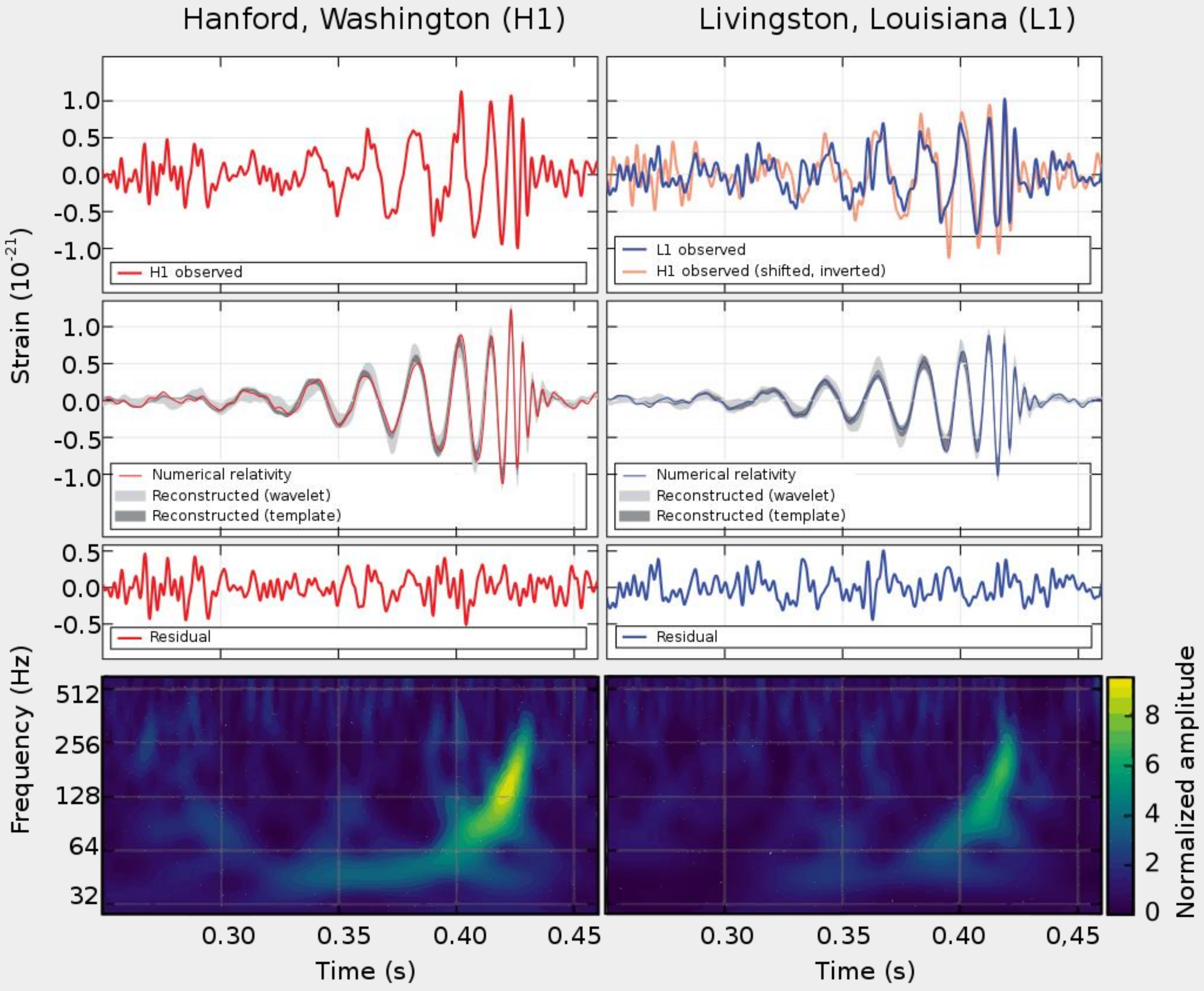
# Advances in gravitational-wave predictions through quantum field theory methods

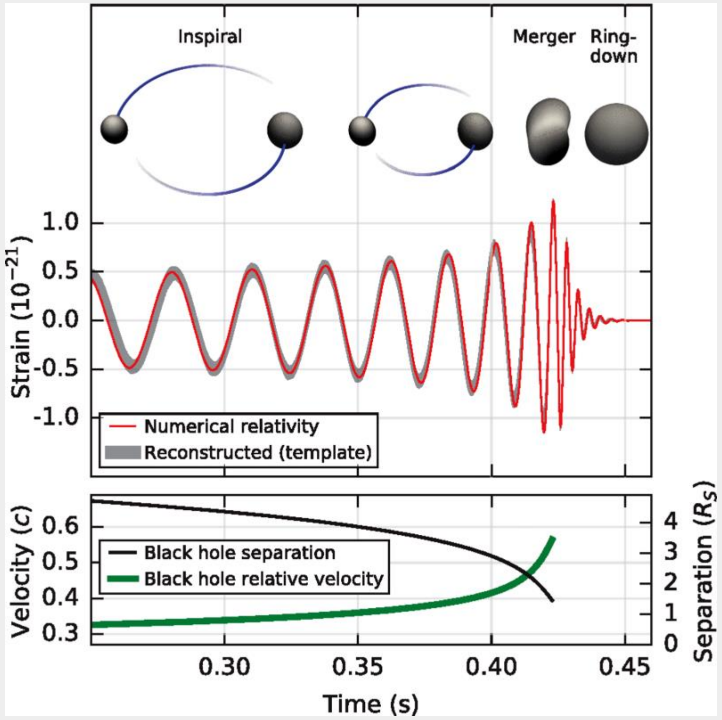
Christoph Dlapa

# Outline

- Background
  - First gravitational wave detections
  - Detectors (2G and 3G)
  - Theory approaches
- Analytic methods
  - Terminology
  - Computation
  - Results

# First detected gravitational wave

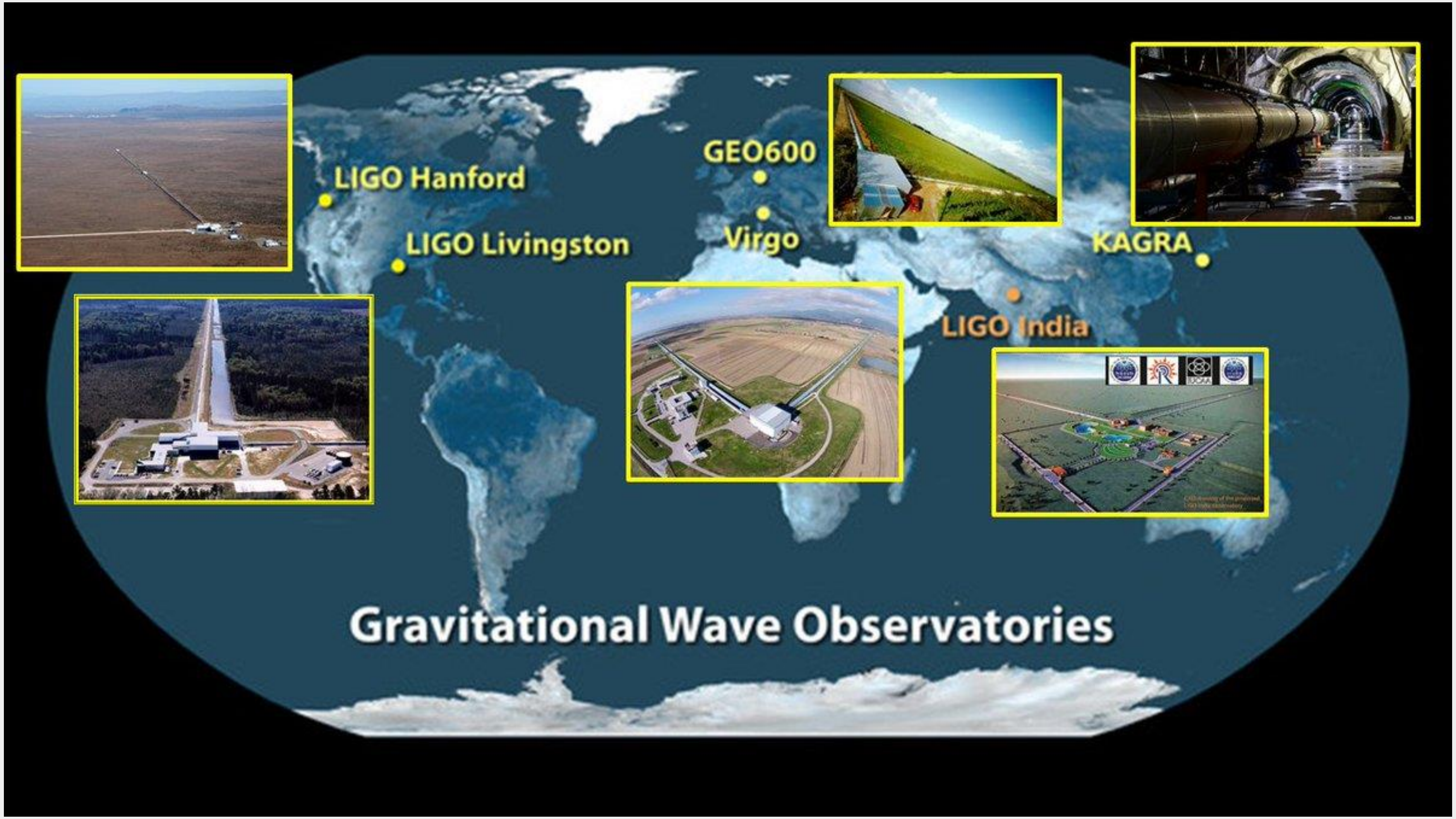




[LIGO]

