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ARXIV:2309.15929 ,
ARXIV:2309.07343 ,
ARXIV:2402.13307 ,
ARXIV:2402.14769 ,
ARXIV:2511.05446 .

Heavy Particle Methods For Precision Beta Decay

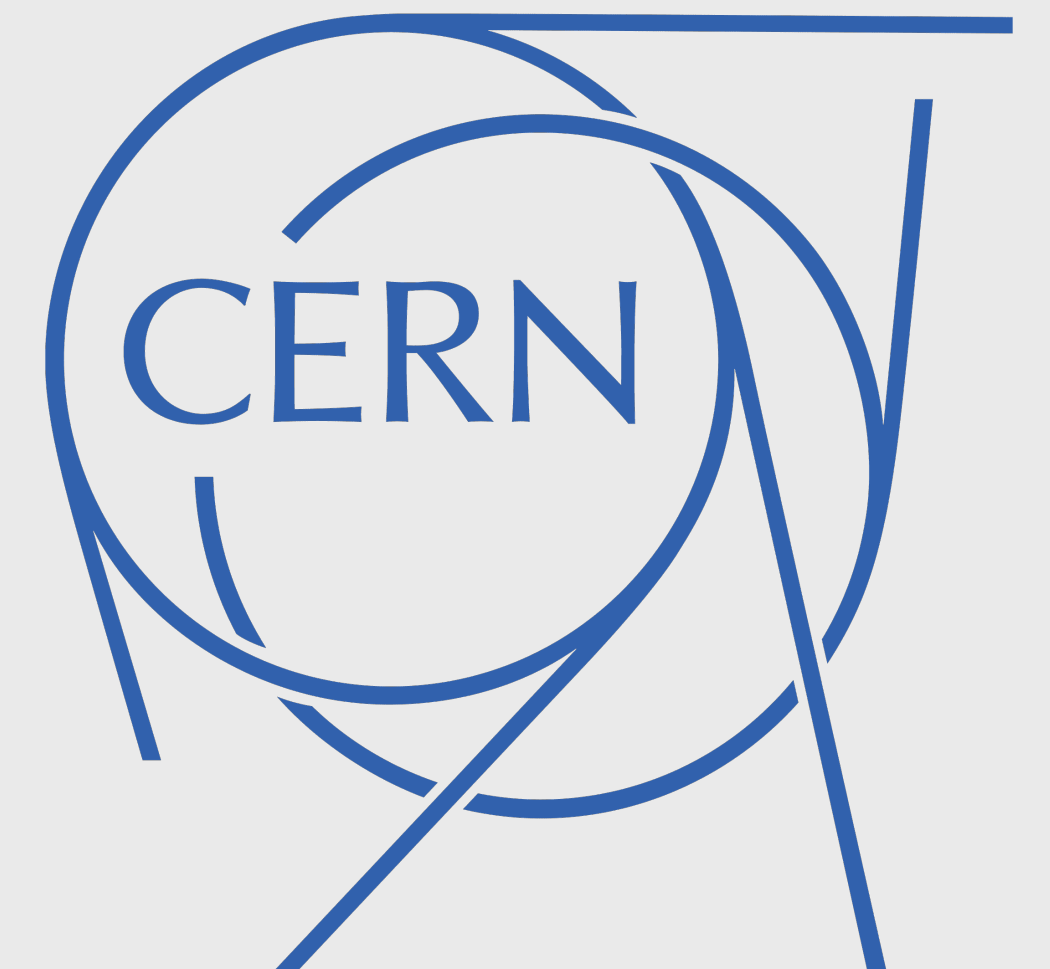
RYAN PLESTID

CERN → OHIO STATE (2027)

COLLABORATORS

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ZEHUA CAO, KAUSHIK BORAH

PARTICLE PHYSICS SEMINAR | UNIVERSITAT WIEN | NOVEMBER 2025

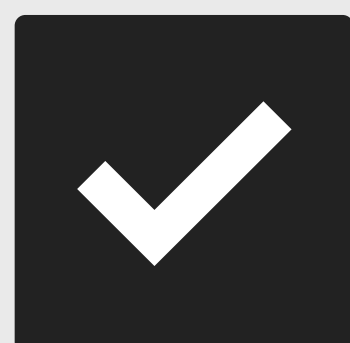




PART 1

EFT & β DECAY

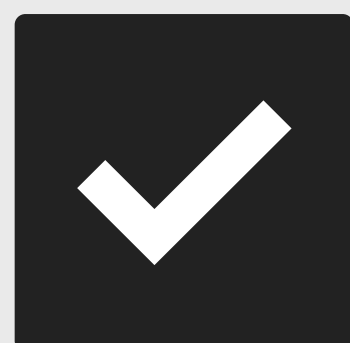
- Motivation & relevance for **fundamental physics**.
- Necessary **precision**, and requisite **loop orders**.



PART 2

FERMI FUNC.

- **Point-like** EFT of nuclei and leptons.
- The **Fermi function** from loops.



PART 3

RAD. CORR.

- Structure of **radiative corrections** from EFT.
- Renormalization group **resummation of logarithms**.

Quark Mixing In The SM

FUNDAMENTAL
CONSTANTS
OF NATURE

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$



CKM $\equiv$ CABIBBO-KOBAYASHI-MASKAWA



CKM Unitarity

FIRST ROW UNITARITY

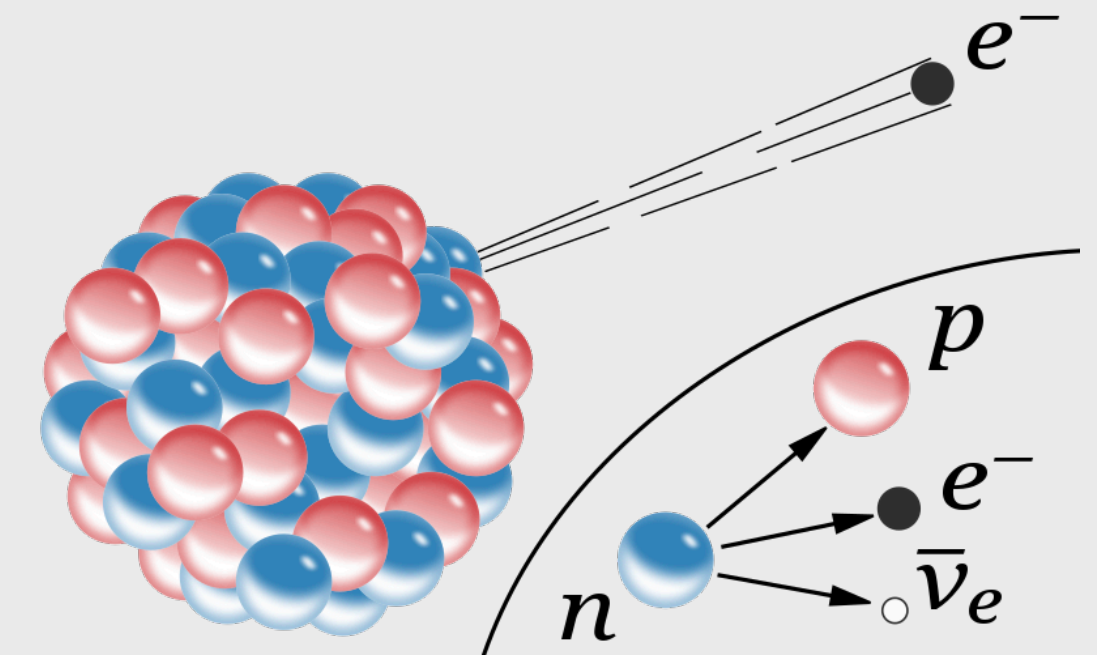
$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$

IN WOLFENSTEIN NOTATION

$$1 - \lambda_{ud}^2 + \lambda_{us}^2 + O(\lambda^6) = 1$$

- Percent-level accuracy in Kaon decay demands 100 ppm accuracy in $0^+ \rightarrow 0^+$ beta decays

$$|V_{ud}|^2$$



$$|V_{us}|^2$$

$$\frac{\Gamma(K \rightarrow \mu\nu(\gamma))}{\Gamma(\pi \rightarrow \mu\nu(\gamma))}$$

CKM Unitarity

FIRST ROW UNITARITY

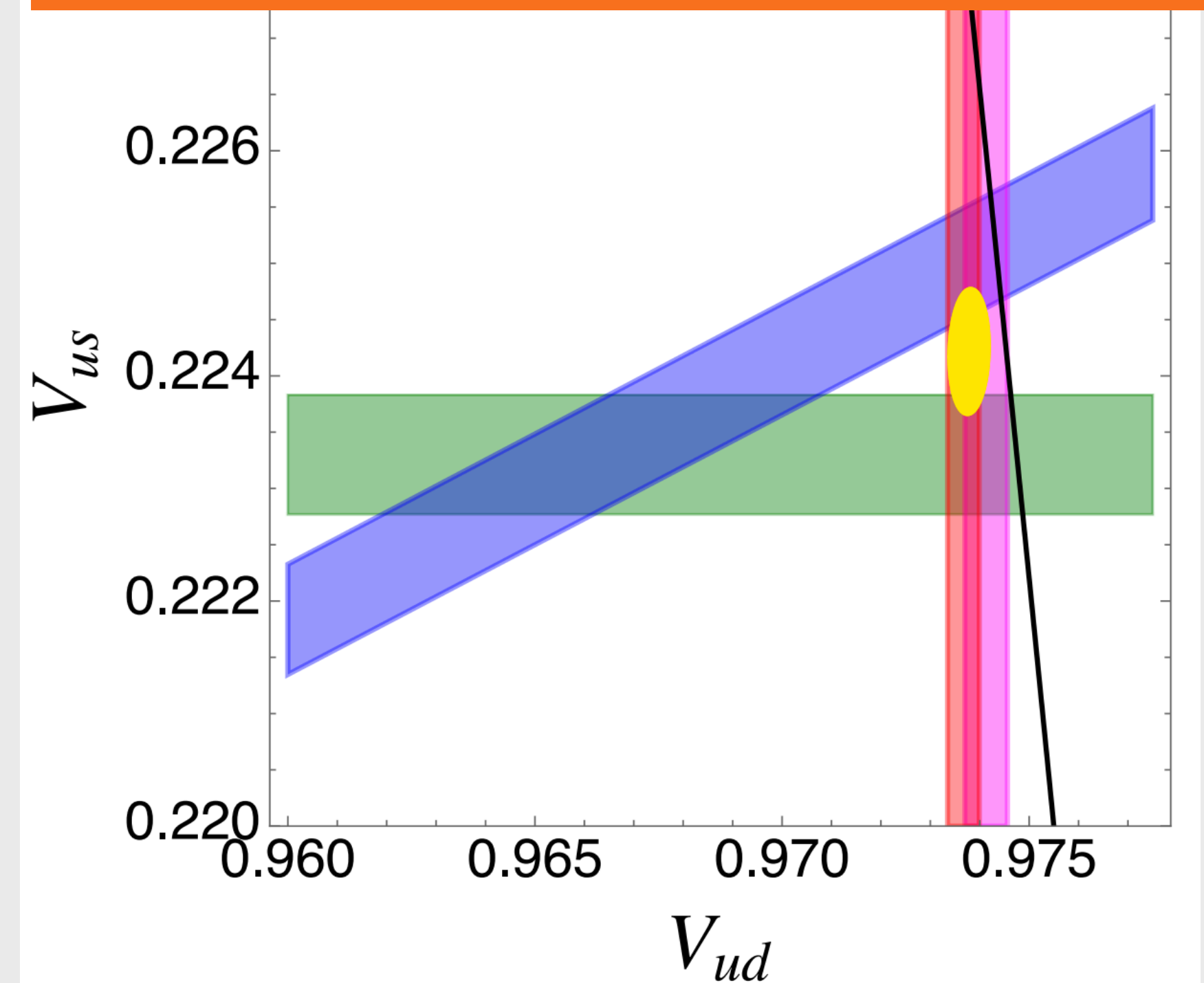
$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$

IN WOLFENSTEIN NOTATION

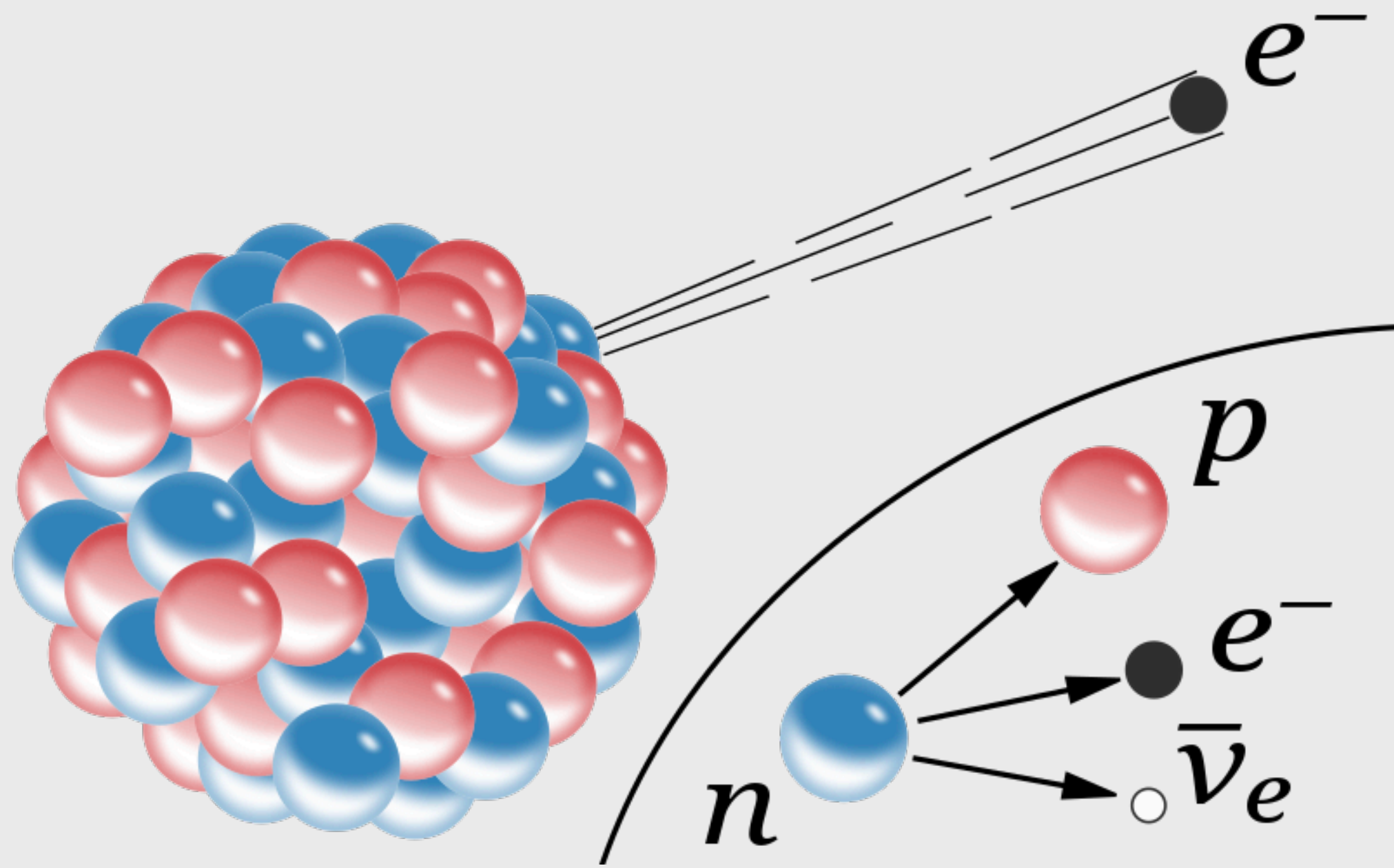
$$1 - \lambda_{ud}^2 + \lambda_{us}^2 + O(\lambda^6) = 1$$

- Tension in first-row CKM unitarity.
- **If** theory is under control: new physics discovered!

CIRIGLIANO +++ ARXIV:2208.11707



Superaligned At Leading Order



$$\Gamma_{\beta} \propto |V_{ud}|^2 \times |\mathcal{M}_{\text{nuc}}|^2$$

$0^+ \rightarrow 0^+$ only J_V^{μ} contributes

$$\partial_{\mu} J_V^{\mu} \approx 0 \quad \text{CVC}$$

$$\langle A | \tau^+ | B \rangle = \sqrt{2} \langle A || 1 || A \rangle$$

Isospin Symmetry

$$|\mathcal{M}_{\text{nuc}}|^2 = 2$$

Analogy With B-Physics: Heavy + Weak + Gauge Theory

$$^{10}\text{C} \rightarrow ^{10}\text{B} e^+ \nu_e \qquad \bar{B} \rightarrow D \ell \nu$$

QCD

Easy! – Isospin

Hard

QED

Big – Large $Z\alpha$

Small

Legs

Three (QED)

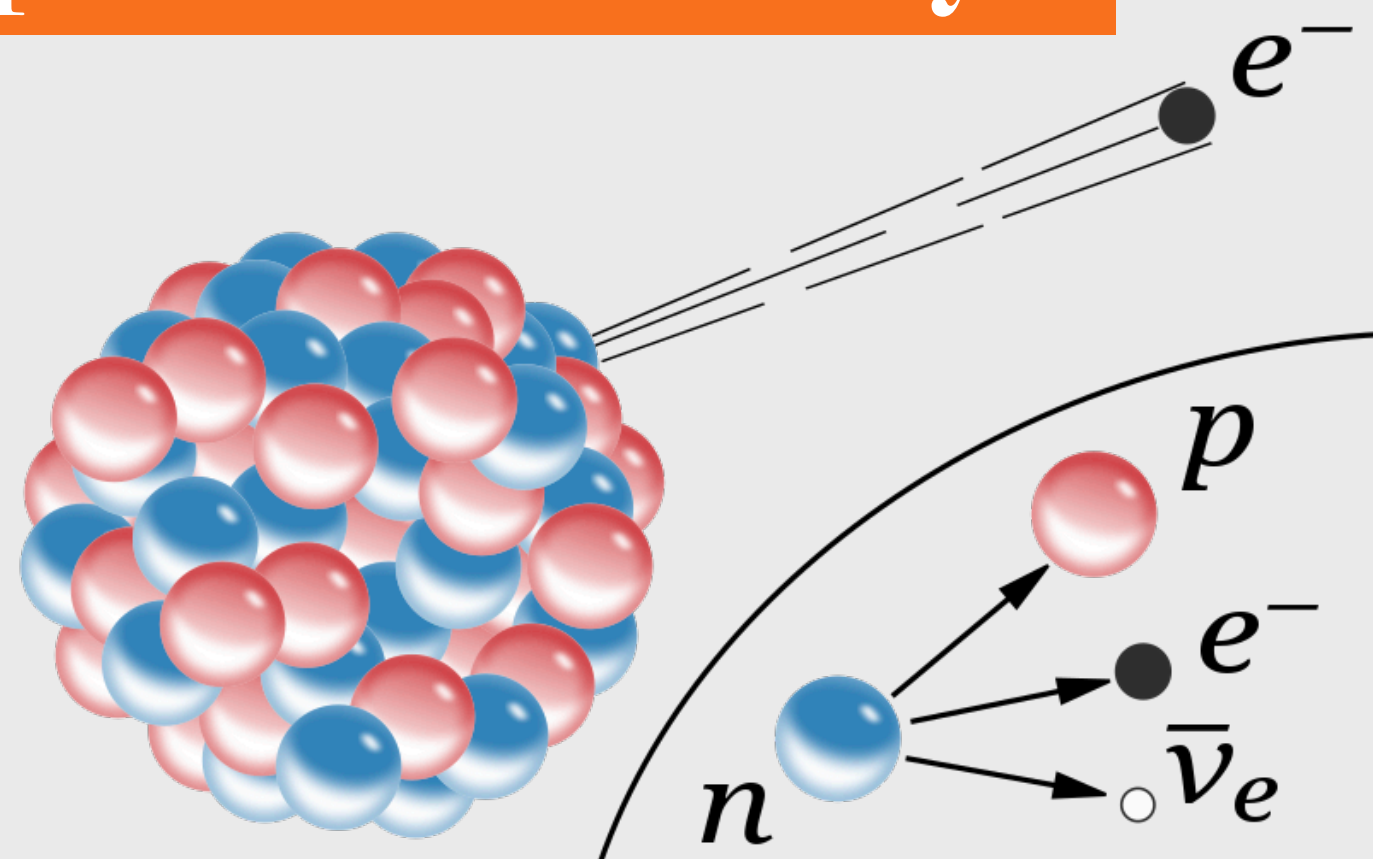
Two (QCD)

How To Measure $|V_{ud}|$

Hardy & Towner 2020

100 PPM PRECISION

Superallowed Decays



$$|V_{ud}|^2 = 0.94815 \pm 0.00060. \quad (24)$$

The uncertainty attached to $|V_{ud}|^2$ in Eq. (24) includes contributions from many sources but is completely dominated by those originating from the theoretical correction terms, with experiment contributing a mere 0.00009 to the 0.00060 total.

