Quantum Mechanics vs. Quantum Field Theory

What is different for infinitely many degrees of freedom?

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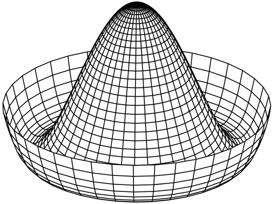
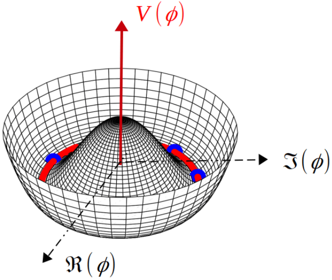
Contents

Introduction

Once upon a time ... when I was a student, I started to study Quantum Theory
... and some things really confused me: Here are two of them:

- Spontaneous Symmetry Breaking and the 2-dim anharmonic oscillator





- Classical electromagnetic fields and QED IR Problem

Quantum Mechanics with One Degree of Freedom

Quantum Mechanics for One Degree of Freedom

Technically speaking: Algebra of three operators
Canonical Commutation Relations (CCR)

$$x,\ p\,,\mathbf{1} \quad \text{with} \quad [x,p] = i\mathbf{1}, \quad [x,\mathbf{1}] = 0 = [p,\mathbf{1}]$$

Task: Find a Hilbert space representation of this algebra

We note

- Assume a finite-dimensional representation space (n dimensions, $n > 0$):

$$\mathrm{Tr}\left([x,p]\right) = 0 \quad \text{but} \quad \mathrm{Tr}\left(\mathbf{1}\right) = n$$

- No finite dimensional representations

Construction of the Hilbert Space $\mathcal{H}$

- Define creation and anihilation operators

$$x = \frac{1}{\sqrt{2}}(a^\dagger + a) \qquad p = \frac{i}{\sqrt{2}}(a^\dagger - a)$$

$$a = \frac{1}{\sqrt{2}}(x + ip) \qquad a^\dagger = \frac{1}{\sqrt{2}}(x - ip)$$

- The canonical commutator implies

$$[a, a^\dagger] = \mathbf{1}$$

- Postulate a "vacuum state" $|0\rangle$ with $a|0\rangle = 0$
- Construct an orthonormal basis of $\mathcal{H}$

$$|0\rangle\,, \quad |1\rangle = a^\dagger|0\rangle\,, \quad |2\rangle = \frac{1}{\sqrt{2}}(a^\dagger)^2|0\rangle\,, \quad \cdots \quad |n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \cdots$$

Equivalent Commutation Relations

Any set of operators satisfying the CCR is "unitary equivalent"
i.e. Corresponds to a unitary transformation within the Hilbert space $\mathcal{H}$

- Relevant Example: New Operators:

$$\xi = x + \lambda \mathbf{1} = x + \lambda \qquad \pi = p \quad \text{with} \quad \lambda \in \mathbb{R}$$

- Satisfy the same CCR
- Unitary Tranformation:

$$U(\lambda) = \exp(-i\lambda p) \quad U(\lambda)^\dagger x U(\lambda) = x + \lambda = \xi \quad U(\lambda)^\dagger p U(\lambda) = p = \pi$$

Proof by the BCH relation 1: $e^{-A} B e^{A} = B + [B, A] + \frac{1}{2}[[B, A], A] + \frac{1}{3!}[[[B, A], A], A] \cdots$

Quantum Mechanics with $N < \infty$ Degrees of Freedom

More Degrees of Freedom

Generalize the CCR: $(k, l = 1, \cdots, N)$

$$x_k,\ p_k,\ ,\mathbf{1} \quad \text{with} \quad [x_k, p_l] = i\delta_{kl}\mathbf{1}, \quad [x_k, x_l] = 0 = [p_k, p_l] = [x_k, \mathbf{1}] = [p_k, \mathbf{1}]$$