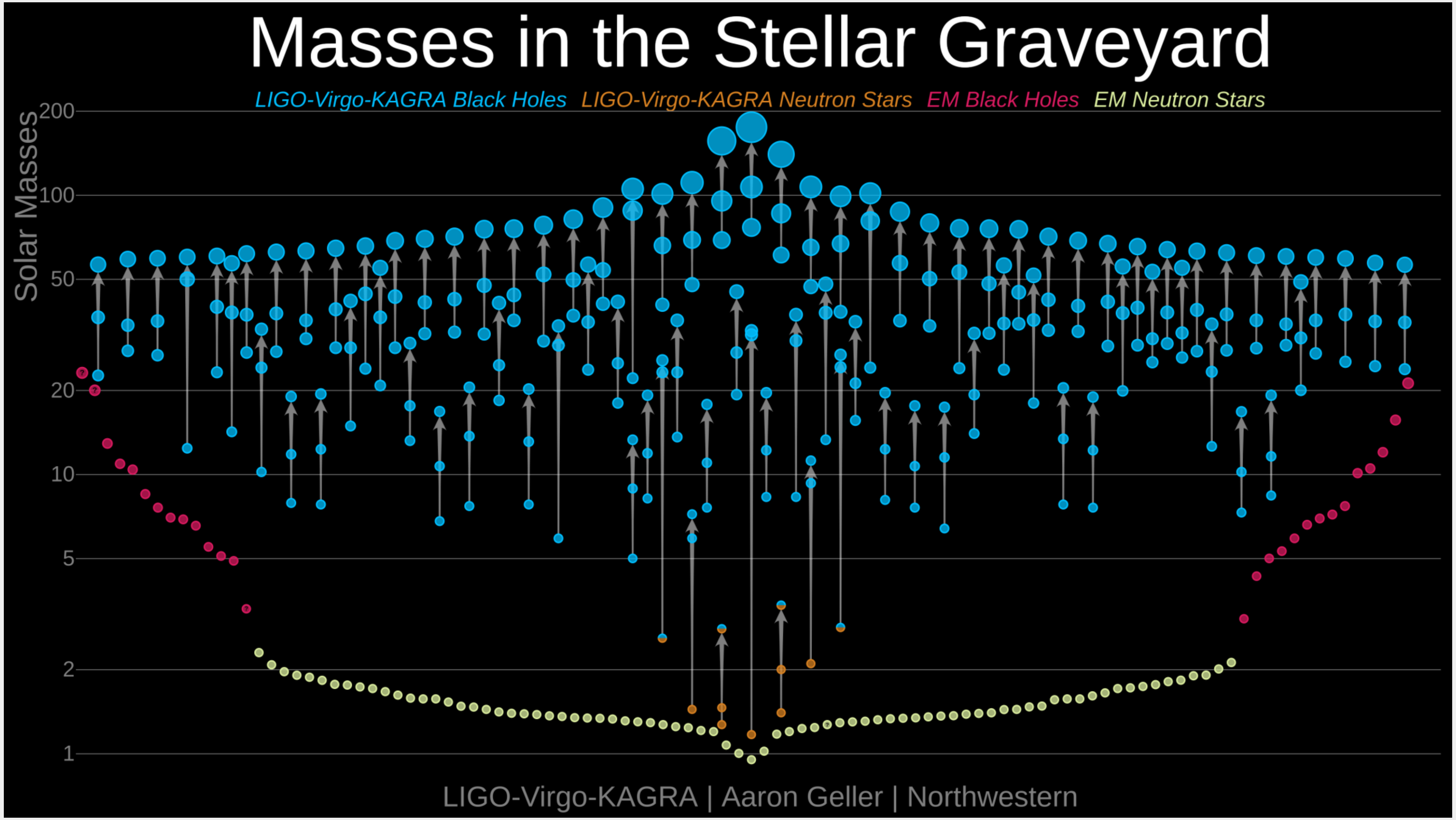
2017 Nobel Prize in Physics:
Weiss, Barish and Thorne

# Global Network of 2G detectors



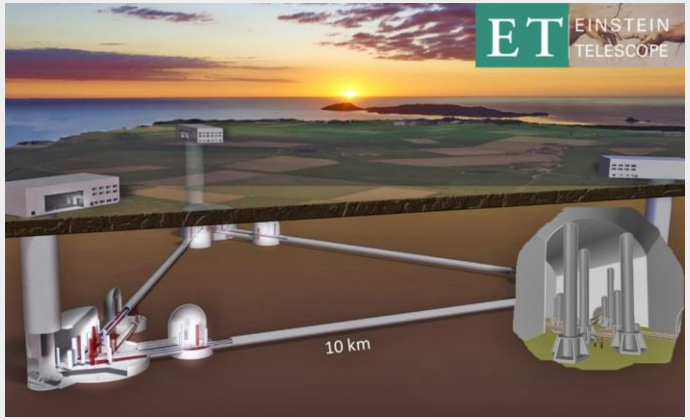
[LIGO]

# Detections since then



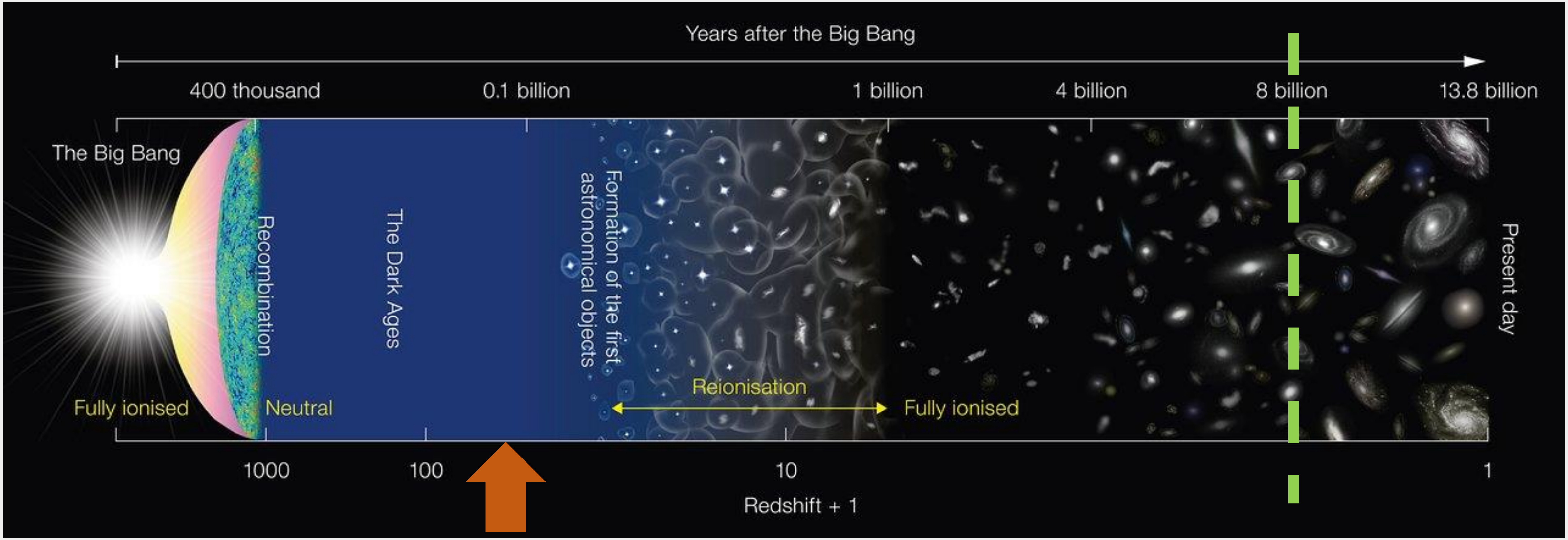
# Proposed third-generation detectors

- Einstein Telescope (ET)
  - Belgium-Netherlands-Germany border
  - Sardinia (Italy)
  - underground, triangular shape, 10 km
- Cosmic Explorer (CE)
  - US
  - above-ground, L-shaped, 40 km + 20 km
- Laser Interferometer Space Antenna (LISA)
  - space-based (ESA), heliocentric orbit
  - triangular shape, 2.5 million km




Artist: Eddie Anaya, Cal State Fullerton

# Probing the Universe



3G Target

2G

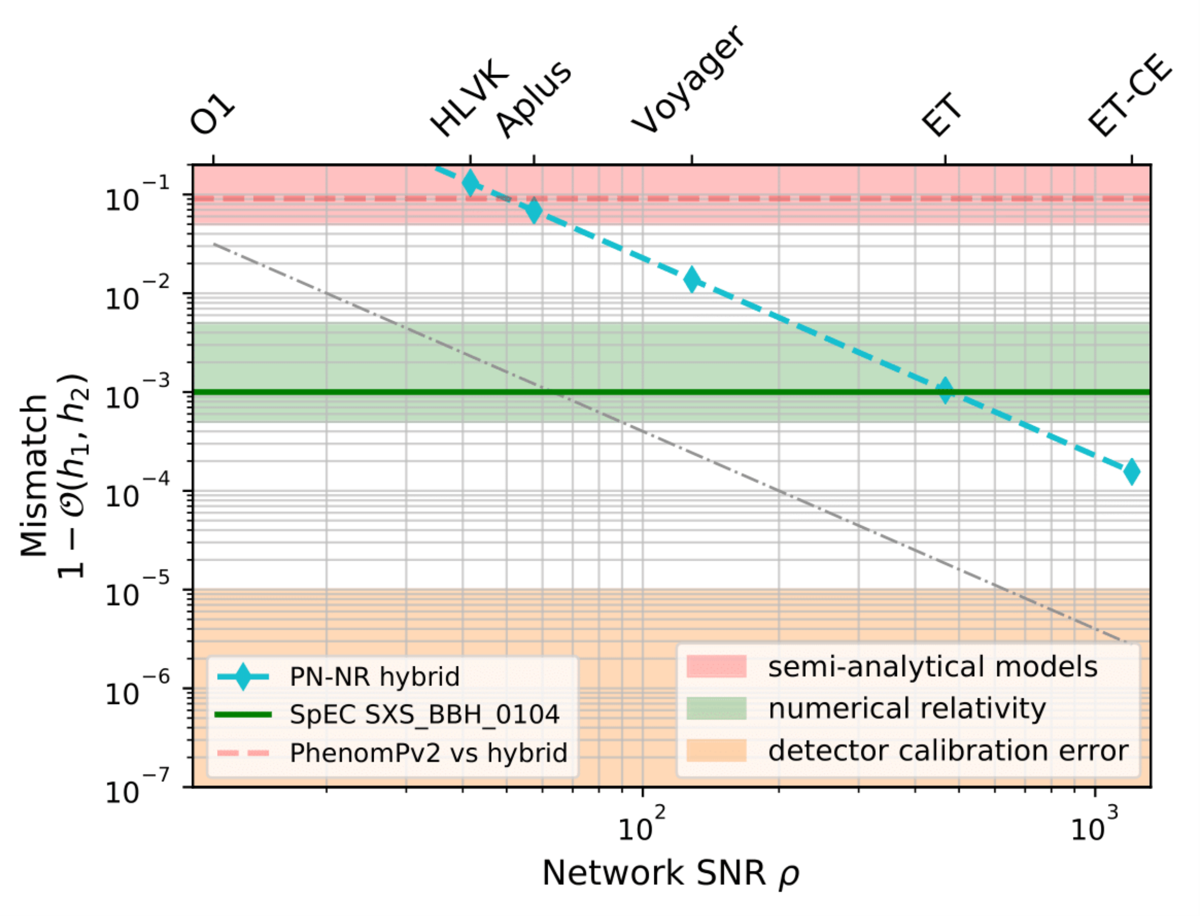
[ALMA collaboration]