Frac. Uncertainty (%)

0.04

0.02

Underestimated

Exp

δ'_R

δ_C

δ_{NS}

Δ_R^V

Historical Approach

$$\mathcal{F}t \equiv ft(1 + \delta'_R)(1 + \delta_{\text{NS}} - \delta_C) = \frac{K}{2G_V^2(1 + \Delta_R^V)},$$

- The “ ft ” value includes the Fermi function (Dirac w.f.) .
- Includes regulated (but not renormalized) UV divergence.
- RCs are **assumed** to factorize (ansatz) from Fermi function.
- RCs are computed in the "independent particle model".

Historical Approach

$$\mathcal{F}t \equiv ft(1 + \delta'_R)(1 + \delta_{\text{NS}} - \delta_C) = \frac{K}{2G_V^2(1 + \Delta_R^V)},$$

- The “ ft ” value includes the Fermi function (Dirac w.f.) .
 - Include radiative corrections (RCs) .
 - RCs are computed in the “independent particle model” .
- THIS IS NOT SUFFICIENT
FOR PRECISION GOALS
OF 100 PPM!**

New Approach With EFT

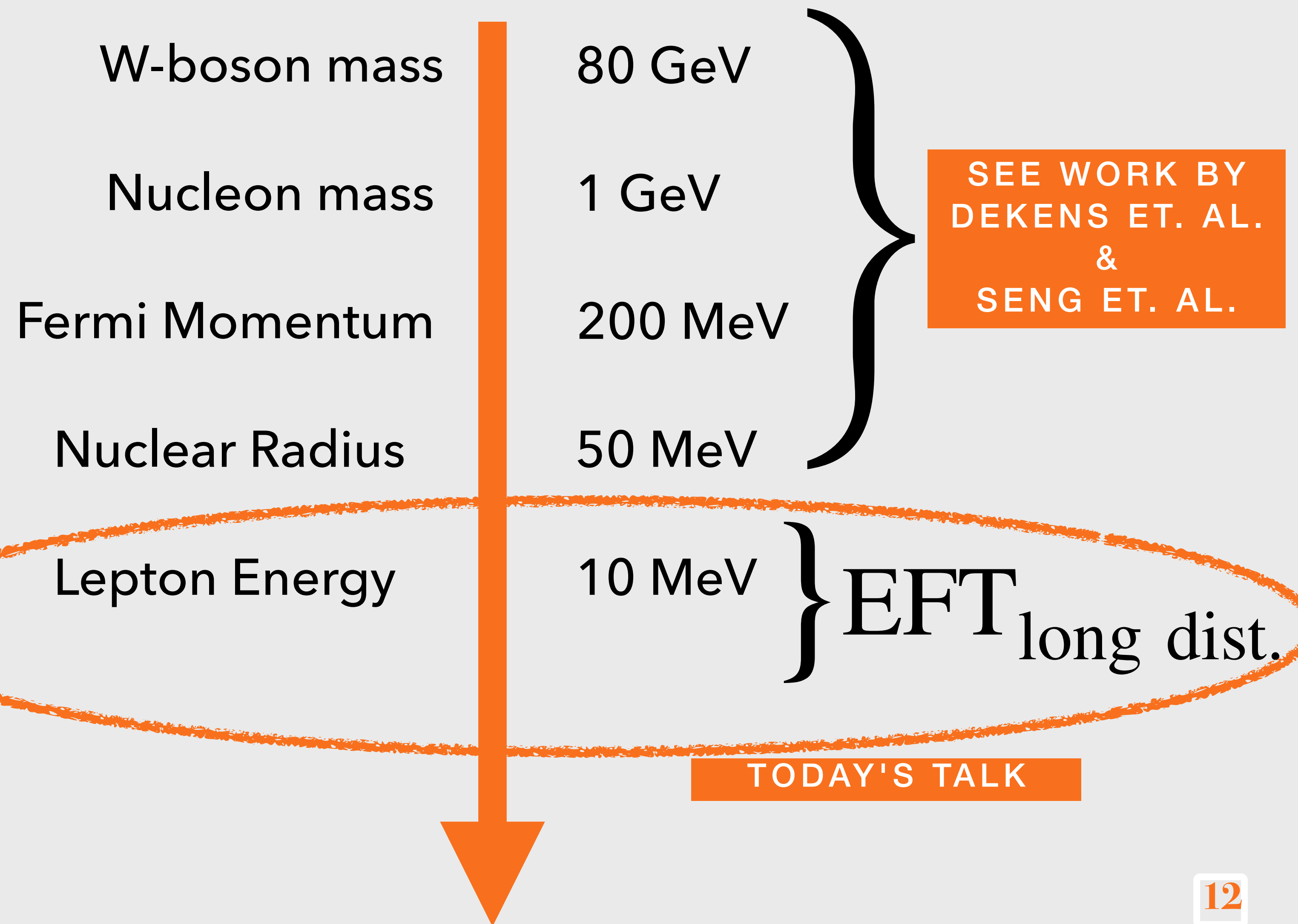
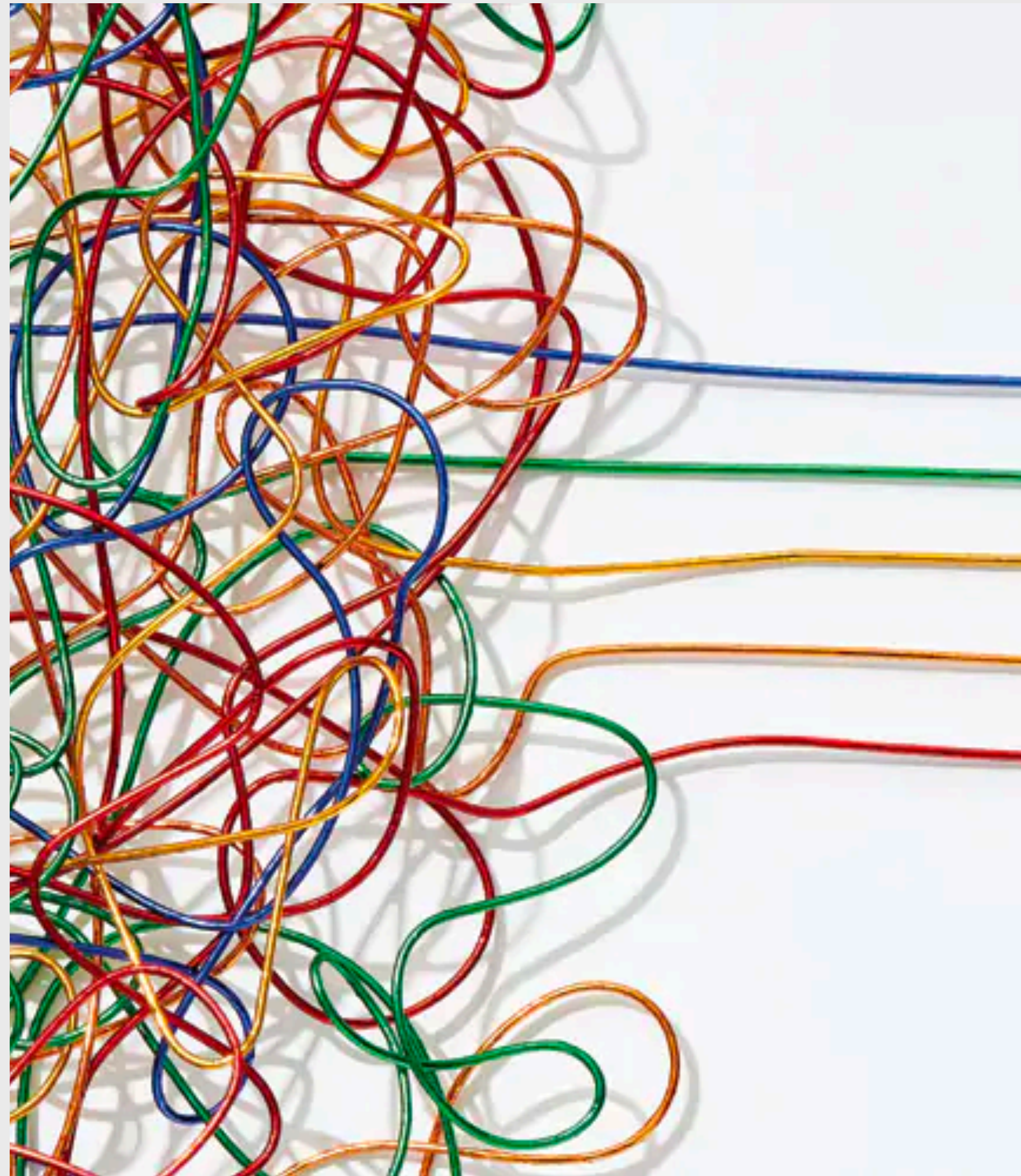
Factorization Theorem At Leading Power + Corrections Of $\mathcal{O}((pR)^2)$

$$\Gamma = \int d\Pi_e d\Pi_\nu \underbrace{\langle |\mathcal{M}|^2 \rangle}_{\text{Matrix Element}} \times \underbrace{|C|^2}_{\text{Wilson Coefficient}} (2\pi)\delta(\Sigma E)$$

Long Distance

Short Distance

Tower Of EFTs





PART 1

EFT & β DECAY

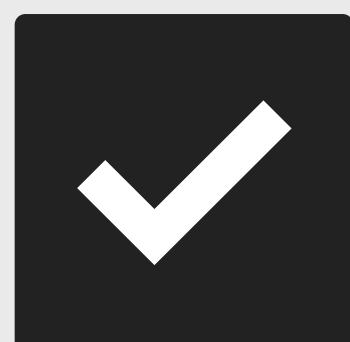
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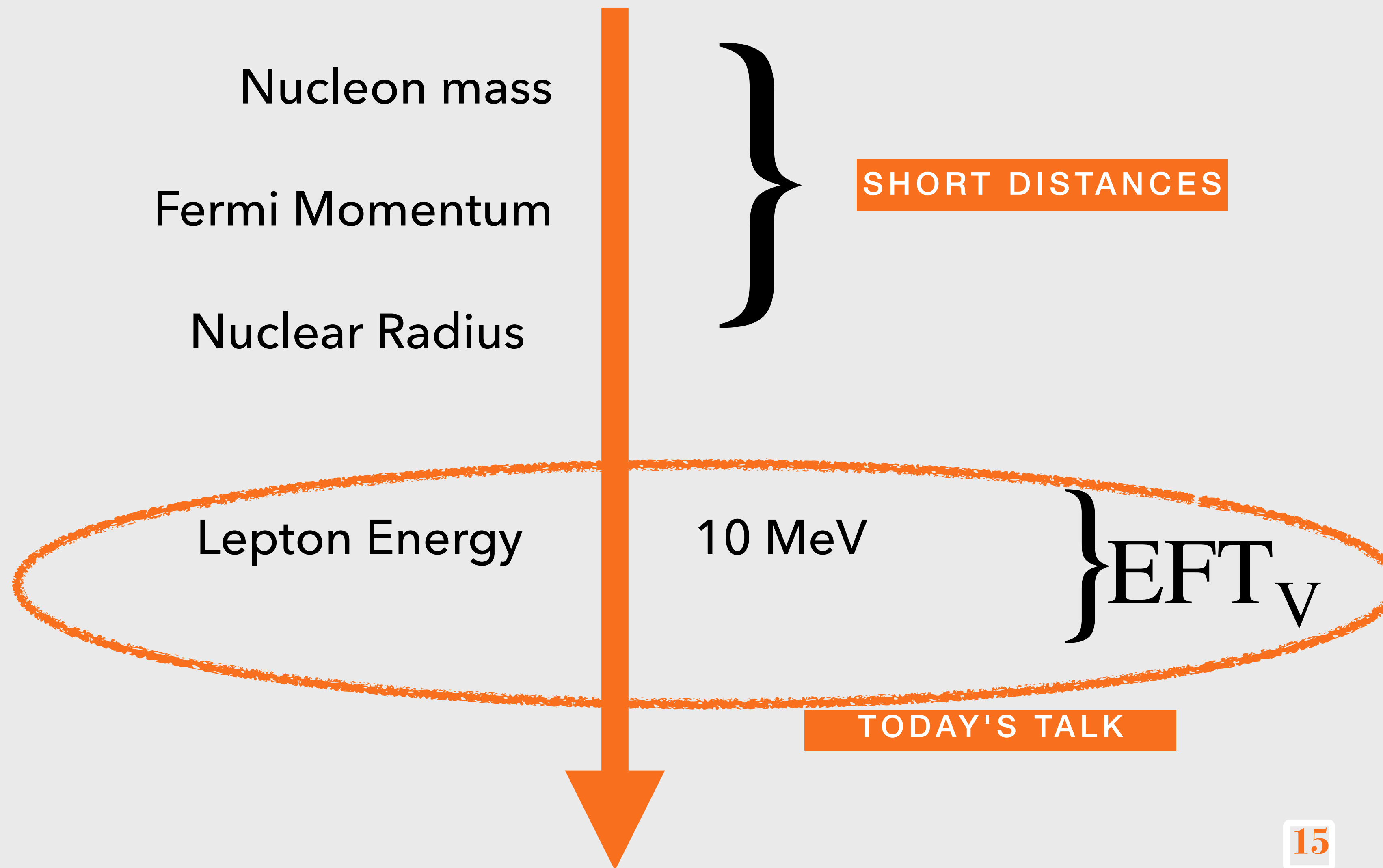


Point-Like EFT Of Nuclei

A Lagrangian For Low-Energy Beta Decay

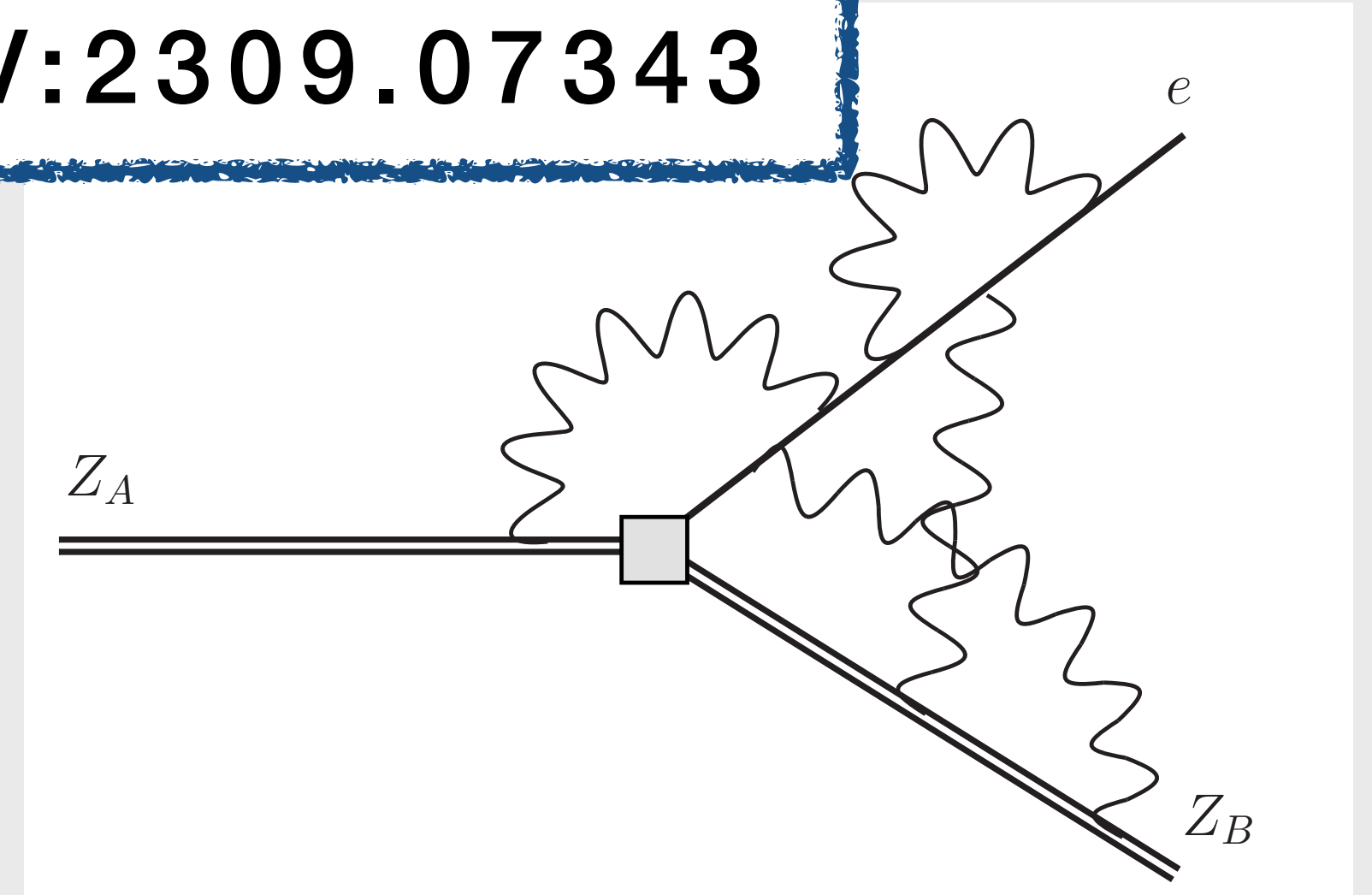
EFT For $0^+ \rightarrow 0^+$

- Largest corrections come from long distance scales.
- Need to work to higher orders in perturbation theory.



EFT For $0^+ \rightarrow 0^+$

$$\begin{aligned} \mathcal{L} = & h_A^\dagger (\nu \cdot D) h_A + h_B^\dagger (\nu \cdot D) h_B \\ & - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{e} (\gamma_\mu D^\mu + m) e + \bar{\nu} \gamma^\mu \partial_\mu \nu \\ & + C(\mu) \times \left[\bar{e} \gamma_\mu P_L \nu \right] \times \left[h_B^\dagger \nu^\mu h_A \right] \end{aligned}$$



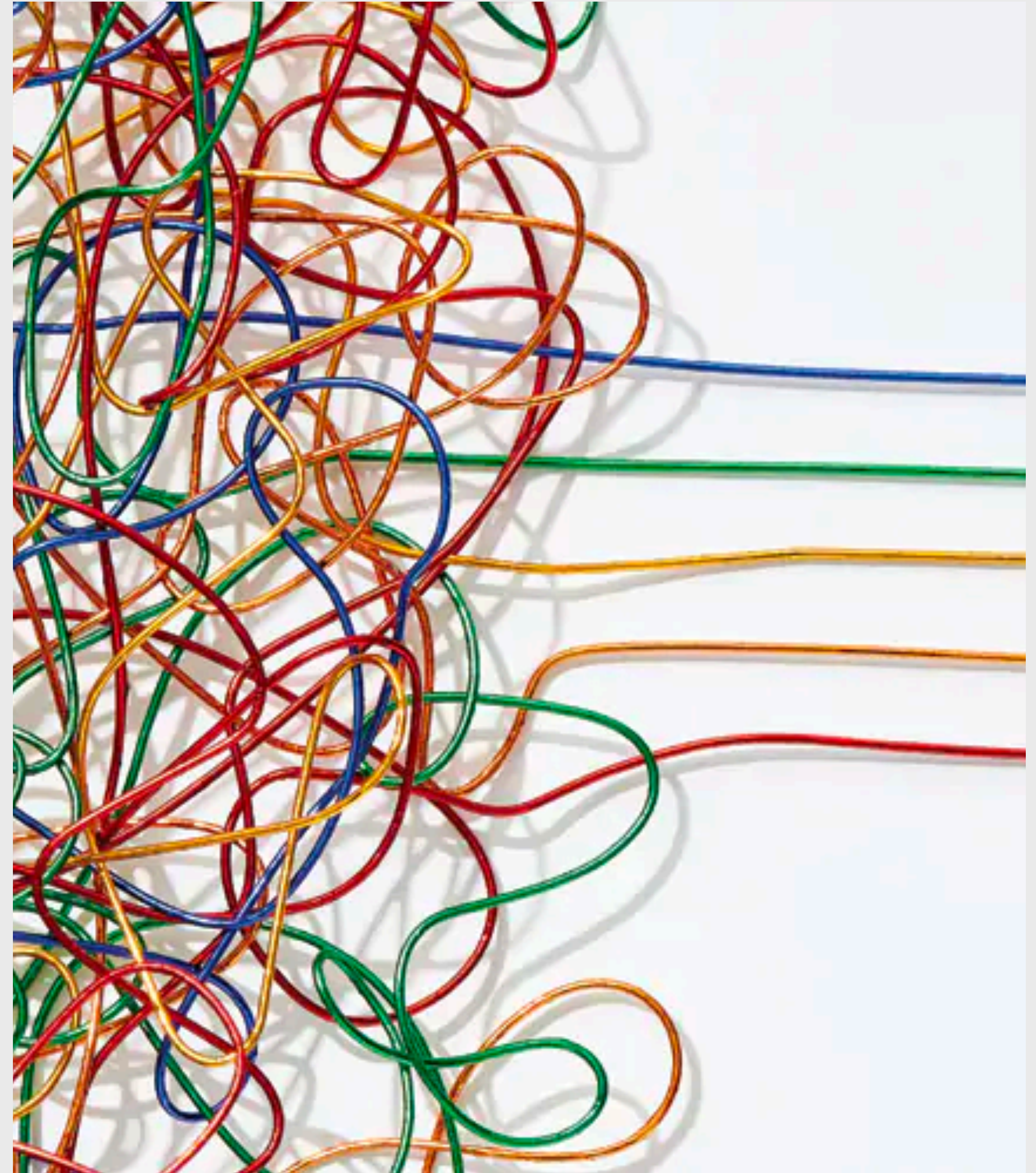
Treat Nuclei Like They Are Heavy 'Quarks'

Heavy Particle EFTs

$$\frac{2M}{(p+k)^2 - M^2} \rightarrow \frac{1}{v \cdot k}$$

$$v = p/M$$

- This simplifies amplitudes.
- Heavy mass never appears.