The same tasks

- Define creation and anihilation operators:

$$x_k = \frac{1}{\sqrt{2}}(a_k^\dagger + a_k) \quad p_k = \frac{i}{\sqrt{2}}(a_k^\dagger - a_k)$$

$$a_k = \frac{1}{\sqrt{2}}(x_k + ip_k) \quad a_k^\dagger = \frac{1}{\sqrt{2}}(x_k - ip_k)$$

- This implies $[a_k, a_l^\dagger] = \mathbf{1}\delta_{kl} \qquad [a_k, a_l] = 0 = [a_k\dagger, a_l^\dagger]$

- Construct $\mathcal{H}$: $|n_1, n_2, \ldots n_N\rangle = \frac{(a_1^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(a_2^\dagger)^{n_2}}{\sqrt{n_2!}} \cdots \frac{(a_N^\dagger)^{n_N}}{\sqrt{n_N!}} |0\rangle$

Again: Equivalent Commutation Relations

- Stick to our relevant example:

$$\xi_k = x_k + \lambda_k \quad \pi_k = p_k \quad \text{with} \quad \lambda_k \in \mathbb{R}$$

- Satisfy the same CCR
- Unitary Transformation:

$$U(\{\lambda\}) = \exp\left(\frac{1}{\sqrt{2}}\sum_{k=0}^{N}\lambda_k\left[a_k^\dagger - a_k\right]\right)$$

$$= \exp\left(\frac{1}{\sqrt{2}}\sum_{k=0}^{N}\lambda_k a_k^\dagger\right)\exp\left(-\frac{1}{\sqrt{2}}\sum_{k=0}^{N}\lambda_k a_k\right)\exp\left(-\frac{1}{4}\sum_{k=0}^{N}\lambda_k^2\right)$$

Proof by BCH Relation 2: $e^{A+B} = e^A\, e^B \exp\left(-\frac{1}{2}[A,B]\right)$ for $[[A,B],B] = 0 = [[A,B],A]$

Summary on QM with finite number of d.o.f.

- For a finite number of d.o.f. the unitary map exists, so for a finite number od d.o.f. all representations of the CCR are unitary equivalent
- However, if $N \to \infty$ it depends on the convergence of the sum

$$\sum_{k=0}^{\infty} \lambda_k^2$$

- In case the sum does not converge, $U(\{\lambda\})$ ceases to be a unitary operator,
- ... so the different represetations of the CCR become inequivalent.

Quantum Mechanics with infinitely many Degrees of Freedom

or in other words

Quantum Field Theory

Quantum Fields

consider a simplified case of a real (scalar) field:

$$\Phi(\vec{x}) = \sum_s u_s(\vec{x}) x_s = \frac{1}{\sqrt{2}} \sum_s u_s(\vec{x})(a_s + a_s^\dagger)$$

- $u_s(\vec{x})$: Set of real basis functions, satisfying

$$\sum_s u_s(\vec{x}) u_s(\vec{y}) = \delta^3(\vec{x} - \vec{y}) \qquad \text{and} \qquad \int d^3x \, u_r(\vec{x}) u_s(\vec{x}) = \delta_{rs}$$

- a_s and $a_s^\dagger$: anihilation and creation operators
- $\Phi(\vec{x})$: “generalized coordinate”

Canonically conjugated momentum

$$\Pi(\vec{x}) = \sum_s u_s(\vec{x}) p_s = \frac{i}{\sqrt{2}} \sum_s u_s(\vec{x})(a_s^\dagger - a_s)$$

Canonical Commutation Relations for Quantum Fields

$$[\Phi(\vec{x}), \Pi(\vec{y})] = i\delta^3(\vec{x} - \vec{y})\mathbf{1}$$

Remarks:

- This is the "equal time" commutation relation
- We cannot consider the time evolution without knowing the Hamiltonian
- .. so this is valid for any QFT system, i.e. in principle also for interacting fields.

