# A precise $\alpha_s$ determination from the R-improved QCD Static Energy

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# Outline


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# Introduction

- $\alpha_s$ most important parameter in QCD $\rightarrow$ should be determined with high precision.
- Our strategy $\rightarrow$ comparing the QCD Static Energy obtained in lattice simulations with highly accurate perturbative results.
- $V_{\rm QCD} \propto \alpha_s \rightarrow$ very sensitive.
- We improve this method building on previous analyses [A. Bazavov, N. Brambilla, X. Garcia i Tormo et al, 2012, A. Bazavov, N. Brambilla, X. Garcia i Tormo et al, 2014] in several ways:
  - Leading renormalon subtraction $\rightarrow$ short-distance scheme (MSR).
  - Resummation of associated large logs with R-evolution.
  - Profiles functions for the renormalization scales.
- We can fit (for the first time) lattice data up to $r \sim 0.5\,\text{fm}$.

# Static Energy and Static Potential

- The Static Energy is defined as the potential energy between an infinitely massive quark anti-quark pair at a distance $r$, corrected by ultra-soft effects.
- The usual computation is

$$V_s(r) = \lim_{T\to\infty} \frac{1}{iT} \ln \left\langle P \exp\left( ig \oint dx_\mu A^\mu \right) \right\rangle = \lim_{T\to\infty} \frac{1}{iT} \ln \langle W_\square \rangle$$

- It is useful to compute it in pNRQCD [N. Brambilla, A. Pineda, J. Soto, A. Vairo, 1999] $\rightarrow$ separation of scales:

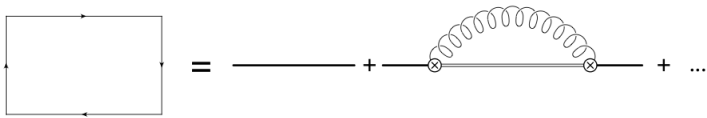
$$m \gg mv \gg mv^2 \;\longleftrightarrow\; \mu_h \gg \mu_s \gg \mu_{\rm us}$$

- Computed in pNRQCD it is just the sum of a matching coefficient and an ultra-soft contribution.

$$E_{\rm s}(r) = V_{\rm s}(r,\mu) + \delta_{\rm us}(r,\mu).$$

- Matching NRQCD and pNRQCD 2-fermion Green functions

# Matching




- In NRQCD

$$I = \delta^3(\mathbf{x}_1 - \mathbf{y}_1)\delta^3(\mathbf{x}_2 - \mathbf{y}_2)\langle W_\square \rangle$$

- In pNRQCD

$$\begin{aligned} I =& Z_s(r)\delta^3(\mathbf{x}_1 - \mathbf{y}_1)\delta^3(\mathbf{x}_2 - \mathbf{y}_2)e^{-iTV_s(r)} \\ &\times \left(1 + iTC_F\frac{\alpha_s}{\pi}\frac{r^2}{3}(V_o - V_s)^3\left\{\frac{1}{\epsilon} + \gamma_E + \ln\frac{(V_o - V_s)^2}{4\pi\mu^2} + \text{constant}\right\}\right. \\ &\left. + C_F\frac{\alpha_s}{\pi}r^2(V_o - V_s)^2\left\{\frac{1}{\epsilon} + \gamma_E + \ln\frac{(V_o - V_s)^2}{4\pi\mu^2} + \text{constant}\right\} + O\left(\frac{1}{T}\right)\right). \end{aligned}$$

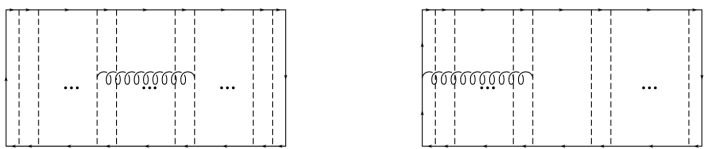
- At first order:

$$Z_s(r)e^{-iTV_s(r)} = \langle W_\square \rangle$$

# Matching

- Since $V_o - V_s \approx C_A \alpha_s / 2r \to \ln(\alpha_s) \to$ IR divergence
- This IR divergences is a perturbation theory artifact and cancel when adding US terms. [C. Anzai, Y. Kiyo and Y. Sumino, 2009]

*a)* *b)*



a)

$$\delta \frac{i}{T} \ln\langle W_\square \rangle = -\frac{C_F C_A^3}{24} \frac{\alpha_s}{r} \frac{\alpha_s^3}{\pi} \ln((V_o - V_s)^2 r^2) + \frac{i}{T} \frac{C_F C_A^2}{12} \frac{\alpha_s^3}{\pi} \ln((V_o - V_s)^2 r^2) + O\left(\frac{1}{T^2}\right),$$

b)

$$\delta \frac{i}{T} \ln \langle W_\square \rangle = \frac{i}{T} \frac{C_F C_A^2}{6} \frac{\alpha_s^3}{\pi} \ln((V_o - V_s)^2 r^2) + O\left(\frac{1}{T^2}\right).$$

# Matching

- If we add this US contribution we obtain the static potential.

$$V_s(r,\mu) = \lim_{T\to\infty} \frac{i}{T} \ln \langle W_\square \rangle + C_F C_A^2 \frac{\alpha_s^3}{12\pi} (V_o - V_s) \ln \frac{(V_o - V_s)^2}{4\pi\mu^2},$$

$$\lim_{T\to\infty} \frac{i}{T} \ln \delta \langle W_\square \rangle = -C_F C_A^2 \frac{\alpha_s^3}{12\pi} (V_o - V_s) \ln \left( (V_o - V_s)^2 r^2 \right).$$

- We find that the static potential does not simply coincide with the Wilson loop.
- Beyond perturbation theory the IR divergence is absent and regularized by $V_o - V_s$.
- The US part is the difference between the static potential and its perturbative expansion

$$[V_{\rm QCD}(r,\mu)]_{\rm US} = V_{\rm QCD}(r,\mu) - [V_{\rm QCD}(r,\mu)]_{\rm PT} \to \delta_{\rm us}(r,\mu) = E_s(r,\mu) - V_s(r,\mu)$$

# Static Potential

- The Static Potential is the basic object to understand the behavior of non-relativistic QCD:

$$V_{\rm s}(r,\mu) = V_{\rm s}^{\rm soft}(r,\mu) + V_{\rm s}^{\rm us}(r,\mu),$$
$$V_{\rm s}^{\rm soft}(r) = -C_F \frac{\alpha_s(\mu)}{r} \sum_{i=0}^{3} \left[\frac{\alpha_s(\mu)}{4\pi}\right]^i \sum_{j=0}^{i} a_{ij} \log^j \left(r\mu e^{\gamma_E}\right).$$

- Coefficients $a_{i0}$ are known to three loops. $a_{ij\geq 0}$ obtained with RGE. $\alpha_s(\mu) = \alpha_s^{(n_\ell=3)}(\mu)$.
- At $\mathcal{O}(\alpha_s^4)$ ultra-soft contributions show up for the first time

$$V_{\rm s}^{\rm us}(r,\mu) = -\frac{C_A^3 C_F}{12\pi} \frac{\alpha_s^4(\mu)}{r} \log\left(\mu r e^{\gamma_E}\right).$$

# Ultra-soft term and resummation

- $\mu_{us}$ appears both in the ultrasoft static potential $V_s^{us}$ and in the matrix element $\delta_{us}$

$$\delta_{\rm us}\left(\mu_s,\mu_{\rm us}\right) = -\frac{C_A^3 C_F}{12\pi}\frac{\alpha_s^3\left(\mu_s\right)\alpha_s\left(\mu_{\rm us}\right)}{r}\log\left[\frac{C_A\alpha_s\left(\mu_s\right)e^{-5/6}}{\mu_{\rm us} r}\right]$$

$$V_{\rm s}^{\rm us}(r,\mu_s,\mu_{\rm us}) = -\frac{C_A^3 C_F}{12\pi}\frac{\alpha_s^3(\mu_s)\alpha_s(\mu_{\rm us})}{r}\log\left(\mu_{\rm us} r e^{\gamma_E}\right)$$

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- One can see the necessity of resummation with these two (incompatible) choices that minimize logs.
  - $\mu_{us} \sim \alpha_s/r$ [use this one: no large logs in $\delta_{\rm us}$]
  - $\mu_{us} \sim 1/r$ [discarded: sum up logs in $V_s^{\rm us}$]

