The Regge Limit of Gravity

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Regge Limit $s \gg -t$

Famously studied by Regge who considered analytically continuing in L. Partial wave amplitudes can have poles in complex l plane

$$A_l \sim \frac{\beta(t)}{l - \alpha(t)}$$

The pole at largest alpha dominates the scattering amplitude as $s \to \infty$

$$M \sim s^{\alpha(t)}$$

Seemed to look like low energy data

$$l = \alpha(E) \qquad l = \alpha(t) m^2$$

Interesting coincidence?

In a relativistic field theory if you sum a tower of t-channel spin J exchanges the cross section takes the form:

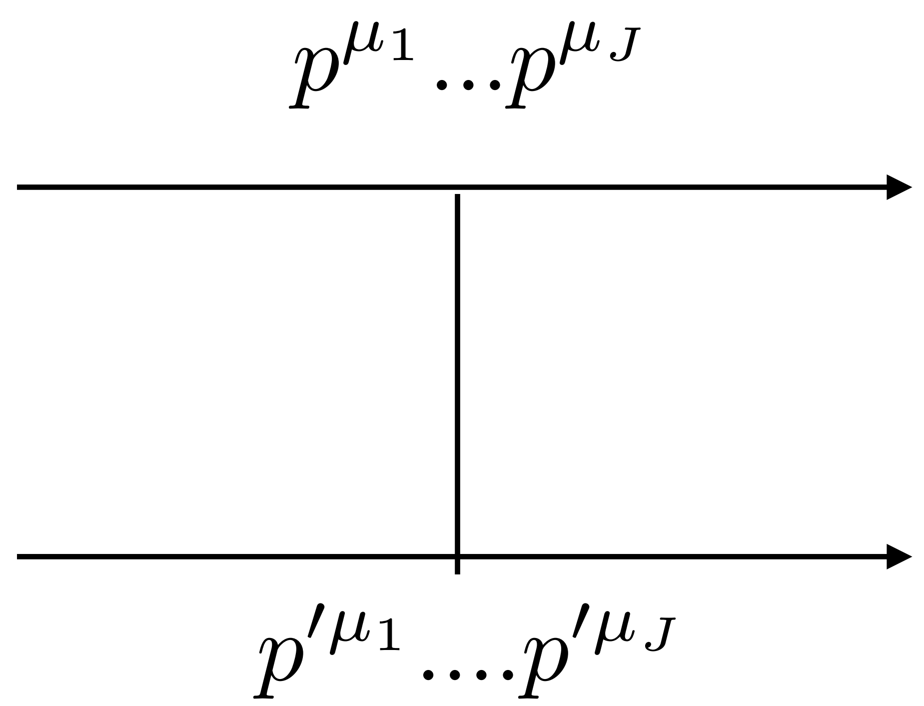
$$\sigma \sim s^{(\alpha-1)}$$

Violates Froissart bound: $\alpha \sim 1.08$ Something must unitarize

What happens in gauge theory?

Exchange of spin J particle in t-channel

$$M_J \sim s^J$$



QCD: $M \sim \frac{\alpha C_F s}{t}$

Gravity: $M \sim \frac{G_N s^2}{t}$

No Froissart but gravity violates partial wave unitarity

What happens once we calculate radiative corrections?

Generate large rapidity $\log(s/t)$

QCD Reggeization: $\frac{sg^2}{t} \to \frac{sg^2}{t}(s/t)^\gamma$ Regge trajectory = Rapidity anomalous dimensions

What happens in gravity?

The graviton also Reggeizes but the problem is that it no longer makes sense to resum logs (at least w/o more careful consideration)

Regge Limit of Gravity $\quad s \gg t$

GR, away from Regge, is understood as an EFT for $\quad (s,t,u) \ll M_{pl}^2$

Assuming sufficiently smooth background metrics (Asymptotically flat for this talk)

$$S_{EFT} = \int d^4x \sqrt{g}(R + c_3 R^3 + ...)$$

Every log is associated with a counter-term with some unknown Wilson coefficient

Loss of systematics

But this CAN NOT be the whole story: After all we know how to calculate planetary scattering

This corresponds to the Regge limit $$s \gg M_{pl}^2 \gg | \, t \, |$$

Crucial distinction between QCD and Gravity

This limit correspnods to large charge for GR

Leads to a VERY sharp saddle point in the path integral

Power Counting

$$\alpha_Q \equiv \frac{t}{M_{pl}^2} < 1 \qquad \alpha_C \equiv \frac{st}{M_{pl}^4} < 1$$

Leads to a clear distinction between classical and quantum corrections (as opposed to QCD)

Puzzle: Theory becomes strongly coupled, yet at the same time the process is semi-classical. Near forward Eikonal scattering.

So the peturbative series must organize itself into a controlled form

The amp must resum (generate Shock Wave Solution)

SCET

- Only keep relevant IR degrees of freedom

- Fields have definate scaling λ

- Write down every terms in the action consistent with the symmetry to chosen order of accuracy.

IR modes

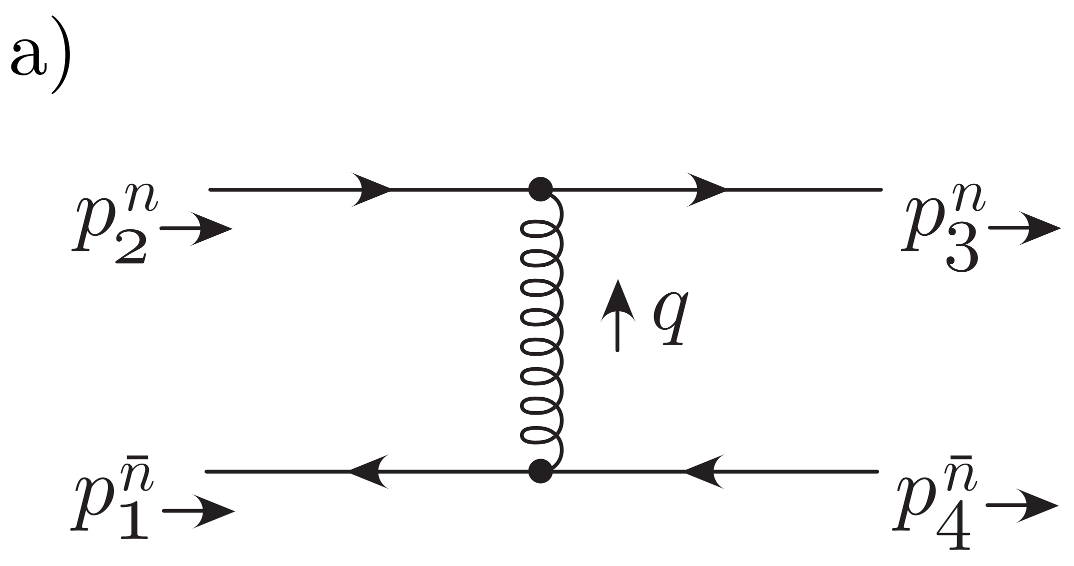
Collinear modes: $p_n^\mu \sim (1, \lambda^2, \lambda)$ and $p_{\bar{n}}^\mu \sim (\lambda^2, 1, \lambda)$

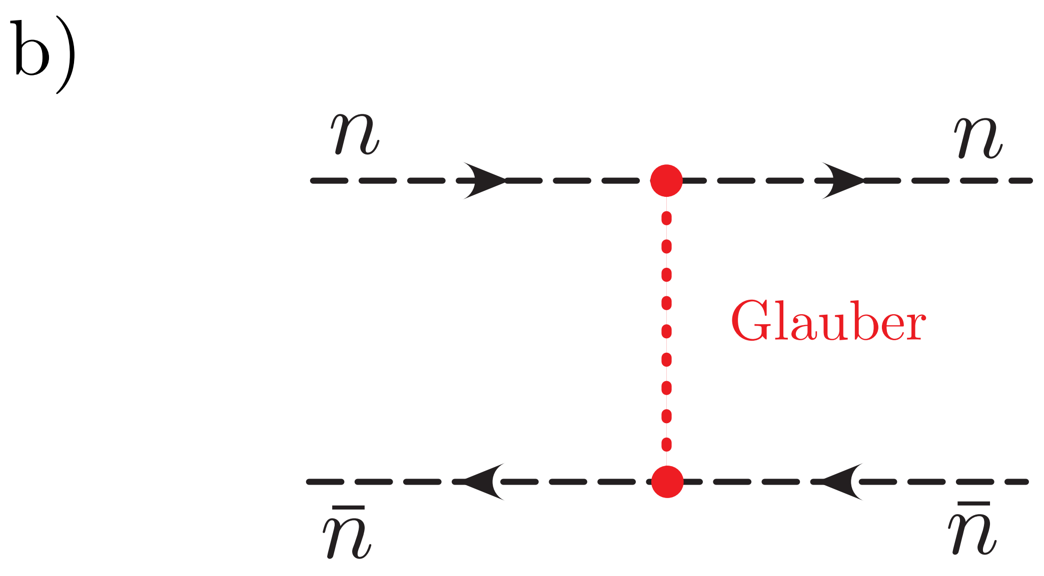
Soft modes: $p_s^\mu \sim (\lambda, \lambda, \lambda)$

EFT for Forward Scattering: (izr+Iain Stewart 2016)

(izr + M. Saavedra 2024)