Heavy Particle EFTs

Simplifications

$$\nu_\mu \quad \text{VS} \quad \gamma_\mu$$

$$\mathcal{L} = h_v^\dagger (\not{v} \cdot D) h_v$$

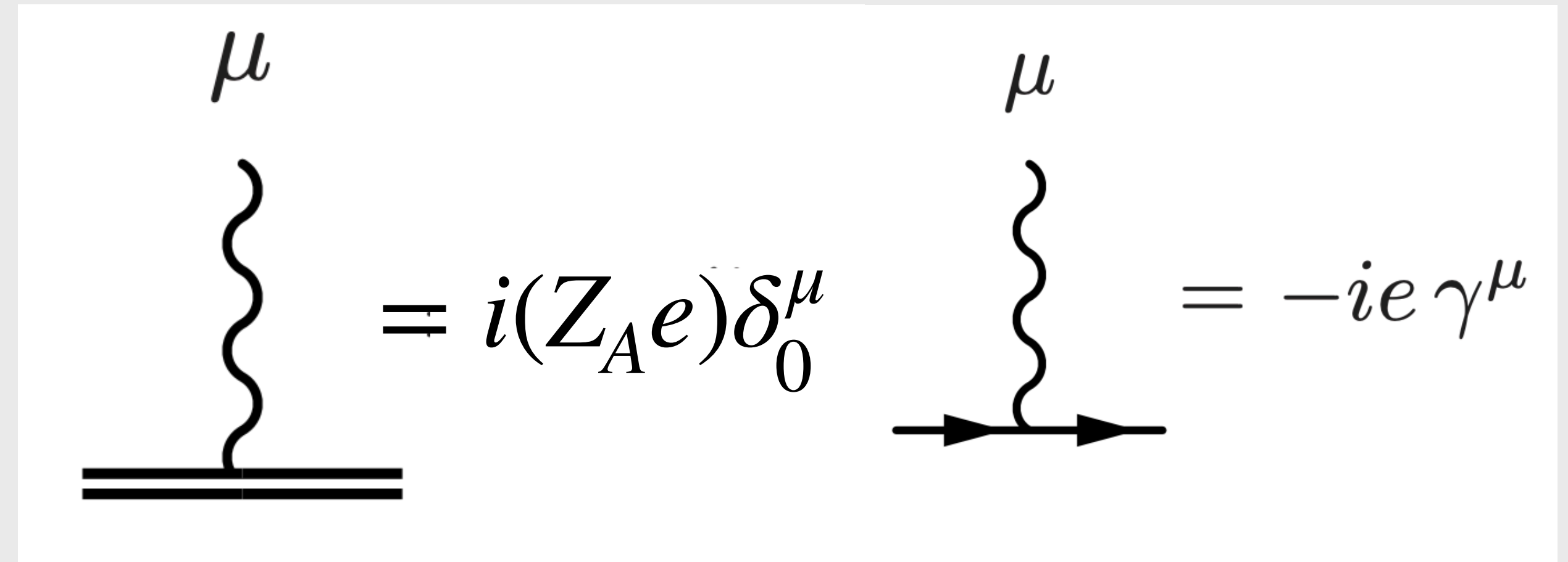


Diagram 1: A heavy particle line (double line) with an incoming wavy line labeled μ . This is equal to $i(Z_A e) \delta_0^\mu$.

Diagram 2: A heavy particle line (double line) with an incoming wavy line labeled μ and an outgoing wavy line labeled μ . This is equal to $-ie \gamma^\mu$.

$$\not{v} \cdot \not{q} \quad \text{VS} \quad q^2 - m^2$$

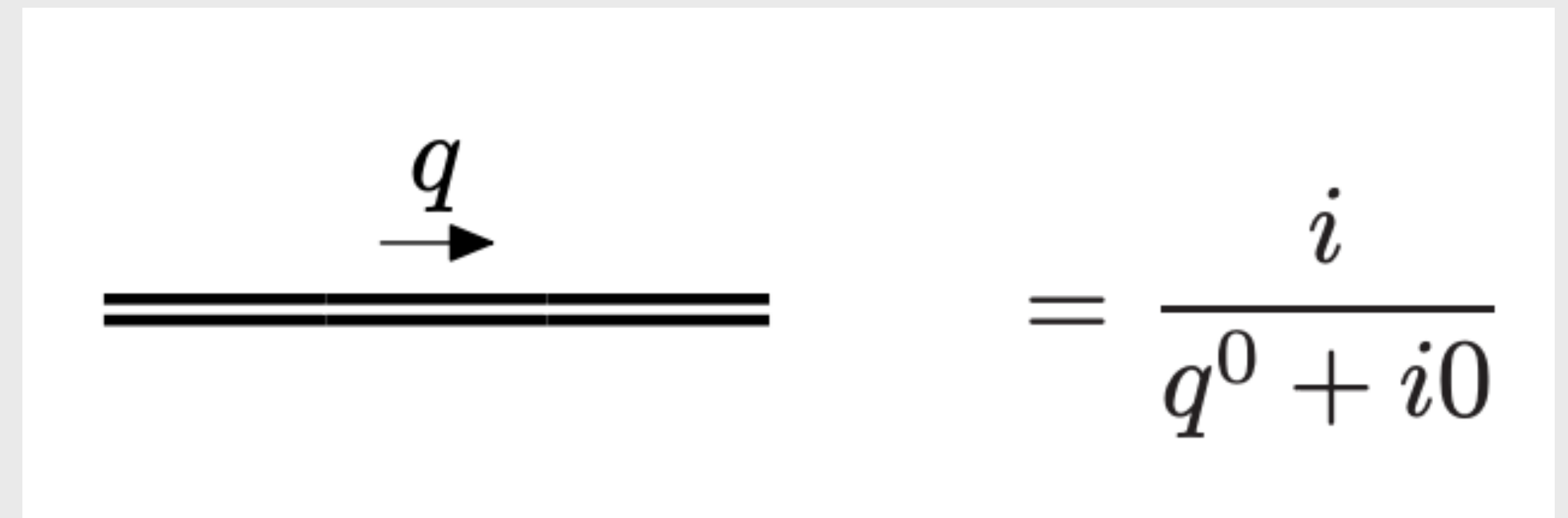
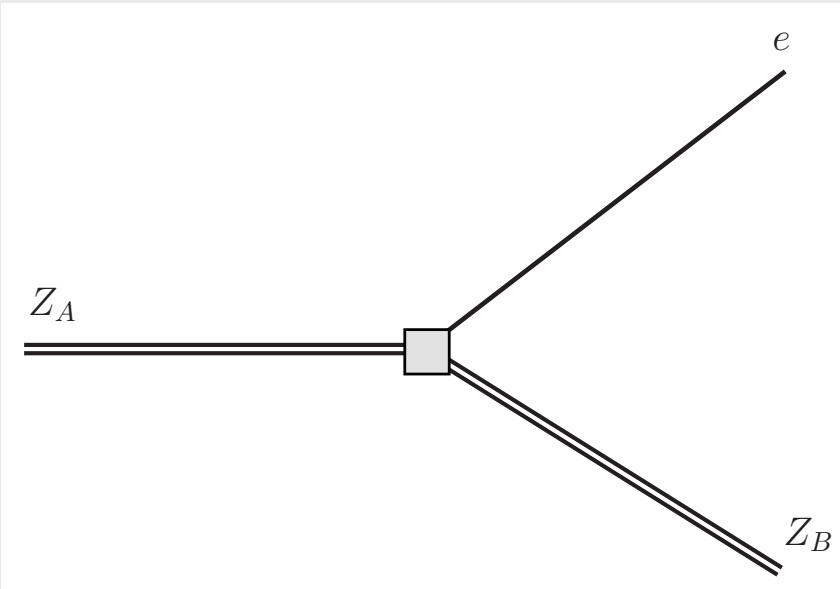


Diagram: A heavy particle line (double line) with an incoming arrow labeled q . This is equal to $\frac{i}{q^0 + i0}$.

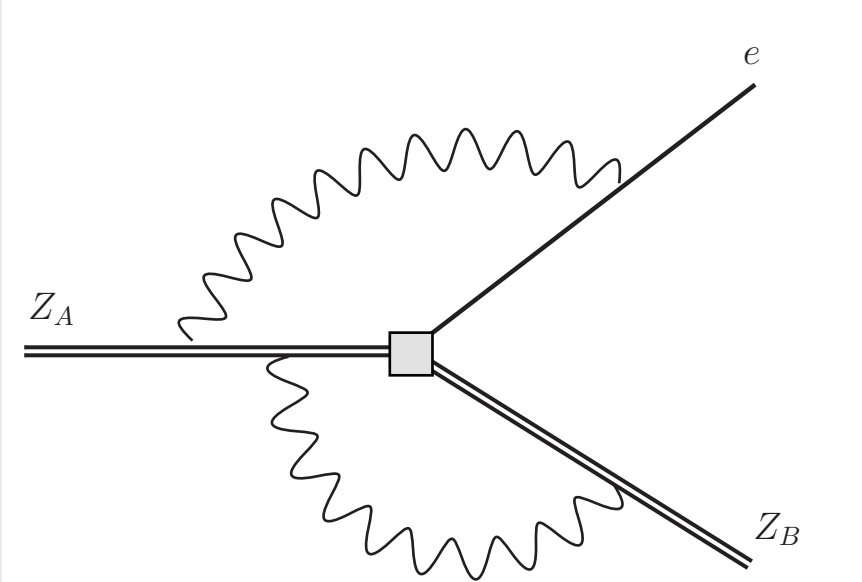
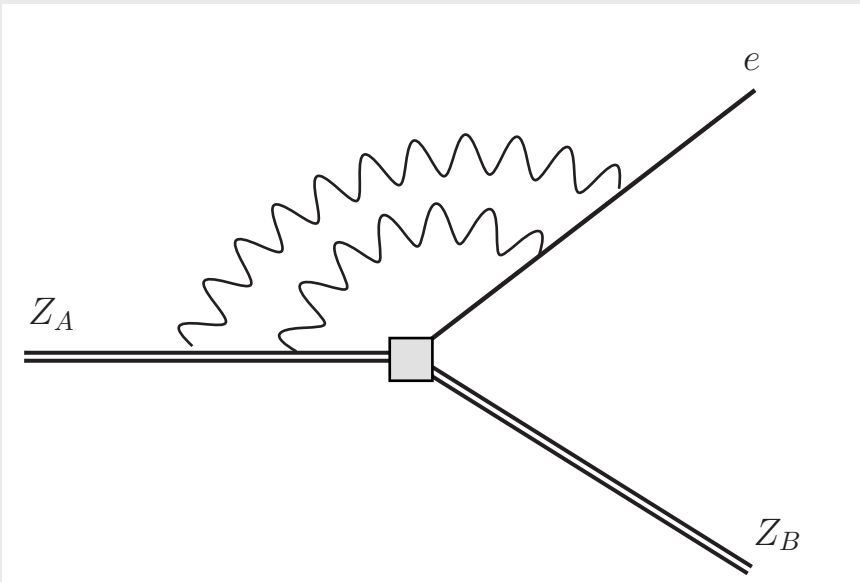
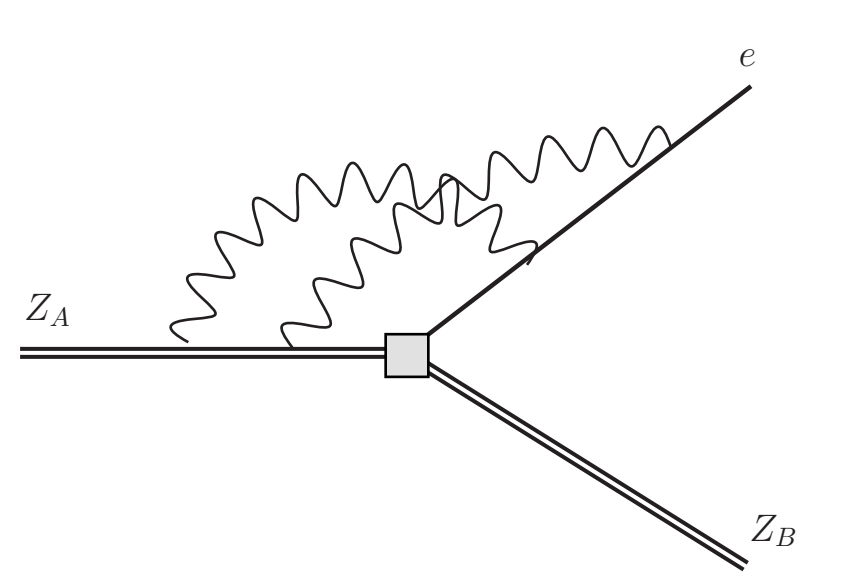
Now We Just Compute Diagrams