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- We perform resummation using pNRQCD RGE

$$\mu\frac{\mathrm{d}V_s\left(r,\mu_s,\mu\right)}{\mathrm{d}\mu}=-\frac{2C_FC_A^3}{24r}\frac{\alpha_s(\mu)}{\pi}\left[1+B\frac{\alpha_s(\mu)}{\pi}\right]\alpha_s^3\left(\mu_s\right)\left\{1+3\frac{\alpha_s\left(\mu_s\right)}{4\pi}\left[a_{1,0}+2\beta_0\log\left(r\mu_s e^{\gamma_E}\right)\right]\right\}.$$

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- Integrating from $\mu=\mu_s\sim 1/r$ to $\mu=\mu_{us}\sim\alpha_s/r$

$$\begin{aligned}V_s\left(r,\mu_s,\mu_{\mathrm{us}}\right)=&V_s\left(r,\mu_s\right)+U_{\mathrm{us}}\left(r,\mu_s,\mu_{\mathrm{us}}\right),\\ U_{\mathrm{us}}\left(r,\mu_s,,\mu_{us}\right)=&\frac{C_A^3C_F}{6\beta_0 r}\alpha_s^3\left(\mu_s\right)\left\{\left(1+3\frac{\alpha_s\left(\mu_s\right)}{4\pi}\left[a_{1,0}+2\beta_0\log\left(r\mu_s e^{\gamma_E}\right)\right]\right)\log\left[\frac{\alpha_s\left(\mu_{\mathrm{us}}\right)}{\alpha_s\left(\mu_s\right)}\right]\right.\\ &\left.+\left(B-\frac{\beta_1}{4\beta_0}\right)\left[\frac{\alpha_s\left(\mu_{\mathrm{us}}\right)}{\pi}-\frac{\alpha_s\left(\mu_s\right)}{\pi}\right]\right\}\end{aligned}$$

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- Replacing in the static energy $\rightarrow$ no large logs for $\mu_s\sim 1/r$ and $\mu_{us}\sim\alpha_s/r$

$$\begin{aligned}E_{\mathrm{s}}(r)=&V_{\mathrm{s}}^{\mathrm{soft}}\left(r,\mu_s\right)+\frac{C_A^3C_F}{12}\frac{\alpha_s^3\left(\mu_s\right)}{r}\left\{\frac{2}{\beta_0}\left(B-\frac{\beta_1}{4\beta_0}\right)\left[\frac{\alpha_s\left(\mu_{\mathrm{us}}\right)}{\pi}-\frac{\alpha_s\left(\mu_s\right)}{\pi}\right]\right.\\&+\frac{2}{\beta_0}\left(1+3\frac{\alpha_s\left(\mu_s\right)}{4\pi}\left[a_{1,0}+2\beta_0\log\left(r\mu_s e^{\gamma_E}\right)\right]\right)\log\left[\frac{\alpha_s\left(\mu_{\mathrm{us}}\right)}{\alpha_s\left(\mu_s\right)}\right]\\&\left.-\frac{\alpha_s\left(\mu_{\mathrm{us}}\right)}{\pi}\log\left[\frac{C_A\alpha_s\left(\mu_s\right)e^{-5/6}}{r\mu_{\mathrm{us}}}\right]-\frac{\alpha_s\left(\mu_s\right)}{\pi}\log(r\mu_s e^{\gamma_E})\right\}.\end{aligned}$$

# R-improvement, renormalon substractions

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- To cancel the renormalon we need a short-distance mass scheme $\rightarrow$ MSR mass [A. H. Hoang, A. Jain, C. Lepenik, V. Mateu, I. Scimemi and I. W. Stewart, 2018]

$$\delta m_Q^{\rm MSR}(\mu, R) \equiv m_Q^{\rm pole} - m_Q^{\rm MSR}(R) = R\sum_{n=1}^{\infty}\delta_n^R\left[\frac{\alpha_s(R)}{4\pi}\right]^n$$

$$= R\sum_{n=1}^{\infty}\left[\frac{\alpha_s(\mu)}{4\pi}\right]^n\sum_{j=0}^{n-1}\delta_{nj}^R\log^j\left(\frac{\mu}{R}\right).$$

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- New type of large logs $\rightarrow$ resummation using R-evolution [A. H. Hoang, A. Jain, I. Scimemi and I. W. Stewart, 2008].

# Previous works

- To see the advantages of R-evolution we go back to the first work that used the static energy to obtain $\alpha_s$ [A. Bazavov, N. Brambilla, X. Garcia i Tormo et al, 2012]

$$E_s(r) = V_s(r, \mu_s, \mu_{us}) + \delta_{US}(r, \mu_s, \mu_{us}) + RS(\rho),$$

  - $V_s(r, \mu_s, \mu_{us}) \to \ln(r\mu_{us})$
  - $RS(\rho) \to \ln(\rho/\mu_{us})$
- There is no right choice for $\mu_{us} \to$ fits only posible for small values of $r$.
- The force was used in [TUMQCD collaboration, 2019]

$$F_{\rm s}(r) = \frac{{\rm d}E_{\rm s}(r)}{{\rm d}r}$$

- Force avoids explicit substraction.
- Using fully canonical scales $\sim 1/r \to$ limits fit-range to $r \sim 0.076$ fm.

# R-improvement, renormalon substractions

- The next goal is to obtain a renormalon-free potential.

$$E_{Q\overline{Q}}(r) = 2m_Q^{\rm MSR}(R_0) + 2\delta m_Q^{\rm MSR}(R_0,\mu) + E_s(r) \equiv 2m_Q^{\rm MSR}(R_0) + V_s^{\rm MSR}(r,\mu,R_0).$$

- We sum up large logs of $\mu/R_0$.

$$\begin{aligned}\delta m_Q^{\rm MSR}(R_0) &= \delta m_Q^{\rm MSR}(R_0) + \delta m_Q^{\rm MSR}(R) - \delta m_Q^{\rm MSR}(R) \\ &= \delta m_Q^{\rm MSR}(R) + m_Q^{\rm MSR}(R) - m_Q^{\rm MSR}(R_0) \\ &= \delta m_Q^{\rm MSR}(R) + \Delta^{\rm MSR}(R,R_0).\end{aligned}$$

- $\Delta^{\rm MSR}(R,R_0) = \int_{R_0}^{R} {\rm d}R' \gamma^R(R')$: solution to MSR mass R-RGE $\rightarrow$ sums up logs of $R/R_0$.
- We have $\delta m_Q^{\rm MSR}(R) \sim \log(\mu/R)$. By choosing $\mu \sim R \rightarrow$ no large logs.
- We define the R-improved static potential:

$$V_s^{\rm MSR}(r,\mu,R_0) = V_s(r,\mu) + 2\delta^{\rm MSR}(R,\mu) + 2\Delta^{\rm MSR}(R,R_0)\,.$$

# Force-type subtractions

- Integrating the Force is equal to perform R-evolution.
- The renormalon doesn't depend on r $\rightarrow$ we can subtract the potential at $r_f$.

$$E_{\rm s}^F(r,r_0) \equiv E_{\rm s}(r) - E_{\rm s}(r_0) = E_{\rm s}(r) - E_{\rm s}(r_f) + [E_{\rm s}(r_f) - E_{\rm s}(r_0)]$$
$$= E_{\rm s}(r) - E_{\rm s}(r_f) + \int_{r_0}^{r_f} {\rm d}r' F_{\rm s}(r') \equiv E_{\rm s}(r) - E_{\rm s}(r_f) + 2\Delta_F(r_0, r_f)$$

- $r_f = r \rightarrow$ reproduce TUMQCD approach
- Connecting with R-evolution, the substraction term is (choosing $R = 1/r_1$ and $\mu = R$):

$$E_s(r_f) \equiv \delta^F_{\rm soft}(R) = -\frac{1}{2} V_{\rm s}^{\rm soft}\left(\frac{1}{R}\right) = 2\pi C_F R \sum_{i=1} \left[\frac{\alpha_s(R)}{4\pi}\right]^i \sum_{j=0}^{i-1} a_{i-1,j}\gamma_E^j \equiv R\sum_{i=1}\left[\frac{\alpha_s(R)}{4\pi}\right]^i \delta_i^F .$$

- We can express $\Delta_F$ as an R-evolution integral:

$$\gamma^F_{\rm soft}(R) = \frac{\rm d}{{\rm d}R}\delta^F_{\rm soft}(R) = -\frac{1}{2}\left[r^2 F_{\rm s}^{\rm soft}(r)\right]_{r=1/R},$$
$$\Delta_F^{\rm soft}(r_0, r_f) = -\frac{1}{2}\int_{r_0}^{r_f} \frac{{\rm d}r'}{(r')^2}\left[(r')^2 F_{\rm s}^{\rm soft}(r')\right] = \int_{1/r_0}^{1/r_f} {\rm d}R' \gamma^F_{\rm soft}(R') .$$