Introduce Glauber Mode

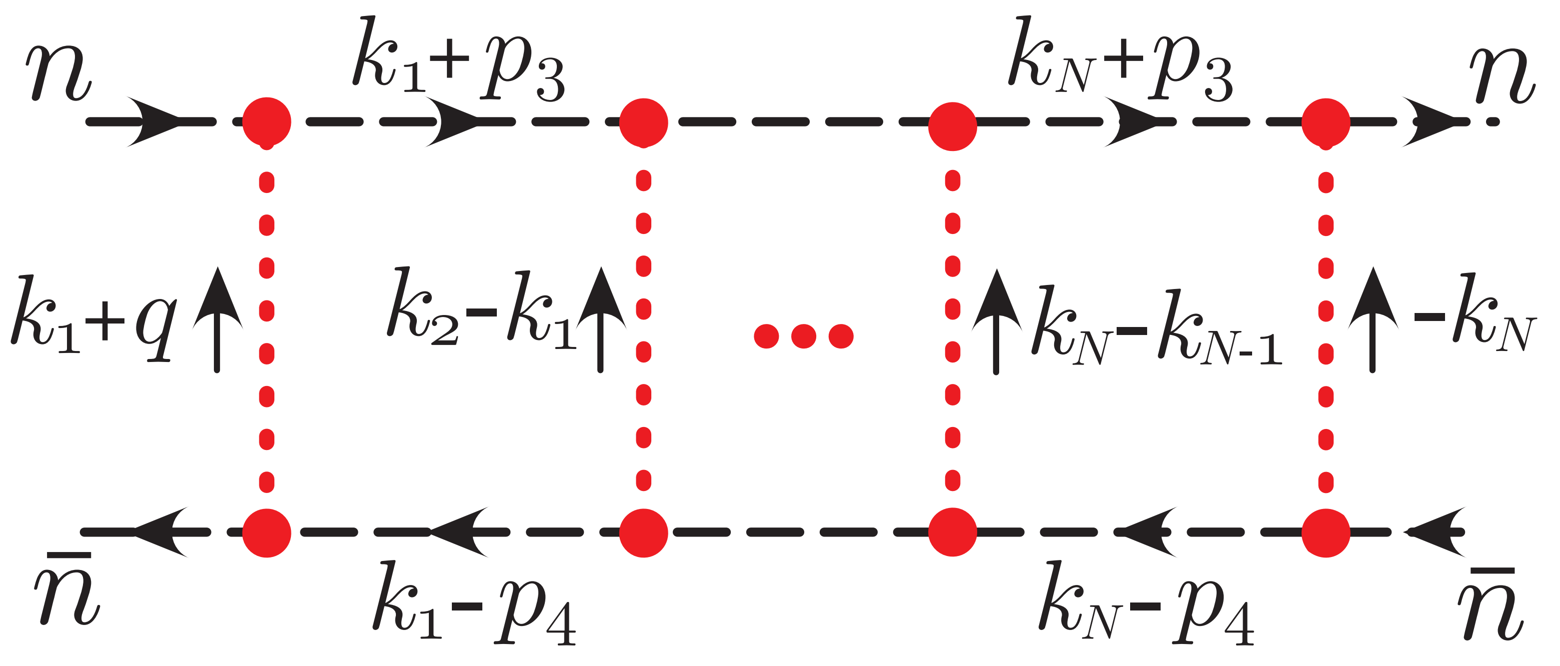




Generates Glauber operator basis

$$S_G = \int d^4x O_n O_s O_{\bar{n}}.$$

Glauber Resummation



$$= 2(-ig^2)^{N+1}\mathcal{S}^{n\bar{n}}_{(N+1)} I_\perp^{(N)}(q_\perp) \; \frac{1}{(N+1)!}$$

$$I_\perp^{(N)}(q_\perp) = \int \frac{d\!\!{}^-{}^{d-2}k_{1\perp}\cdots d\!\!{}^-{}^{d-2}k_{N\perp}\,(\iota^\epsilon\mu^{2\epsilon})^{N+1}}{(\vec{k}_{1\perp}+\vec{q}_\perp)^2(\vec{k}_{2\perp}-\vec{k}_{1\perp})^2\cdots(\vec{k}_{N\perp}-\vec{k}_{(N-1)\perp})^2\,\vec{k}^{\,2}_{N\perp}}$$

$$\underset{\mathrm{F.T.}_\perp}{\Longrightarrow} \quad \frac{1}{(N+1)!}\left[i\phi(b_\perp)\right]^{N+1}\; 2\mathcal{S}^{n\bar{n}}$$

$$\phi(b_\perp) = -\mathbf{T}_1^A\otimes\mathbf{T}_2^A\, g^2(\mu)\int\frac{d\!\!{}^-{}^{d-2}q_\perp\,(\iota^\epsilon\mu^{2\epsilon})}{\vec{q}^{\,2}_\perp}\,e^{i\vec{q}_\perp\cdot\vec{b}_\perp} \qquad \delta^{(0)}_{\mathrm{Cl}} = Gs\pi^\epsilon(\bar{\mu}^2b^2)^\epsilon\Gamma(-\epsilon)$$

$$= -\mathbf{T}_1^A\otimes\mathbf{T}_2^A\, g^2(\mu)\,\frac{\Gamma(-\epsilon)}{4\pi}\left(\frac{\mu|\vec{b}_\perp|e^{\gamma_E/2}}{2}\right)^{2\epsilon}.$$