WAVEFUNCTION RENORMALIZATION NOT SHOWN

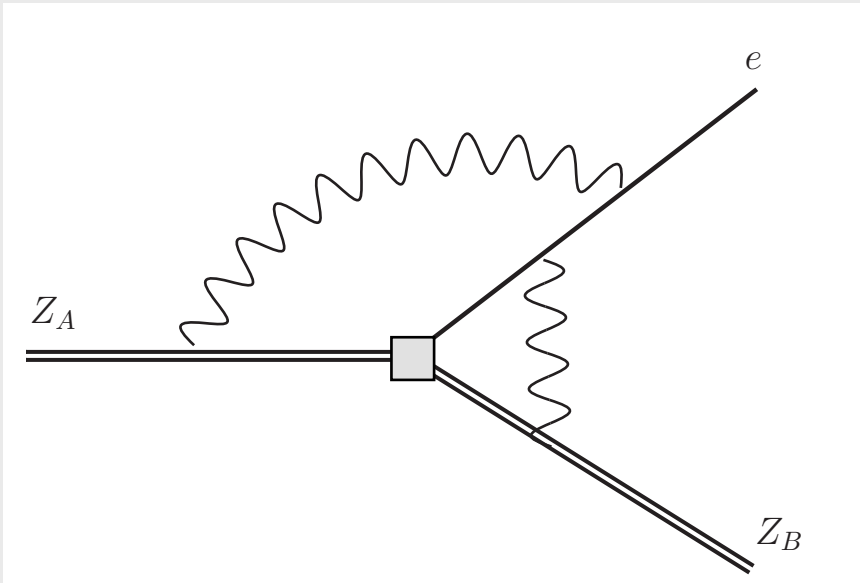
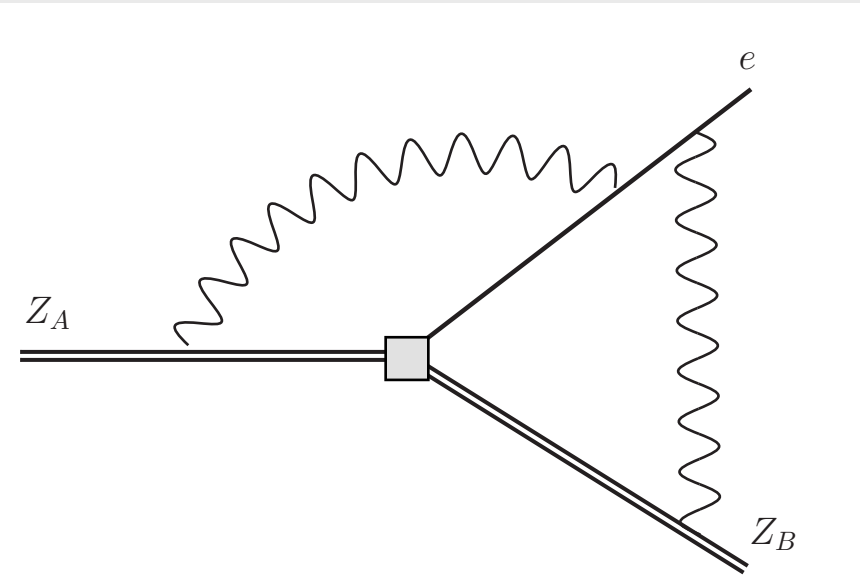
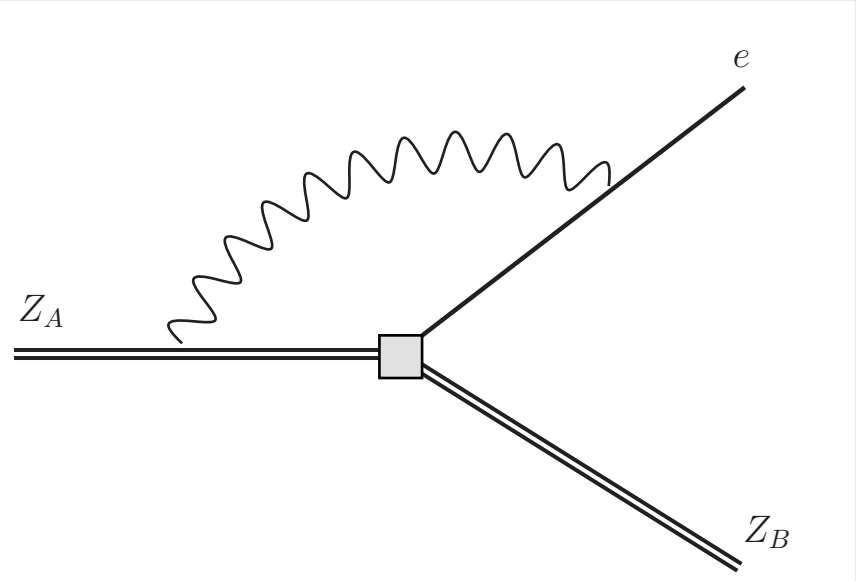
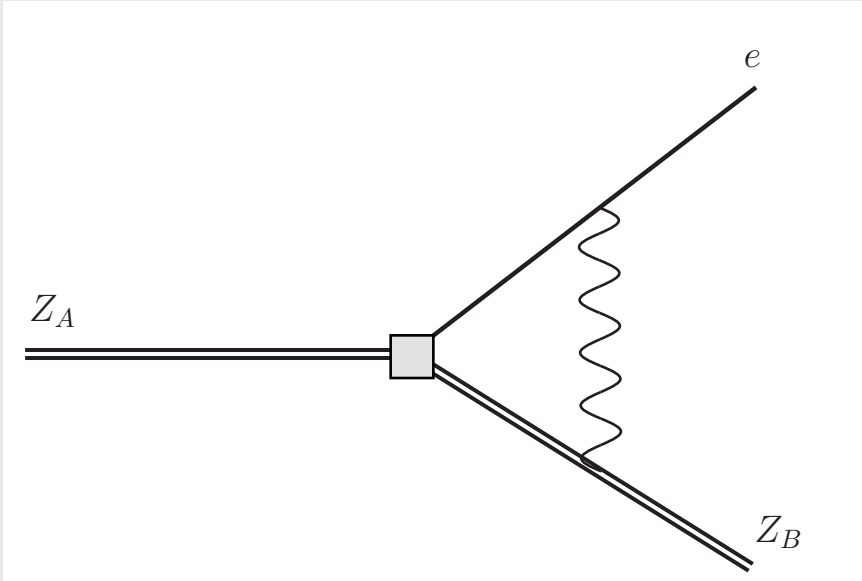
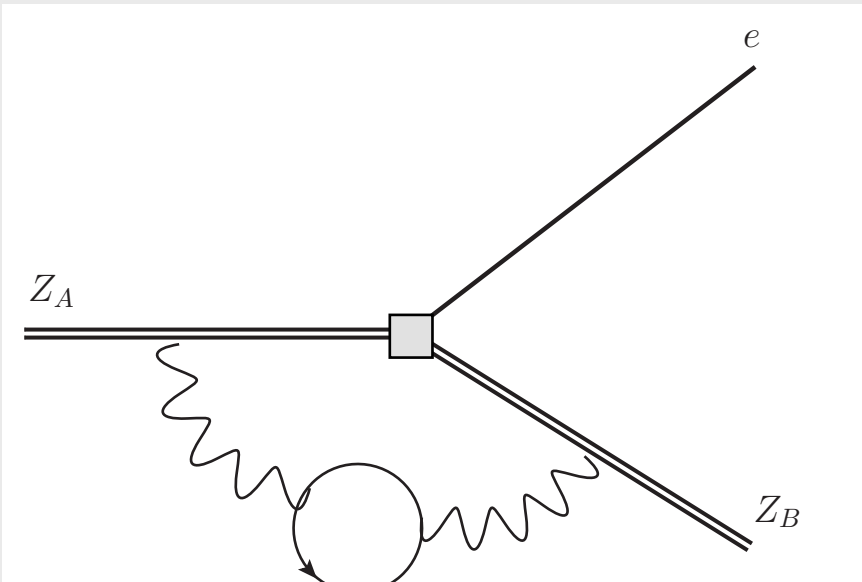
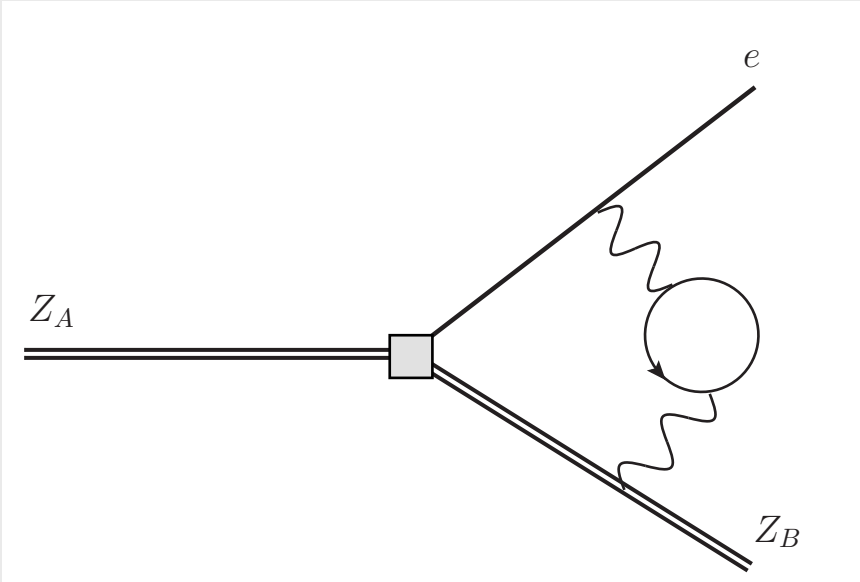
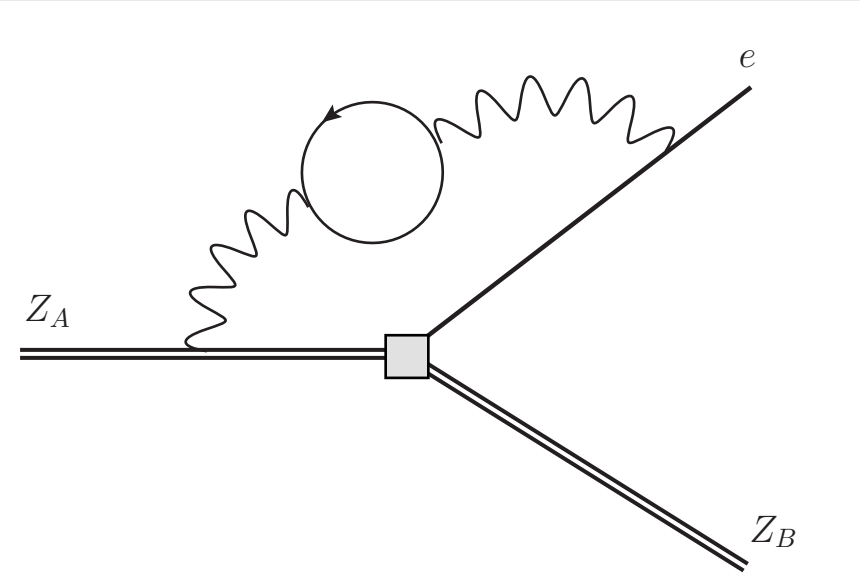
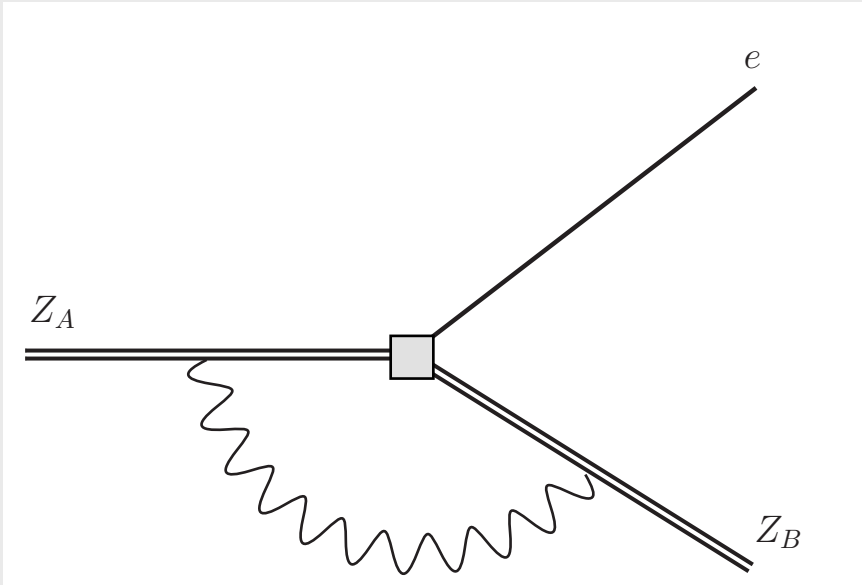
TREE-LEVEL



TWO LOOP



ONE LOOP



Equivalent Feynman Rules

TREE-LEVEL

- 1 diagram.

ONE LOOP

- 2 diagrams.

TWO LOOP

- 5 diagrams.

THREE LOOP

- 10 diagrams.

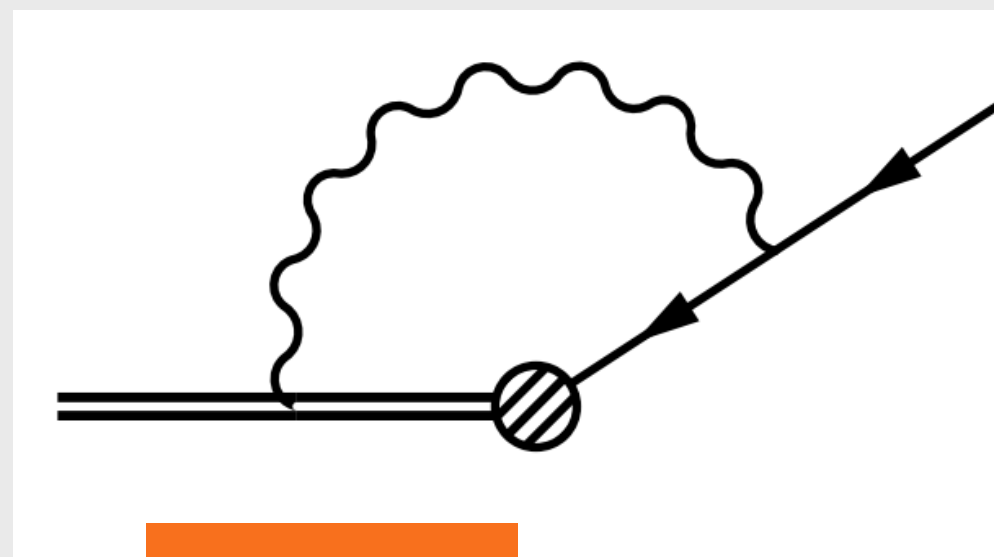
1 NUCLEUS WITH UNIT CHARGE

+ A BACKGROUND COULOMB FIELD

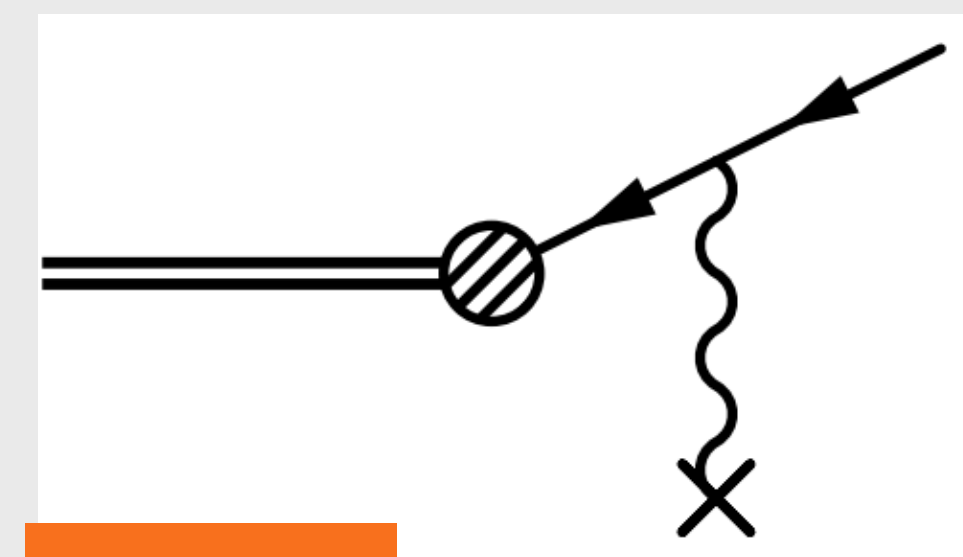
$$\begin{array}{c} \mu \\ \text{wavy line} \\ \times \end{array} = iZe \delta_0^\mu 2\pi \delta(q^0)$$

$$\begin{array}{c} \mu \\ \text{wavy line} \\ \text{double line} \end{array} = ie \delta_0^\mu$$

ONE LOOP



$\mathcal{O}(\alpha)$



$\mathcal{O}(Z\alpha)$

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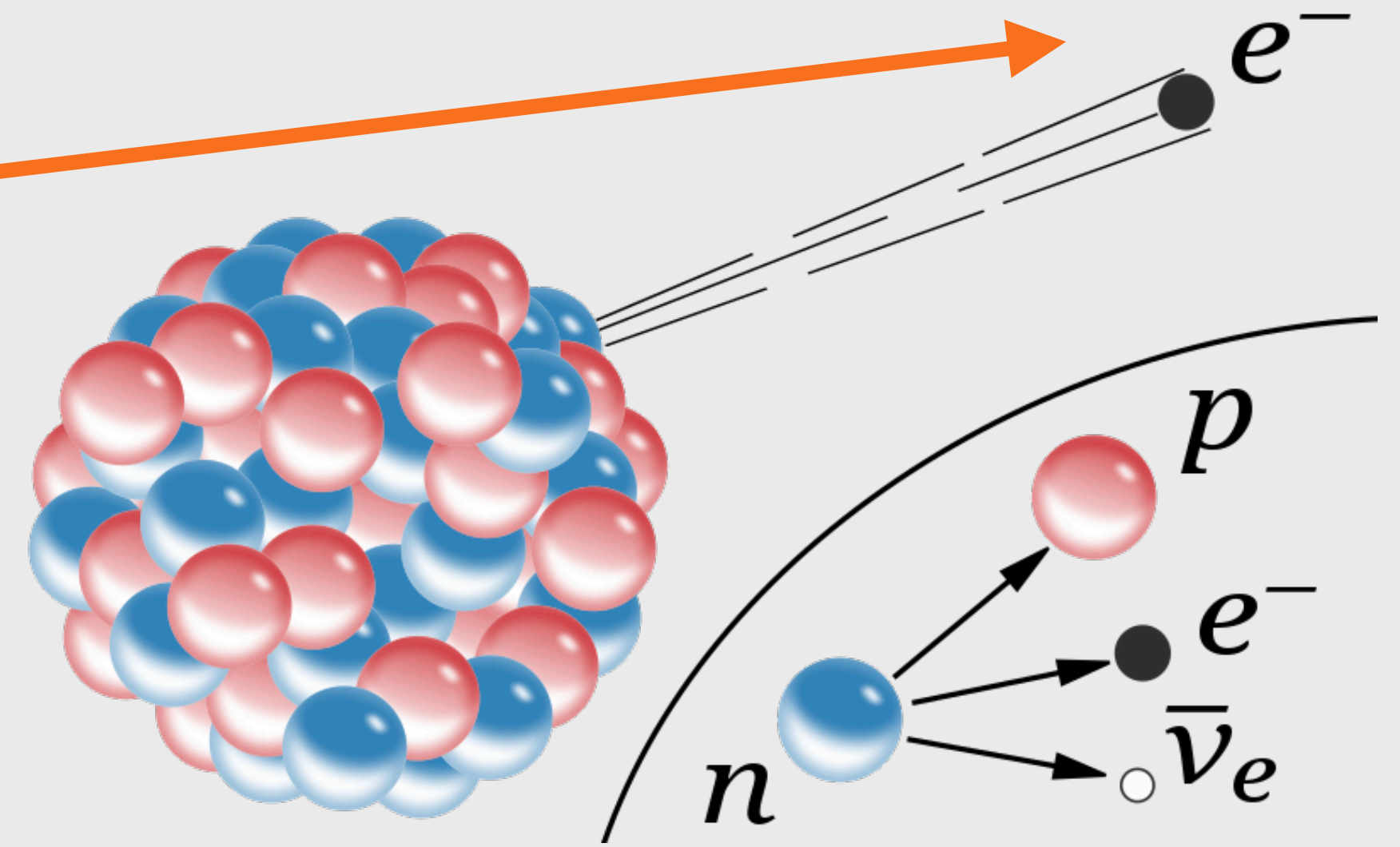


Field Theory Of The Fermi Function

Leading-Z Resummation

Fermi Function

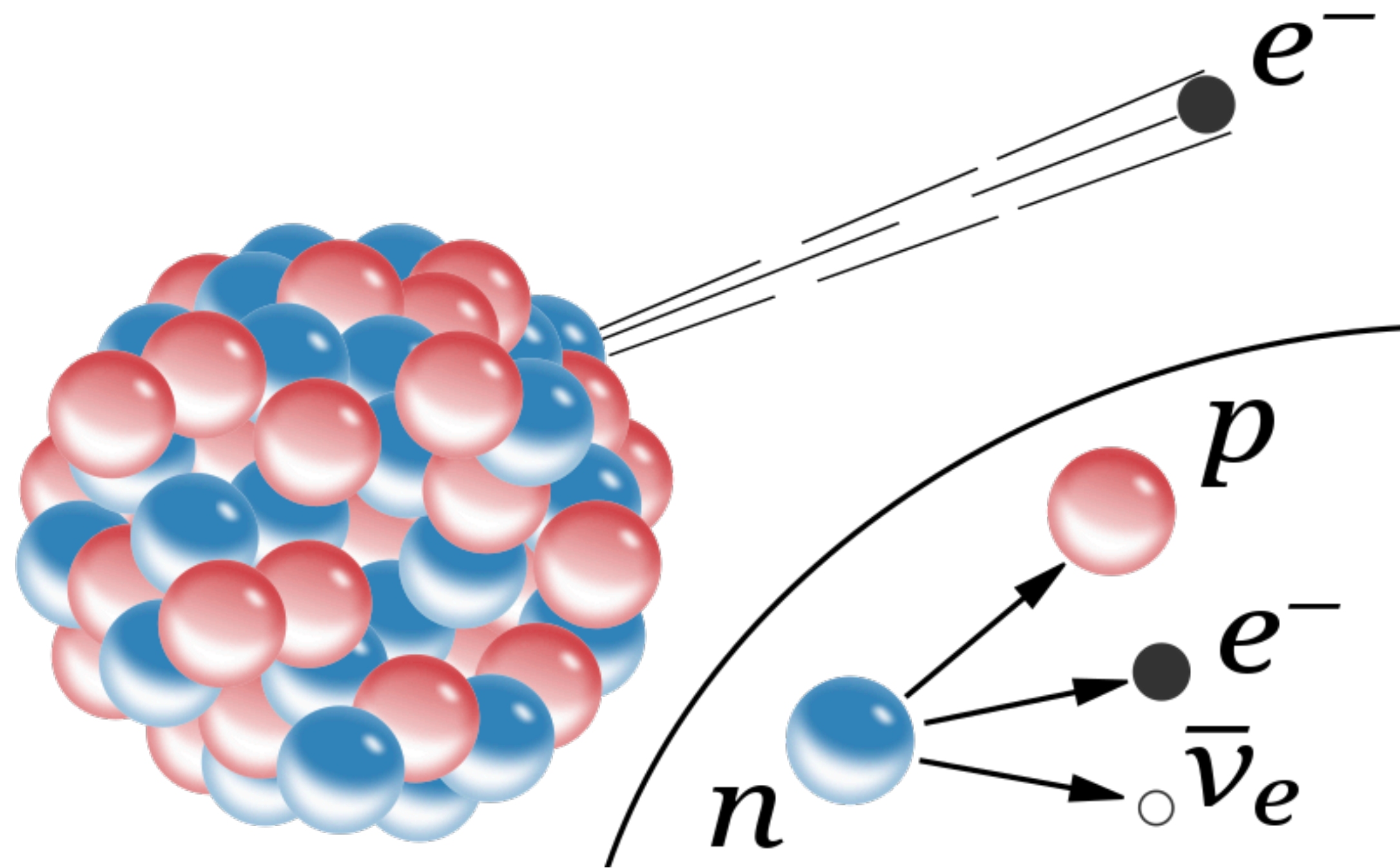
ATTRACTED TO NUCLEUS



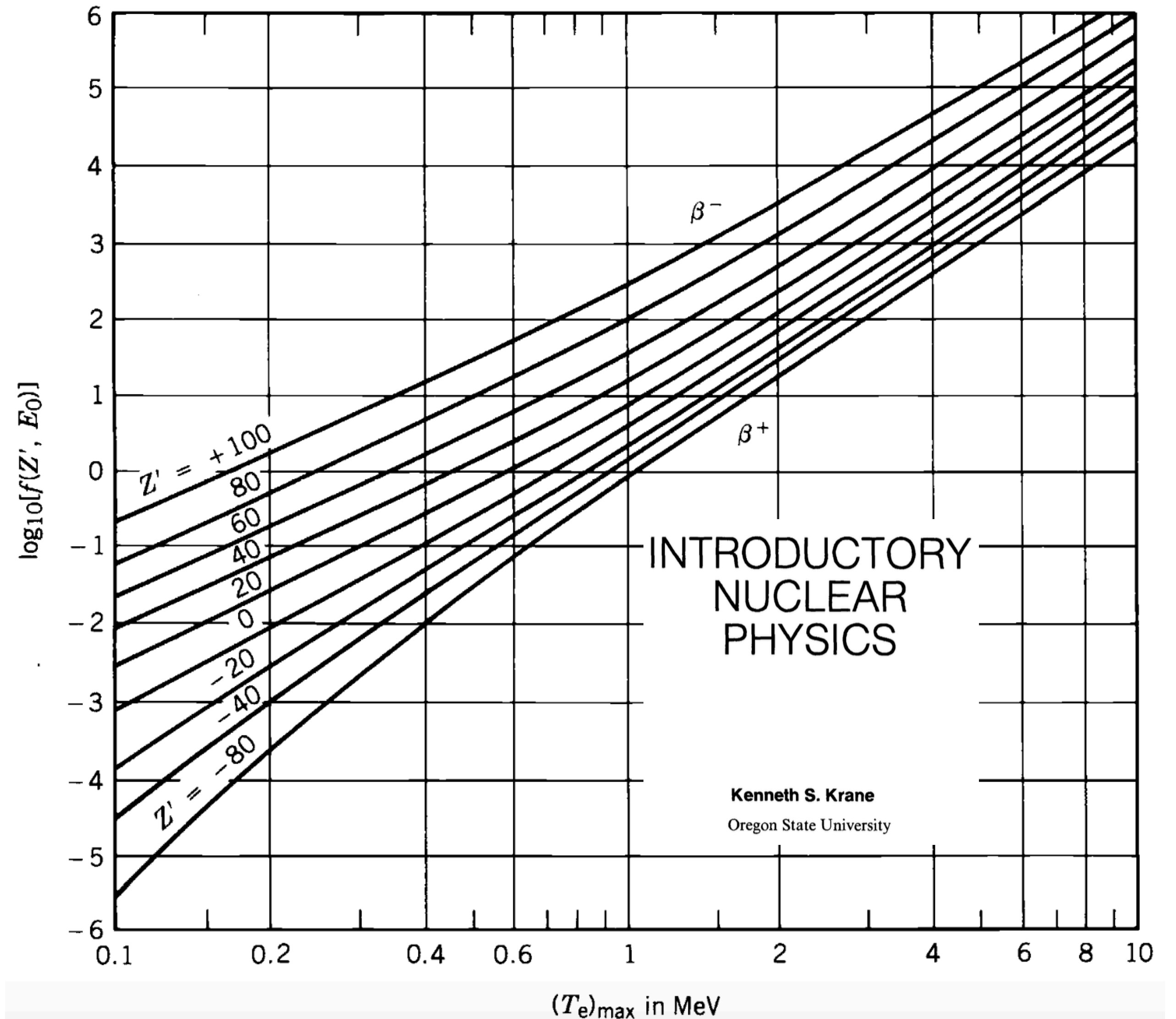
- Largest effects are a series in $Z\alpha$
- Historically done with finite-distance regulator

$$\langle e^- | \bar{\psi}(\mathbf{x}) | 0 \rangle \sim \left(\frac{1}{|\mathbf{x}|} \right)^\nu \quad \nu = \sqrt{1 - Z^2 \alpha^2} - 1$$