# Potential-type subtractions

- If we choose $\mu = R$ and $R = \frac{e^{-\gamma_E}}{r}$

$$\delta^V_{\rm soft}(R) \equiv -\frac{1}{2} V_s^{\rm soft}\left(\frac{e^{-\gamma_E}}{R}\right) = 2\pi e^{\gamma_E} C_F R \sum_{i=1}\left[\frac{\alpha_s(R)}{4\pi}\right]^i a_{i-1,0} \equiv R\sum_{i=1}\left[\frac{\alpha_s(R)}{4\pi}\right]^i \delta_i^V.$$

- We can obtain their anomalous dimensions

$$\gamma^V_{\rm soft}(R) = -\frac{1}{2}\left[r^2 V_s^{\rm soft}(r)\right]_{r=1/R}$$

$$\Delta_V^{\rm soft}(r_0, r_1) = \int_{r_0}^{r_1} \frac{{\rm d}r'}{(r')^2}\left[(r')^2 V_s^{\rm soft}(r')\right] = \frac{1}{2}\int_{1/r_0}^{1/r_1} {\rm d}R' \gamma^V_{\rm soft}(R')$$

# Subtraction schemes: PS scheme

- Defined from its relation to the pole mass: $m_p - m^{\rm PS}(R) \equiv \delta_{\rm PS}(R)$

$$\delta^{\rm PS}_{\rm soft}(R) = -\frac{1}{2}\int_{|\vec{q}|<R}\frac{{\rm d}^3\vec{q}}{(2\pi)^3}\tilde{V}^{\rm soft}_{\rm s}(q,R) \equiv C_F R\frac{\alpha_s(R)}{\pi}\sum_{i=0}\left[\frac{\alpha_s(R)}{4\pi}\right]^i c_i,$$

$$c_i = \sum_{j=0}^{i} a_{i,j}h_j, \qquad \text{with} \qquad h_j = j!\sum_{\ell=0}^{j}\sum_{k=0}^{\text{floor}\left[\frac{\ell}{2}\right]}\kappa_{\ell-2k}\left(\frac{\pi}{2}\right)^{2k}\frac{(-1)^k}{(2k)!}.$$

# Subtraction schemes: ultrasoft terms

- Since all these three schemes come from the potential, they have ultrasoft terms in their R-evoluition.

$$\gamma^F(R) = \gamma^F_{\rm soft}\,(R) - \frac{C_A^3 C_F}{24\pi}\,[\alpha_s(R)]^4 \log\left[C_A \alpha_s(R) e^{\gamma_E - 5/6}\right],$$

$$\gamma^V(R) = \gamma^V_{\rm soft}\,(R) - \frac{C_A^3 C_F}{24\pi} e^{\gamma_E}\,[\alpha_s(R)]^4 \log\left[C_A \alpha_s(R) e^{\gamma_E - 5/6}\right],$$

$$\gamma^{\rm PS}(R) = \gamma^{\rm PS}_{\rm soft}(R) - \frac{C_F C_A^3}{12\pi^2}\,[\alpha_s(R)]^4 \log\left[C_A \alpha_s(R) e^{\gamma E - 5/6}\right].$$

- They inherit the infrared problem of the static potential.

# RS scheme

- It is defined from the pole mass by subtracting its leading asymptotic behavior [Antonio Pineda, 2001].

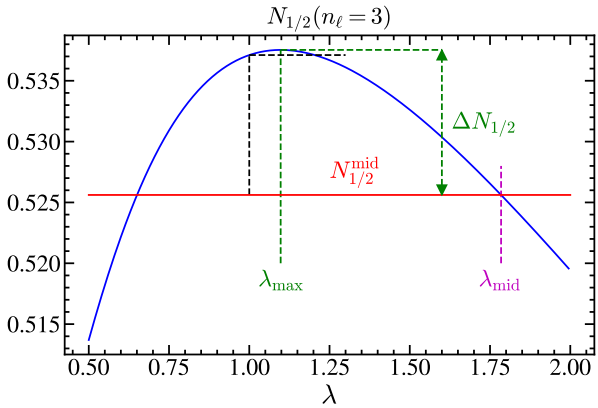
$$m_Q^{\rm pole} - m_Q^{\rm RS}(R) = \frac{2\pi}{\beta_0} R N_{1/2} \sum_{n=1}^{\infty} \left[\frac{\beta_0 \alpha_s(R)}{2\pi}\right]^n \sum_{\ell=0}^{\infty} g_\ell \left(1+\hat{b}_1\right)_{n-1-\ell},$$

- $N_{1/2}$ is the normalization of the leading renormalon

$$N_{1/2}^{(n)} = \frac{\beta_0}{2\pi} \sum_{k=0}^{n} \frac{S_k}{\left(1+\hat{b}_1\right)_k} \qquad S_k = \sum_{k=0}^{j} \tilde{\gamma}_k^R \sum_{i=0}^{j-k} (-1)^i \tilde{b}_i^N \tilde{g}_{j-i-k}^N,$$

- $N_{1/2}$ depends on $\lambda$, it reshuffles higher perturbative orders. We vary it to estimate $N_{1/2}$ (similar to scale variation).

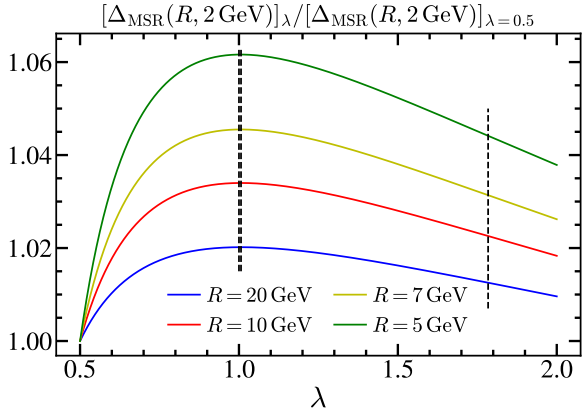
# $N_{1/2}$




- The canonical value of $\lambda = 1$ is clearly biased.
- We pick $\lambda = 1.784$ and vary it from 1.5 to 2.1.

# Dependence on $\lambda$

- We don't know the R-evolution kernel up to infinite order $\rightarrow$ can estimate the truncation error with a dimensionless parameter $\lambda$, its variation account for higher-order remnantes (similar to renormalization scale variation).



# Profiles

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- Solution: use profile functions that ensure series convergence:

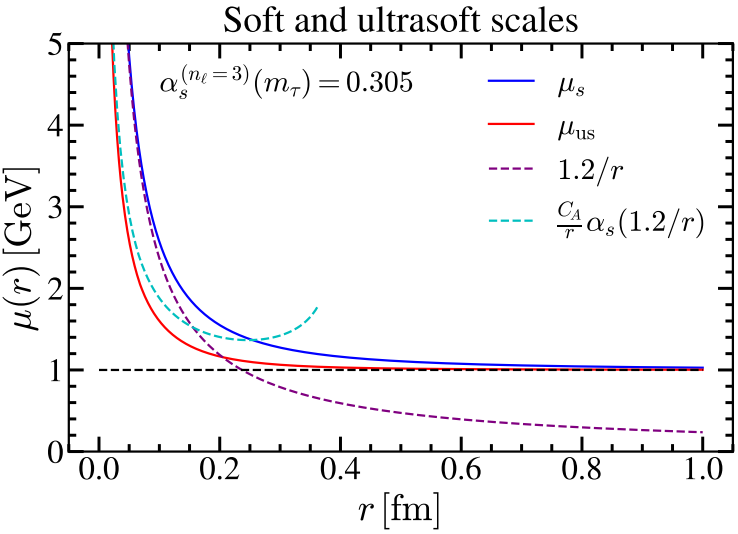
$$\mu_s(r,\xi,\mu_\infty,\Delta,b) = \sqrt{\left(\frac{\xi}{r}\right)^2 + \frac{b}{r} + (\mu_\infty - \Delta)^2} - \Delta = \begin{cases} \frac{\xi}{r} & \text{for } r \to 0 \\ \mu_\infty & \text{for } r \to \infty \end{cases}$$

$$R(r,\beta,R_\infty,\Delta,b) = \mu_s(r,\beta,R_\infty,\Delta,b)$$

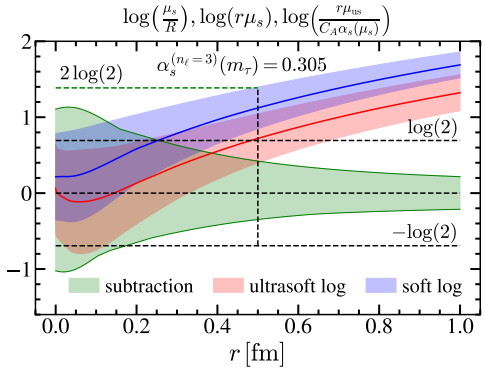
$$\mu_{\mathrm{us}}(r,\xi,\kappa,\mu_\infty,\Delta,b) = C_A\Big\{\mu_s(r,\kappa\xi,\mu_\infty,\Delta,b)\alpha_s\big[\mu_s(r,\kappa\xi,\mu_\infty,\Delta,b)\big] - \mu_\infty\alpha_s(\mu_\infty)\Big\} + \mu_\infty$$