$\delta(b)$ Classical Eikonal

This is a highly valuable object for Gravitational wave physics

Can Extract the two body Hamiltonian in GR from the eikonal

Scattering angle: $$\chi \sim \frac{d}{d\log b}\delta(b)$$

From which we can extract H order by order in G

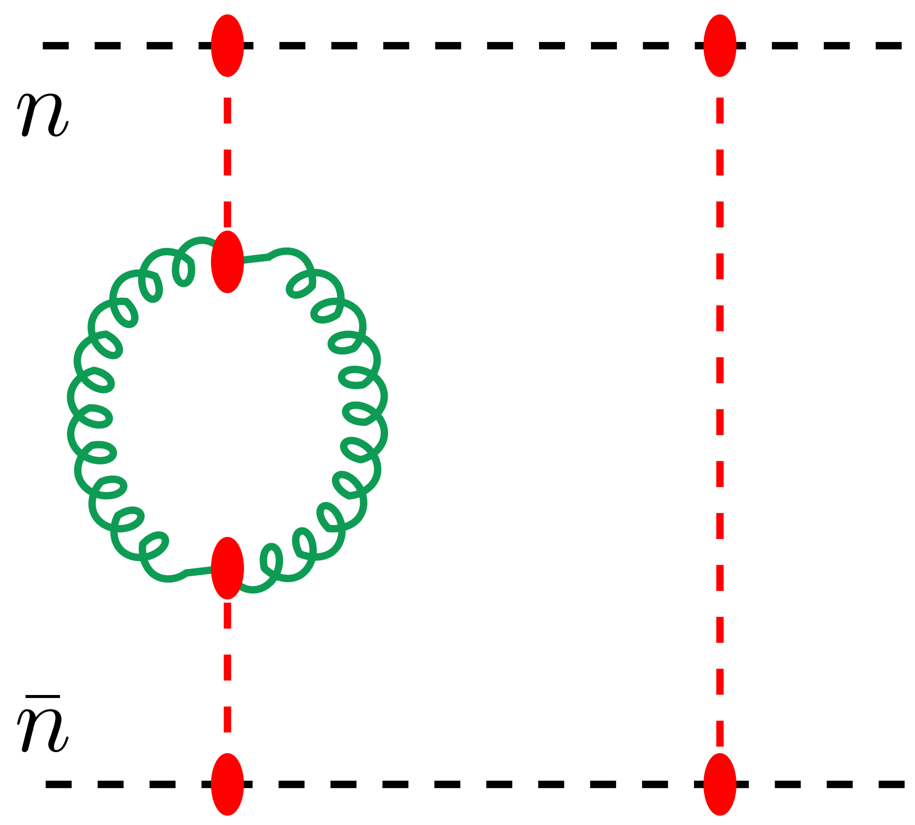
Crucial distinction between QCD + GR

Glauber exchange is SUPER-LEADING

But we just saw that ladder exchanges scale as $Gs \sim \alpha_C/\alpha_Q$

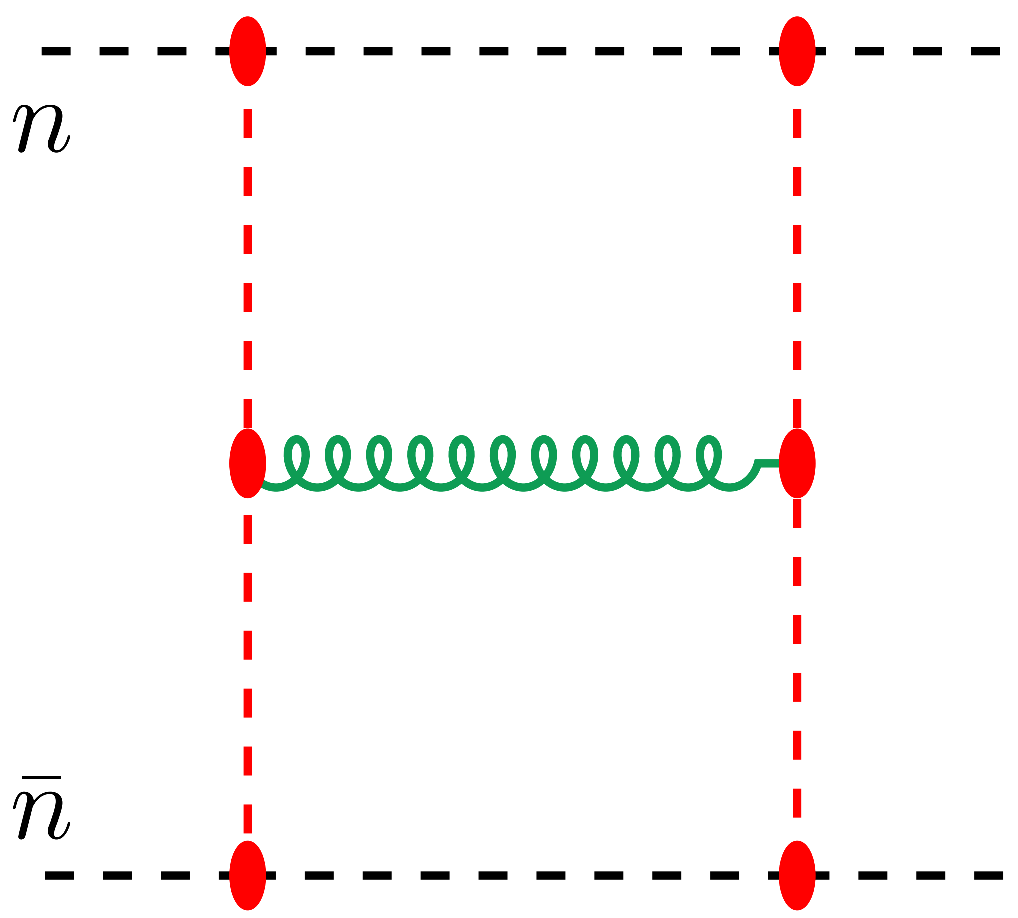
$$\alpha_Q \equiv \frac{t}{M_{pl}^2} < 1 \qquad \alpha_C \equiv \frac{st}{M_{pl}^4} < 1$$

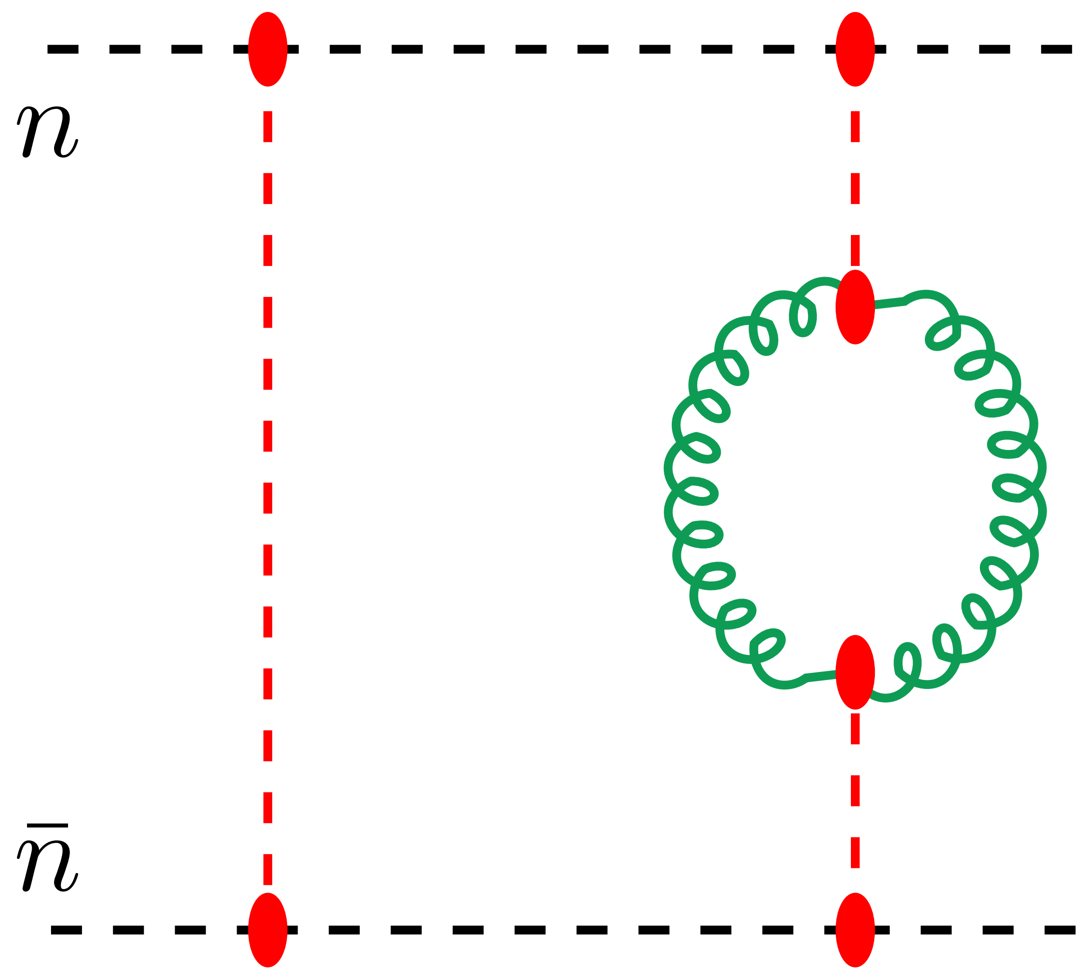
So for a given digram which is quantum we can make it classical by adding a Glauber!



How do we consistently include radiative corrections to this leading order result? There should be both classical and quantum corrections

e.g.





$$(GS)(\alpha_Q) \sim \alpha_C$$

Should contribute to classical Eikonal

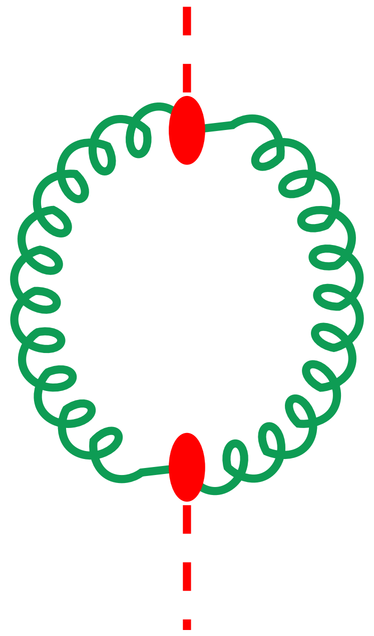
Should not contribute to classical Eikonal

DeVecchia et all proposed the following generalized form:

1911.11716v2

$$\frac{iA(k_i,\ldots)}{2s} \simeq \hat{A}^{(0)}(k_i,\ldots)\int d^{D-2}b\ \mathrm{e}^{-ibq/\hbar}\left[\Big(1+2i\Delta(s,b)\Big)\,\mathrm{e}^{2i\delta(s,b)}-1\right]$$

$$\Delta = \sum \alpha_Q^n C_i(b) \qquad \delta = \sum \alpha_C^n \tilde{C}_i(b)$$



Δ_1

Works at two loops

Subsequently found that this was not consistent with explicit 3 loop calculations

in N=8 SUGRA (Henn and Mistlberger)

How do we go about systematically finding the correct form for the amplitude?

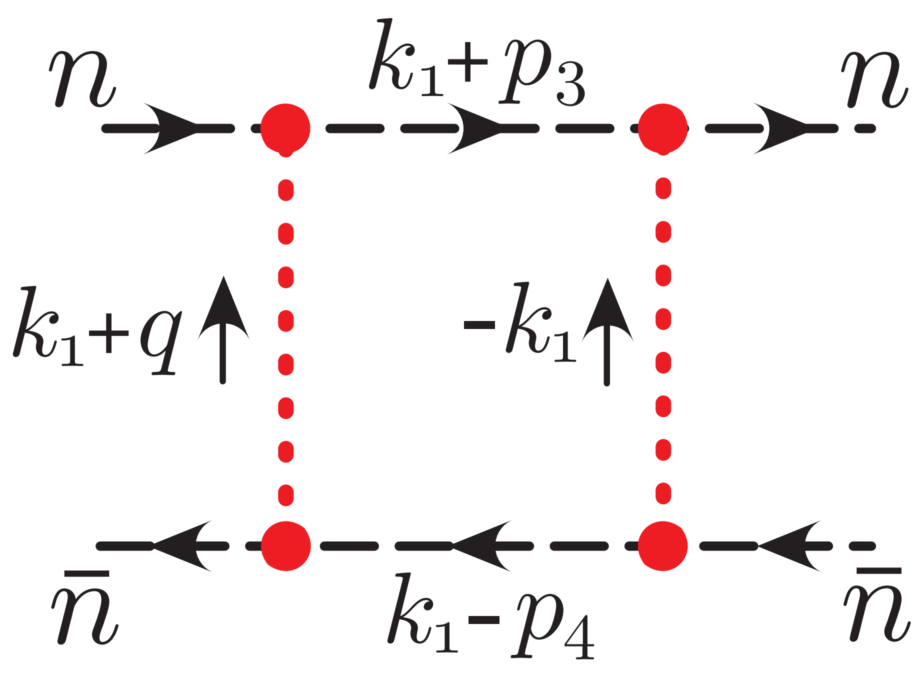
$$\mathcal{L} = \mathcal{L}_n + \mathcal{L}_{\bar{n}} + \mathcal{L}_s \cdot + O(t/s)$$

This is a highly non-trivial statement as it implies that the Hilbert space factorizes

$$\mid \psi\rangle = \mid n\rangle \otimes \mid \bar{n}\rangle \otimes \mid s\rangle$$