Construct the Hilbert space:

- Assume a vacuum state $|0\rangle$ (no particles)
- Build up the Hibert space

$$|s\rangle = a_s^\dagger |0\rangle : \text{ One particle state} \qquad |s\, s'\rangle = a_s^\dagger a_{s'}^\dagger |0\rangle : \text{ Two particle state} \qquad \dots$$

Inequivalent representations ...

Now let us change the operators:

$$\xi_s = x_s + \lambda_s \qquad \text{with} \qquad \lambda_s \in \mathbb{R} \qquad \pi_s = p_s \quad \text{(unchanged)}$$

from which we get

$$\Phi'(\vec{x}) = \Phi(\vec{x}) + f(\vec{x})\mathbf{1} \qquad \text{with} \qquad f(\vec{x}) \in \mathbb{R}$$

$$\lambda_s = \int d^3x\, f(\vec{x}) u_s(\vec{x}) \qquad \sum_{s=0}^{\infty} \lambda_s^2 = \int d^3x\, [f(\vec{x})]^2$$

"Unitary" Transformation:

$$U(\{f(\vec{x})\}) = \exp\left((-i) \int d^3x\, f(\vec{x}) \Pi(\vec{x}) \right)$$

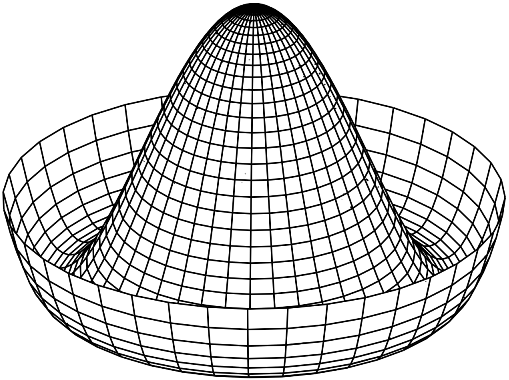
$$= \exp\left(\sum_{s=0}^{\infty} a_s^\dagger \int d^3x\, f(\vec{x}) u_s(\vec{x}) \right) \exp\left(- \sum_{s=0}^{\infty} a_s \int d^3x\, f(\vec{x}) u_s(\vec{x}) \right) \exp\left(-\frac{1}{4} \int d^3x\, [f(\vec{x})]^2 \right)$$

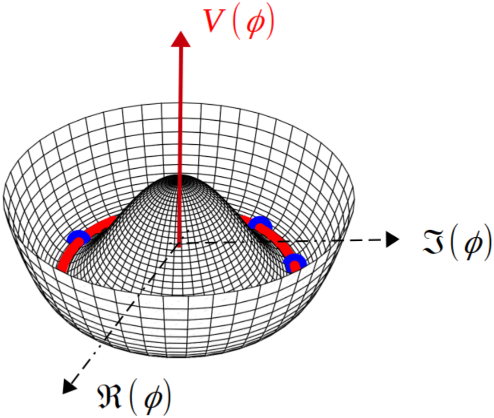
However

- $\langle 0|\Phi(\vec{x})|0\rangle = 0$ from the construction of the Hilbert space.
- With the new field, using the original operators, we get $\langle 0|\Phi'(\vec{x})|0\rangle = f(\vec{x})$
- But the vacuum should be translationally invariant, so $f(\vec{x}) = const \neq 0$
- Hence $\exp\left(-\frac{1}{4}\int d^3x\,[f(\vec{x})]^2\right) \to 0$
- Zero overlapp between the to Hilbert spaces
- This is actually why we can have spontaneous symmetry breaking

Spontaneous Symmetry Breaking

Spontaneous Symmetry Breaking





Modes of Symmetry

Wigner Weyl mode

- The Hilbert space becomes a representation space of the symmetry
- States fall into degenerate multiplets of the symmetry group
- The symmetry is explicitely visible ...

Nambu Goldstone mode

- No degenerate multiplets of the symmetry group
- The symmetry is “hidden”
- A typical spectrum of massless particles appears

The Nambu Goldstone mode requires a quantum Field theory ...