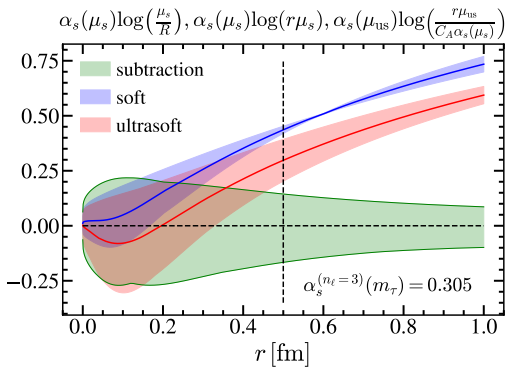
with $\xi = \mathcal{O}(1)$, $\mu_\infty \sim 1\,\mathrm{GeV}$ and $\mu_s \gg \mu_{\mathrm{us}}$ for small distances. This makes sure the series is stable and convergent over the entire spectrum.

# Profiles


Soft and ultrasoft scales
$\alpha_s^{(n_\ell=3)}(m_\tau) = 0.305$
$\mu_s$
$\mu_{\rm us}$
$1.2/r$
$\frac{C_A}{r}\alpha_s(1.2/r)$
$\mu(r)$ [GeV]
$r$ [fm]
0
1
2
3
4
5
0.0
0.2
0.4
0.6
0.8
1.0

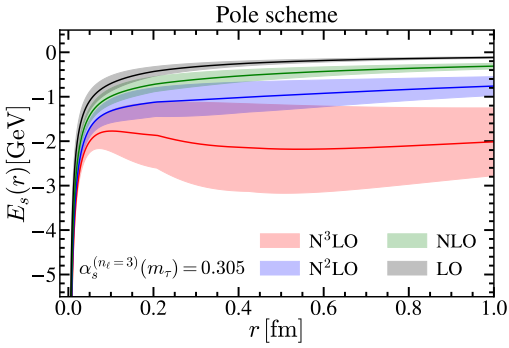
# Profiles

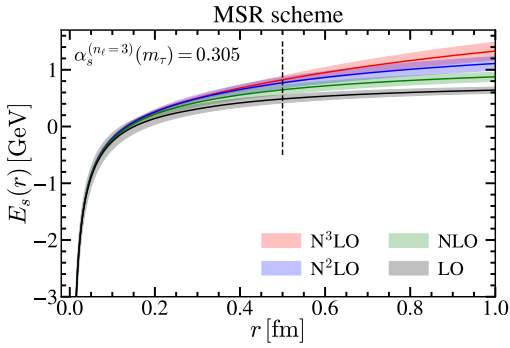



- Ultrasoft logarithm smaller than the soft one and $\mathcal{O}(1)$ in the fit range.
- The product of $\alpha_s(\mu)\times\log(\mu)$ is smaller than 1.

# Perturbative analysis



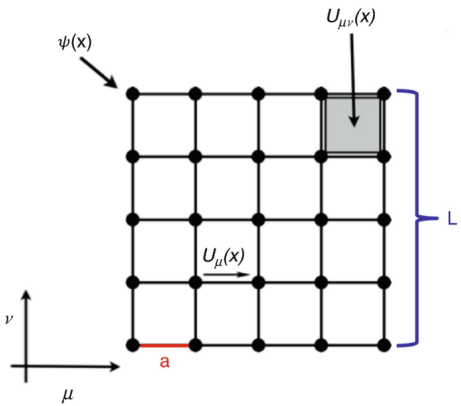

At $r = 0.3\,\mathrm{fm}$ we have the following uncertainties for LO, NLO, N$^2$LO and N$^3$LO:

- Pole scheme $\rightarrow [0.16, 0.31, 0.71, 1.86]\,\mathrm{GeV}$.
- MSR scheme $\rightarrow [0.17, 0.15, 0.13, 0.07]\,\mathrm{GeV}$.

# Lattice data

- Compare lattice data and perturbative expression. Are they the same?
- They discretisize the space time

$$U(x) \equiv \mathrm{P} \exp\left\{ \mathrm{ig} \int_x A_\mu \mathrm{d}x^\mu \right\}$$




- Computing the potential we get

$$\langle\psi|\mathrm{e}^{-TH}|\psi\rangle \underset{T\to\infty}{\sim} \mathrm{e}^{-TV(R)} = \frac{1}{Z}\int \mathrm{d}U \,\mathrm{Tr}\, U\left(C_{R,T}\right) e^{-S(U)}$$

$$U\left(C_{R,T}\right) = P \exp\left(\mathrm{ig} \oint_{C_{R,T}} dx_\mu A^\mu\right)$$

- Which is just the definition of the Wilson loop

$$\mathrm{e}^{-TV(R)} = \langle W_\square \rangle \to V(R) \equiv -\lim_{T\to\infty} \frac{1}{T} \log\langle W_\square \rangle$$

# Lattice data

- Lattice QCD data from HotQCD. $(2+1)$ flavor simulations. We have 9 lattices sets adding up to 2512 data points.
- Only diagonal part of statistic covariance matrix publicly available.
- Small-r data super precise but, potentially affected by large discretisation errors.
- To change from lattice to physical units we use $r_1 = 0.3093(20)\,\text{fm}$, average of [A. Bazavov et al, 2014] and [R. Larsen, S. Mukherjee, P. Petreczky et al, arXiv 2502.08061].

# Fits

- Each set $n$ has a different origin of the static energy $\rightarrow$ offset $A_n$, included in the $\chi^2$ function:

$$\chi^2(\alpha_s, \{A_k\}) = \sum_{k=1}^{N_s} \chi^2_k(\alpha_s, A_k) \qquad \chi^2_k(\alpha_s, A_k) = \sum_{i=1}^{n^k_{\text{data}}} \frac{\left[V_{ik}(\alpha_s) + A_k - V^{\text{exp}}_{ik}\right]^2}{[\sigma^2_k]_i}$$

- We can marginalize analytically first with respect to the offsets.

$$\frac{\partial \chi^2}{\partial A_k} = 0 \Longrightarrow \tilde{A}_k(\alpha_s) = \frac{\sum_{i=1}^{n^k_{\text{data}}} [\sigma_k^{-1}]^2_i \left[V^{\text{exp}}_{ik} - V_{ik}(\alpha_s)\right]}{\sum_{i=1}^{n^k_{\text{data}}} [\sigma_k^{-1}]^2_i},$$

$$\Longrightarrow \tilde{\chi}^2_k(\alpha_s) = \chi^2_k(\alpha_s, \tilde{A}_k(\alpha_s)) - \frac{\left\{\sum_{i=1}^{n^k_{\text{data}}} [\sigma_k^{-1}]_i \left[V_{ik}(\alpha_s) - V^{\text{exp}}_{ik}\right]\right\}^2}{\sum_{i=1}^{n^k_{\text{data}}} [\sigma_k^{-1}]_i}.$$

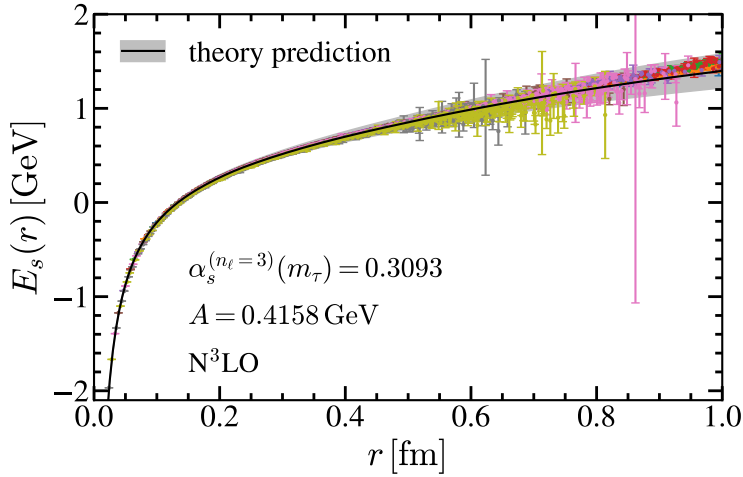
# Fits

- Minimizing we obtain $\alpha_s^{\rm BF}$ that verifies $\tilde{\chi}^2(\alpha_s^{\rm BF}) = \chi^2(\alpha_s^{\rm BF}, A^{\rm BF}) = \chi^2_{\rm min}$.
- The fit uncertainty is defined as $\tilde{\chi}^2(\alpha_s^{\rm BF} \pm \Delta_{\rm fit}\alpha_s) = \chi^2_{\rm min} + 1$.
- Theory uncertainties are estimated through a random scan.
- We perform the fit for 500 random profiles $\rightarrow \alpha_s^{\rm BF}$ and $\Delta_{\rm fit}\alpha_s$ for all of them.
- The theory (or perturbative) uncertainty is computed as