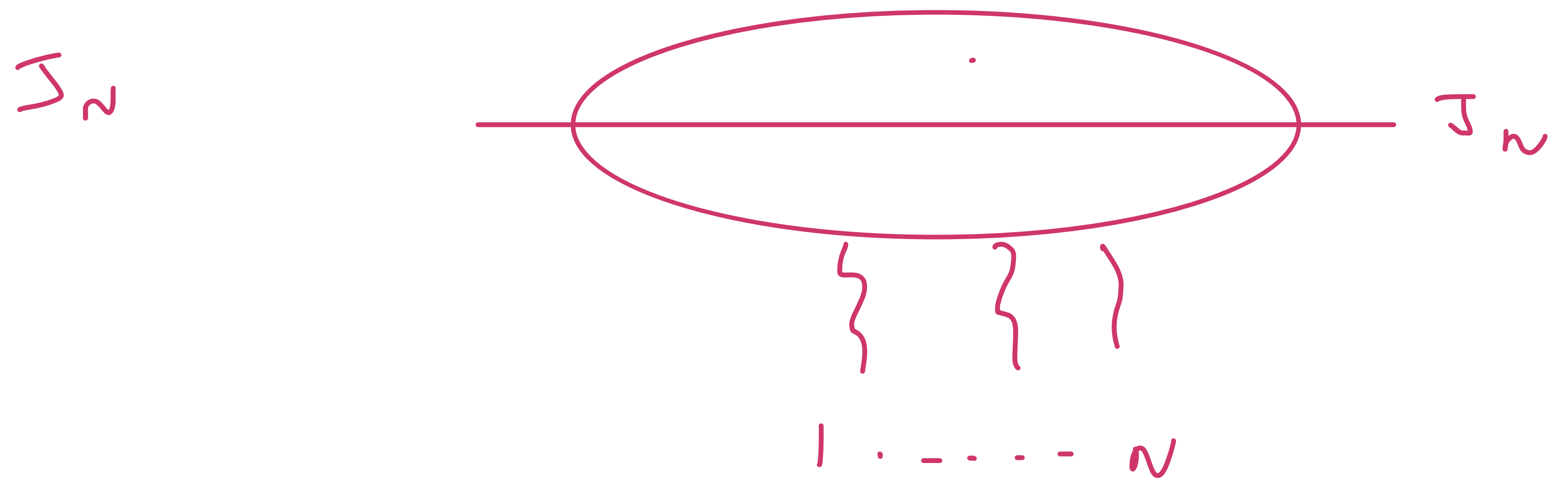
Symmetry group $\quad G = n_{diff} \otimes \bar{n}_{diff} \otimes s_{diff}$

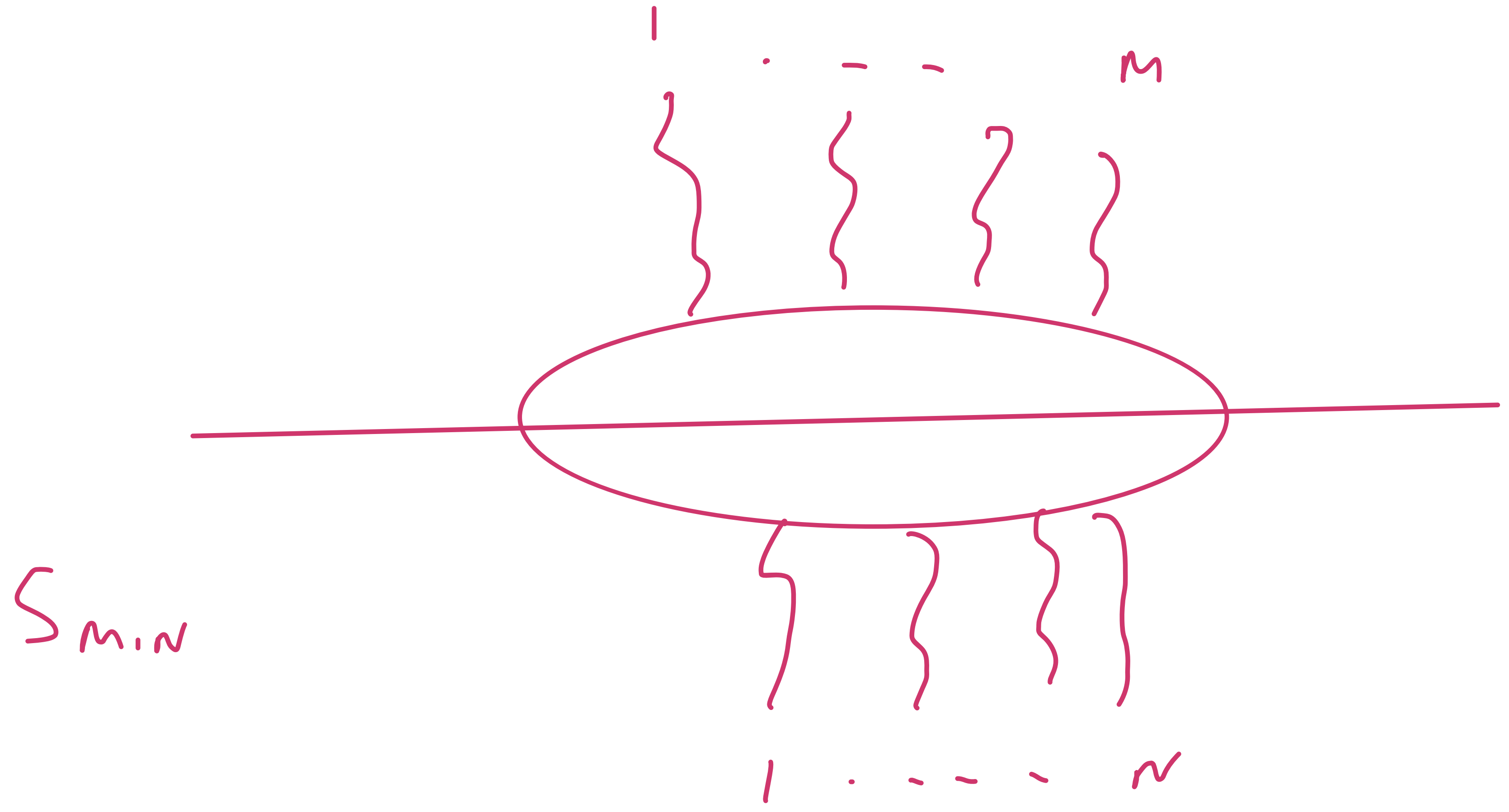
However, this can not be the end of the story as we know that factorization is broken by box graph topologies



Generalized factorization theorem. (Gao et al)

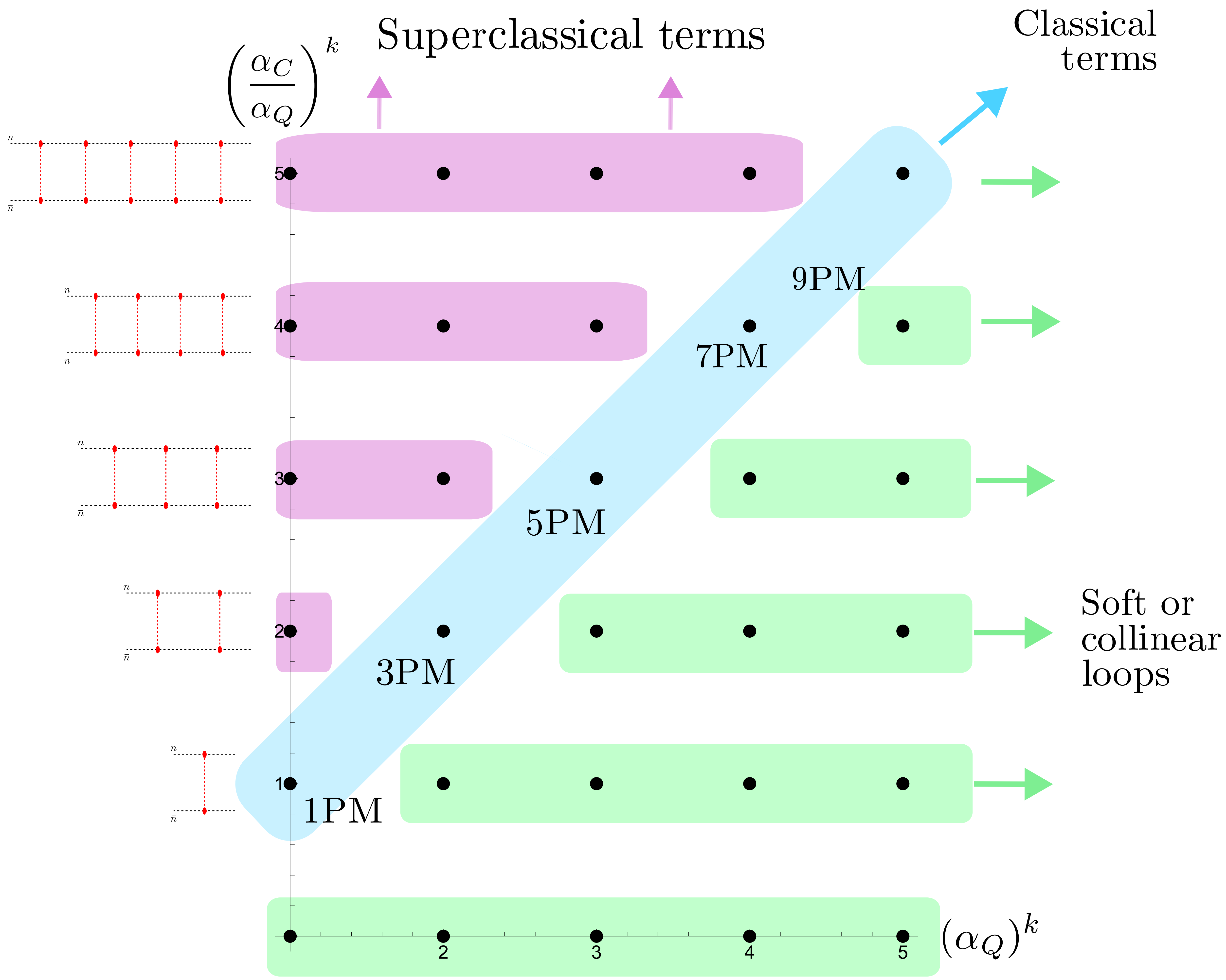
$$iM^{\kappa\kappa'} = -\sum_{MN} \int \int_{\perp(N,M)} J_{\kappa N}(\{l_{\perp i}\}, \epsilon, \eta) S_{(N,M)}(\{l_{\perp i}\}; \{l'_{\perp i}\}, \epsilon, \eta) \bar{J}_{\kappa' M}(\{l'_{\perp i}\}, \epsilon, \eta)$$