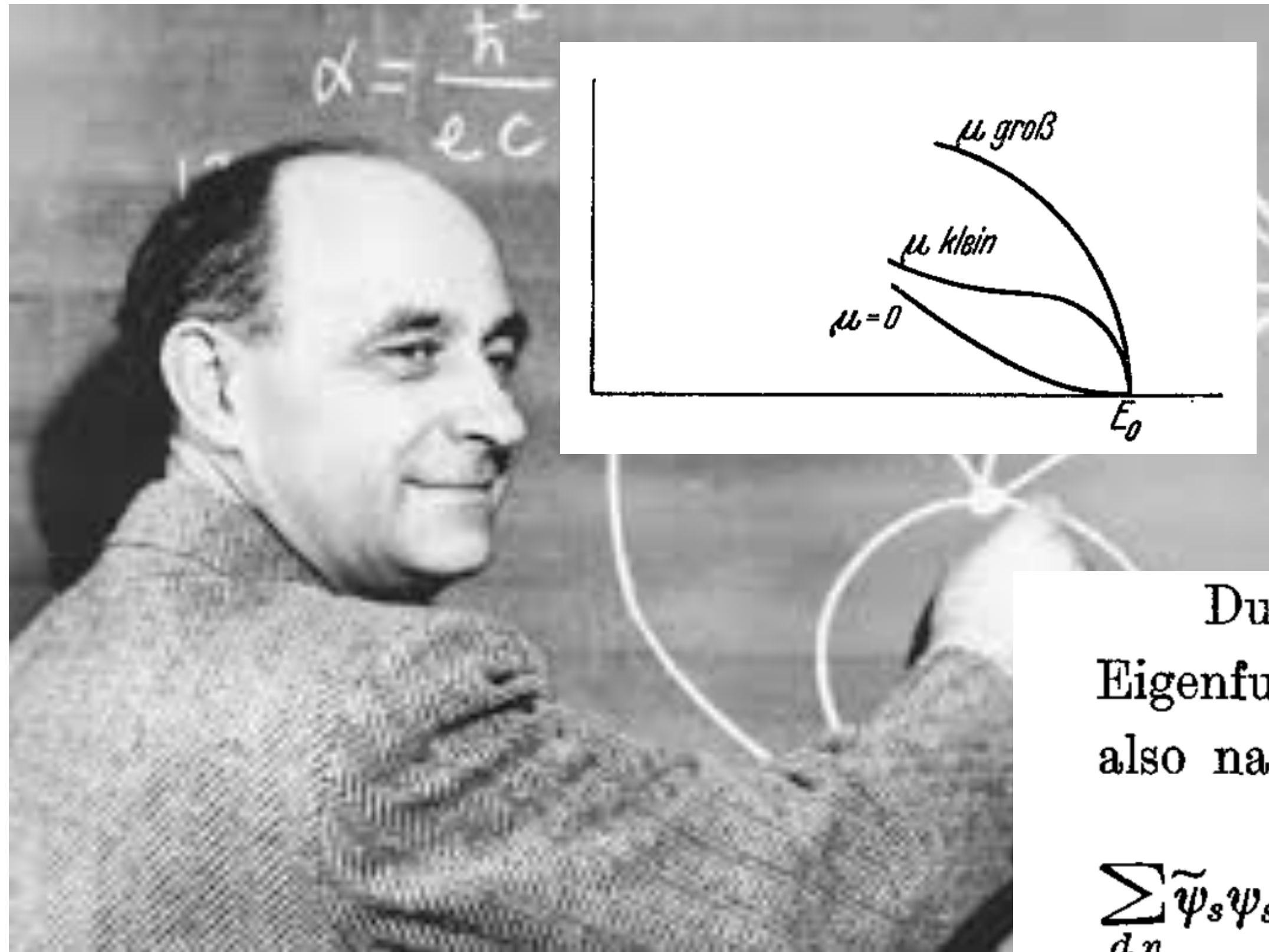
The Fermi Function Is Big



EFFECT IS MASSIVE!
VISIBLE ON LOG-SCALE



Enrico Fermi & Beta Decay



Versuch einer Theorie der β -Strahlen. I¹⁾.

Von E. Fermi in Rom.

Mit 3 Abbildungen. (Eingegangen am 16. Januar 1934.)

Eine quantitative Theorie des β -Zerfalls wird vorgeschlagen, in welcher man die Existenz des Neutrinos annimmt, und die Emission der Elektronen und Neutrinos aus einem Kern beim β -Zerfall mit einer ähnlichen Methode behandelt, wie die Emission eines Lichtquants aus einem angeregten Atom in der Strahlungstheorie. Formeln für die Lebensdauer und für die Form des emittierten kontinuierlichen β -Strahlenspektrums werden abgeleitet und mit der Erfahrung verglichen.

Durch Heranziehung der bekannten Formeln¹⁾ für die relativistischen Eigenfunktionen des Kontinuums im wasserstoffähnlichen Falle findet man also nach einer ziemlich langwierigen Rechnung

$$\sum_{d\eta} \tilde{\psi}_s \psi_s = d\eta \cdot \frac{32 \pi m^3 c^3}{h^3 [\Gamma(3+2S)]^2} \left(\frac{4\pi m c \varrho}{h} \right)^{2S} \eta^{2+2S} e^{\pi\gamma \frac{\sqrt{1+\eta^2}}{\eta}} \left| \Gamma \left(1+S+i\gamma \frac{\sqrt{1+\eta^2}}{\eta} \right) \right|^2, \quad (40)$$

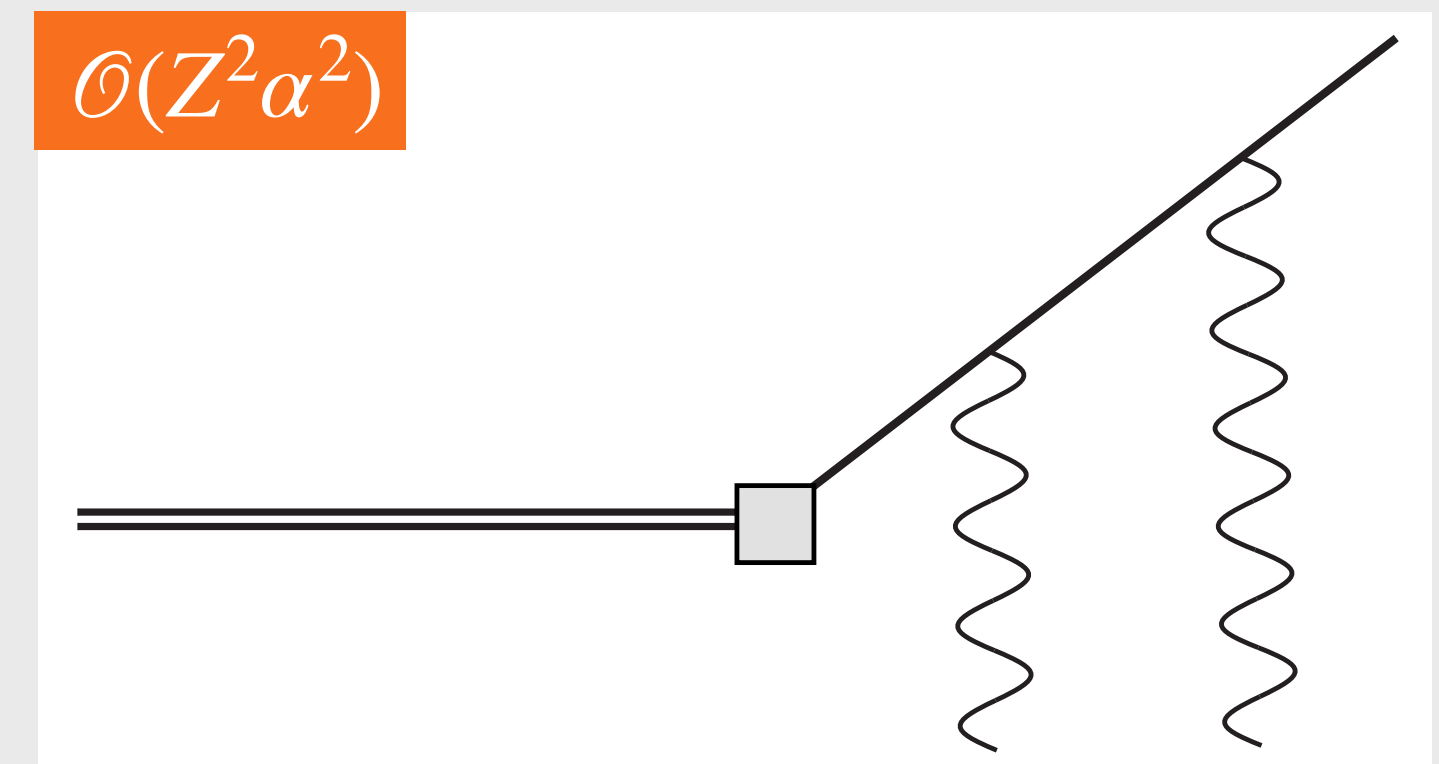
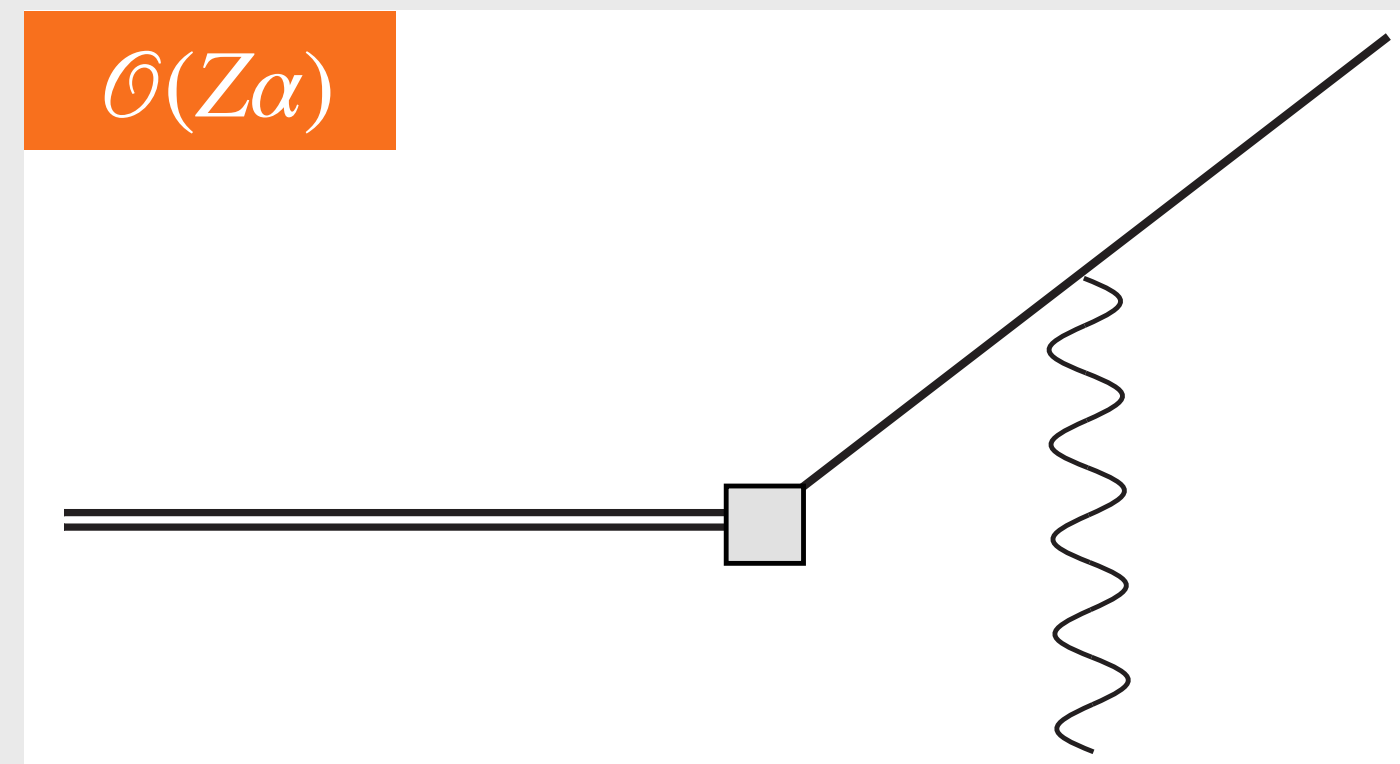
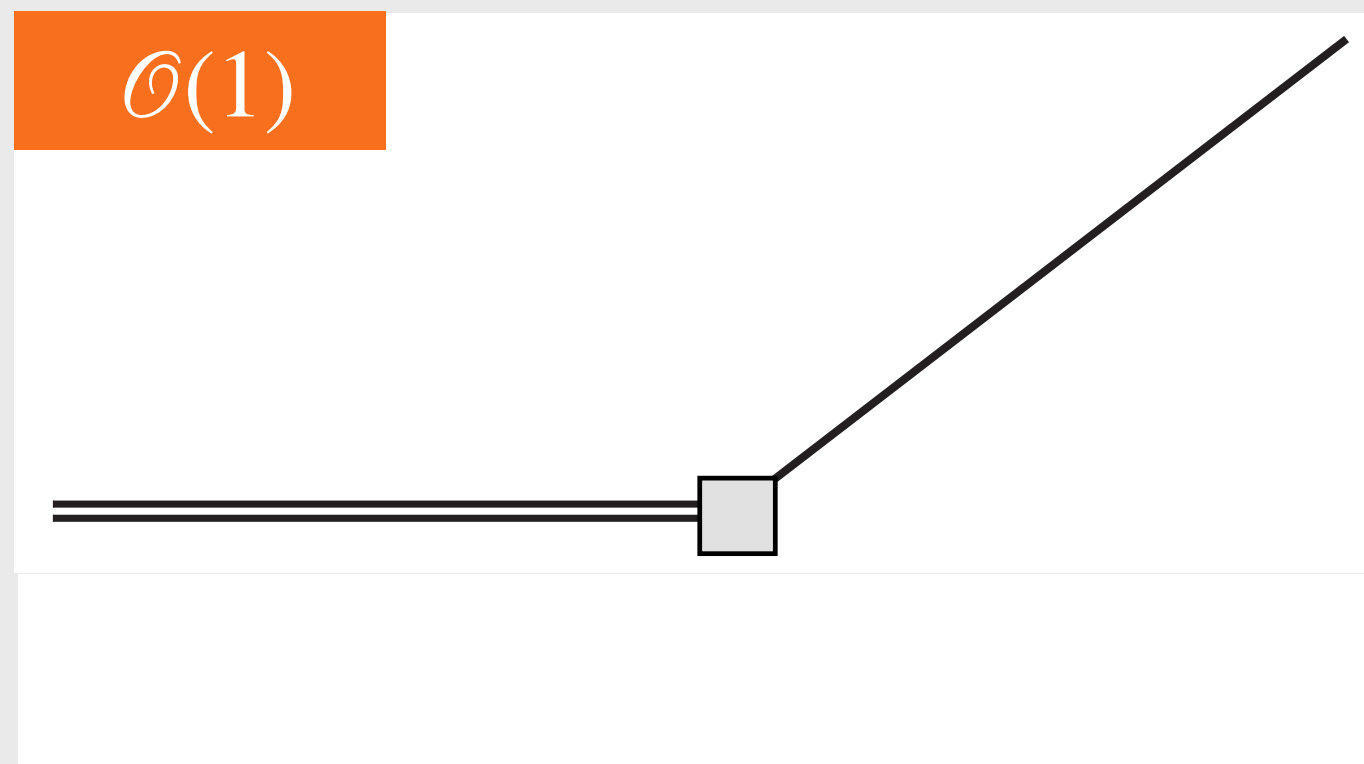
wo

$$\gamma = Z/137; \quad S = \sqrt{1-\gamma^2} - 1. \quad (41)$$

FOUNDATIONAL WORK (1934)

THE "FERMI FUNCTION"

Diagrammatic Expansion $\mathcal{M}_H$



- With modified Feynman rules counting Z is easy.
- Keep only the "leading-in- Z " terms.

+ ...

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Wavefunctions And Feynman Diagrams

- One can try to explicitly compute loops, but it is hard work.
- Can extract information from Dirac Equation with a Coulomb field.

Wavefunction Satisfies Lippmann-Schwinger Equation

$$|\psi_p^{(\pm)}\rangle = |\phi_p\rangle + \frac{1}{H - E_p \pm i\varepsilon} V |\phi_p\rangle + \frac{1}{H - E_p \pm i\varepsilon} V \frac{1}{H - E_p \pm i\varepsilon} V |\phi_p\rangle + \dots$$

- One-to-one correspondence between loops and expansion of the Dirac Coulomb wavefunction.

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Factorization Of Dirac Wavefunction

WHAT WE WANT

$$\mathcal{M} = \mathcal{M}_S(\mu_S) \mathcal{M}_H(\mu_S, \mu_H) \mathcal{M}_{UV}(\mu_H, \Lambda)$$

SAME

DIFFERENT

$$\Psi(\mathbf{x}) = \mathcal{M}_S(\mu_S) \mathcal{M}_H(\mu_S, \mu_H) \tilde{\mathcal{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

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All Orders Calculation

SEE BACKUP SLIDES FOR EQUATIONS

$$\tilde{\mathcal{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

- Finite distance $\mathbf{x}$ acts as regulator.
- Can be computed in the $p_e, m_e \rightarrow 0$ limit.
- All orders in $Z\alpha$ solution can be obtained.