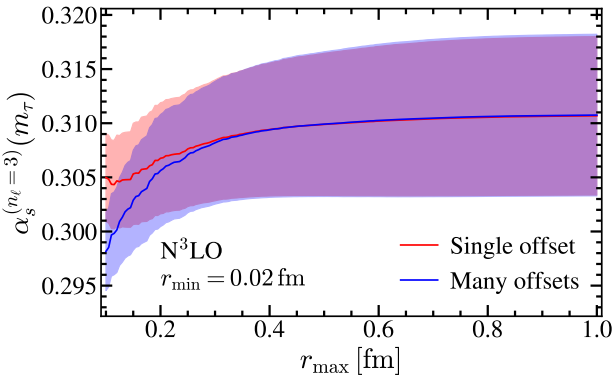
$$\Delta_{\rm pert}\alpha_s = (\alpha_{\rm max} - \alpha_{\rm min})/2$$

- The final result is the mean of the 500 $\alpha_s^{\rm BF}$ and $\Delta_{\rm fit}\alpha_s$.
- We obtain $\mathcal{O}(100)$ $\chi^2$ values $\rightarrow$ inflate lattice error by $\sqrt{\chi^2_{\rm min}/{\rm d.o.f.}}$.
- Fit for $r \in [r_{\rm min}, r_{\rm max}]$. $r_{\rm data} \geq 0.023$ fm.
- $r_{\rm min} \in [0.02, 0.045]$ fm and $r_{\rm max} \in [0.1, 1]$ fm with 0.05 fm spacing $\rightarrow$ 1086 different datasets.

# Static Energy

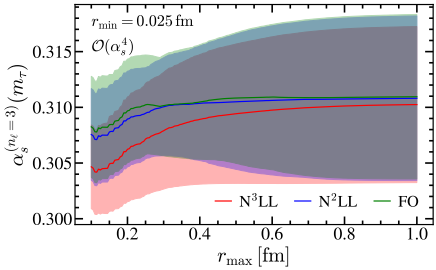


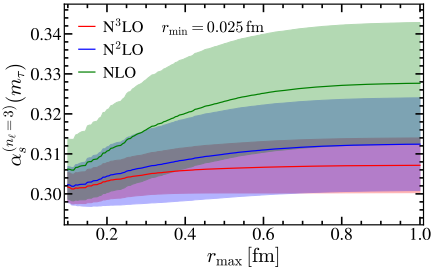
# Offset approach

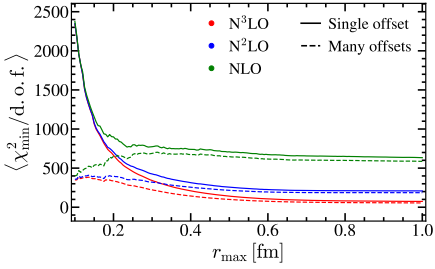



- We consider two offsets approaches:
  - Each ensemble has a different offset.
  - Common offset for all the ensembles.
- Both agree for $r_{\rm max} > 0.3$ fm.
- Overparametrization of $\chi^2$ at small distances for the many offset case.
- We consider $r_{\rm max} \geq 0.35$ fm.

# Order-by-order agreement

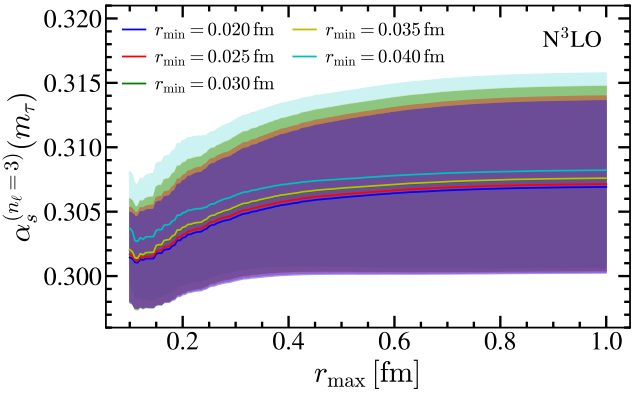


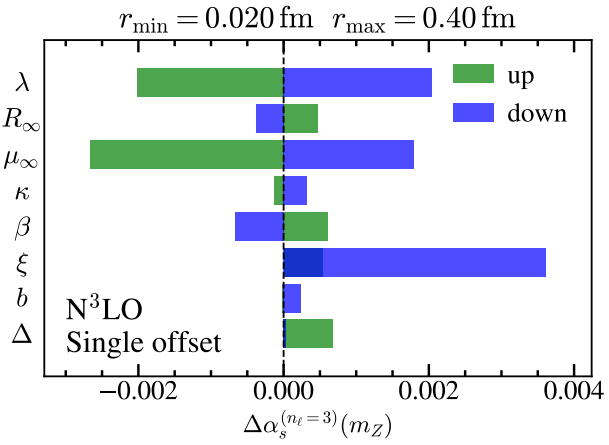

- Uncertainty bands nested for $r_{\max} < 0.5$ fm.
- We choose $r_{\max} \leq 0.45$ fm

# Study of the dataset




- We consider the range $0.02 \leq r_{min} \leq 0.04$ fm .

# Variation of profiles parameters one at a time




| Parameter | Default | Range |
|---|---|---|
| $\xi$ | $1.2 \times \hbar c$ | $[0.7, 2.2] \times \hbar c$ |
| $\beta$ | $1.2 \times \hbar c$ | $[0.7, 2.2] \times \hbar c$ |
| $b$ | 0 | $[-0.3, 0.3] \times \hbar c$ |
| $\Delta$ | 0 | $[-0.6, 0.6]$ GeV |
| $\mu_\infty$ | 1 GeV | $[0.9, 1.1]$ GeV |
| $R_\infty$ | 1 GeV | $[0.9, 1.1]$ GeV |
| $\kappa$ | 1 | $[0.8, 1.2]$ |
| $\lambda$ | 1.8 | $[1.5, 2.1]$ |

# Final results

- Requirements: $r_{\min} \in [0.02, 0.04]$ fm and $r_{\max} \in [0.35, 0.45]$ fm $\rightarrow$ 105-element dataset.
- We perform a fit in all of them:

$$\alpha_s^{(n_f=3)}(m_\tau) = 0.3093 \pm 0.00001_{\text{lattice}} \pm 0.0061_{\text{th}} \pm 0.0011_{\text{set}} \pm 0.0011_{r_1}$$

$$\downarrow$$

$$\alpha_s^{(n_f=3)}(m_\tau) = 0.3093 \pm 0.0063$$

$$\alpha_s^{(n_f=5)}(m_Z) = 0.1170 \pm 0.0008_{\text{th}} \pm 0.0001_{\text{set}} \pm 0.0001_{r_1} \pm 0.0003_{\mu_c} \pm 0.0002_{\mu_b}$$

$$\downarrow$$

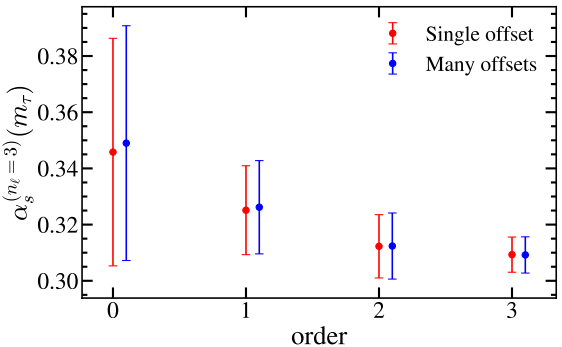
$$\alpha_s^{(n_f=5)}(m_Z) = 0.1170 \pm 0.0009$$