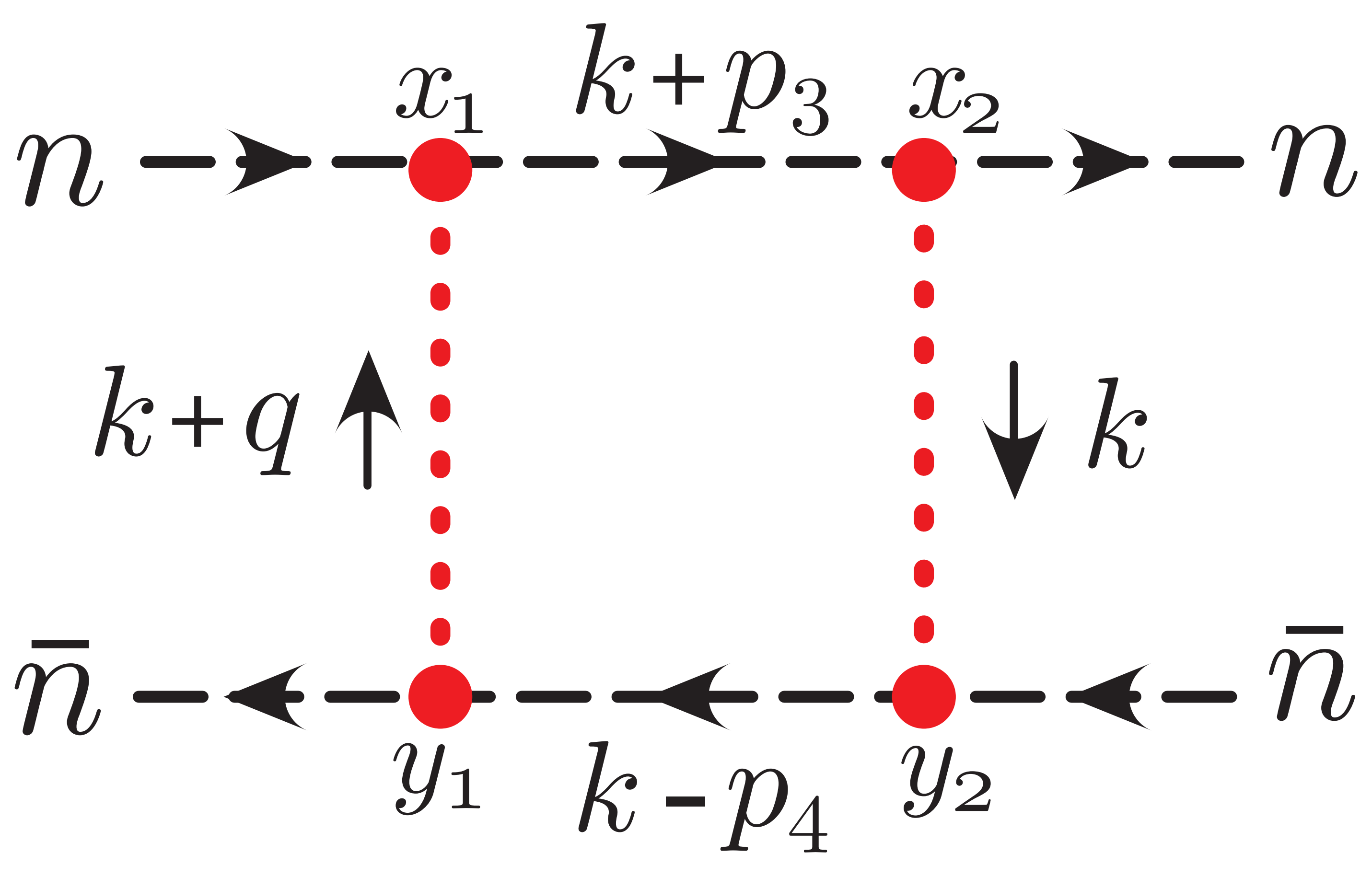
$$S^{(0)}_{(i,j)}(l_{i\perp}; l'_{i\perp}) = 2\delta_{ij} i^j j! \prod_{a=1}^{j} l'^2_{i\perp} \prod_{n=1}^{j-1} \delta^{d'}(l_{n\perp} - l'_{n\perp})$$

Factorization Theorem Implies the Generalized structure of the Amplitude



No classical contribution at even PM order. Also predicts the exitence of Classical Regge log(s/t) at each odd order.

To understand the Failure of the Ansatz we first must see how Exponentiation works in the EFT



$$I_{\mathrm{Gbox}} = \int \frac{\text{đ}^{d-2}k_\perp \; \text{đ}k^+ \; \text{đ}k^-}{2(\vec{k}_\perp^{\,2})(\vec{k}_\perp+\vec{q}_\perp)^2\Big(k^+ + p_3^+ - (\vec{k}_\perp+\vec{q}_\perp/2)^{\,2}/p_2^- + i0\Big)\Big(-k^- + p_4^- - (\vec{k}_\perp+\vec{q}_\perp/2)^{\,2}/p_1^+ + i0\Big)},$$

These are NOT eikonal propagators

Yet still behaves semi-classically why?

$$I_{\mathrm{Gbox}} = \int \frac{đ^{d-2}k_\perp \; đk^+ \; đk^-}{2(\vec{k}_\perp^{\,2})(\vec{k}_\perp+\vec{q}_\perp)^2\Big(k^+ + p_3^+ - (\vec{k}_\perp+\vec{q}_\perp/2)^{\,2}/p_2^- + i0\Big)\Big(-k^- + p_4^- - (\vec{k}_\perp+\vec{q}_\perp/2)^{\,2}/p_1^+ + i0\Big)} \,,$$

Integral is not well defined it needs to be regulated (even in d-dimensions)