# Prospects

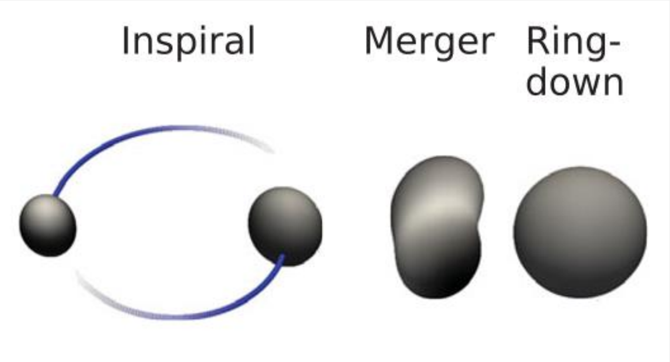
- Black hole properties
- Neutron star properties
- New astrophysical sources
- Dark matter (e.g. accreting on compact objects)
- Testing Einstein gravity (modified GW propagation)

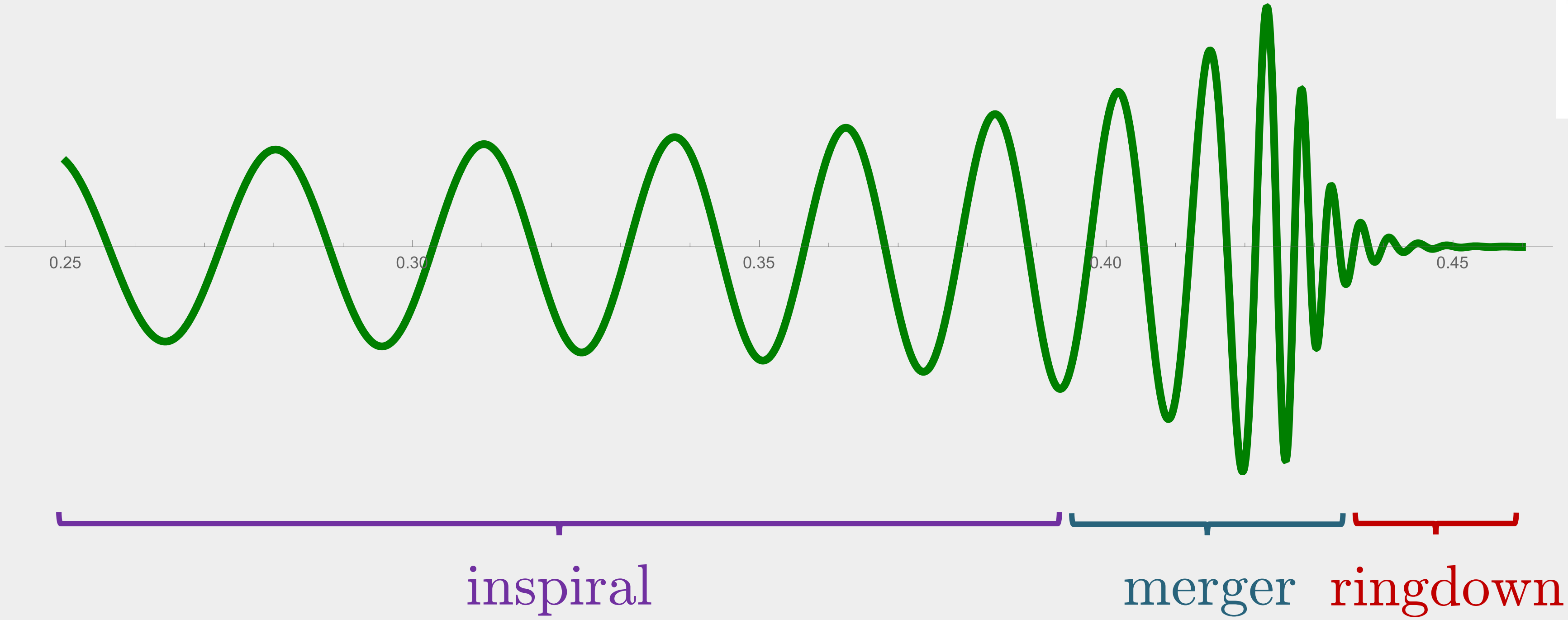
# 3G Detectors: Higher accuracy needed

[Pürrer, Haster, ‘20]



# Gravitational two-body problem





- Inspiral: analytic models for parameter estimation studies (PES)
- Merger: numerical relativity (NR)
- Ringdown: BH perturbation theory

Effective one-body (EOB) theory for inspiral-merger-ringdown waveforms

# Terminology

non-spinning ⟷ spinning

black hole ⟷ neutron star

Post Newtonian (PN) ⟷ Post Minkowskian (PM)

bound (elliptic) ⟷ unbound (hyperbolic)

conservative ⟷ dissipative

potential modes ⟷ radiation modes

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bookkeeping
in integrand

# PN vs PM

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  - integrals are non-relativistic (3d)
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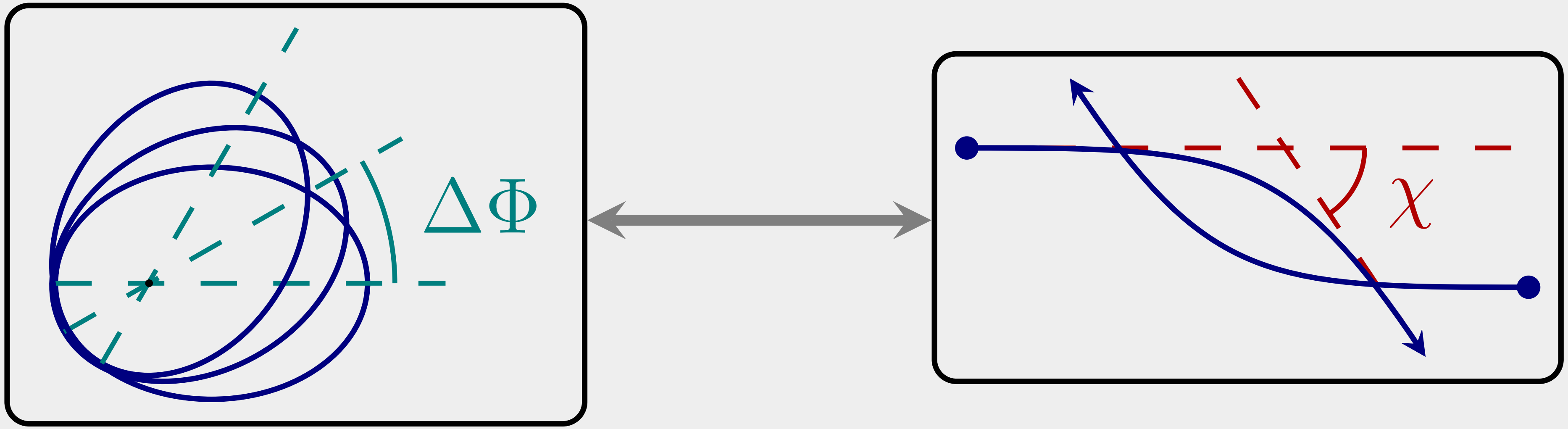
- Post-Minkowskian (PM): $G \ll 1$
  - expansion in coupling
  - ➢ natural expansion for pQFT tools
  - incorporates all orders in velocities (more general)
  - results are naturally for scattering events

# Bound vs Unbound motion

[Kälin, Porto, ‘20]



$\mathcal{E} > 0$

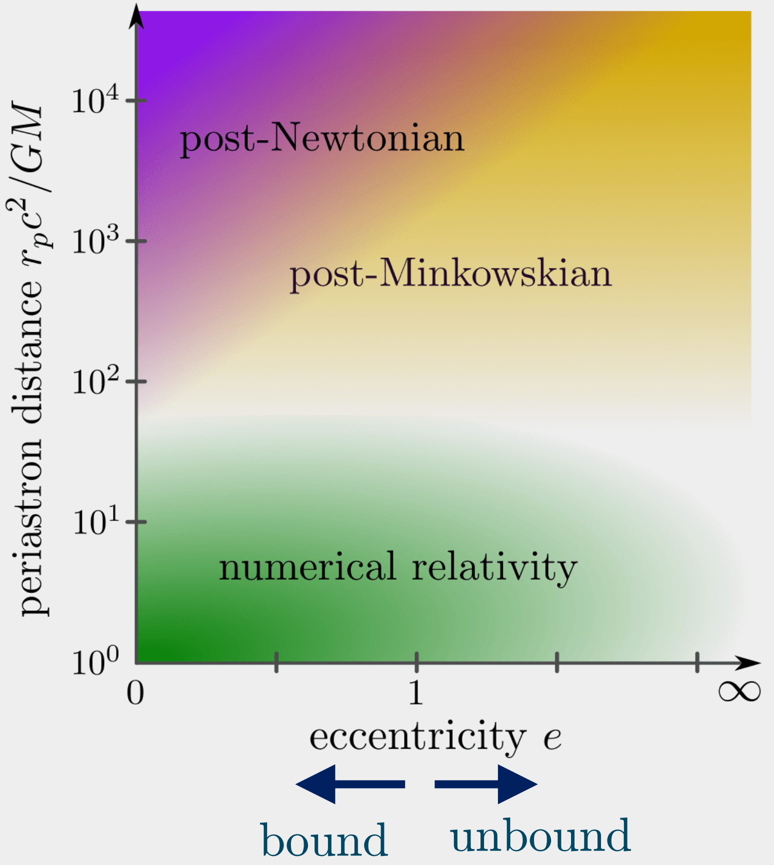
analytic continuation
in binding energy

$\mathcal{E} < 0$

Eccentricity e

# Post-Minkowskian (PM) expansion

- PM incorporates PN $(v^2 \ll 1)$
- Cleaner results: understand structure of observables, e.g. singularities, symmetries,…
- PM velocity resummation increases accuracy for scattering-like events
  - detection rate is expected to increase in 3G detectors
  - [Khalil, Buonanno, Steinhoff, Vines, ‘22; Damour, Rettegno, ‘23]



[Khalil, Buonanno, Steinhoff, Vines, ‘22]

# PN vs PM

|  | 0PN | 1PN | 2PN | 3PN | 4PN | 5PN | 6PN |  |
|---|---|---|---|---|---|---|---|---|
| 1PM | $1$ | $v^2$ | $v^4$ | $v^6$ | $v^8$ | $v^{10}$ | $v^{12}$ | $\times G^1$ |
| 2PM |  | $1$ | $v^2$ | $v^4$ | $v^6$ | $v^8$ | $v^{10}$ | $\times G^2$ |
| 3PM |  |  | $1$ | $v^2$ | $v^4$ | $v^6$ | $v^8$ | $\times G^3$ |
| 4PM |  |  |  | $1$ | $v^2$ | $v^4$ | $v^6$ | $\times G^4$ |
| 5PM |  |  |  |  | $1$ | $v^2$ | $v^4$ | $\times G^5$ |
| 6PM |  |  |  |  |  | $1$ | $v^2$ | $\times G^6$ |