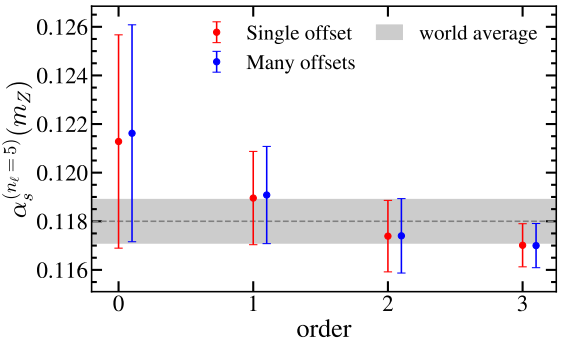
Competitive with the w.a. $\alpha_s^{(5)}(m_Z) = 0.1180 \pm 0.0009$ and compatible at 0.5-$\sigma$ level.
Comparing with previous analyses:

$\alpha_s^{(n_f=5)}(m_Z) = 0.11660^{+0.00110}_{-0.00056} \rightarrow \alpha_s^{(n_f=5)}(m_Z) = 0.1162(8)$ [TUMQCD collaboration, 2019]

$\alpha_s^{(n_f=5)}(m_Z) = 0.1181 \pm 0.0009 \rightarrow \alpha_s^{(n_f=5)}(m_Z) = 0.1177(9)$ [C. Ayala, X. Lobregat, A. Pineda, 2020]

# Final results



# Conclusions

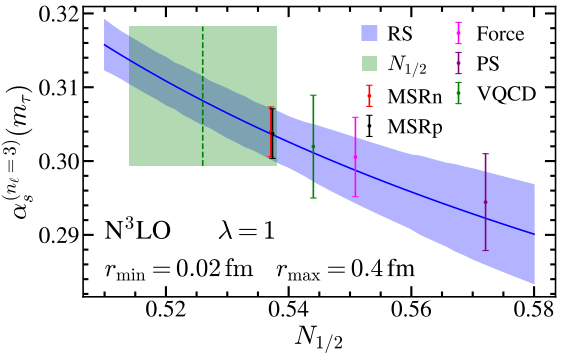
- Used the static energy to determine $\alpha_s$, building (and improving) on previous analyses.
- Performed ultra-soft large-log resummation up to $\mathrm{N^3LL}$.
- Employed the MSR mass and R-evolution to improve the static potential.
- Designed profile functions to increase the validity of the potential up to $r \sim 0.5\,\mathrm{fm}$.
- Studied carefully the dependence on the dataset.
- Carried out fits to lattice data to obtain a very competitive result for $\alpha_s$.

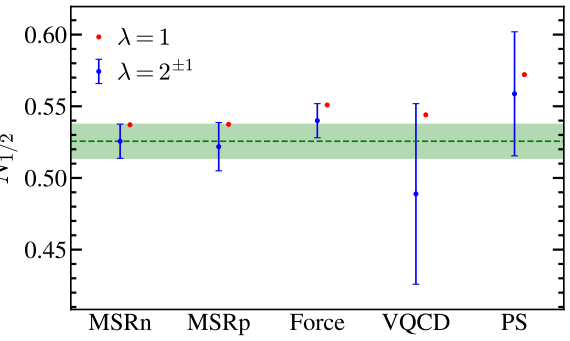
$$\alpha_s^{(n_f=3)}(m_\tau) = 0.3093 \pm 0.0063 \qquad \alpha_s^{(n_f=5)}(m_Z) = 0.1170 \pm 0.0009$$