Extraction Of Hard Matrix Element

SEE BACKUP SLIDES FOR EQUATIONS

$$\Psi(\mathbf{x}) = \mathcal{M}_S(\mu_S) \mathcal{M}_H(\mu_S, \mu_H) \tilde{\mathcal{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

KNOWN TO ALL ORDERS IN $Z\alpha$

Extraction Of Hard Matrix Element

SEE BACKUP SLIDES FOR EQUATIONS

$$\Psi(\mathbf{x}) = \mathcal{M}_S(\mu_S) \mathcal{M}_H(\mu_S, \mu_H) \tilde{\mathcal{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

KNOWN TO ALL ORDERS IN $Z\alpha$

$$\mathcal{M}_H(\mu_S, \mu_H) = \frac{\Psi(\mathbf{x})}{\tilde{\mathcal{M}}_{\mathbf{x}}(\mu_H, \mathbf{x}) \mathcal{M}_S(\mu_S)}$$

Factorization Of Dirac Wavefunction

$$\tilde{\mathcal{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

RESULT CAN BE RENORMALIZED
AT ALL-ORDERS IN $Z\alpha$

$$\mathcal{M}_{\text{UV}}^R(\mu) = (\mu r e^{\gamma_E})^{\eta-1} \frac{1+\eta}{2\sqrt{\eta}} \left[1 + \frac{Z\alpha}{1+\eta} \frac{i\gamma_0 \boldsymbol{\gamma} \cdot \mathbf{x}}{|\mathbf{x}|} \right],$$

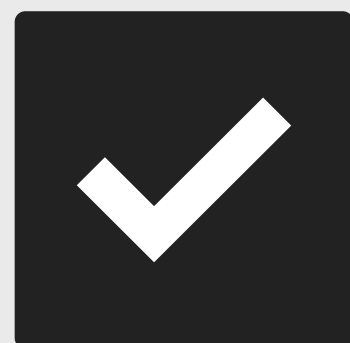
$$\eta = \sqrt{1 - (Z\alpha)^2}$$

All-Orders Hard Matrix Element

$$\mathcal{M}_H(\mu_S, \mu_H) = e^{\frac{\pi\xi}{2} + i\xi \left(\log \frac{2p}{\mu_S} - \gamma_E \right) - i(\eta-1) \frac{\pi}{2}} \frac{2\Gamma(\eta - i\xi)}{\Gamma(2\eta + 1)}$$

$$\sqrt{\frac{\eta - i\xi}{1 - i\xi \frac{m}{E}}} \sqrt{\frac{E + \eta m}{E + m}} \sqrt{\frac{2\eta}{1 + \eta}} \left(\frac{2pe^{-\gamma_E}}{\mu_H} \right)^{\eta-1} \times \left[\frac{1 + M^*}{2} + \frac{1 - M^*}{2} \gamma^0 \right]$$

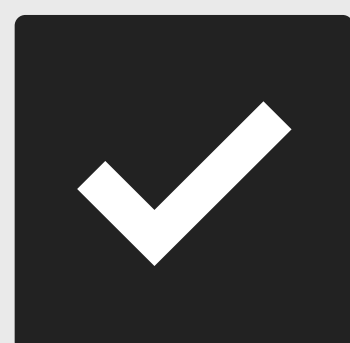
• $\eta = \sqrt{1 - Z^2 \alpha^2}$ • $\xi = Z\alpha/\beta$ • $M = (E + m)(1 + i\xi m/E)/(E + \eta m)$



PART 1

EFT & β DECAY

- Motivation & relevance for **fundamental physics**.
- Necessary **precision**, and requisite **loop orders**.



PART 2

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- **Point-like** EFT of nuclei and leptons.
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RAD. CORR.

- Structure of **radiative corrections** from EFT.
- Renormalization group **resummation of logarithms**.



Long-Distance Radiative Corrections

A Low Energy Perspective

Factorization Theorem

ARXIV:2309.07343

- Amplitude depends on Wilson coefficient and matrix element.

$$d\Gamma \propto |C(\mu)|^2 |\mathcal{M}|^2(\mu) + \mathcal{O}((pR)^2)$$

- Implies that all ***short-distances*** factorize from ***long-distances***.



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RG Analysis & Anomalous Dim.

Resumming Logs

Relating Different Scales

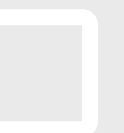
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$$d\Gamma \propto |C(\mu)|^2 |\mathcal{M}|^2(\mu)$$

No Large Logs

$$= C(\mu_H) \left[\frac{|C(\mu_L)|^2}{|C(\mu_H)|^2} \right] |\mathcal{M}|^2(\mu_L)$$

Calculate With Renormalization Group



Resummation With RG + EFT

Factorize & Run

- Need beta function in QED
- Need anomalous dimension

$$\left[\frac{|C(\mu_L)|^2}{|C(\mu_H)|^2} \right] = \exp \left[\int \frac{\gamma(Z, \alpha)}{\beta(\alpha)} d\alpha \right]$$

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$$\mathcal{M} = C(\mu) \mathcal{M}_H(\mu, p)$$

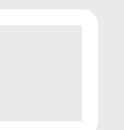
Nuclear Radius
50 MeV

$C(\Lambda)$

RG EVOLUTION

Electron Energy
5 MeV

$C(\mu) \mathcal{M}_H(\mu, p)$



Anomalous Dimension

ARXIV:2402.14769

$$\gamma_C = \frac{dC(\mu)}{d \log \mu}$$

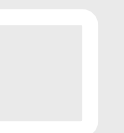
SOLVE DIRAC EQ'N

SYMMETRY IN MASSLESS LIMIT

$$(Z, Z - Q, Q) \longleftrightarrow (Z + Q, Z, -Q)$$

$$\begin{aligned} \gamma_C = & \alpha \left(Z\gamma^{(1,1)} + \gamma^{(1,0)} \right) + \alpha^2 \left(Z^2\gamma^{(2,2)} + Z\gamma^{(2,1)} + \gamma^{(2,0)} \right) \\ & + \alpha^3 \left(Z^3\gamma^{(3,3)} + Z^2\gamma^{(3,2)} + Z\gamma^{(3,1)} + \gamma^{(3,0)} \right) + \dots \end{aligned}$$

Z=0 REDUCES TO HEAVY-LIGHT CURRENT IN HQET



TAKE FROM HQET LIT.

SOLVE DIRAC EQ'N

↕ SYMMETRY

	α^0	α^1	α^2	α^3	α^4
Z^0	0	$\gamma^{(1,0)}$ ✓	$\gamma^{(2,0)}$ ✓	$\gamma^{(3,0)}$ ✓	$\gamma^{(4,0)}$ ✓
Z^1	—	0	$\gamma^{(2,1)}$	$\gamma^{(3,1)}$	$\gamma^{(4,1)}$
Z^2	—	—	$\gamma^{(2,2)}$ ✓	$\gamma^{(3,2)}$ ✓	$\gamma^{(4,2)}$
Z^3	—	—	—	0	$\gamma^{(4,3)}$
Z^4	—	—	—	—	$\gamma^{(4,4)}$ ✓

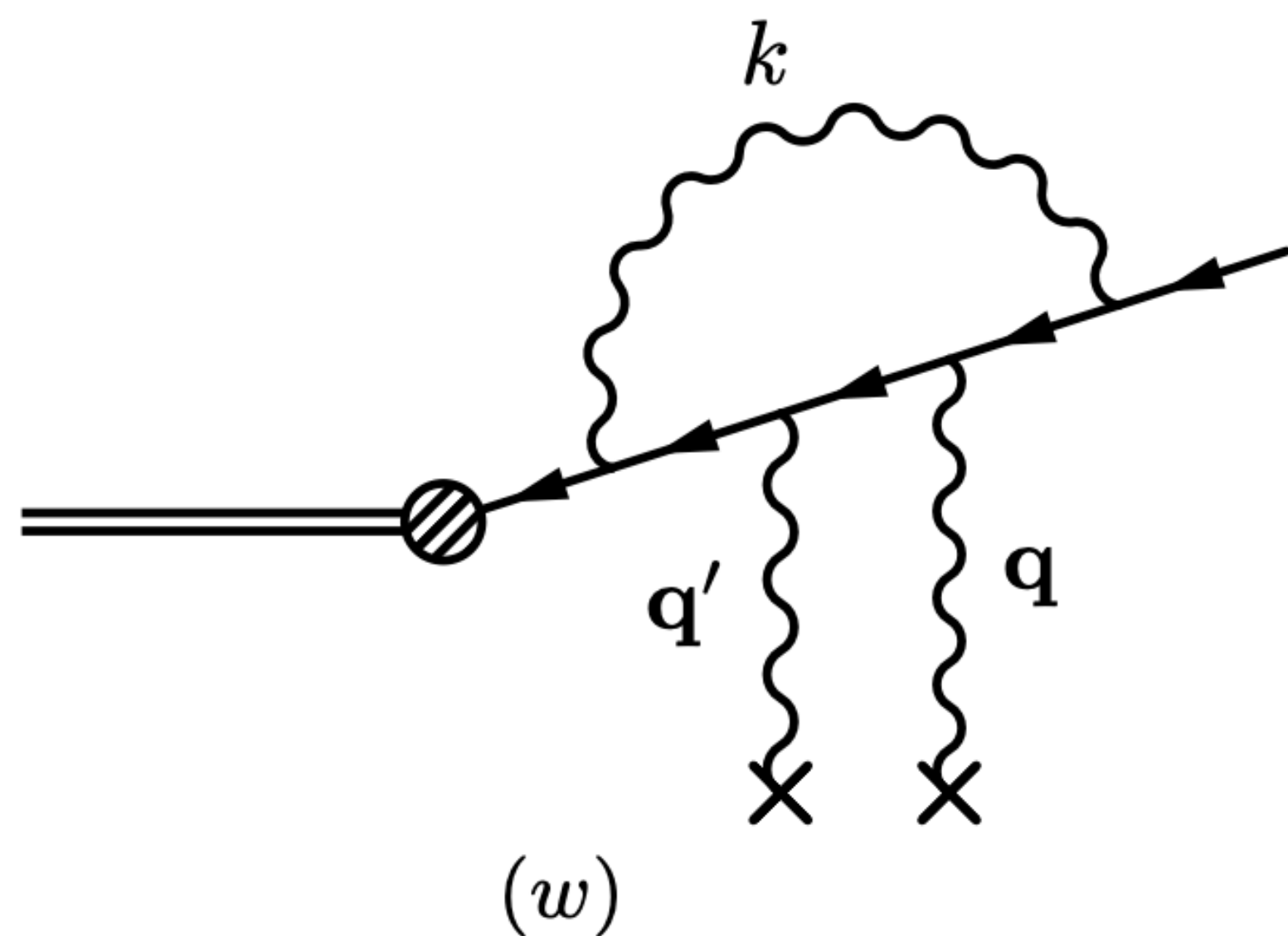
ARXIV:2402.14769

NEW INPUT!

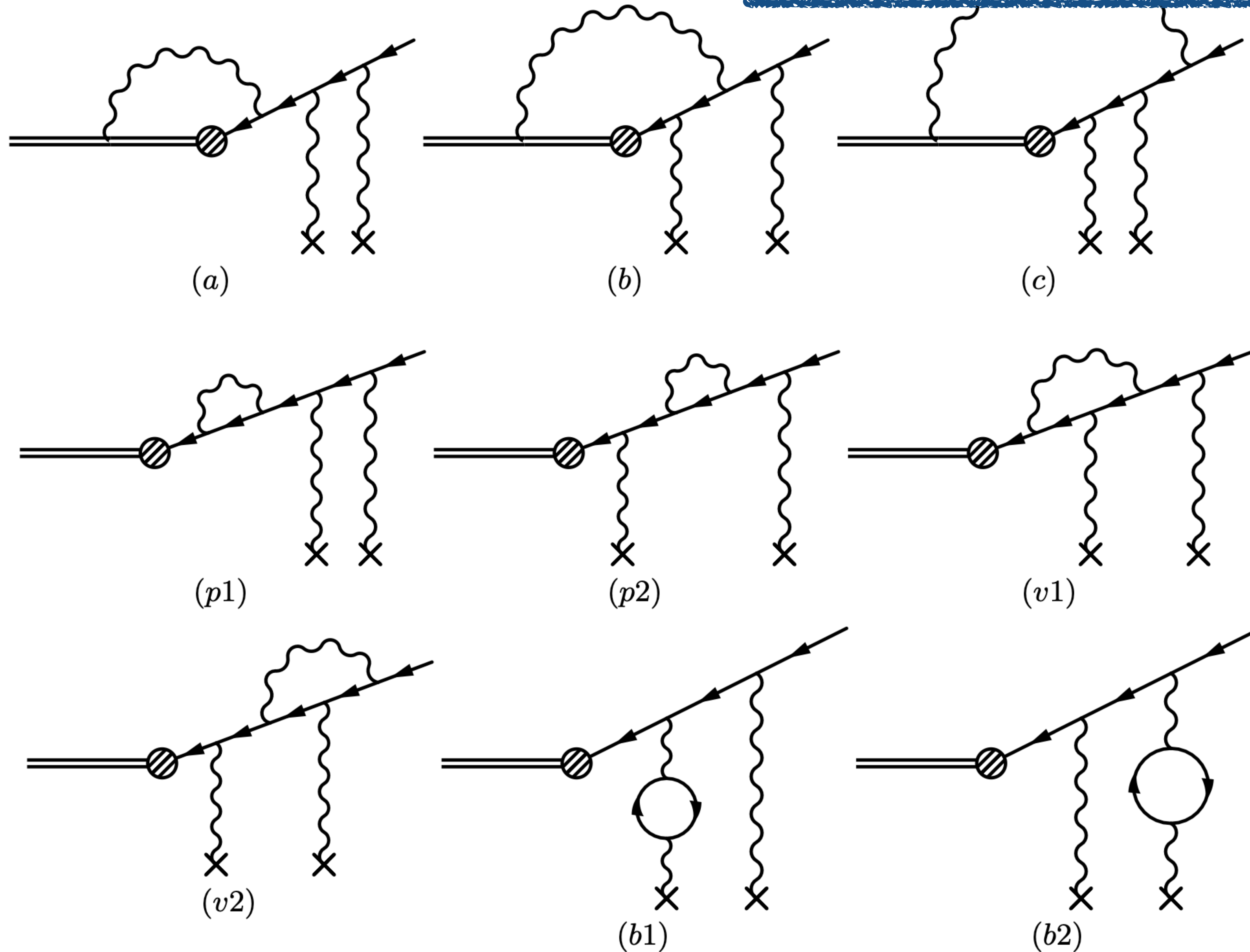


USE EIKONAL ALGEBRA TO REDUCE DIAGRAMS

$$\gamma_2^{(1)} = 16\pi^2 \left(6 - \frac{\pi^2}{3}\right)$$



MIXED EUCLIDEAN + LORENTZIAN INTEGRALS



New Result For Anomalous Dimension

Loops Z^n	1-loop	2-loop	3-loop	4-loop
Z^0	$\gamma_0^{(1)} = -3$	$\gamma_1^{(2)} = -16\zeta_2 + \frac{5}{2} + \frac{10}{3}n_e$	$\gamma_2^{(3)} = \text{GROZIN 2003}$	$\gamma_3^{(4)} = \text{GROZIN 2023}$
Z^1	$\gamma_0^{(0)} = 0$	$\gamma_1^{(1)} = \gamma_2^{(2)}$	$\gamma_2^{(2)} = \gamma_2^{(1)}$	$\gamma_3^{(3)} = \gamma_3^{(2)} - \gamma_3^{(0)}$
Z^2	—	$\gamma_1^{(0)} = -8\pi^2$	$\gamma_2^{(1)} = 16\pi^2 \left(6 - \frac{\pi^2}{3}\right)$	$\gamma_3^{(2)} = ?$
Z^3	—	—	$\gamma_2^{(0)} = 0$	$\gamma_3^{(1)} = 2\gamma_3^{(0)}$
Z^4			—	$\gamma_3^{(0)} = -32\pi^4$

RESUMMATION COMPLETE
THROUGH 3-LOOPS!

ARXIV:2402.14769

NEW INPUT





ARXIV:2309.07343

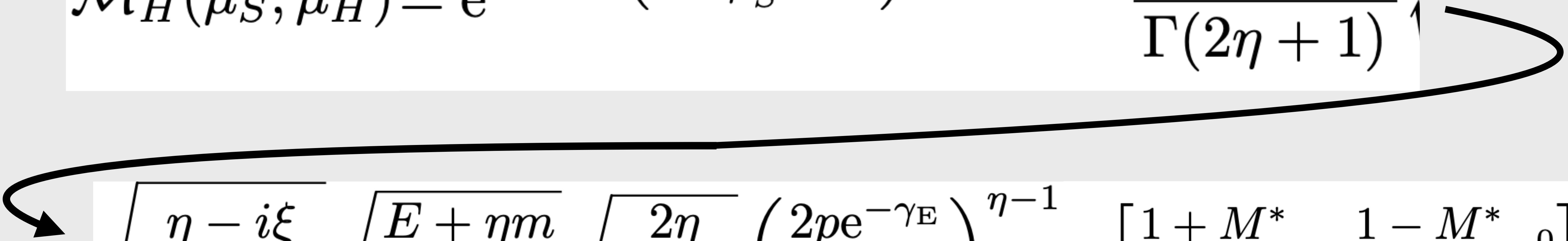
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Long-Distance Matrix Element

Resumming Logs

Recall We Have The Leading- $Z\alpha$ Result

$$\mathcal{M}_H(\mu_S, \mu_H) = e^{\frac{\pi\xi}{2} + i\xi \left(\log \frac{2p}{\mu_S} - \gamma_E \right) - i(\eta-1) \frac{\pi}{2}} \frac{2\Gamma(\eta - i\xi)}{\Gamma(2\eta + 1)}$$


$$\sqrt{\frac{\eta - i\xi}{1 - i\xi \frac{m}{E}}} \sqrt{\frac{E + \eta m}{E + m}} \sqrt{\frac{2\eta}{1 + \eta}} \left(\frac{2pe^{-\gamma_E}}{\mu_H} \right)^{\eta-1} \times \left[\frac{1 + M^*}{2} + \frac{1 - M^*}{2} \gamma^0 \right]$$

• $\eta = \sqrt{1 - Z^2\alpha^2}$ • $\xi = Z\alpha/\beta$ • $M = (E + m)(1 + i\xi m/E)/(E + \eta m)$

Next We Need Subleading Corrections

$$\langle |\mathcal{M}_H|^2 \rangle = F(Z\alpha, E, \mu) \left[1 + \delta^{(1,0)}(E) + \delta^{(2,1)}(E) + \dots \right]$$

- Order- α result in "Sirlin's Scheme" in 1967 & in $\overline{\text{MS}}$ in 2004.
- Order- $Z\alpha^2$ result in "Sirlin's Scheme" in 1986
- As of one month ago, $\overline{\text{MS}}$ calculation did not exist for $Z\alpha^2$ terms.

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$$\langle |\mathcal{M}_H|^2 \rangle = F(Z\alpha, E, \mu) \left[1 + \delta^{(1,0)}(E) + \delta^{(2,1)}(E) + \dots \right]$$

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Radiative corrections to superallowed beta decays at $\mathcal{O}(\alpha^2 Z)$.

Ö. L. Crosas^{a,b,c} E. Mereghetti^b

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CALT-TH-2025-032
FERMILAB-PUB-25-0773-T
CERN-TH-2025-20

The $Z\alpha^2$ correction to superallowed beta decays
in effective field theory and implications for $|V_{ud}|$

ZEHUA CAO¹, RICHARD J. HILL^{1,2}, RYAN PLESTID^{3,4}, AND PETER VANDER GRIEND^{1,2}

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Pasadena, CA 91125 USA

⁴Theoretical Physics Department, CERN, 1 Esplanade des Particules, CH-1211 Geneva 23,
Switzerland

● $\overline{\text{MS}}$ calculation for $Z\alpha^2$ terms confirmed by two groups (2025).

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Kinoshita-Lee-Naueberg Constraints

A Collinear-Safe Observable

Inclusive Lifetime Is IR-Collinear Safe

PHYSICAL REVIEW

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Degenerate Systems and Mass Singularities*

T. D. LEE† AND M. NAUENBERG‡

Columbia University, New York, New York

(Received 24 October 1963)

For a system with degenerate energies, the power series expansions of the S -matrix elements may become singular. An elementary theorem in quantum mechanics is proved which shows that under certain general conditions such singularities do not appear in the power series expansions of the transition probabilities, provided these are averaged over an appropriate ensemble of degenerate states. Application of this theorem leads to the cancellations of mass singularities and infrared divergences in quantum electrodynamics. The question of whether a charged particle can have zero mass is studied.

- This implies further constraints on the amplitude.

IR-Collinear Safe & $\overline{\text{MS}}$

- Study limit of $m_e \rightarrow 0$ with $\bar{\alpha}(\mu)$ held fixed.

$$\Gamma = \Gamma^{(0)} + \bar{\alpha}\Gamma^{(1)} + \bar{\alpha}^2\Gamma^{(2)} + \bar{\alpha}^3\Gamma^{(3)} + \dots$$

- KLN tells us these coefficients have *no logarithms of* $\log(m_e)$.
- Only holds *after intergrating over phase space*.

