]} = \frac{\lambda}{32\pi^2} + \mathcal{O}(\lambda^2)$$

RG-Equation for ϕ at 1-loop

$$\frac{d}{d\log[\mu^2]}\,\phi_0^2 \overset{!}{=} 0 = \frac{d}{d\log[\mu^2]}\left[Z_\phi \phi^2\right] = \frac{d}{d\log[\mu^2]}\,\phi^2$$

$$\gamma_\phi = \frac{1}{\phi^2}\frac{d\phi^2}{d\log[\mu^2]} = 0 + \mathcal{O}(\lambda^2)$$

Field is RG-invariant at 1-loop level (not anymore at higher orders)

Solutions to the RGE´s

Straighforward solution with initial conditions at scale μ_0:

$$\lambda(\mu) = \frac{\lambda(\mu_0)}{1 - \frac{3}{32\pi^2}\lambda(\mu_0)\log\left[\frac{\mu^2}{\mu_0^2}\right]}$$

$$m^2(\mu) = m^2(\mu_0)\left(\frac{\lambda(\mu)}{\lambda(\mu_0)}\right)^{\gamma_m/\beta_\lambda}$$

$$\phi(\mu) = \phi(\mu_0)$$

Relation between $\overline{\text{MS}}$- and On-Shell mass m^2_{on}

$$m_0^2 = Z_m m^2, \quad m_0^2 = Z_m^{\text{on}} m^2_{\text{on}} \quad \Rightarrow \quad m^2_{\text{on}} = m^2 Z_m (Z_m^{\text{on}})^{-1}$$

Plug in + expand:

On-shell mass

$$m^2_{\text{on}} = m^2(\mu) \left(1 - \frac{\lambda}{32\pi^2} \left(\log\left[\frac{\mu^2}{m^2}\right] + 1\right) + \mathcal{O}(\lambda^2)\right)$$

- m^2_{on} is actually RG-invariant (like every observable)
- $\Rightarrow$ can be evaluated at any convenient scale μ

m^2_{on} at scale $\mu \sim m$

$$m^2_{\text{on}} = m^2(m) \left(1 - \frac{\lambda}{32\pi^2} + \mathcal{O}(\lambda^2)\right)$$

Running Coupling

- Coupling obviously energy dependend:

$$\lambda(\mu) = \frac{\lambda_0}{1 - \frac{3}{32\pi^2}\,\lambda_0 \log\left[\frac{\mu^2}{\mu_0^2}\right]}$$

- RG interpolates between processes measured at different energies
- Expansion:

$$\lambda(\mu) = \lambda_0 \left(1 + \frac{3}{32\pi^2}\,\lambda_0 \log\left[\frac{\mu^2}{\mu_0^2}\right] + \left(\frac{3}{32\pi^2}\right)^2 \lambda_0^2 \log^2\left[\frac{\mu^2}{\mu_0^2}\right] \right.$$
$$\left. + \ldots \right)$$

- Ideal scale: $\mu \sim \mu_0 \quad \Rightarrow \quad$ no large log's

- Diverges if denominator vanishes:

$$1 - \frac{3}{32\pi^2}\lambda(\mu_0)\log\left[\frac{\mu^2}{\mu_0^2}\right] \stackrel{!}{=} 0 \quad \Rightarrow \quad \mu_\infty^2 = (\mu_0)^2 e^{\frac{32\pi^2}{3\lambda(\mu_0)}}$$

$\rightarrow$ needs input from experiment: $\lambda(\mu_0)$

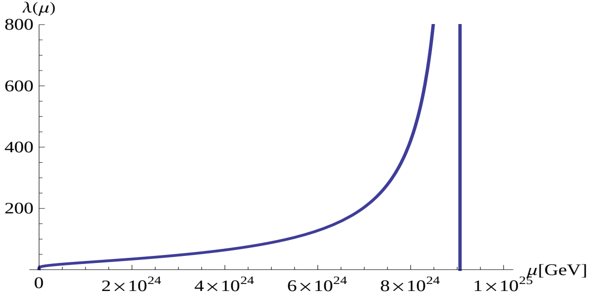


Figure: Running coupling of ϕ^4-theory at 1-loop ($\mu_0 = 125$ GeV, $\lambda_0 = 1$)

Running Coupling

- Conclusion: although introduced just for dimensional reasons, μ plays the role of a cutoff!!
- At fixed loop-level predictions give reliable answers if scales are of the order of μ
- Theory breaks down for $\mu_0^2 \gg\ll \mu^2$ (perturbative expansion invalid, large log's)
- Smart choice simplifies computation of quantum corrections
- Problematic scenario: process with widely varying scales

Example: $e^+e^- \to$ hadrons at $E_{cm} = \sqrt{s} \gg m_q$

- RG-equation for the strong coupling α_s:

$$\frac{d\alpha_s}{d\log[\mu^2]} = -\frac{\alpha_s^2}{4\pi}\,\beta_0 - \frac{\alpha_s^3}{(4\imath)^2}\,\beta_1 - \frac{\alpha_s^4}{(4\pi)^3}\,\beta_2 + \ldots$$

- Solution at 1-loop:

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \frac{\alpha_s(\mu_0)}{4\pi}\beta_0\,\log\left[\frac{\mu^2}{\mu_0^2}\right]}$$

Example: $e^+e^- \to$ hadrons at $E_{cm} = \sqrt{s} \gg m_q$

$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} =$$

$$= N_c \sum_q Q_q^2 \left[1 + \frac{\alpha_s(\mu_0)}{\pi} + \frac{\alpha_s^2(\mu_0)}{\pi^2} \left(f_3 - \frac{\beta_0}{4} \log\left[\frac{s}{\mu_0^2}\right] \right) + \ldots \right]$$

- $\mu_0^2 \gg, \ll s$ would spoil perturbative expansion
- Reliable pert. prediction only for $\mu_0^2 \sim s$
- Way around this potential problem: Use the RG-solution to evolve $\alpha_s(\mu_0)$ to $\alpha_s(\sqrt{s})$
- Then: $\mu_0 = \sqrt{s}$, and $\log\left[\frac{s}{\mu_0^2}\right] = 0$!

Summary

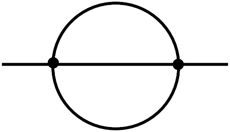
Summarizing

We saw/achieved:

- Origin of UV-divergences $\longrightarrow$ locality
- Classification of UV-divergences
- Explicit renormalization of ϕ^4-theory:
  - Regulariztion schemes (Dim. reg.)
  - Mulitplicative Renormalization: $q_0 = Z_q\, q$
  - Counterterms to cancel infinities
  - Renormalization (Subtraction) schemes
    $\longrightarrow$ $\overline{\text{MS}}$ vs. On-shell
- Derivation RGE's by μ-dependence
- Solution $\longrightarrow$ energy-dependend (*running*) parameters
- μ as "cutoff" (determines quality of pert. expansion)

Further Intricacies

- Renormalization of theories with different particle species (QED,...)
- Symmetries (e.g. gauge-invariance) influencing the procedure
- Multi-loop calculations such as



$\longrightarrow$ overlapping (*nested, non-local*) divergences

- Non-renormalizable theories

References

- Lecture Notes Prof. Hoang
- M. Peskin, D. Schroeder: *An Introduction to Quantum Field Theory*,
- J. Collins: *Renormalization*
- M. Schwartz: *QFT and the Standard Model*