# THANK YOU

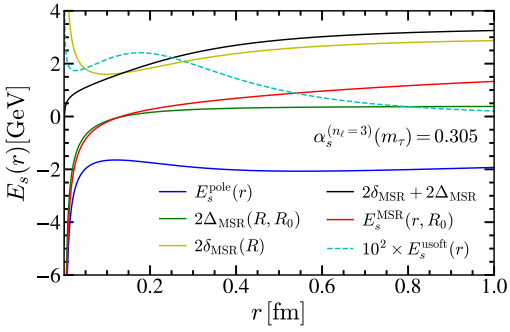
# BACKUP

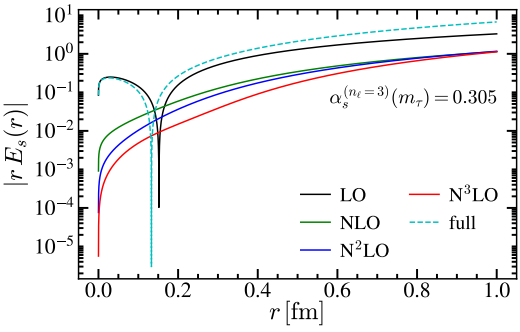
# Offset analysis



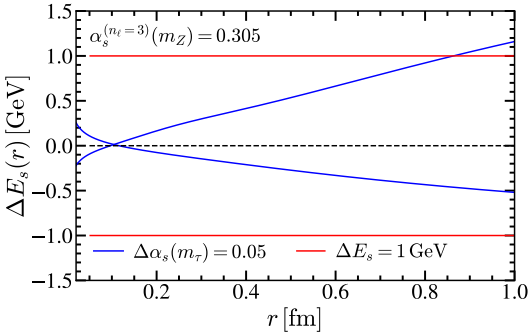



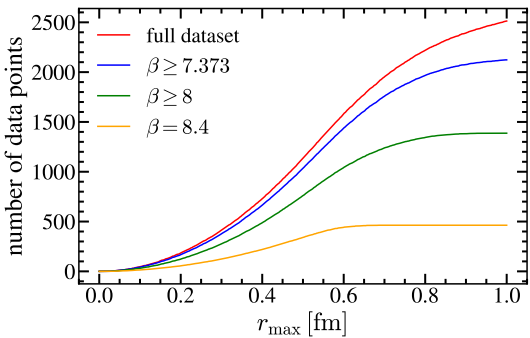
# Perturbative analysis



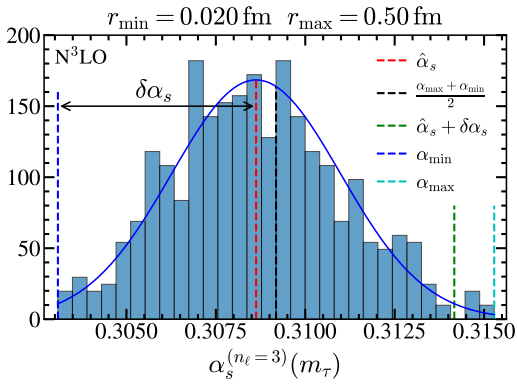



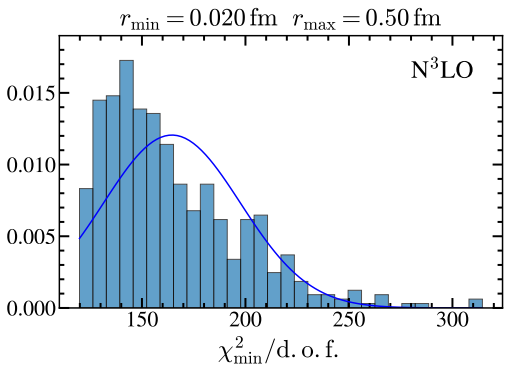
# Auxiliar figures



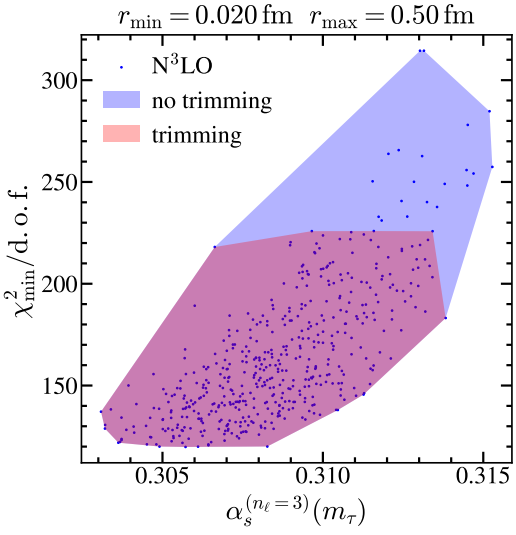



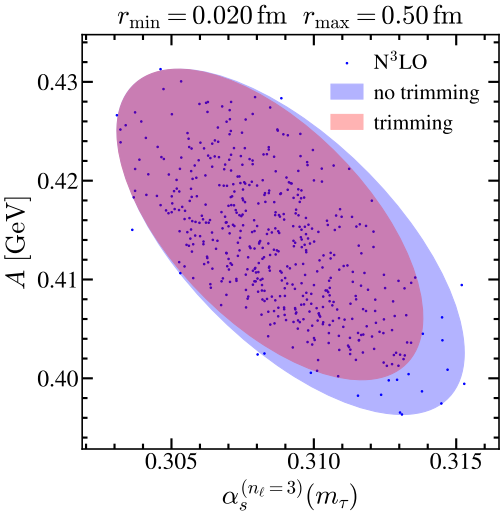
# Random scan histograms



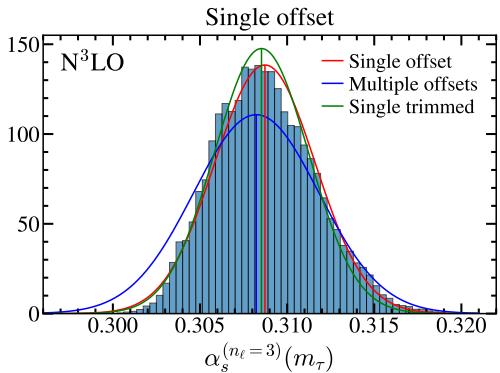



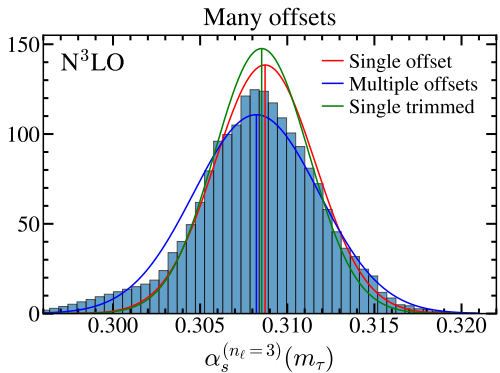
# $\chi^2$ trimming



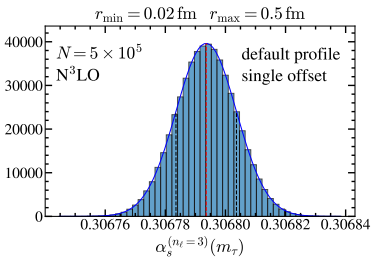



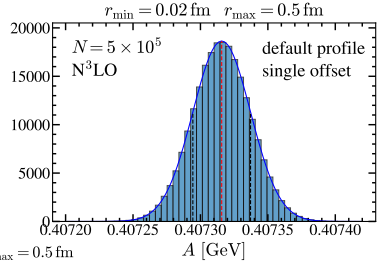
# Fits histograms

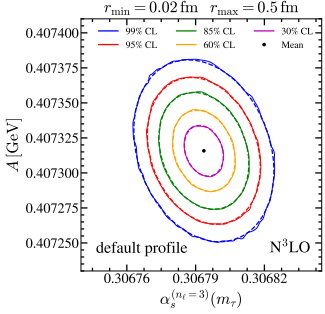


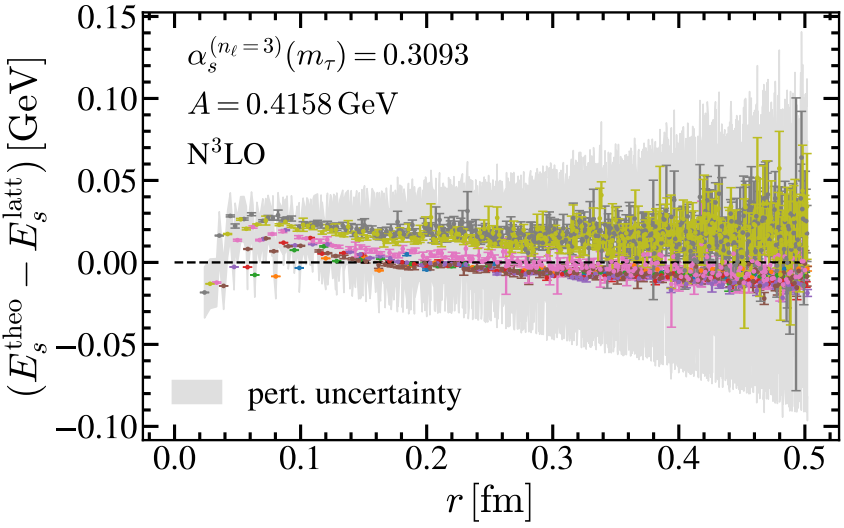
# Replica method



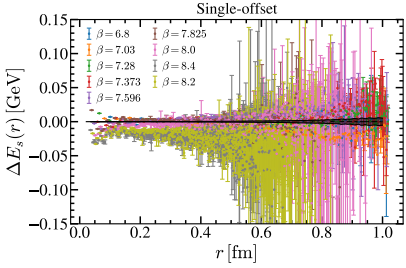



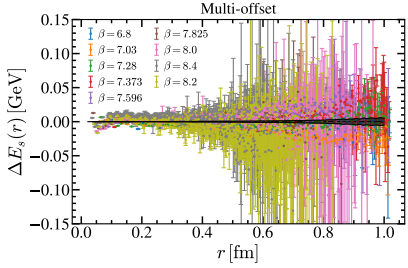



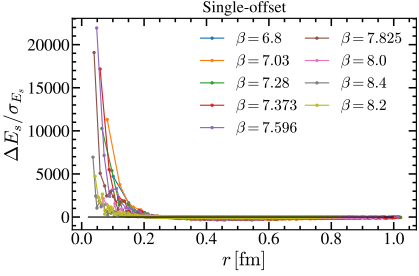
# Lattice data

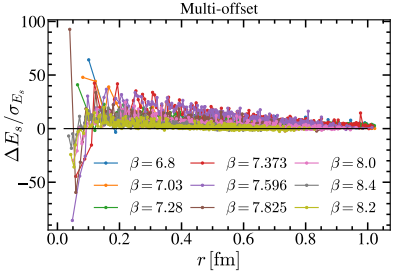


# Lattice data

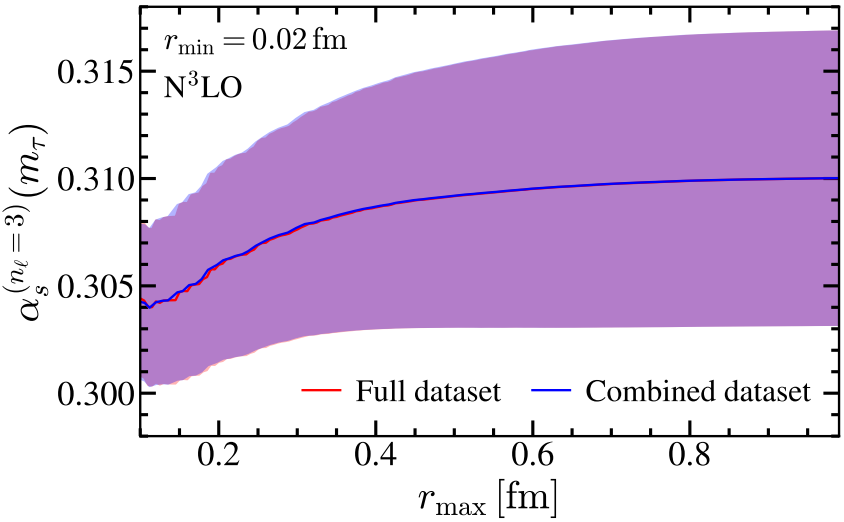


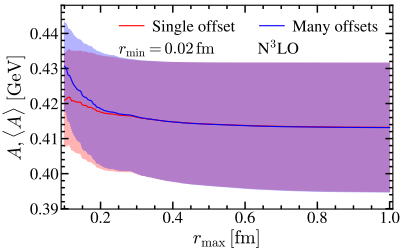



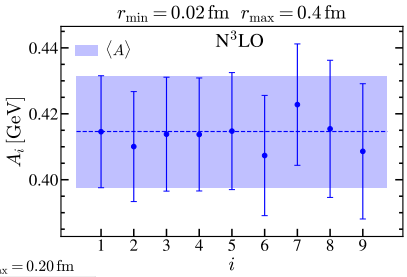



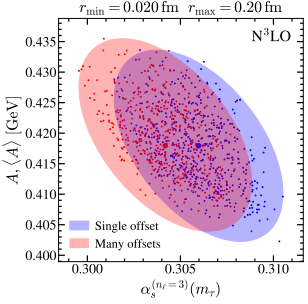
# Data clustering



# Offset analysis



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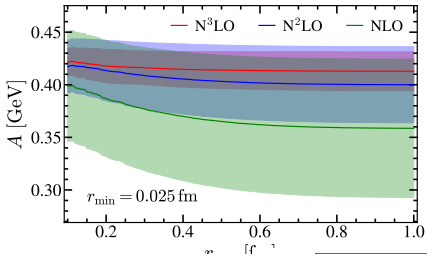
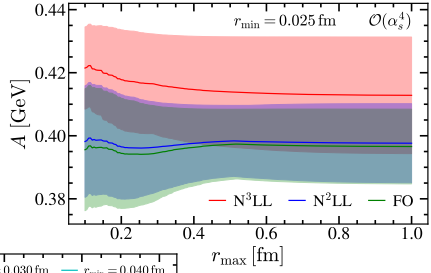