- State of the art for non-spinning BH:
  - 4PM conservative [Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, Zeng, '21; CD, Kälin, Liu, Porto, '21, ...]
  - 4PM dissipative [CD, Kälin, Liu, Neef, Porto, '22, ...]
  - 5PM conservative [Driesse et al, '26, ...]
  - 5PN & 6PN (conservative): [Foffa, Mastrolia, Sturani, Sturm Bobadilla, '19; Blümlein, Maier, Marquard, '19; Bini, Damour, Geralico, Laporta, Mastrolia, '20]

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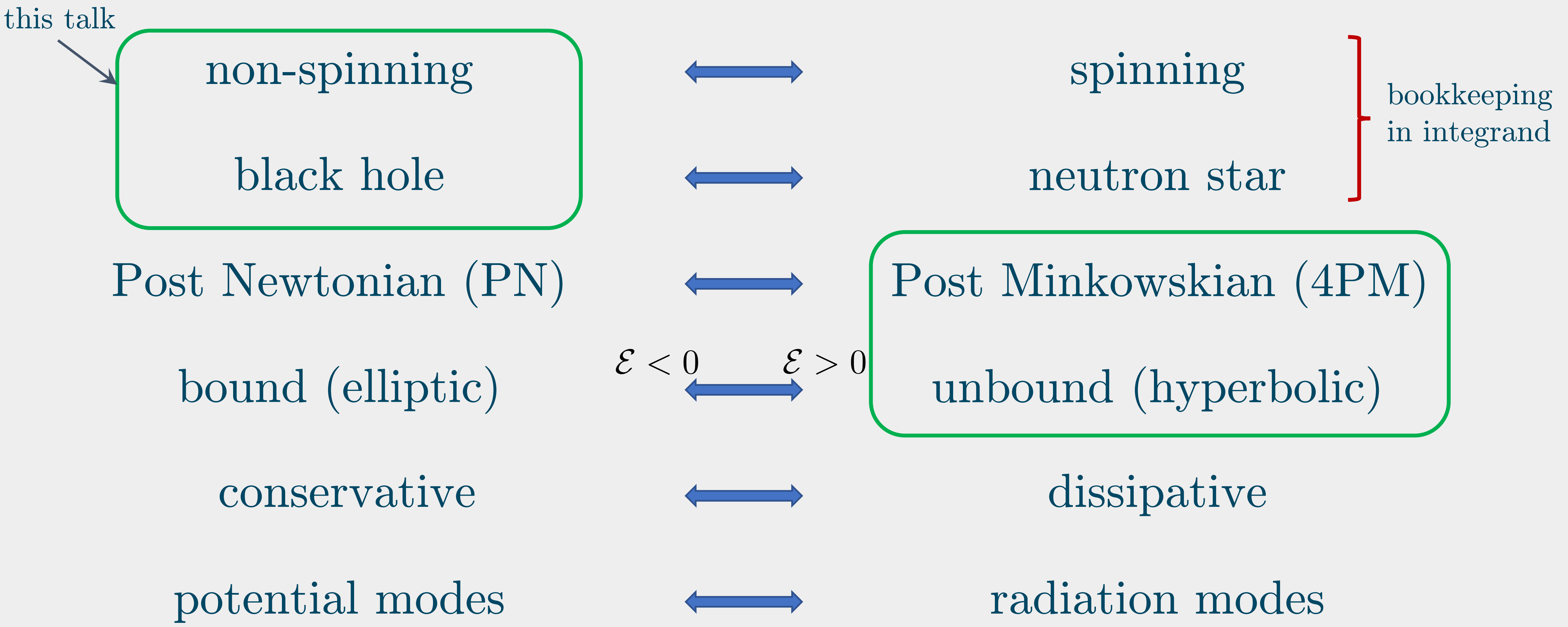
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# Conservative vs. dissipative

- Conservative: standard Feynman propagators

$$\Delta_{\mathrm{F}}(\ell) = \frac{1}{\ell^2 + i0}$$

related to in-out formalism ⇒ vacuum-to-vacuum amplitudes

- Dissipative: retarded/advanced propagators

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  - alternatively: view as virtual + real

$$\Delta_{\mathrm{ret/adv}}(\ell) = \Delta_{\mathrm{F}}(\ell) + 2\pi i\, \Theta(\mp\ell_0)\delta(\ell^2)$$

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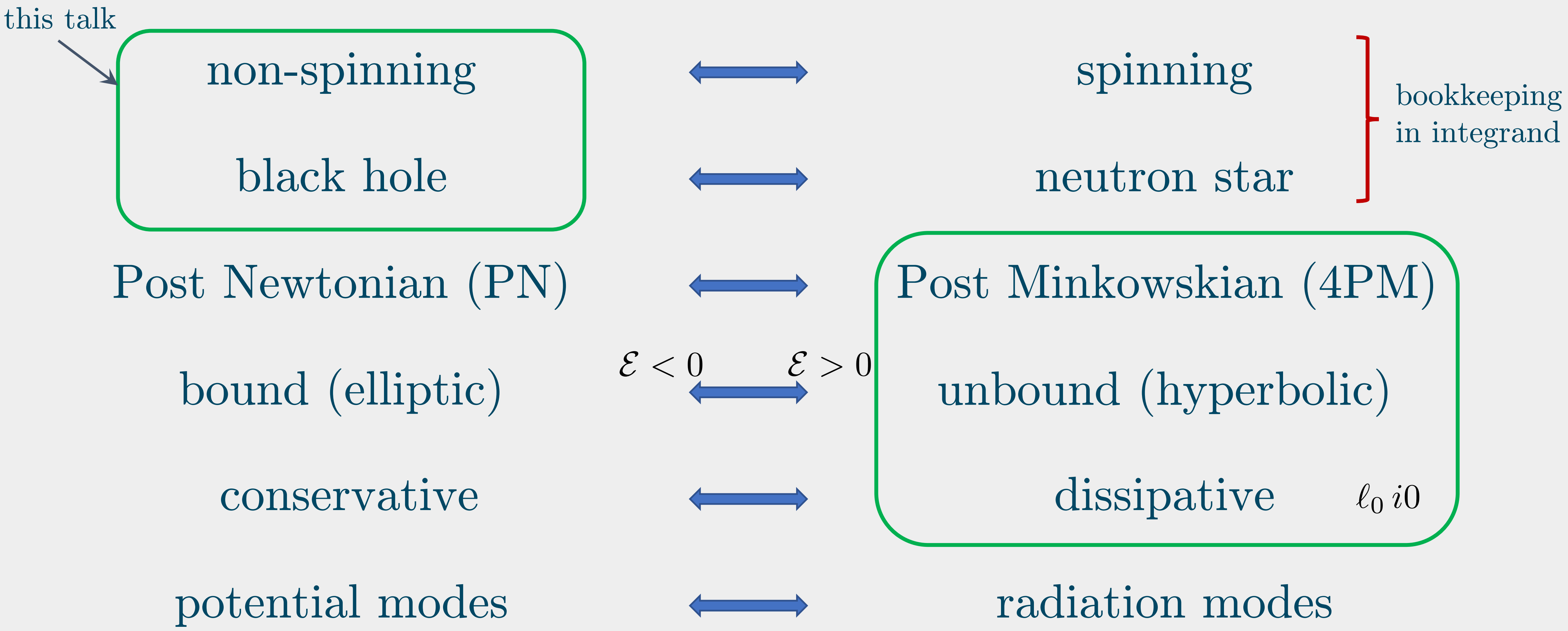

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# Potential vs. radiation modes

- Potential modes:
  - off-shell integration region
  - i0 prescription irrelevant
- Radiation modes:
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- Radiation modes:
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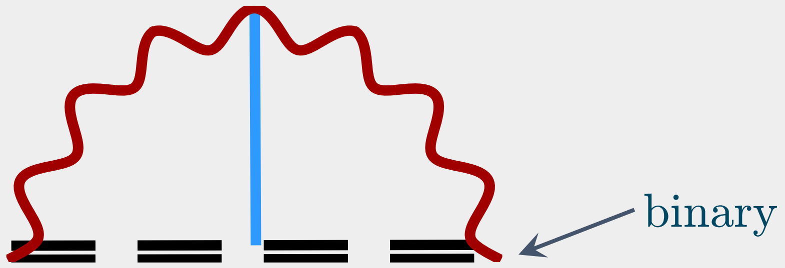
  - Radiation modes possible in conservative computation!
  - ➢ complicates continuation to elliptic

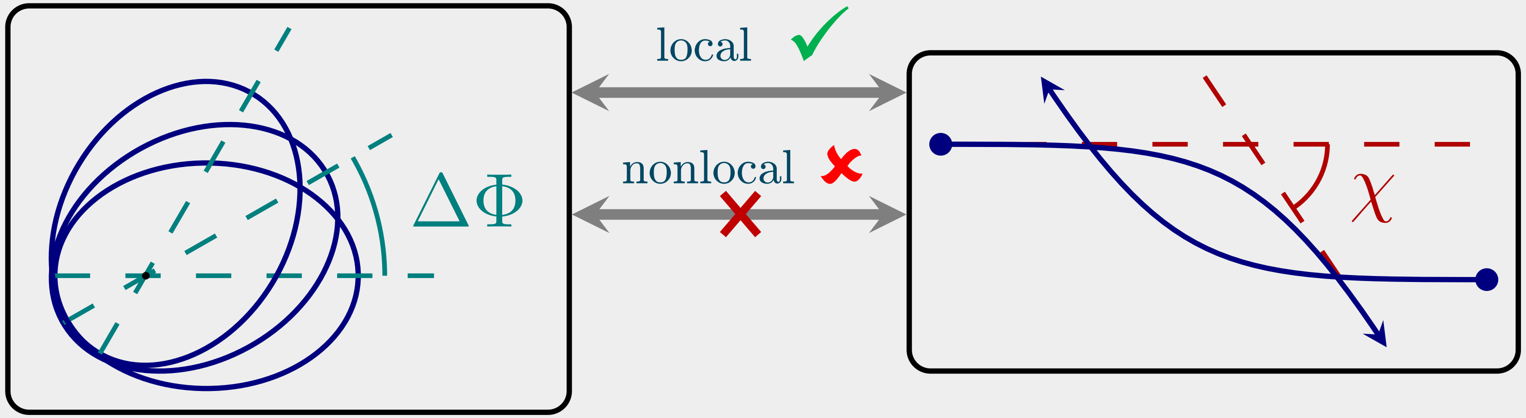


tail effect
[Galley, Leibovich, Porto, Ross, ‘15]

# Nonlocal-in-time contributions

[CD, Kälin, Liu, Porto, 24']

[CD, Kälin, Liu, Porto, 25']



local ✓

nonlocal ✗

[CD, Kälin, Liu, Porto, 24’]