Rapidity Regulator

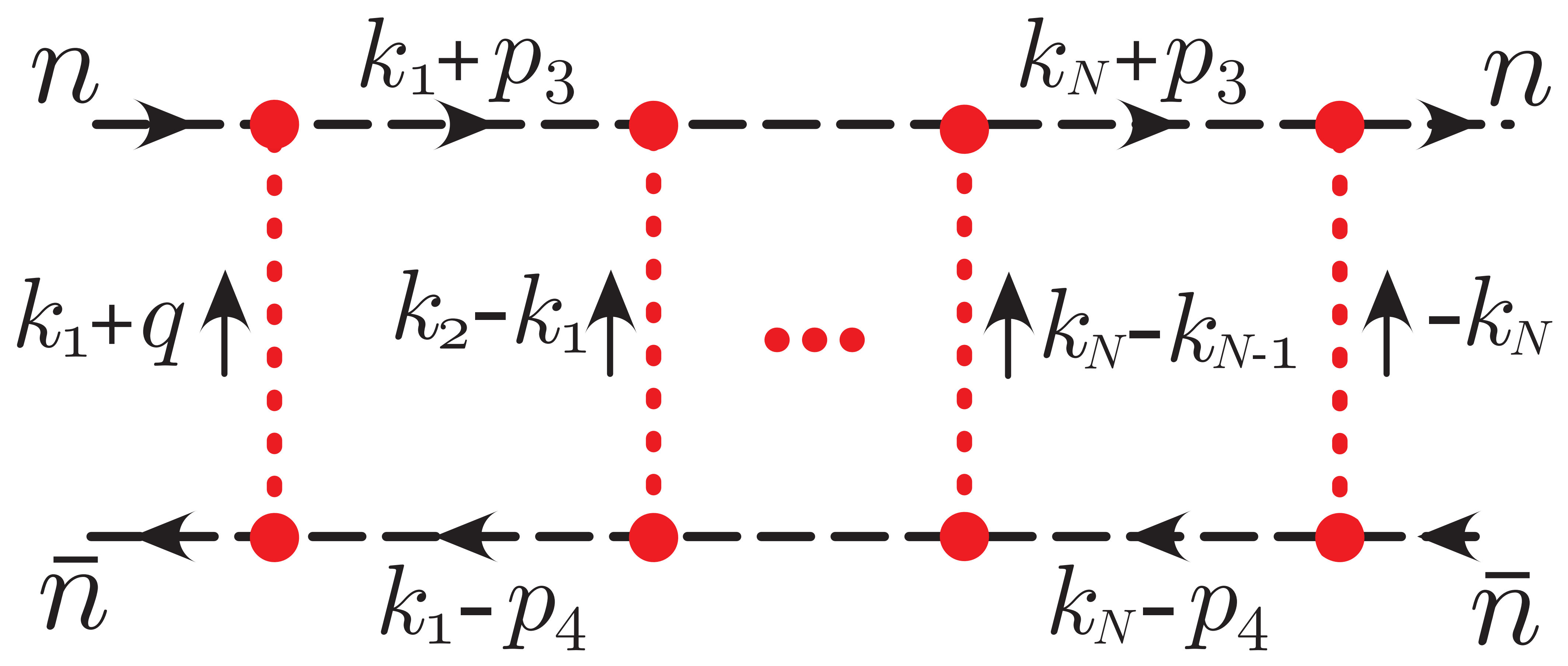
Rapidity Factorization Scale

$$I_{\mathrm{Gbox}} = \int \frac{đ^{d-2}k_\perp \;\, đk^0 \;\, đk^z \;\, |k^z|^{-2\eta}\, (\nu/2)^{2\eta}}{(\vec{k}_\perp^{\,2})(\vec{k}_\perp+\vec{q}_\perp)^2\Big(k^0 - k^z + p_3^+ - (\vec{k}_\perp + \frac{\vec{q}_\perp}{2})^{\,2}/p_2^- + i0\Big)\Big(-k^0 - k^z + p_4^- - (\vec{k}_\perp + \frac{\vec{q}_\perp}{2})^{\,2}/p_1^+ + i0\Big)}$$

$$= \Big(\frac{-i}{4\pi}\Big) \int \frac{đ^{d-2}k_\perp}{(\vec{k}_\perp^{\,2})(\vec{k}_\perp+\vec{q}_\perp)^2} \Big[- i\pi + \mathcal{O}(\eta) \Big]$$

Can't drop

Independent of external perp momenta, effective eikonalization. The box behaves semi-classically despite that the power counting does not fix it to be so.



$$= 2(-ig^2)^{N+1}\mathcal{S}^{n\bar{n}}_{(N+1)} I^{(N)}_{\perp}(q_\perp)\ \frac{1}{(N+1)!}\Big[1+\mathcal{O}(\eta)\Big]$$

Consequence of Rapidity Regulator

$$I^{(N)}_{\perp}(q_\perp) = \int \frac{đ^{d-2}k_{1\perp}\cdots đ^{d-2}k_{N\perp}\,(\iota^{\epsilon}\mu^{2\epsilon})^{N+1}}{(\vec{k}_{1\perp}+\vec{q}_\perp)^2(\vec{k}_{2\perp}-\vec{k}_{1\perp})^2\cdots(\vec{k}_{N\perp}-\vec{k}_{(N-1)\perp})^2\,\vec{k}^{\,2}_{N\perp}}$$

$$\int đ^{d-2}q_\perp\, e^{i\vec{q}_\perp\cdot\vec{b}_\perp}\sum_{N=0}^{\infty}\mathrm{G.Box}^{\,2\to 2}_{N}(q_\perp) = \big(\tilde{G}(b_\perp)-1\big) \qquad \tilde{G}(b_\perp) = e^{i\phi(b_\perp)}$$

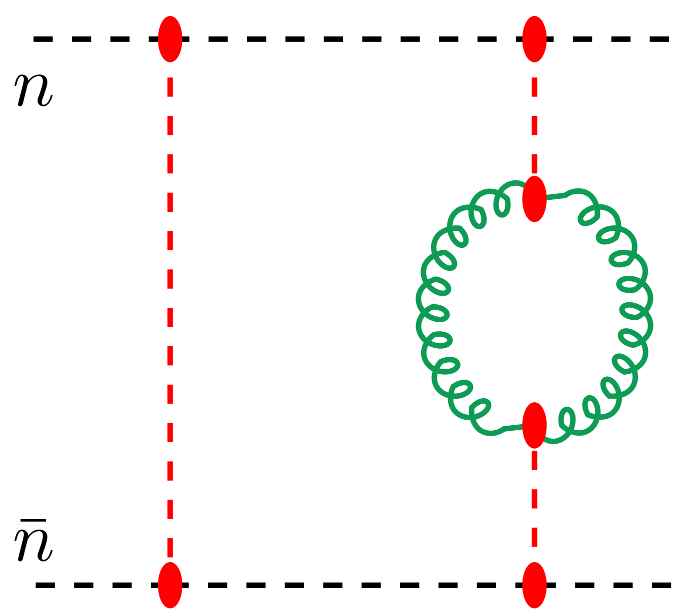
$$\delta^{(0)}_{\mathrm{Cl}} = Gs\pi^{\epsilon}(\bar{\mu}^2b^2)^{\epsilon}\Gamma(-\epsilon)$$

What are the necessary conditions for ACV ansatz to apply?

$$\frac{iA(k_i,\ldots)}{2s} \simeq \hat{A}^{(0)}(k_i,\ldots)\int d^{D-2}b\ \mathrm{e}^{-ibq/\hbar}\left[\Big(1+2i\Delta(s,b)\Big)\,\mathrm{e}^{2i\delta(s,b)}-1\right]$$

- All energetic line eikonalize (at least effectively)

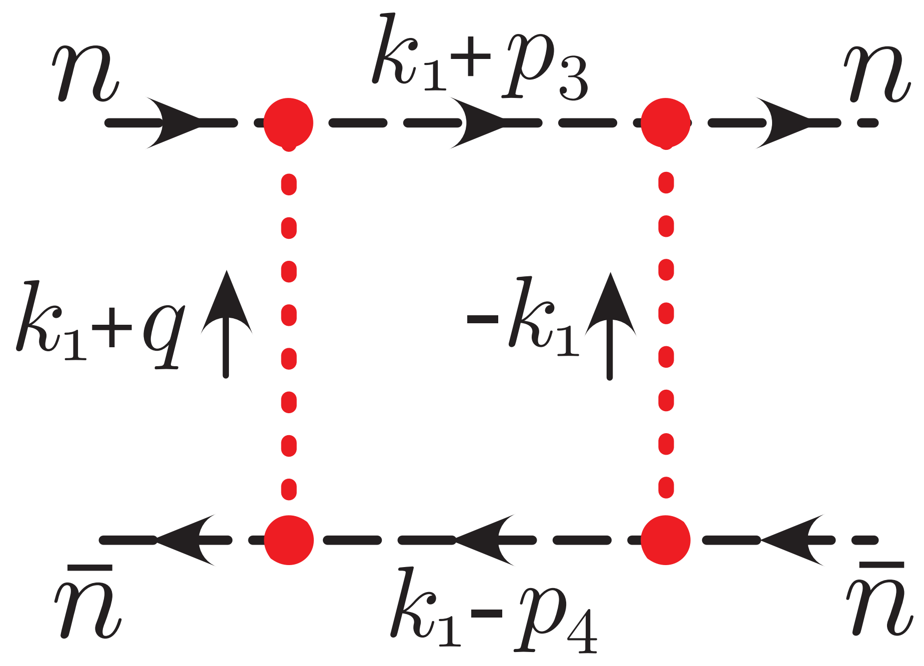
- Quantum corrections should only modify a single Glauber exchanges. Corrections be should be absorbable into the convolution in momentum space.



The EFT shows us that both of these conditions fail at three loops

-When do the energetic lines NOT behave semi-classically?