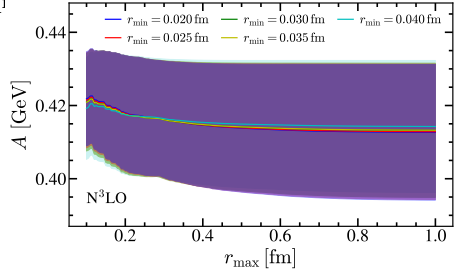
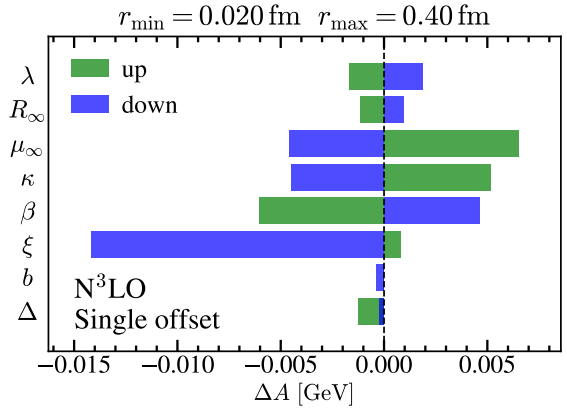
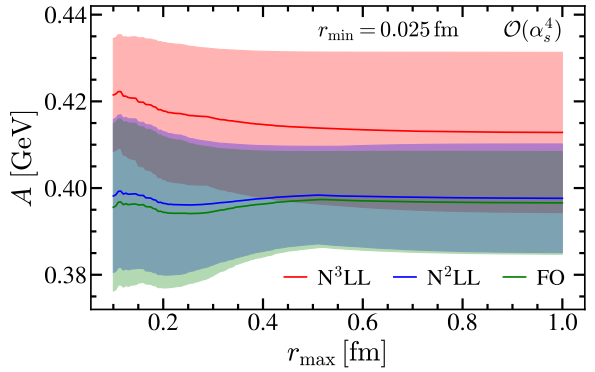


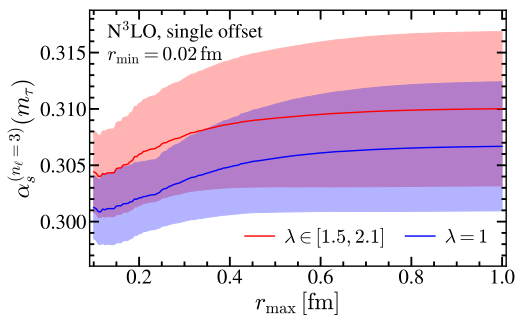



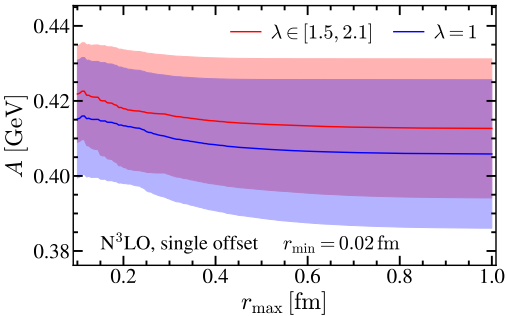
# Offset analysis



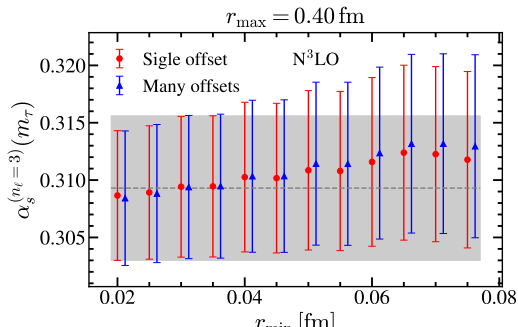



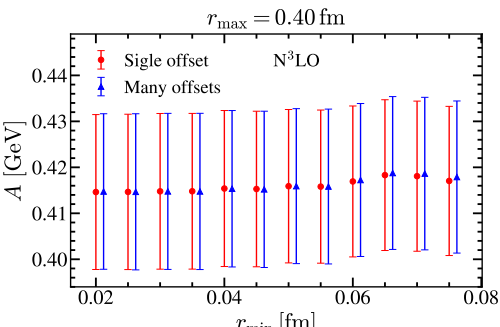
# $\lambda$ variation



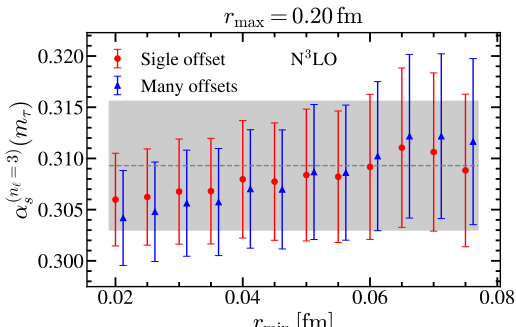



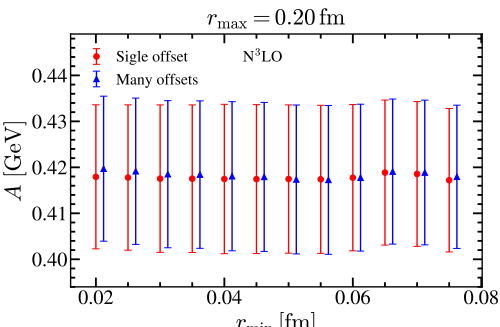
# Further dependence check on $r_{min}$



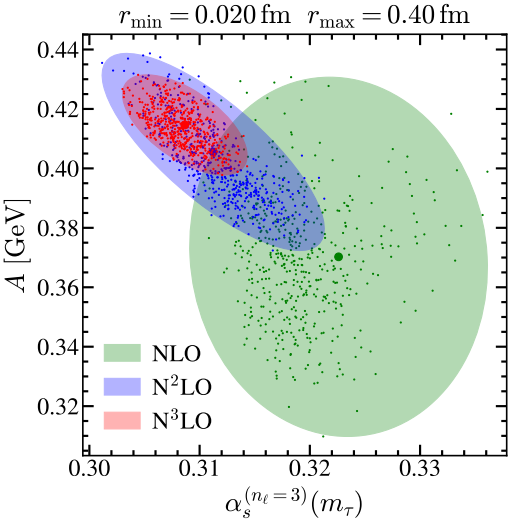



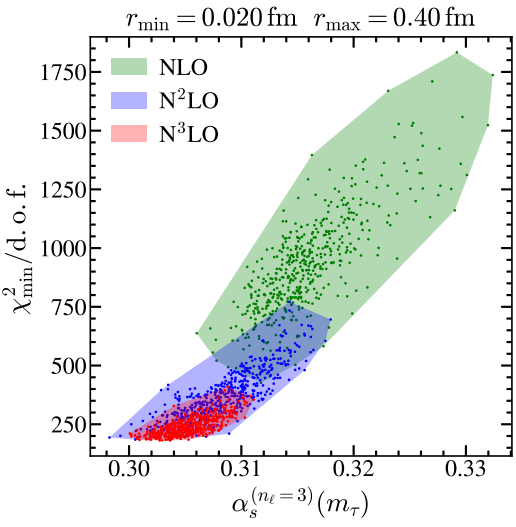
# Further dependence check on $r_{min}$



# Scattered fit points



# Aditional fits

| | full | $\beta \geq 7.737$ | $\beta \geq 8$ | $\beta = 8.4$ |
|---|---|---|---|---|
| $\alpha_s^{(n_\ell=3)}(m_\tau)$ | $0.3093 \pm 0.0063$ | $0.3089 \pm 0.0061$ | $0.3089 \pm 0.0070$ | $0.3080 \pm 0.0072$ |
| $\alpha_s^{(n_\ell=5)}(m_Z)$ | $0.1170 \pm 0.0009$ | $0.1170 \pm 0.0009$ | $0.1170 \pm 0.0010$ | $0.1169 \pm 0.0010$ |

| | $\mu_c = \overline{m}_c$ | $\lambda = 1$ | FO | many offsets |
|---|---|---|---|---|
| $\alpha_s^{(n_\ell=3)}(m_\tau)$ | $0.3093 \pm 0.0063$ | $0.3055 \pm 0.0049$ | $0.3108 \pm 0.0057$ | $0.3092 \pm 0.0064$ |
| $\alpha_s^{(n_\ell=5)}(m_Z)$ | $0.1174 \pm 0.0009$ | $0.1165 \pm 0.0007$ | $0.1172 \pm 0.0008$ | $0.1170 \pm 0.0009$ |