Higher Order Constraints

- Order- α correction has non-trivial cancellation.

$$\log \left(\frac{2E_m}{m} \right) \left[\frac{4(1-x)}{3x} - 3 + \frac{(1-x)^2}{6x^2} + 4 \log \left(\frac{1-x}{x} \right) \right]$$

- Only vanishes after phase space integration.

$$\int_0^1 dx \, x^2 (1-x)^2 \left[\frac{4(1-x)}{3x} - 3 + \frac{(1-x)^2}{6x^2} + 4 \log \left(\frac{1-x}{x} \right) \right] = 0$$

Higher Order Constraints

- Fermi-function introduces extra energy dependence.

$$\int_0^1 dx \, x^2 (1-x)^2 \left[\frac{4(1-x)}{3x} - 3 + \frac{(1-x)^2}{6x^2} + 4 \log \left(\frac{1-x}{x} \right) \right] \times [-\log(x)] = \frac{\pi^2}{45} - \frac{1}{12} .$$

- Fixes logarithmic enhancement at $O(Z^2 \alpha^3)$.

$$\Gamma = |\mathcal{C}(\mu_H)|^2 \left| \frac{\mathcal{C}(\mu_L)}{\mathcal{C}(\mu_H)} \right|^2 \int d\Phi_e \tilde{F}(Z\bar{\alpha}_L, E, \mu_L) \left[1 + \frac{\bar{\alpha}_L}{2\pi} \left(\hat{g}(E, \mu) + \frac{11}{4} \right) + Z\bar{\alpha}_L^2 \left(-\frac{5}{3} L_E + \frac{8}{9} + \frac{11}{4} - \zeta_2 \right) + \frac{1}{4\pi} Z^2 \bar{\alpha}_L^3 L_m \left(\frac{2\pi^2}{3} - \frac{5}{2} \right) + O(\lambda^4) \right] .$$



P. VANDER GRIEND
ON THE MARKET!

Results For Superalloyed Decays

New Shifts In Old Places

Long-Distance Corrections

Beta emitter	Z_{daughter}	Λ [MeV]	$2E_m$ [MeV]	$R(\mu_H, \mu_L)$	$M(\mu_L)$	$1 + \delta'_R(\mu_H)$
^{10}C	5	199.1	2.794	1.01634	0.99793	1.01424(5)
^{14}O	7	188.9	4.641	1.01464	0.99805	1.01266(3)
^{26m}Al	12	159.3	7.443	1.01309	0.99826	1.01133(1)
^{34}Cl	16	147.2	9.961	1.01226	0.99842	1.01066(2)
^{34}Ar	17	142.6	11.102	1.01181	0.99844	1.01023(2)
^{38m}K	18	142.0	11.066	1.01199	0.99851	1.01048(3)
^{42}Sc	20	137.8	11.831	1.01191	0.99860	1.01050(4)
^{46}V	22	134.0	13.083	1.01166	0.99869	1.01033(6)
^{50}Mn	24	132.1	14.247	1.01152	0.99878	1.01029(7)
^{54}Co	26	130.9	15.467	1.01141	0.99888	1.01027(9)

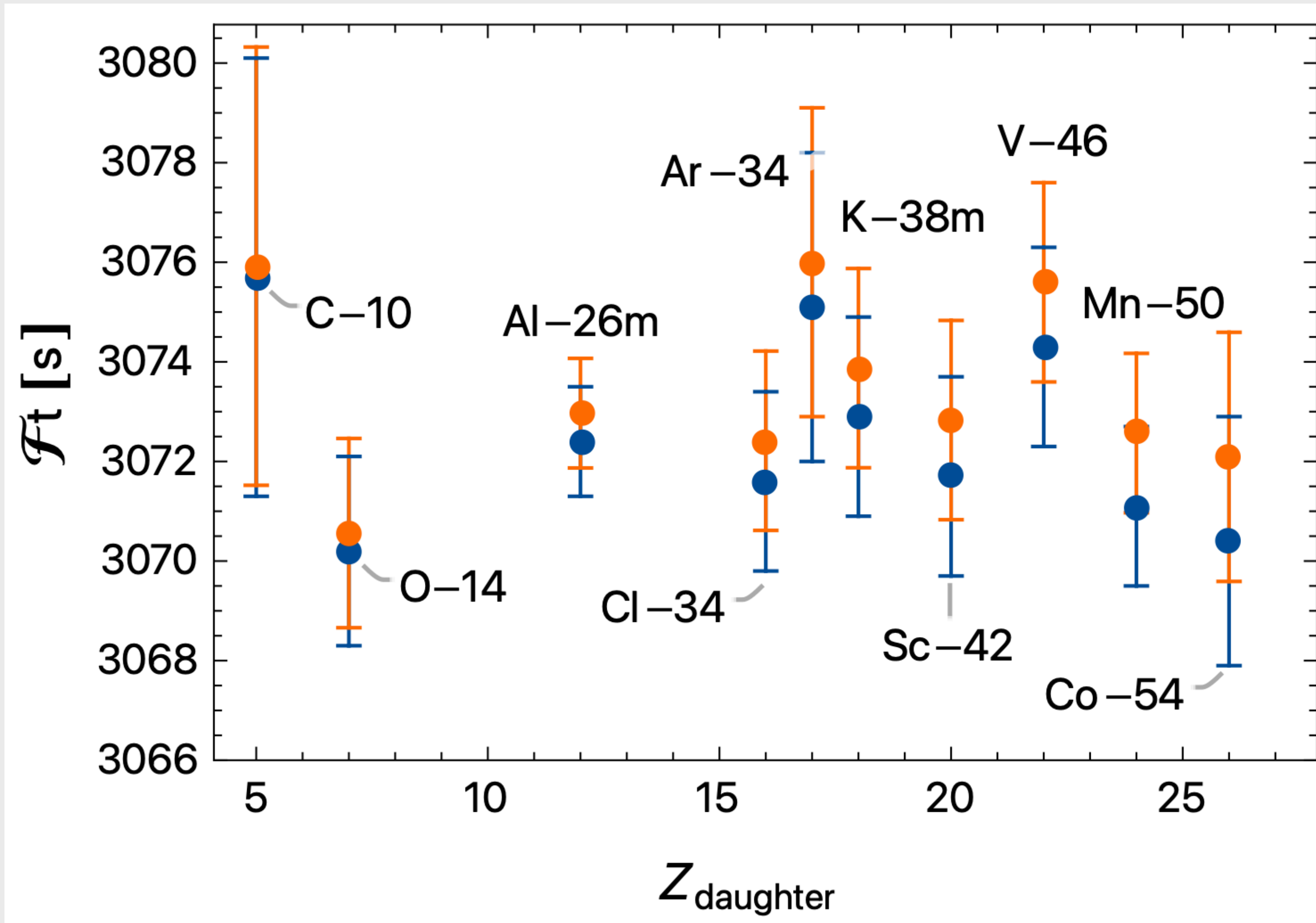
- Scale variation uncertainties below error budget goal!

Corrections To Literature

Beta emitter	Z_{daughter}	Convention shift	Long-distance shift	Total-shift	$\delta_{\text{NS,TH}}^{(2,1)} [10^{-4}]$
^{10}C	5	1.000105	0.999967	1.00007	2.1
^{14}O	7	1.00012	1.00000	1.00012	2.9
^{26m}Al	12	1.00013	1.00006	1.00019	4.3
^{34}Cl	16	1.00013	1.00014	1.00027	5.5
^{34}Ar	17	1.00013	1.00016	1.00029	5.8
^{38m}K	18	1.00015	1.00017	1.00032	6.2
^{42}Sc	20	1.000154	1.000214	1.00037	6.8
^{46}V	22	1.00015	1.00027	1.00042	7.2
^{50}Mn	24	1.00015	1.00033	1.00048	7.6
^{54}Co	26	1.00016	1.00039	1.00055	8.0

-  Reorganize existing result
-  New input
-  Full Result
-  Short-distance

Implications For $|V_{ud}|$



$$|V_{ud}|^2 \sim 1/\mathcal{F}t$$

- Shift upwards of $\mathcal{F}t$ by about $\sim 1\sigma$.
- Exacerbates the first-row CKM unitarity tension.
- More to do at short-distances.



Neutron Beta Decay

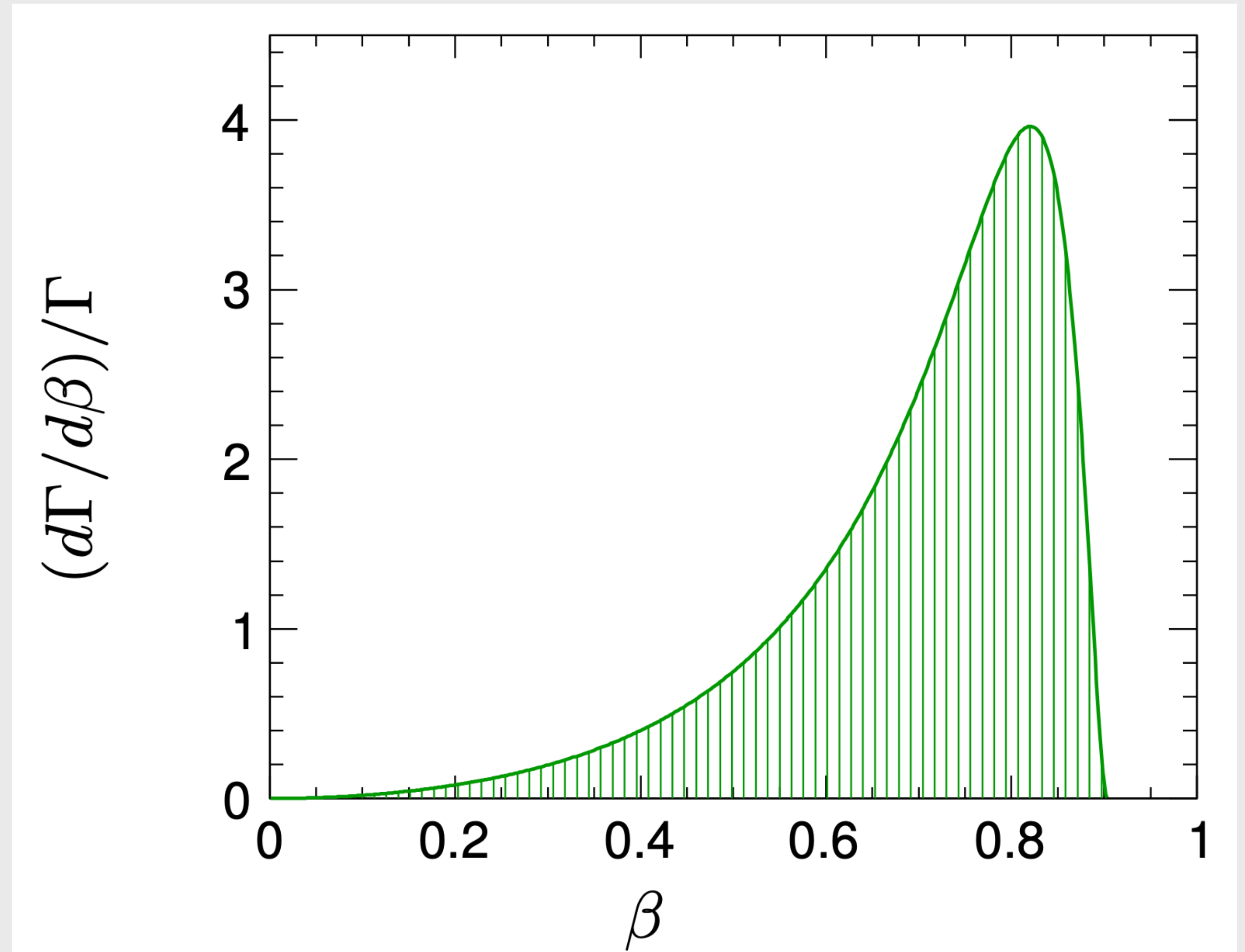
The Fermi Function & Neutron Beta Decay

The Fermi-Function In Neutron Decay?

- In historical treatment largest corrections to neutron decay come from Fermi-function.

- Fermi function resums $Z\alpha/\beta$ but....

$$Z = 1 \quad \beta \sim O(1)$$



Factorization Of Amplitude

$$\mathcal{M} = \mathcal{M}_S(\mu_S) \mathcal{M}_H(\mu_S, \mu_H) \mathcal{M}_{UV}(\mu_H, \Lambda)$$

ORIGIN

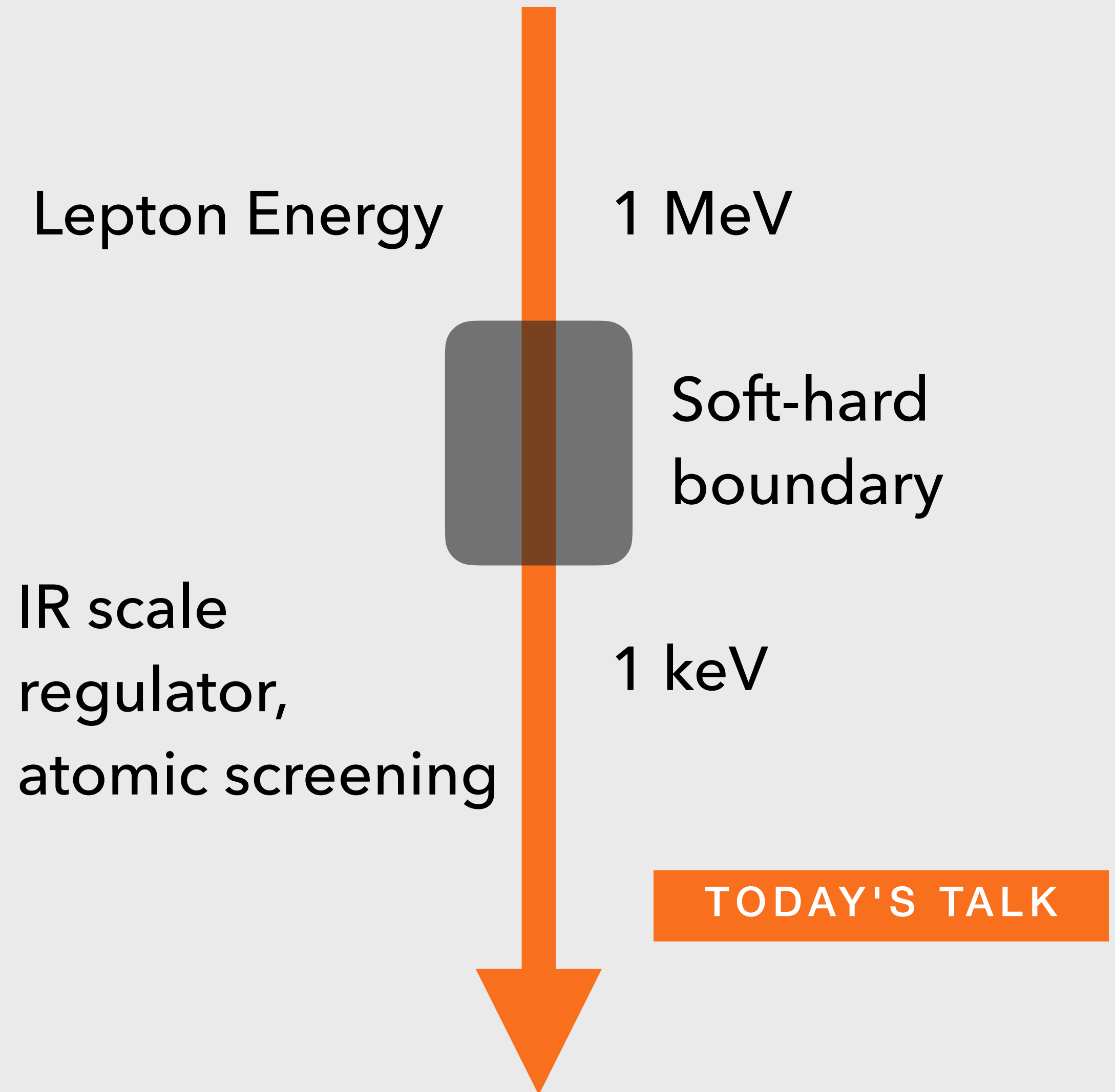
$$\frac{Z\alpha}{4\pi}(2\pi i) \times \log \frac{-p^2 - i0}{\mu^2} = \frac{Z\alpha}{4\pi}(2\pi i) \times \left(\log \frac{p^2}{\mu^2} - i\pi \right)$$

3-D MODES

IR DIV.

Source Of Enhancements

- In a conventional calculation the large enhancements are from the "hard" region.
- They come from IR divergences in the hard region.
- We can make a different choice and associate them with UV divergences in the soft region.



Results : Reorganizing Theory

	Without Resummation	With Resummation
1	0.3 ± 3.5 ± 2.1	34.5 ± 3.6 ± 2.2
1 + $H_V^{(1)}$	32.6 ± 0.1 ± 2.2	33.2 ± 0.004 ± 2.2
1 + $H^{(1)}$	28.8 ± 0.08 ± 0.05	29.32 ± 0.02 ± 0.01
1 + $H^{(1)}$ + $H_V^{(2)}$	29.04 ± 0.05 ± 0.05	29.31 ± 0.02 ± 0.01

Quantity	Value [10^{-3}]
Δ_R	45.37 ± 0.27
$\delta_{R,\text{static}}$	29.18 ± 0.07
δ_{recoil}	$- 2.06$
$\delta_{\text{rad.rec.}}$	$- 0.08$

- Can resum factors of $(2\pi i)^2$ with RG.
- Improves convergence, and gives most precise prediction.

$$\tau_n \times |V_{ud}|^2 (1 + 3\lambda^2) \left[1 + \Delta_R(\mu_{UV} = \Delta) \right] \left[1 + 27.04(7) \times 10^{-3} \right] = \frac{2\pi^3 \hbar}{G_F^2 \Delta^5 f_{\text{static}}} = 5263.284(17) \text{ s} .$$

Conclusions & Outlook

Executive Summary

- CKM unitarity demands precise theory.
- Key players: superallowed and neutron beta decays.
- Heavy particle methods give new theoretical control.
- Many new results in the past two years.
- Theory being systematized across length scales.

Backup Slides



Wavefunctions & Diagrammatics

Wavefunctions And Feynman Diagrams

- Coulomb effects historically handled with “distorted waves”
- What are the equivalent effects in Feynman diagrams?

Use Lippmann-Schwinger Equation!

$$|\psi_p^{(\pm)}\rangle = |\phi_p\rangle + \frac{1}{H - E_p \pm i\epsilon} V |\phi_p\rangle + \frac{1}{H - E_p \pm i\epsilon} V \frac{1}{H - E_p \pm i\epsilon} V |\phi_p\rangle + \dots$$

Wavefunctions And Feynman Diagrams

- Coulomb effects historically handled with "distorted waves"
- What are the equivalent effects in Feynman diagrams?

Loop With A Phase Factor!



$$\langle x | \psi_p^{(\pm)} \rangle = e^{i\mathbf{p} \cdot \mathbf{x}} \left(1 + \int \frac{d^3 Q}{(2\pi)^3} \frac{1}{2\mathbf{P} \cdot \mathbf{Q} + Q^2 \pm i\epsilon} \frac{Z\alpha}{Q^2} e^{i\mathbf{Q} \cdot \mathbf{x}} + \dots \right)$$



Two-Loop Expressions At $\mathcal{O}(Z^2\alpha^2)$

Brute Force 2-Loop Calculation

- Compute Coulomb corrections explicitly through 2-loops.
- Dim-reg + renormalization. Well defined amplitude.

$$\mathcal{M}_H(\mu_S, \mu_H) = 1 + \frac{Z\alpha}{\beta} \left[i \left(\log \frac{2p}{\mu_S} - \frac{i\pi}{2} \right) + \frac{i}{2} \left(\frac{m}{E} \gamma^0 - 1 \right) \right] + \left(\frac{Z\alpha}{\beta} \right)^2 \left\{ \frac{-\pi^2}{12} - \frac{1}{2} \left(\log \frac{2p}{\mu_S} - \frac{i\pi}{2} \right)^2 \right. \\ \left. - \frac{1}{2} \left(\log \frac{2p}{\mu_S} - \frac{i\pi}{2} \right) \left(\frac{m}{E} \gamma^0 - 1 \right) + \left[\frac{5}{4} - \frac{1}{2} \left(\log \frac{2p}{\mu_H} - \frac{i\pi}{2} \right) \right] \beta^2 \right\} + \mathcal{O}(\alpha^3),$$

- No obvious pattern. Resummation impossible by brute force.