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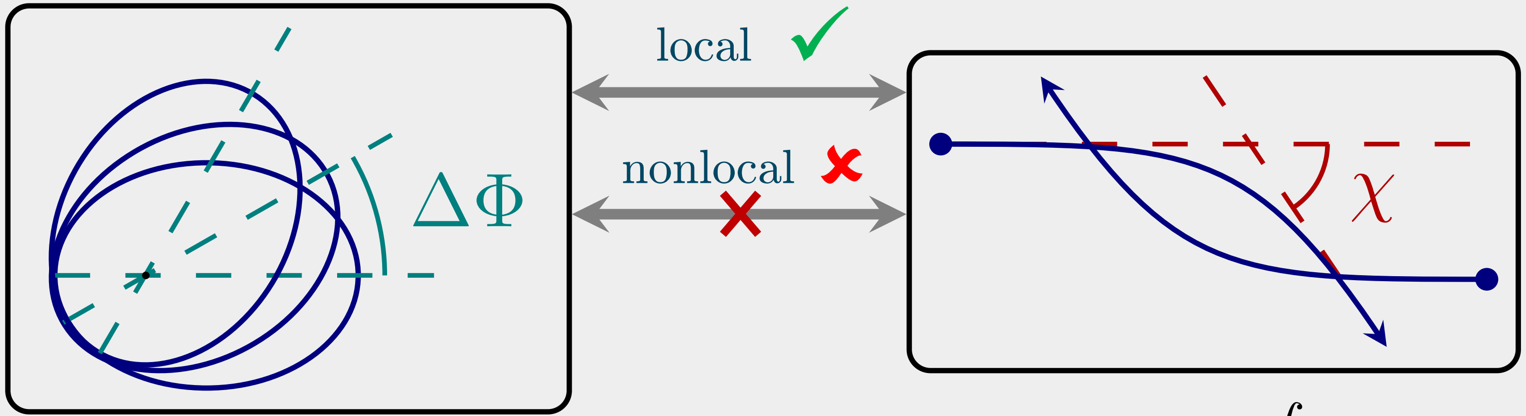

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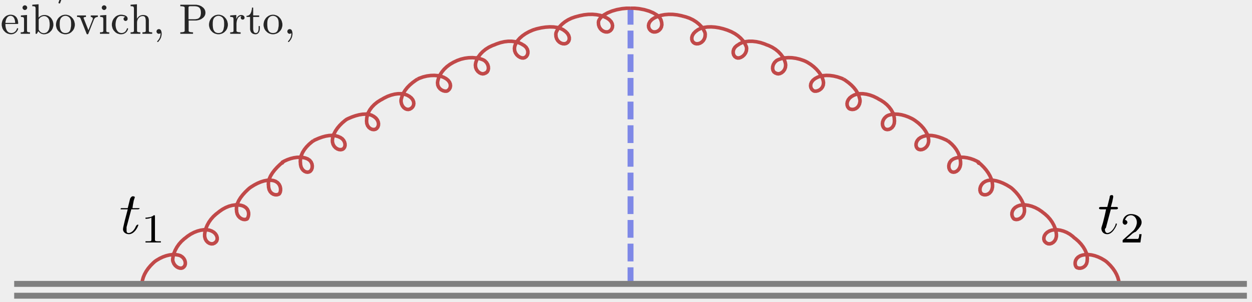

$$\mathcal{S}_r^{\mathrm{nloc}} \sim GE \int \mathcal{F}(t_1, t_2) \frac{\mathrm{d}t_1 \mathrm{d}t_2}{|t_1 - t_2|},$$

$$\sim GE \int \frac{\mathrm{d}E}{\mathrm{d}\omega} \log(\omega^2)\, \mathrm{d}\omega$$

$$\chi \sim \partial_j \mathcal{S}_r$$

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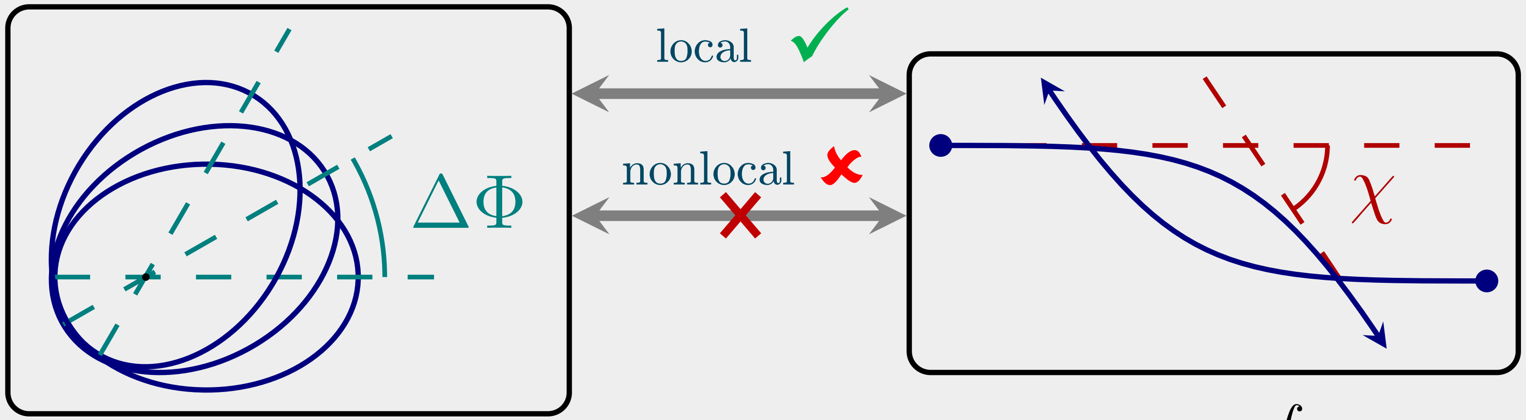

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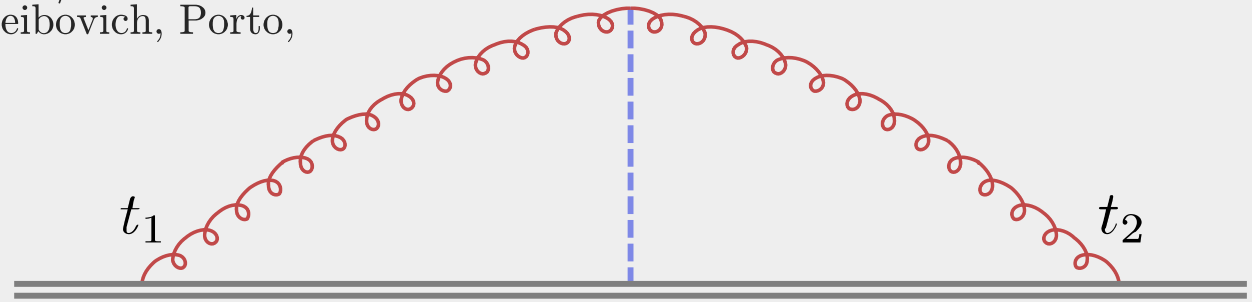

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$$\chi \sim \partial_j \mathcal{S}_r$$

$$\Delta p_1 \to \Delta E$$

- Related to previous computation:

$$\Delta E = \int \frac{\mathrm{d}E}{\mathrm{d}\omega} \mathrm{d}\omega,$$

⇒ add log to integrand

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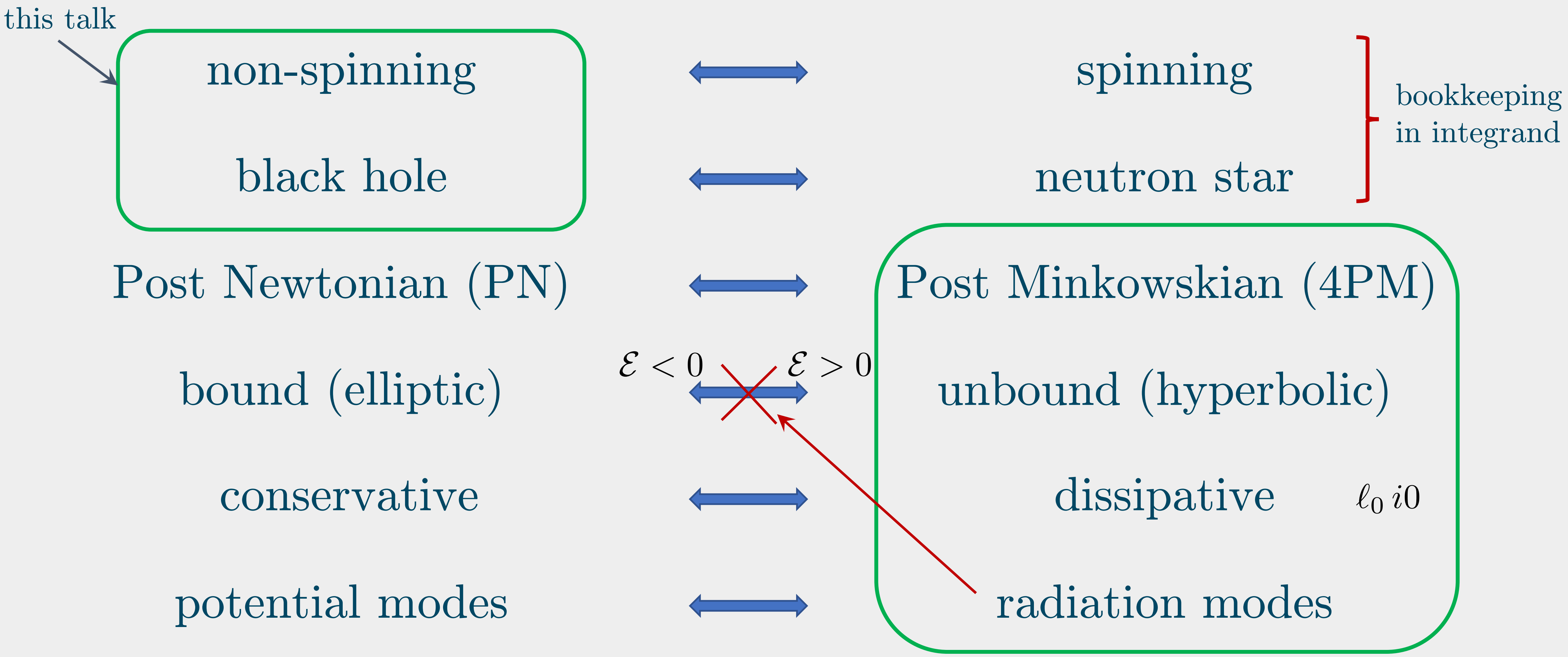

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# Analytic computation methods

- Setup
  - Effective theory of point particles
- Integral computation
  - Example diagram
  - Method of differential equations (+ elliptic)
  - Boundary conditions