Glauber mode exchange: $$FT(1/k_\perp^2) \sim \delta(x_+)\delta(x_-)$$

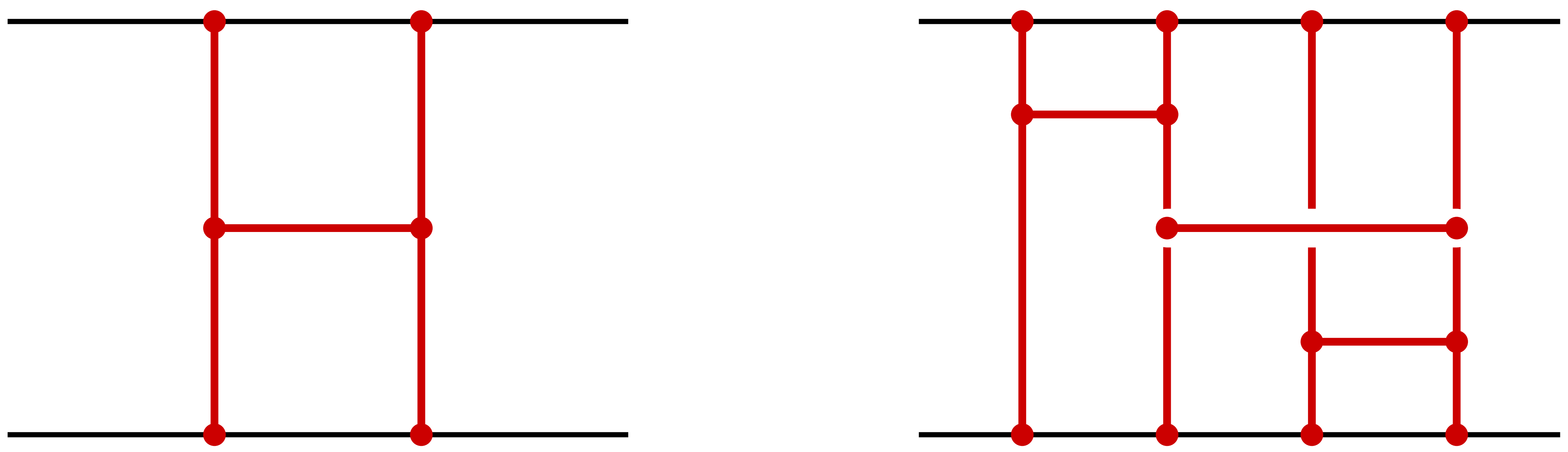


Exchange is instantaneous

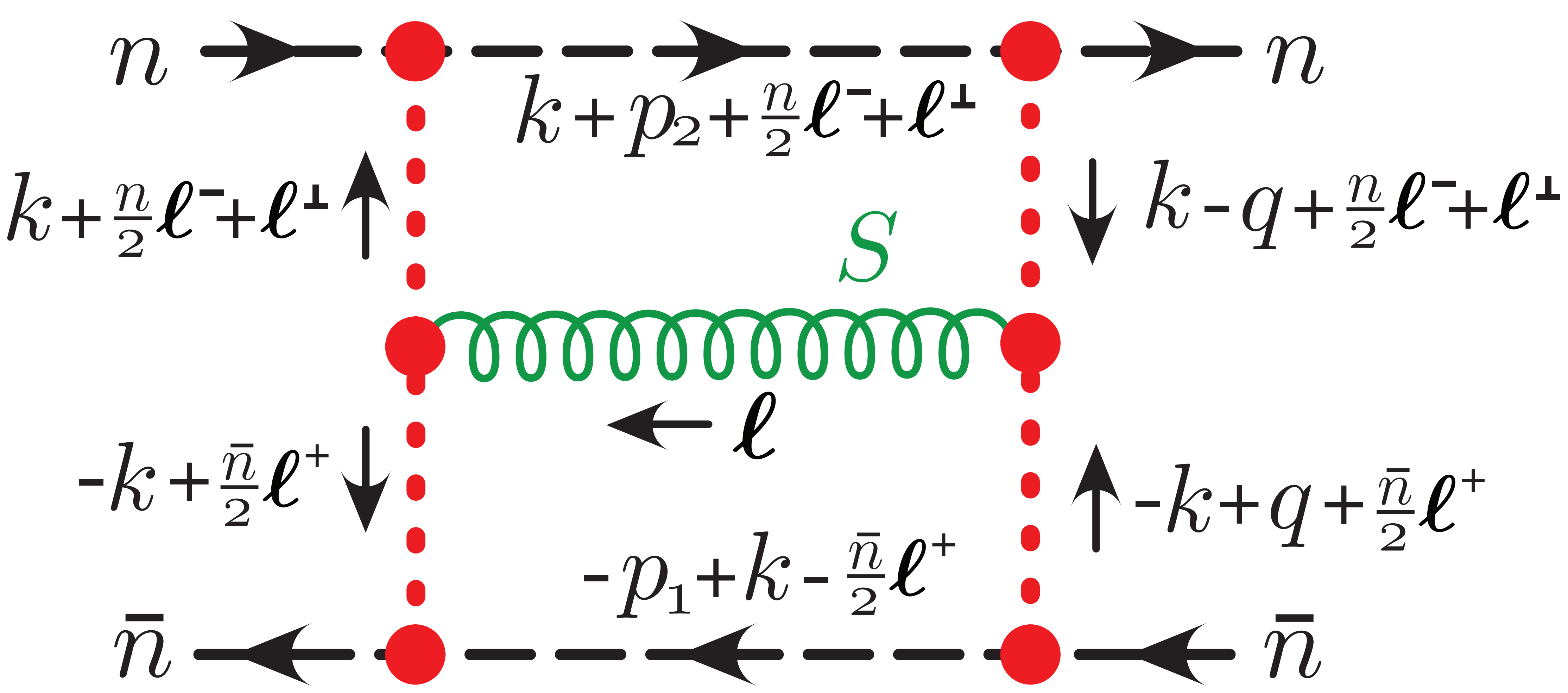
Glaubers form ``bursts" which build up the Shock wave (AS) metric. As long as nothing interupts the burst it will eikonalize.

If we have only soft corrections this would indeed be true.

Proof: Consider first all possible corrections that are two particle irreduciible which scale classically. These correspond to H graph topologies:



Remarkably all such digrams cane be resummed in the EFT as a consequence of the fact that the EFT enforces homogenous scaling of integral



$l_S \sim (\lambda, \lambda, \lambda)$

By routing the light cone momenta appropriately $l_S^\pm$
Does not show up in the box propagators. All external box light cone integrals can be done trivially, leaving only two-d transverse momentum integrals.

Lipatov/Verlinde+Verlinde 1991 (Two d field theory?)

Consider A unattached Glaubers and B insertions of H's

$$\sum_{A,B} N(A,B) = \frac{1}{(A+2B)!}\binom{A+2B}{A}(2B-1)!! = \frac{1}{A!2^B B!} \longrightarrow e^{i\delta = i(\delta_0+\delta_2)}$$

Box integrations, consequence of rapidity regulator

Similar arguments show that

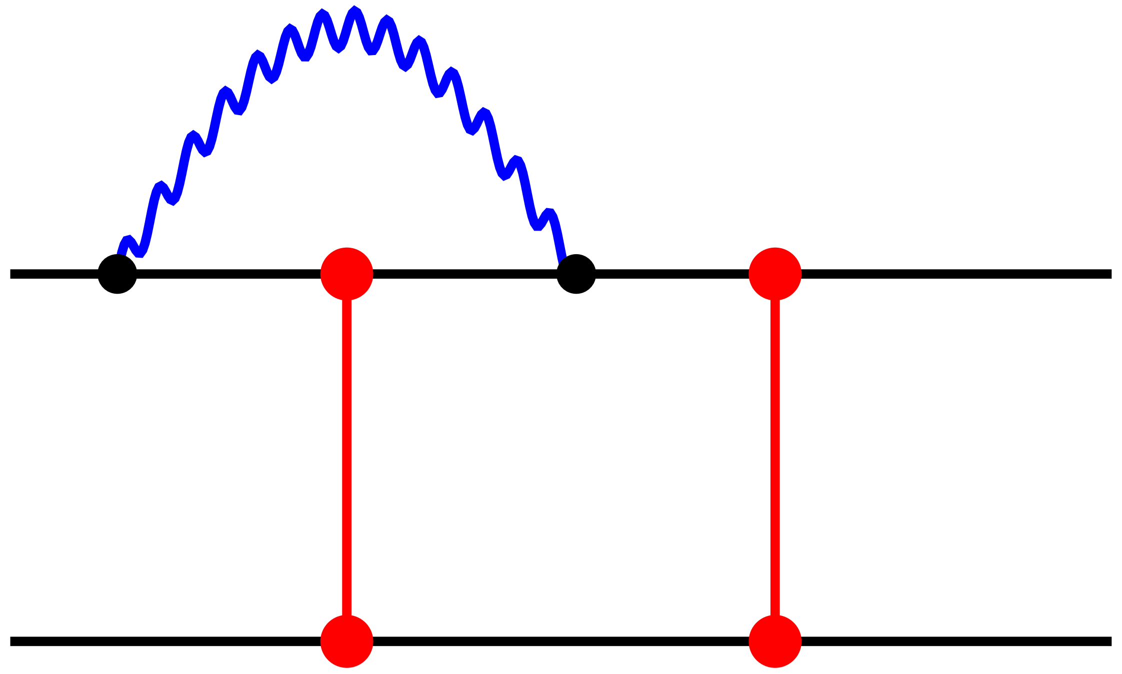
$$\delta_{\rm cl}(b,s) = \sum_{L=0}^{\infty} \alpha_C^L\, \delta_H^{(L)}(b,s), \qquad \delta_H^{(L)}(b,s) = \frac{1}{S_L}\int \prod_{a=1}^{L+1} \frac{d^{2-2\epsilon}q_a}{(2\pi)^{2-2\epsilon}} \exp\left(ib\cdot\sum_{a=1}^{L+1} q_a\right) \mathcal{H}_L(q_1,\ldots,q_{L+1};s).$$

This shows that to all orders soft emissions are all consistent with the ansatz. The problem is with the collinear emissions.

Weinberg (1965) famous showed that for wide angle scattering there are no collinear divergences at leading power in soft limit.