Eikonal Algebra Identity

New Result

$$\langle B(v) | J_{\mu_1}(q_1) \dots \mathcal{O} \dots J_{\mu_N}(q_N) | A(v) \rangle = v_{\mu_1} \dots v_{\mu_N} G(q_1 \dots q_N)$$

$$\begin{aligned} G(q_1 \dots q_N) = & Z^n \prod_{i=1}^N (2\pi i) \delta(v \cdot q_i) \\ & + Z^{n-1} \sum_j \frac{1}{v \cdot q_j} \prod_{i \neq j} (2\pi i) \delta(v \cdot q_i) \\ & + Z^{n-2} \sum_k \sum_{j \neq k} \frac{1}{v \cdot q_k} \frac{1}{v \cdot q_j} \prod_{i \neq j, k} (2\pi i) \delta(v \cdot q_i) \\ & + \dots \end{aligned}$$

Fermi Function

LOG(2PR)



$$(1 + \delta'_R) := \left[\frac{C(\mu_L)/C(\mu_H)}{\exp \left[(1 - \sqrt{1 - Z^2 \alpha^2}) \log(\mu_H/\mu_L) \right]} \right]^2 \left(\frac{\int d\Pi \langle |\mathcal{M}_H|^2 \rangle}{\int d\Pi F(Z, E) \times \frac{4\eta}{(1+\eta)^2}} \right)_{\mu=\mu_L}$$

- We can define "outer" radiative corrections in the EFT
- Factorize into a RG-running piece, and a low-energy matrix element.
- Fermi function has been factored out.



Explicit Expressions For Fermi Function

Factorization Of Dirac Wavefunction

$$\tilde{\mathcal{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

CLOSED FORM INTEGRALS AT
ARBITRARILY HIGH ORDER

$$\mathcal{I}_1^{(n)} = \left[\prod_{j=1}^{n-1} C(\nu_j) \right] \times \frac{\Gamma(d - \nu_n - 1)}{(4\pi)^d \Gamma(\nu_n)} B\left(\frac{d}{2} - 1, 1 + \frac{d}{2} - \nu_n\right) \left(\frac{\mathbf{x}^2}{4}\right)^{\nu_n + 1 - d},$$
$$\mathcal{I}_2^{(n)} = \left[\prod_{j=1}^n C(\nu_j) \right] \left[\frac{2\Gamma(\frac{d}{2} - \nu_{n+1} + 1)}{(4\pi)^{d/2} \Gamma(\nu_{n+1})} \right] \left[\frac{\mathbf{x}^2}{4} \right]^{\nu_{n+1} - (d+1)/2} \times \frac{i\gamma_0 \boldsymbol{\gamma} \cdot \mathbf{x}}{2|\mathbf{x}|}.$$

Factorization Of Dirac Wavefunction

$$\tilde{\mathcal{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

BARE AMPLITUDE MAY BE
SUMMED TO ALL ORDERS

$$F_1^{\text{bare}} = 2^{\frac{1}{4\epsilon} - \frac{1}{2}} \left(\frac{\sqrt{\tilde{g}}}{\epsilon} \right)^{1 - \frac{1}{2\epsilon}} \Gamma\left(\frac{1}{2\epsilon}\right) J_{\frac{1}{2\epsilon} - 1} \left(\frac{\sqrt{8}\sqrt{\tilde{g}}}{\epsilon} \right) ,$$

$$(Z\tilde{\alpha})^{-1} F_2^{\text{bare}} = 2^{\frac{1}{4\epsilon}} \left(\frac{\sqrt{\tilde{g}}}{\epsilon} \right)^{-\frac{1}{2\epsilon}} \Gamma\left(1 + \frac{1}{2\epsilon}\right) J_{\frac{1}{2\epsilon}} \left(\frac{\sqrt{8}\sqrt{\tilde{g}}}{\epsilon} \right) .$$

Factorization Of Dirac Wavefunction

$$\tilde{\mathcal{M}}_{\mathbf{x}}(\mu_H, \mathbf{x})$$

RESULT CAN BE RENORMALIZED
AT ALL-ORDERS IN $Z\alpha$

$$\mathcal{M}_{\text{UV}}^R(\mu) = (\mu r e^{\gamma_E})^{\eta-1} \frac{1+\eta}{2\sqrt{\eta}} \left[1 + \frac{Z\alpha}{1+\eta} \frac{i\gamma_0 \boldsymbol{\gamma} \cdot \mathbf{x}}{|\mathbf{x}|} \right],$$

$$\eta = \sqrt{1 - (Z\alpha)^2}$$

All-Orders Hard Matrix Element

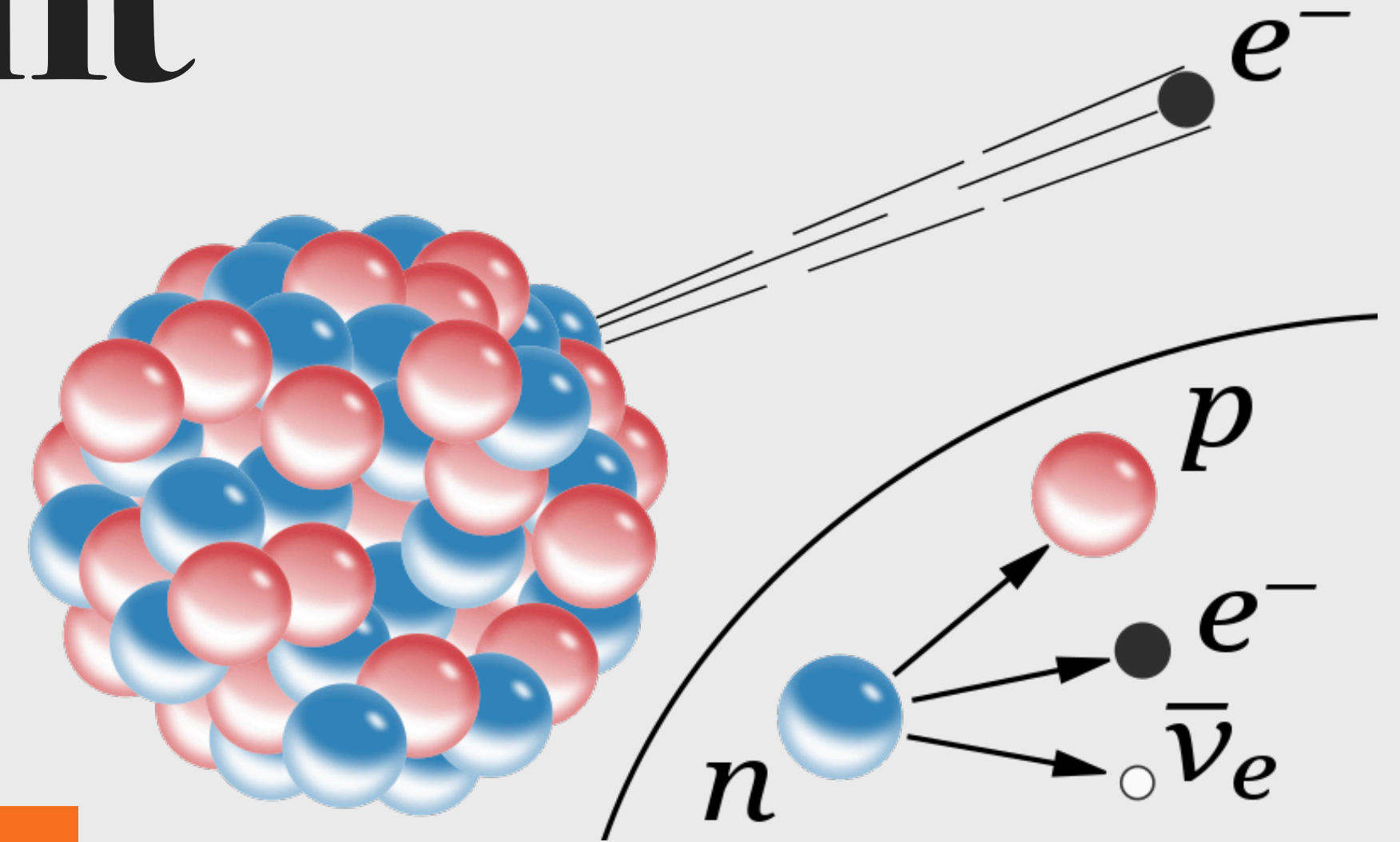
$$\mathcal{M}_H(\mu_S, \mu_H) = e^{\frac{\pi\xi}{2} + i\xi \left(\log \frac{2p}{\mu_S} - \gamma_E \right) - i(\eta-1) \frac{\pi}{2}} \frac{2\Gamma(\eta - i\xi)}{\Gamma(2\eta + 1)}$$

$$\sqrt{\frac{\eta - i\xi}{1 - i\xi \frac{m}{E}}} \sqrt{\frac{E + \eta m}{E + m}} \sqrt{\frac{2\eta}{1 + \eta}} \left(\frac{2pe^{-\gamma_E}}{\mu_H} \right)^{\eta-1} \times \left[\frac{1 + M^*}{2} + \frac{1 - M^*}{2} \gamma^0 \right]$$

• $\eta = \sqrt{1 - Z^2 \alpha^2}$ • $\xi = Z\alpha/\beta$ • $M = (E + m)(1 + i\xi m/E)/(E + \eta m)$

Coulomb Enhancement

- Largest effects are a series in $Z\alpha$



MS-BAR RENORMALIZED

UNIVERSAL RESULT FOR QED

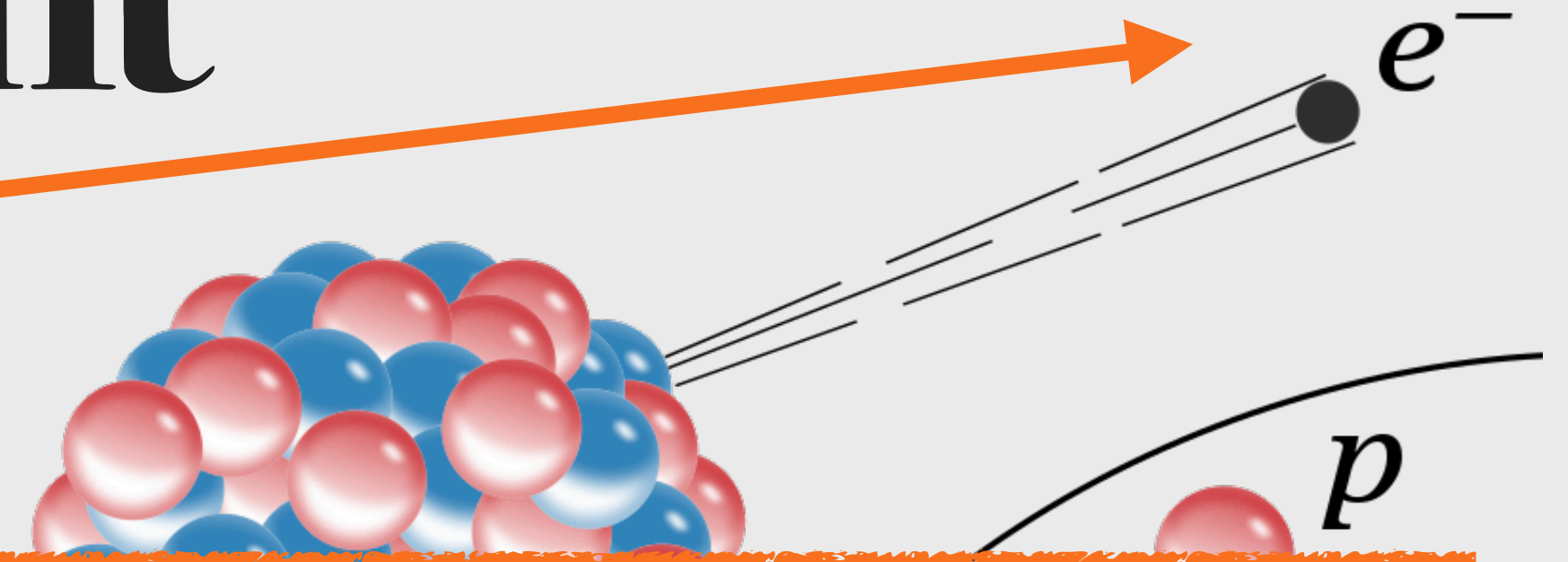
ALL ORDERS IN $Z\alpha$

$$\mathcal{M}_H(\mu_S, \mu_H) = e^{\frac{\pi\xi}{2} + i\xi \left(\log \frac{2p}{\mu_S} - \gamma_E \right) - i(\eta-1)\frac{\pi}{2}} \frac{2\Gamma(\eta - i\xi)}{\Gamma(2\eta + 1)} \sqrt{\frac{\eta - i\xi}{1 - i\xi \frac{m}{E}}} \sqrt{\frac{E + \eta m}{E + m}} \sqrt{\frac{2\eta}{1 + \eta}} \left(\frac{2pe^{-\gamma_E}}{\mu_H} \right)^{\eta-1} \times \left[\frac{1 + M^*}{2} + \frac{1 - M^*}{2} \gamma^0 \right]$$

- $\eta = \sqrt{1 - Z^2\alpha^2}$
- $\xi = Z\alpha/\beta$
- $M = (E + m)(1 + i\xi m/E)/(E + \eta m)$

Coulomb Enhancement

ATTRACTED TO NUCLEUS



- Well defined EFT matrix element. Can be evolved with RG to re-sum logs.

UNIVERSAL RESULT FOR QED

ALL ORDERS IN $Z\alpha$

$$\mathcal{M}_H(\mu_S, \mu_H) = e^{\frac{\pi\xi}{2} + i\xi \left(\log \frac{2p}{\mu_S} - \gamma_E \right) - i(\eta-1)\frac{\pi}{2}} \frac{2\Gamma(\eta - i\xi)}{\Gamma(2\eta + 1)} \sqrt{\frac{\eta - i\xi}{1 - i\xi \frac{m}{E}}} \sqrt{\frac{E + \eta m}{E + m}} \sqrt{\frac{2\eta}{1 + \eta}} \left(\frac{2pe^{-\gamma_E}}{\mu_H} \right)^{\eta-1} \times \left[\frac{1 + M^*}{2} + \frac{1 - M^*}{2} \gamma^0 \right]$$

- $\eta = \sqrt{1 - Z^2\alpha^2}$
- $\xi = Z\alpha/\beta$
- $M = (E + m)(1 + i\xi m/E)/(E + \eta m)$

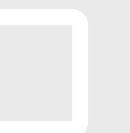


Properties Of The Anomalous Dimension

Anomalous Dimension

$$\gamma_C = \frac{dC(\mu)}{d \log \mu}$$

$$\begin{aligned} \gamma_C = & \alpha \left(Z\gamma^{(1,1)} + \gamma^{(1,0)} \right) + \alpha^2 \left(Z^2\gamma^{(2,2)} + Z\gamma^{(2,1)} + \gamma^{(2,0)} \right) \\ & + \alpha^3 \left(Z^3\gamma^{(3,3)} + Z^2\gamma^{(3,2)} + Z\gamma^{(3,1)} + \gamma^{(3,0)} \right) + \dots \end{aligned}$$



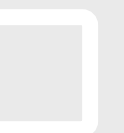
Anomalous Dimension

SOLVE DIRAC EQ'N

$$\gamma_C = \frac{dC(\mu)}{d \log \mu}$$

- Subtlety: Divergent as $x \rightarrow 0$
- New result: All orders result in the $\overline{\text{MS}}$ -scheme (good for RG).

$$\begin{aligned} \gamma_C = & \alpha \left(Z\gamma^{(1,1)} + \gamma^{(1,0)} \right) + \alpha^2 \left(Z^2\gamma^{(2,2)} + Z\gamma^{(2,1)} + \gamma^{(2,0)} \right) \\ & + \alpha^3 \left(Z^3\gamma^{(3,3)} + Z^2\gamma^{(3,2)} + Z\gamma^{(3,1)} + \gamma^{(3,0)} \right) + \dots \end{aligned}$$



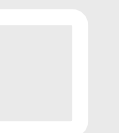
Anomalous Dimension

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$$\gamma_C = \frac{dC(\mu)}{d \log \mu}$$

$$\begin{aligned} \gamma_C = & \alpha \left(Z\gamma^{(1,1)} + \gamma^{(1,0)} \right) + \alpha^2 \left(Z^2\gamma^{(2,2)} + Z\gamma^{(2,1)} + \gamma^{(2,0)} \right) \\ & + \alpha^3 \left(Z^3\gamma^{(3,3)} + Z^2\gamma^{(3,2)} + Z\gamma^{(3,1)} + \gamma^{(3,0)} \right) + \dots \end{aligned}$$

Z=0 REDUCES TO HEAVY-LIGHT CURRENT IN HQET



Anomalous Dimension

$$\gamma_C = \frac{dC(\mu)}{d \log \mu}$$

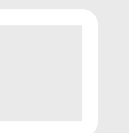
SOLVE DIRAC EQ'N

SYMMETRY IN MASSLESS LIMIT

$$(Z, Z - Q, Q) \longleftrightarrow (Z + Q, Z, -Q)$$

$$\begin{aligned} \gamma_C = & \alpha \left(Z\gamma^{(1,1)} + \gamma^{(1,0)} \right) + \alpha^2 \left(Z^2\gamma^{(2,2)} + Z\gamma^{(2,1)} + \gamma^{(2,0)} \right) \\ & + \alpha^3 \left(Z^3\gamma^{(3,3)} + Z^2\gamma^{(3,2)} + Z\gamma^{(3,1)} + \gamma^{(3,0)} \right) + \dots \end{aligned}$$

Z=0 REDUCES TO HEAVY-LIGHT CURRENT IN HQET

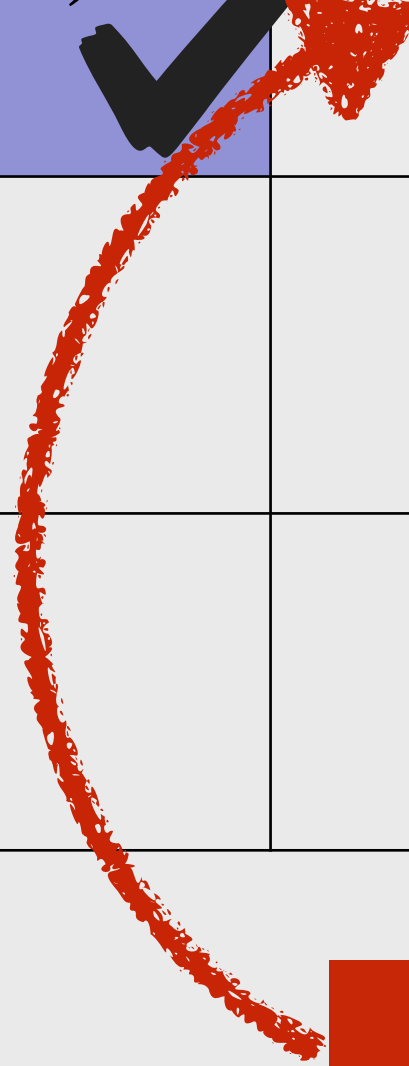


TAKE FROM HQET LIT.

SOLVE DIRAC EQ'N

 SYMMETRY

	α^0	α^1	α^2	α^3	α^4
Z^0	0	$\gamma^{(1,0)}$ ✓	$\gamma^{(2,0)}$ ✓	$\gamma^{(3,0)}$ ✓	$\gamma^{(4,0)}$ ✓
Z^1	—	0	$\gamma^{(2,1)}$	$\gamma^{(3,1)}$	$\gamma^{(4,1)}$
Z^2	—	—	$\gamma^{(2,2)}$ ✓	$\gamma^{(3,2)}$ ✓	$\gamma^{(4,2)}$
Z^3	—	—	—	0	$\gamma^{(4,3)}$
Z^4	—	—	—	—	$\gamma^{(4,4)}$ ✓



NEW INPUT!



Ratio Of Wilson Coefficients

$$Z \sim L \sim \alpha^{-1/2}$$

$$\begin{aligned} \log \left(\frac{C(\mu_L)}{C(\mu_H)} \right) = & \frac{\gamma_0^{(1)}}{2\beta_0} \left\{ \left[\log \frac{a_H}{a_L} + \frac{Z^2 \gamma_1^{(0)}}{\gamma_0^{(1)}} (a_H - a_L) \right] + \left[\frac{Z \gamma_1^{(1)}}{\gamma_0^{(1)}} (a_H - a_L) \right] \right. \\ & \left. + \left[\left(\frac{\gamma_1^{(2)}}{\gamma_0^{(1)}} - \frac{\beta_1}{\beta_0} \right) (a_H - a_L) + \left(\frac{Z^2 \gamma_2^{(1)}}{\gamma_0^{(1)}} - \frac{\beta_1}{\beta_0} \frac{Z^2 \gamma_1^{(0)}}{\gamma_0^{(1)}} \right) \frac{1}{2} (a_H^2 - a_L^2) + \frac{Z^4 \gamma_3^{(0)}}{\gamma_0^{(1)}} \frac{1}{3} (a_H^3 - a_L^3) \right] \right\} \end{aligned}$$