Other approaches:

|  |  |
|---|---|
| Scattering amplitudes | [Cheung, Rothstein, Solon; Bern, Parra-Martinez, Roiban, Ruf, Shen, Zeng, ...] |
| WL-QFT | [Jakobsen, Mogull, Plefka, Steinhoff, '21] |
| Eikonal | [Di Vecchia, Heissenberg, Russo, Veneziano] |

# Gravitational two-body problem

[Kälin, Porto, 19' /
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$$S_{\mathrm{pp}} = -\sum_{a=1,2} \frac{m_a}{2} \int \mathrm{d}\tau \, g_{\mu\nu} v_a^\mu v_a^\nu = -\sum_{a=1,2} \frac{m_a}{2} \int \mathrm{d}\tau \, [\eta_{\mu\nu} v_a^\mu v_a^\nu + h_{\mu\nu} v_a^\mu v_a^\nu]$$

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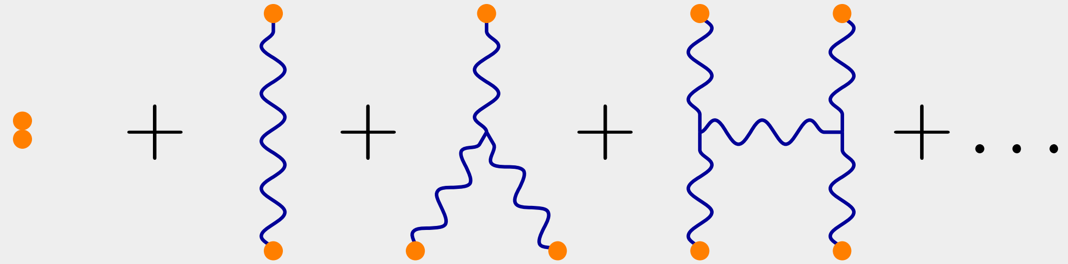

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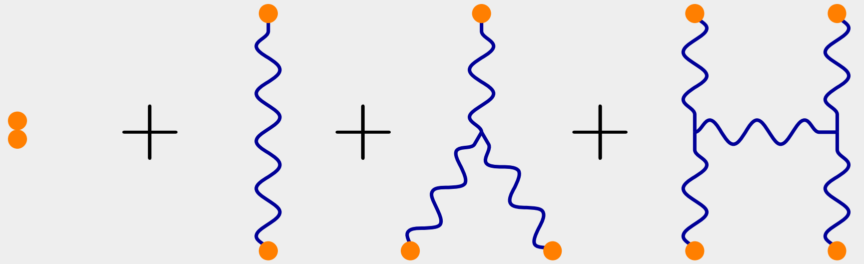

$$+ \dots = -\sum_{a=1,2} \frac{m_a}{2} \int \mathrm{d}\tau\, \eta_{\mu\nu} v_a^\mu v_a^\nu$$

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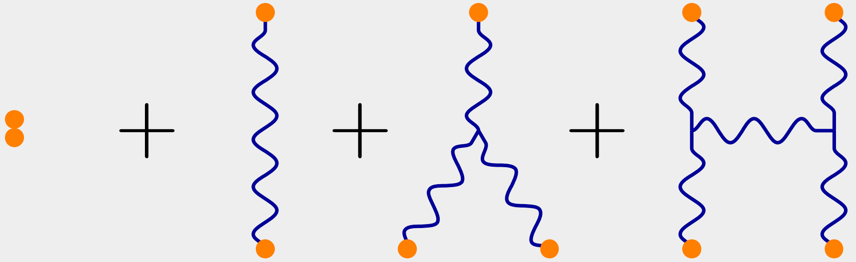

$+\dots$

$$= -\sum_{a=1,2}\frac{m_a}{2}\int \mathrm{d}\tau\,\eta_{\mu\nu}v_a^\mu v_a^\nu$$

$$+G\frac{i}{4}\int \mathrm{d}\tau_1\mathrm{d}\tau_2\, v_1^\alpha v_1^\beta v_2^\mu v_2^\nu \int \mathrm{d}^4k\,\frac{P_{\alpha\beta\mu\nu}}{\ell^2}e^{i\ell\cdot(x_1-x_2)}$$

# Change in momentum

$$S_{\text{eff}} = \sum_{n=0}^{\infty} G^n S_n = -\sum_{a=1,2} \frac{m_a}{2} \int \mathrm{d}\tau\, \eta_{\mu\nu} v_a^\mu v_a^\nu + \dots,$$

$$S_n = \int \mathrm{d}\tau_1 \mathcal{L}_n[x_1, x_2]$$

- Equations of motion: $0 = \sum_{n=0}^{\infty} \left( \frac{\partial \mathcal{L}_n}{\partial x_1^\nu} - \frac{\mathrm{d}}{\mathrm{d}\tau_1} \frac{\partial \mathcal{L}_n}{\partial v_1^\nu} \right)$

$$m_1 \frac{\mathrm{d}v_1^\mu}{\mathrm{d}\tau_1} = -\eta^{\mu\nu} \sum_{n=1}^{\infty} \left( \frac{\partial \mathcal{L}_n}{\partial x_1^\nu} - \frac{\mathrm{d}}{\mathrm{d}\tau_1} \frac{\partial \mathcal{L}_n}{\partial v_1^\nu} \right)$$

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- Corrections to trajectories: $\int_{-\infty}^{\tau_1} \mathrm{d}\tilde{\tau}_1 \longrightarrow v_a^\mu = u_a^\mu + \sum_n G^n \delta^{(n)} v_a^\mu$
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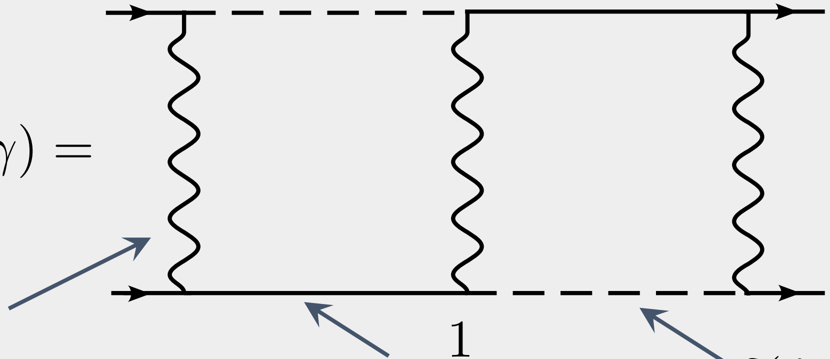
- PP propagators: $$\int_{-\infty}^{\tau_1} \mathrm{d}\tilde{\tau}_1 e^{i\tilde{\tau}_1(\ell\cdot u_1)} = \frac{-i\, e^{i\tau_1(\ell\cdot u_1)}}{\ell \cdot u_1}, \qquad \int_{-\infty}^{\infty} \mathrm{d}\tau_1 e^{i\tau_1(\ell\cdot u_1)} = 2\pi\, \overset{\text{cut propagator}}{\overset{\swarrow}{\delta}}(\ell \cdot u_1)$$

# Loop integrals at 4PM

- Example integral family:

$$D = 4 - 2\epsilon$$

$I(\gamma) =$



$$= \int \mathrm{d}^D\ell_1 \mathrm{d}^D\ell_2 \frac{\delta(\ell_1\cdot u_1)\delta(\ell_2\cdot u_2)}{(\ell_1\cdot u_2)(\ell_2\cdot u_1)} \frac{1}{\ell_1^2\ell_2^2(\ell_1+\ell_2-q)^2}$$

$\frac{1}{\ell_1^2}$

$\frac{1}{\ell_1\cdot u_2}$

$\delta(\ell_2\cdot u_2)$

$$u_1^2 = u_2^2 = 1,$$

$$u_1\cdot q = u_2\cdot q = 0$$

$$u_1\cdot u_2 = \gamma$$

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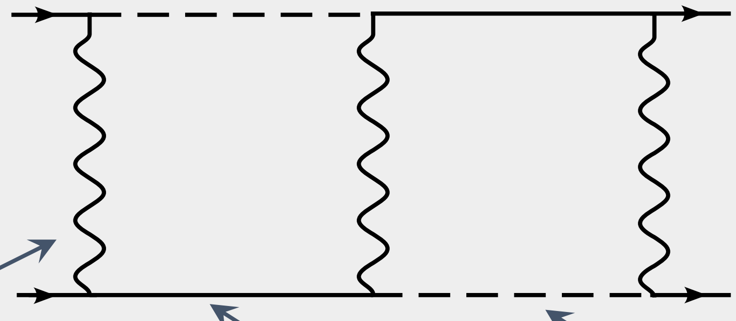

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  - Treat cut propagators just as linear propagators
  - Rationalize square-root:

$$\gamma = \frac{1}{2}(1/x + x), \qquad v \equiv \sqrt{\gamma^2 - 1} = \frac{1}{2}(1/x - x)$$

only one variable

# Differential equations method

- Integration-by-parts reduction to basis

[FIRE6, Smirnov, Chukharev, '19 /
LiteRed, Lee, '13]

$$I_{a_1\ldots a_{15}} = \sum_i c_i(x,\epsilon) f_i$$

ret/adv ⇒ turn off symmetry detection

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ret/adv ⇒ turn off symmetry detection

- Solve by transforming to canonical form: [Henn, '13]

$$\partial_x \vec{f} = A(x,\epsilon)\vec{f} \quad \xrightarrow{\vec{g} = T\vec{f}} \quad \partial_x \vec{g} = \epsilon\, \tilde{A}(x)\vec{g} \qquad D = 4 - 2\epsilon$$

rational

[Lee, '15]
[Libra, epsilon, Fuchsia, CANONICA]

[INITIAL:
CD, Henn, Wagner, '22;
CD, Henn, Yan, '20]

Dyson series

$$\vec{g} = \mathrm{P}\, e^{\epsilon \int_{x_0}^{x} \tilde{A}(\tilde{x})\, \mathrm{d}\tilde{x}}\, \vec{g}_0(\epsilon) = \left[1 + \epsilon \int_{x_0}^{x} \mathrm{d}x_1 \tilde{A}(x_1) + \epsilon^2 \int_{x_0}^{x} \mathrm{d}x_1 \tilde{A}(x_1) \int_{x_0}^{x_1} \mathrm{d}x_2 \tilde{A}(x_2) + \mathcal{O}(\epsilon^3)\right] \vec{g}_0(\epsilon)$$

# Elliptic integrals

- Polylogarithms for most sectors

$$\partial_x \vec{g} = \epsilon\, \tilde{A}(x)\vec{g}, \qquad \tilde{A}(x) = \sum_i \underset{\uparrow}{M_i} a_i(x)$$

constant matrices

$$a_i(x) \in \left\{ \frac{1}{x}, \frac{1}{x-1}, \frac{1}{x+1}, \frac{x}{1+x^2} \right\}$$

new at 4PM

- 4PM: Elliptics integrals: $\mathrm{K}(z) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-zt^2)}},$
- 5PM: Calabi-Yau periods [Driesse et al, Nature 264]