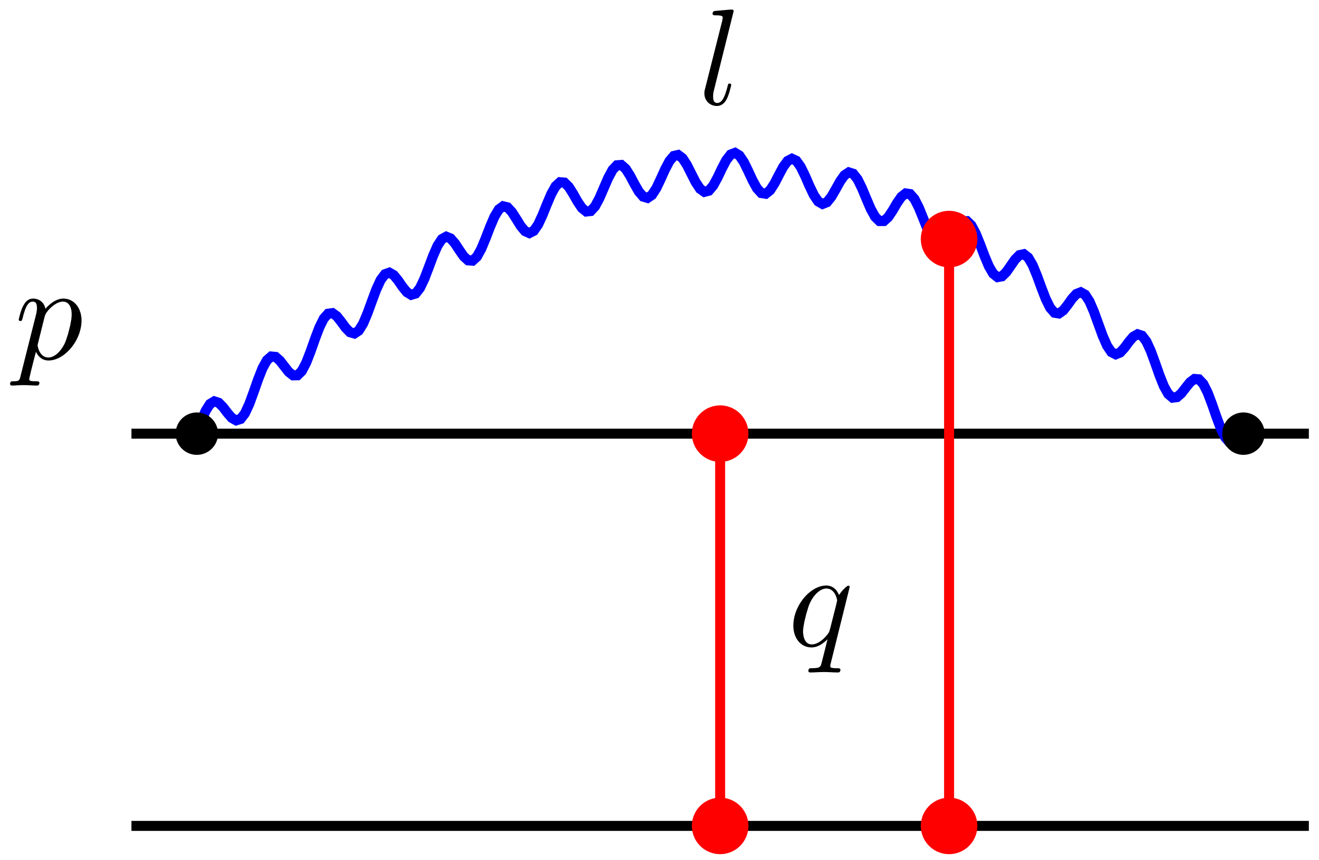
(Generalized to non-eikonal by Akhoury et al)

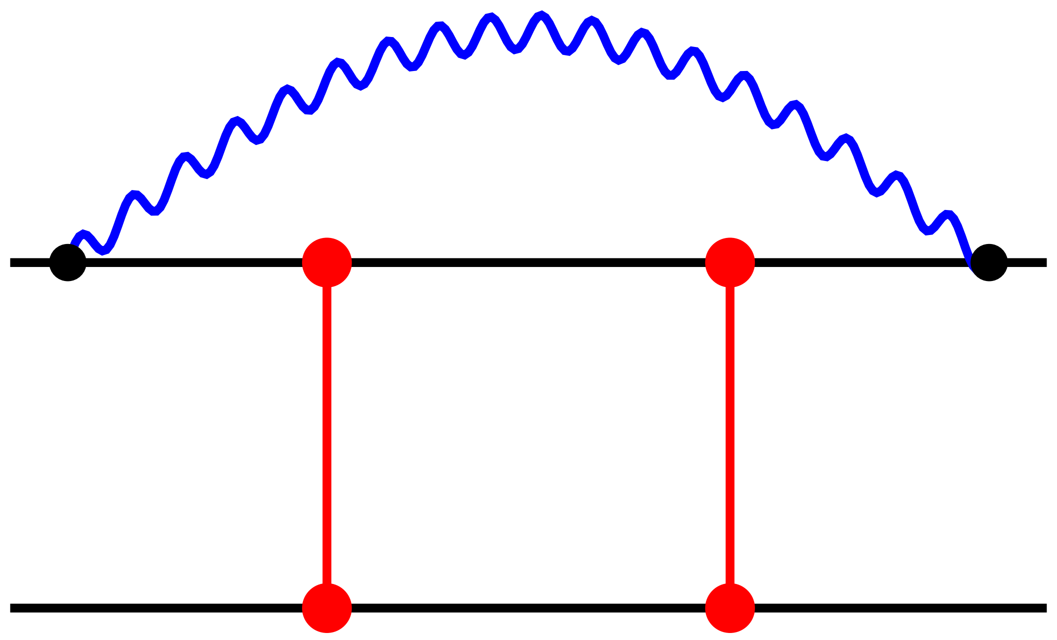
However, the possibile lack of collinear divergences does not imply that there are no collinear (non-analytic) contributions.



=0. Shock waves can
no be disrupted.

However, each collinear partons has its
own Glauber burst





These are form factor like corrections which are NOT
convoluations in momentum space

All orders form of the Superplanckian Amplitude

$$M(b) = \sum_i (1+\Delta_i) \otimes \prod_{i\otimes} e^{i(\delta_{cl}+\delta_Q^s)}$$

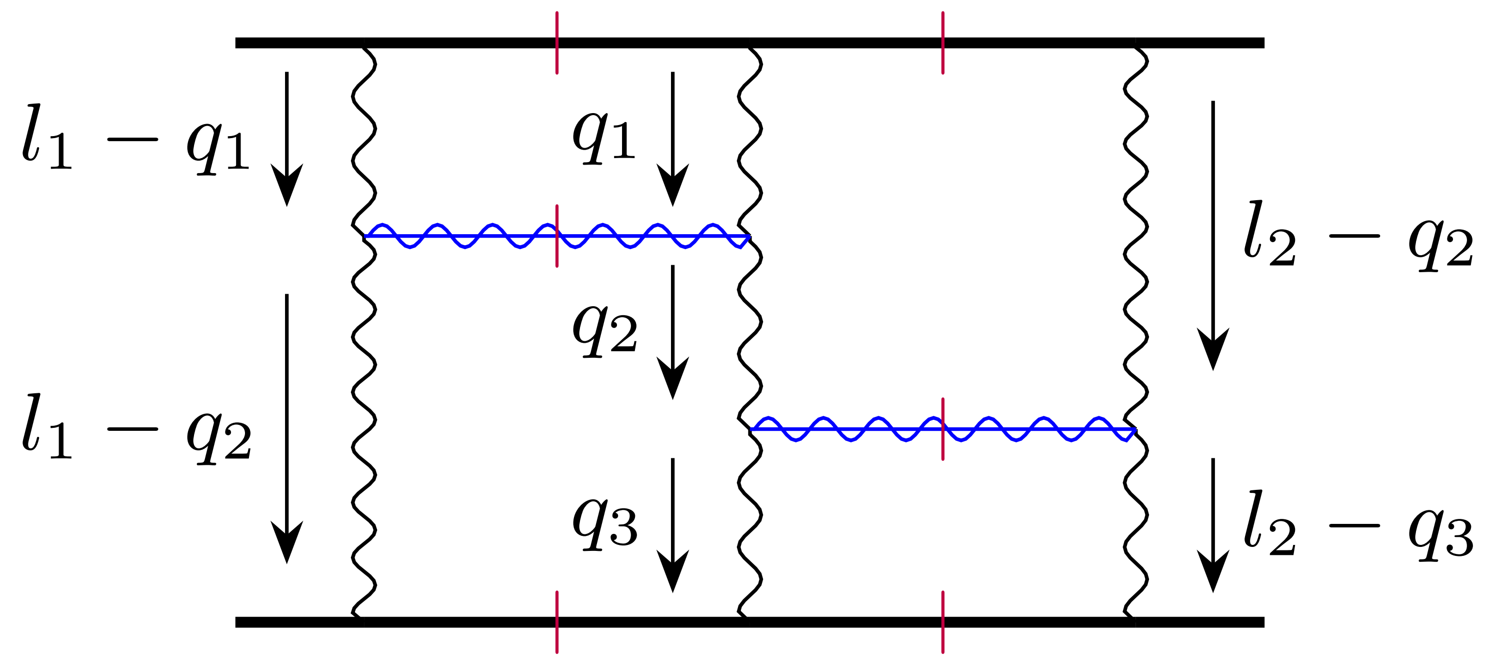
Glauber burst for each Parton

Number of collinear partons

EFT factorization gives all orders form for the amplitude, allows us to consistently separate quantum from classical and strongly constrains the form of the amplitude

Can use it e.g. to calculate higher order CLASSICAL Regge Logs using rapidity RG

(Alessio et al 2025)



Iterate 1-loop rapidity anomalous dimension twice

$$2\mathrm{Re}\tilde{\mathcal{M}}^{(4)}_{2\to 2}(s,b^2) \simeq -256\, G_N^5 \frac{s^3\log^2(s)}{(\pi b^2)^2}\left(\pi b^2 \varsigma e^{\gamma_E}\right)^{5\epsilon} \times \left[\frac{1}{8\epsilon^2} - \frac{1}{\epsilon} + \frac{5}{2} + \frac{5}{16}\zeta_2 - \frac{1}{4}\zeta_3 + O(\epsilon)\right].$$

There are other open problems in this limit having to do with radiation (``energy crisis”) that I think the EFT can shed light on