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input from mathematics, computer science and particle physics

output to particle physics

# Boundary conditions

- From PN-like limit $v^2 \ll 1$
  - method of regions:

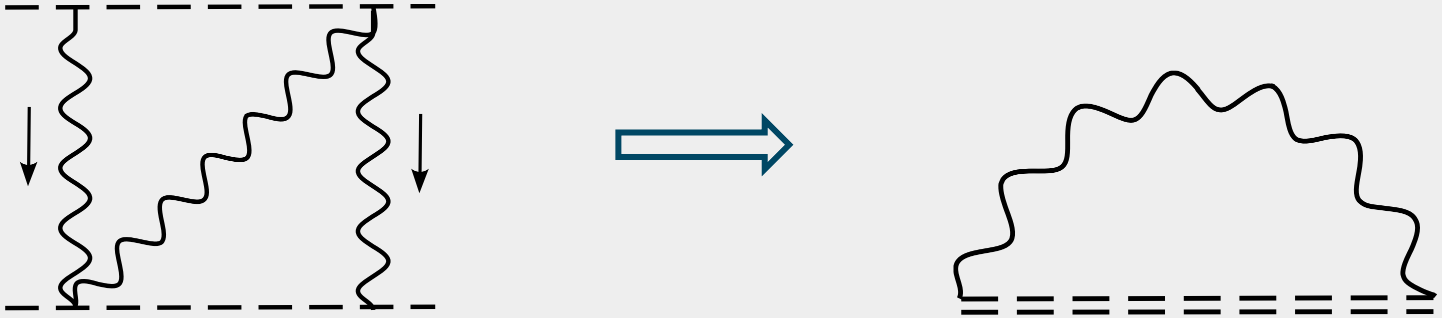
[Beneke, Smirnov, '97]
[asy2.m, Jantzen, Smirnov, Smirnov, '12; Pak, Smirnov, '10]

potential graviton: $(\ell^0, \boldsymbol{\ell}) \sim (1, v)$

radiation graviton: $(\ell^0, \boldsymbol{\ell}) \sim (v, v)$

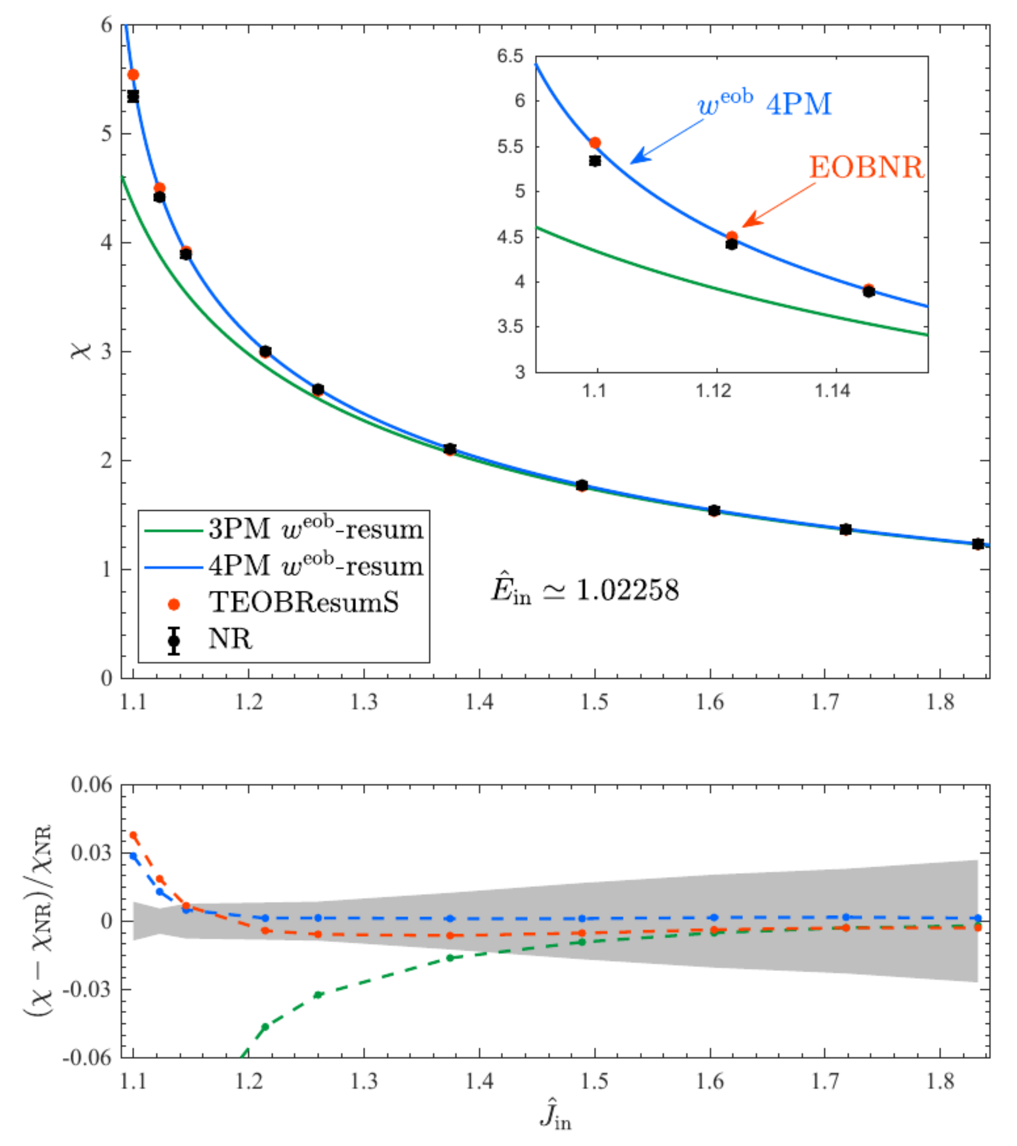
[Goldberger, Rothstein, '06]

  - correspond nicely to PN regions and (3d) integrals



[Galley, Leibovich, Porto, Ross, '15]

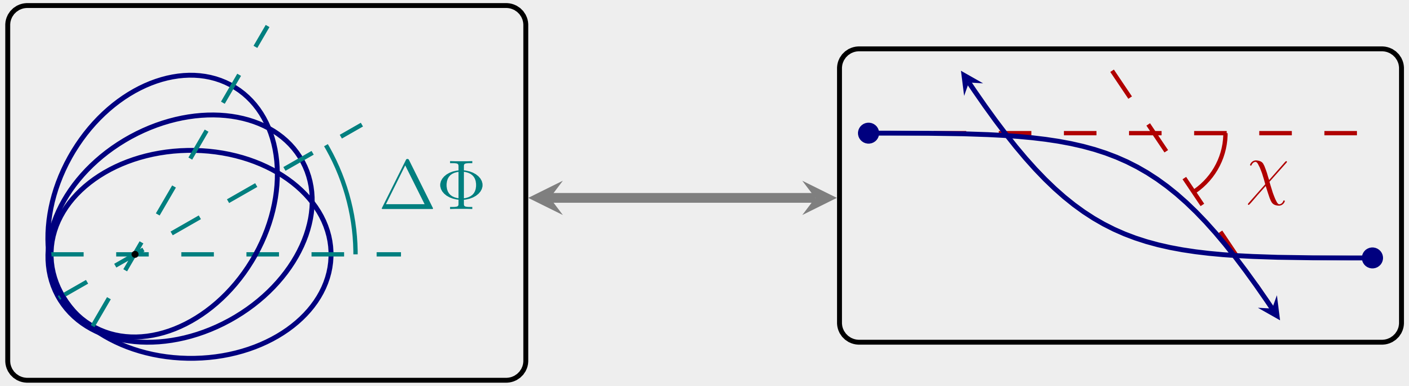
# PM radiative state-of-the-art accuracy



[Damour, Rettegno, '22]

# Summary

- Theory is not ready for 3G GW observatories
- Gravitational two-body problem
  - Post-Minkowskian approximation
  - Use tools from particle physics
  - Result for scattering shows remarkable agreement with NR
- B2B map:
  - Bound from scattering
  - Nonlocality problematic
- Outlook: Waveform



# Challenges

- Detector accuracy
  - Higher PM, spin, internal structure, …
  - Adjust QFT tools more to our needs
  - Continue scattering to bound state data
  - Incorporate PM more directly into waveform generation (EOB)

    [Damour, ‘16; Damgaard, Vanhove, ‘ 21; Khalil, Buonanno, Steinhoff, Vines, ‘22; Damour, Rettegno, ’23; …]

# Boundary conditions

$$\vec{f} = T^{-1}\mathrm{P}\, e^{\epsilon \int \tilde{A}(x)\,\mathrm{d}x}\, \vec{g}_0(\epsilon)$$

- Compare series expansions around singular point
  - Small velocity expansion: $v \equiv \sqrt{\gamma^2 - 1}$ $\quad v \to 0 \Longleftrightarrow \gamma \to 1 \Longleftrightarrow x \to 1$

1) Solution of differential equations:

[Lee, Smirnov, Smirnov, Steinhauser, '19]
[Libra: Lee, '20]

2) Explicit integral expansions:

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$$\partial_x \vec{f} = A(x,\epsilon)\vec{f}, \qquad \vec{f}(v,\epsilon) \simeq \sum_{n_1,n_2,k} v^{n_1+n_2\epsilon} \log^k(v) H_{n_1,n_2,k}(\epsilon)\vec{g}_0(\epsilon), \qquad n_1, n_2, k \in \mathbb{Z}$$

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PN-like integrals

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relations between infinite set of PN-like integrals

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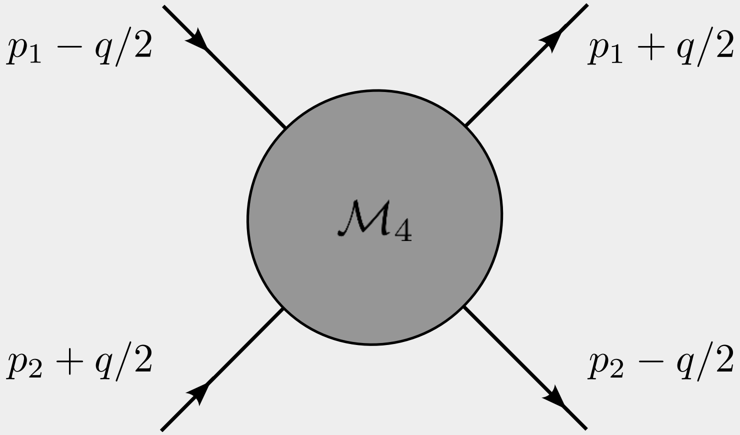
- Identify an independent set: $\vec{g}_0(\epsilon) = H_{\text{indep}}(\epsilon)^{-1}\vec{h}_{\text{indep}}(\epsilon)$

[Bern, Cheung, Roiban, Shen, Solon, Zeng, '20]

# The Scattering Amplitudes Approach

$$S_{\mathrm{GR}} = \int \mathrm{d}^D x \sqrt{-g} \left[ -\frac{1}{16\pi G} R + \frac{1}{2} \sum_{a=1,2} \left( D^\mu \phi_a D_\mu \phi_a - m_a^2 \phi_a^2 \right) \right]$$



$$p_1 \cdot q = p_2 \cdot q = 0$$

$$p_1^\mu = m_1 u_1^\mu, \quad p_2^\mu = m_2 u_2^\mu$$

- Integrand construction:
  - Generalized unitarity $C = \sum M_{(1)}^{\mathrm{tree}} M_{(2)}^{\mathrm{tree}} \dots M_{(m)}^{\mathrm{tree}}$
  - Double copy $M_{\mathrm{tree}} \sim (A_{\mathrm{tree}})^2$

Gravity → $M_{\mathrm{tree}}$ ; Yang-Mills → $A_{\mathrm{tree}}$

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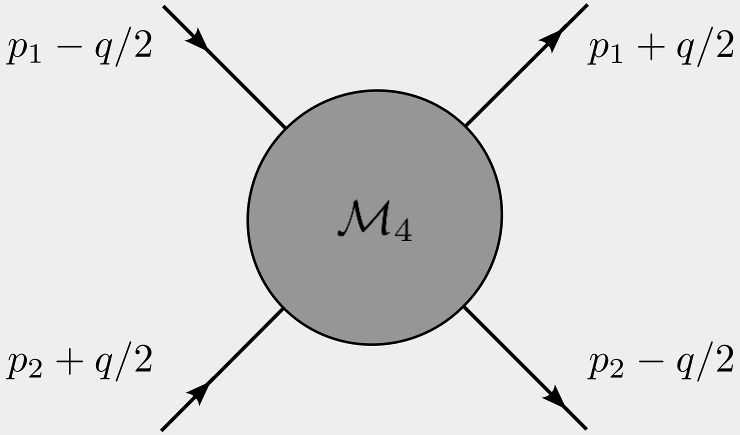

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  $\hbar \longleftrightarrow |q|$ leading term in soft expansion!

$$\frac{1}{(\ell + p_i - q/2)^2 - m_1^2} \simeq \frac{1}{2p_1 \cdot \ell} = \frac{1}{m_1 \, 2u_1 \cdot \ell}$$

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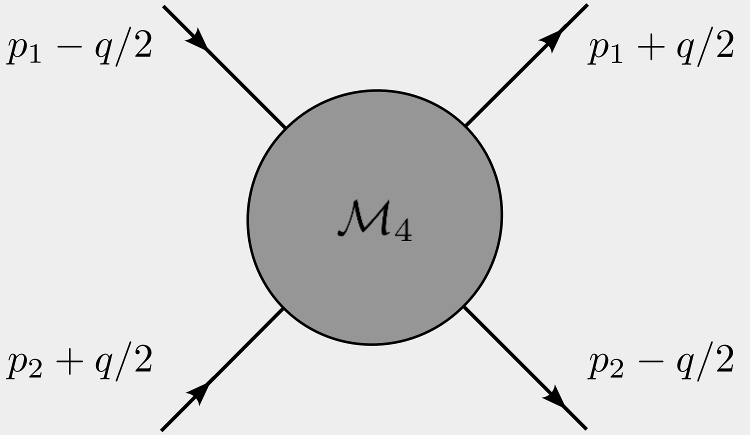

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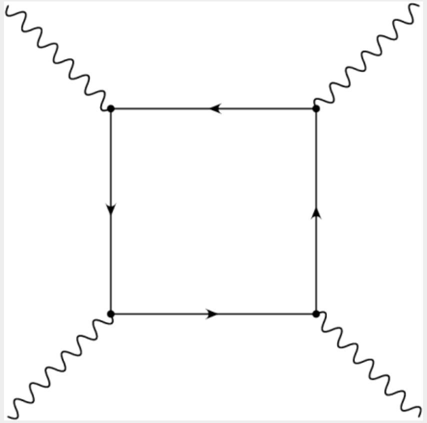
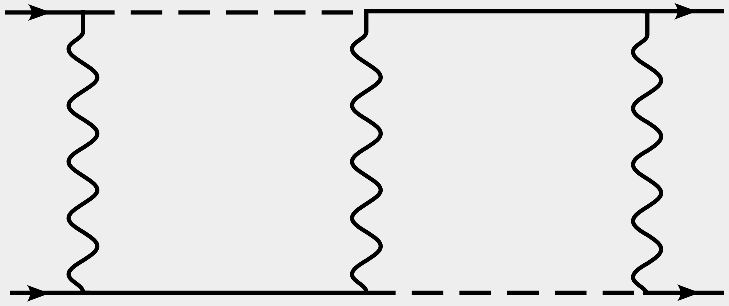
- Potential from EFT matching

# The classical limit

[Damgaard, Haddad, Helset, '21, Holstein, Donoghue, '13]





$$\int \mathcal{D}\phi\, e^{\frac{i}{\hbar}(S_0+gS_3+g^2S_4)} \sim \int \mathcal{D}\phi\, e^{\frac{i}{\hbar g^2}S}$$

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$$

$$p = \hbar k$$

$$m = \frac{\hbar}{\lambda}$$

$$b \gg \lambda$$

$$q \ll m$$

# Elliptic integrals

- Polylogarithms for most sectors

$$\partial_x \vec{g} = \epsilon\, \tilde{A}(x) \vec{g}, \quad \tilde{A}(x) = \sum_i M_i a_i(x)$$

constant matrices

$$a_i(x) \in \left\{ \frac{1}{x}, \frac{1}{x-1}, \frac{1}{x+1}, \frac{x}{1+x^2} \right\}$$

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  - analyze $\epsilon^0$-part:

$$A(x,\epsilon) = A^{(0)}(x) + \epsilon A^{(1)}(x) + \mathcal{O}(\epsilon^2), \qquad \partial_x T^{(0)}(x) = A^{(0)}(x) T^{(0)}(x)$$

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$$\partial_x \vec{f} = A(x,\epsilon)\vec{f}$$

$$\partial_\gamma \vec{f} = A(\gamma,\epsilon)\vec{f} \longrightarrow \sqrt{\gamma^2-1}$$

$$T^{(0)}(\gamma)$$

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- Algorithms do not work for one sector
  - Third-order DE: $$\left[\partial_x^3 - \frac{6x}{1-x^2}\partial_x^2 - \frac{1-4x^2+7x^4}{x^2(1-x^2)^2}\partial_x - \frac{1+x^2}{x^3(1-x^2)}\right]\Psi_{1,2,3} = 0$$
  - Solve with Mathematica: $\Psi_1 = x\mathrm{K}^2(1-x^2), \quad \Psi_2 = x\mathrm{K}(1-x^2)\mathrm{K}(x^2), \quad \Psi_3 = x\mathrm{K}^2(x^2)$

$$\mathrm{K}(z) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-zt^2)}},$$

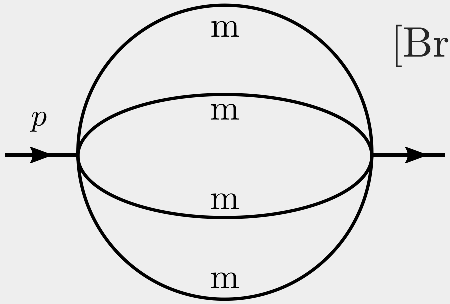
$$\mathrm{E}(z) = \int_0^1 \frac{1-zt^2}{\sqrt{1-t^2}}\mathrm{d}t$$

# Elliptic integrals

- Polylogarithms for most sectors

$$\partial_x \vec{g} = \epsilon\, \tilde{A}(x)\vec{g}, \quad \tilde{A}(x) = \sum_i \underset{\substack{\uparrow \\ \text{constant matrices}}}{M_i} a_i(x)$$

- Algorithms do not work for one sector
  - Third-order DE: $\left[\partial_x^3 - \frac{6x}{1-x^2}\partial_x^2 - \frac{1-4x^2+7x^4}{x^2(1-x^2)^2}\partial_x - \frac{1+x^2}{x^3(1-x^2)}\right]\Psi_{1,2,3} = 0$
  - Solve with Mathematica: $\Psi_1 = x\mathrm{K}^2(1-x^2), \quad \Psi_2 = x\mathrm{K}(1-x^2)\mathrm{K}(x^2), \quad \Psi_3 = x\mathrm{K}^2(x^2)$



[Broedel, Duhr, Dulat, Marzucca, Penante, Tancredi, '19;
Broedel, Duhr, Matthes, '21;
Pögel, Wang, Weinzierl, '22]

$$\mathrm{K}(z) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-zt^2)}},$$

$$\mathrm{E}(z) = \int_0^1 \frac{1-zt^2}{\sqrt{1-t^2}}\mathrm{d}t$$

# $\epsilon$-form using INITIAL

- Elliptic differential equations

68 master integrals:
$\Rightarrow$ constant $68 \times 68$ matrices

$$\partial_x \vec{g} = \epsilon\, \tilde{A}(x) \vec{g}, \qquad \tilde{A}(x) = \sum_i M_i a_i(x)$$

$$a_i(x) \in \Bigg\{ \frac{\pi^2}{x(1-x^2)\mathrm{K}^2(1-x^2)}, \quad \frac{1}{1-x}, \quad \frac{1}{x}, \quad \frac{1}{1+x}, \quad \frac{x}{1+x^2},$$
$$\frac{\mathrm{K}^2(1-x^2)}{\pi^2 x(1-x^2)}, \quad \frac{\mathrm{K}^2(1-x^2)}{\pi^2(1-x^2)}, \quad \frac{\mathrm{K}^2(1-x^2)}{\pi^2 x}, \quad \frac{\mathrm{K}^2(1-x^2)}{\pi^2}, \quad \frac{(1-x^2)\mathrm{K}^2(1-x^2)}{\pi^2 x},$$
$$\frac{\mathrm{K}^4(1-x^2)}{\pi^4 x(1-x^2)}, \quad \frac{\mathrm{K}^4(1-x^2)}{\pi^4 x}, \quad \frac{(1-x^2)\mathrm{K}^4(1-x^2)}{\pi^4 x}, \quad \frac{(1-x^2)^2\mathrm{K}^4(1-x^2)}{\pi^4 x} \Bigg\}$$

  - Differential equations not problematic for us, despite elliptics
  - Canonical form just as in polylogarithmic 3PM case