(QED) Infrared Problem

(QED) Infrared Problem

The QED infrared problem has a similar origin:

- Fock states of QED: Electrons/Positrons and Photons
- Single Electron Fock state: $|p, s\rangle = b^\dagger(p, s)|0\rangle$
- What is the expectation value of the electromagnetic field?

$$\langle p, s|A_\mu(x)|p, s\rangle = 0$$

- “Dress” the electron with soft photons: “Unitary” Møller Operator

$$|e, p, s\rangle = \Omega|p, s\rangle = U(\infty, 0)|p, s\rangle$$

- Ω can be computed in the soft photon limit:

$$\Omega = \exp\left(-e \int \widetilde{dk} \sum_\lambda \frac{p_\mu}{pk}[\epsilon^\mu(k, \lambda)a(k, \lambda) - \epsilon^{*\mu}(k, \lambda)a^\dagger(k, \lambda)]\right)$$

- This dresses the electron with its Couplomb field

$$\langle e, p, s | A_\mu(x) | e, p, s \rangle \propto \frac{e}{|\vec{x}|}$$

- But $|e, p, s\rangle$ lives in a different Hilbert space

$$\Omega \propto \exp\left(-\frac{e^2}{4} \int \widetilde{dk} \frac{m^2}{(pk)^2} \right) \sim \exp\left(-\mathrm{const} \int^m \frac{dk}{k} \right) \to 0$$

- Redefinition of the asymptotic states: chose a different Hilbert space
- The states carry their coulomb fields
- Soft radiation is the change in the Coulomb fields, once the electron suffers a large momentum transfer

Eikonal Approximation

Dress the incomming and the ourgoing electron:

$$\mathcal{A} = \mathcal{A}_0 \langle 0| \exp\left(-e \int \widetilde{dk} \sum_\lambda \left[\frac{p_\mu}{pk} - \frac{p'_\mu}{p'k} \right] [\epsilon^\mu(k,\lambda) a(k,\lambda) - \epsilon^{*\mu}(k,\lambda) a^\dagger(k,\lambda)] \right) |\text{Soft } \gamma\text{'s}\rangle$$

Renormalization

The commutator Function

- Start from interacting fields (depend on x_0)

$$D(x-y) = \langle 0|\Phi(x)\Phi(y)|0\rangle = \sum_X \langle 0|\Phi(x)|X\rangle \langle X|\Phi(y)|0\rangle = \sum_X e^{-ip_X(x-y)} |\langle 0|\Phi(0)|X\rangle|^2$$

- $|\langle 0|\Phi(0)|X\rangle|^2$: probability that $\Phi(0)$ generates the state $|X\rangle$ from the vacuum
- Collect all states with a fixed mass μ

$$D(x-y) = \langle 0|\Phi(x)\Phi(y)|0\rangle = \int \frac{d\mu^2}{2\pi} \int \frac{d^4p}{(2\pi)^4} (2\pi)\delta(p^2-\mu^2) e^{-ip(x-y)} \rho(\mu^2)$$

$$= \int \frac{d\mu^2}{2\pi} \rho(\mu^2) \langle 0|\Phi_0(x)\Phi_0(y)|0\rangle_{\mu^2} \qquad \text{with} \qquad \Phi_0 \text{ a free field with mass } \mu$$

and the spectral function

$$\rho(\mu^2) = \sum_X (2\pi)^4 \delta^4(p-p_X)(2\pi)|\langle 0|\Phi(0)|X\rangle|^2$$

The Lehmann Sum Rule

- Now compute the Commutator function:

$$\Delta(x-y) = \langle 0|[\Phi(x)\,,\,\Phi(y)]|0\rangle = \int \frac{d\mu^2}{2\pi}\rho(\mu^2)\langle 0|[\Phi_0(x)\,,\,\Phi_0(y)]|0\rangle_{\mu^2}$$

- Assume that $\Pi(x) = \partial_0\Phi(x)$ (true for the free field and in many other cases)

$$\partial_0\Delta(x-y) = \langle 0|[\partial_0\Phi(x)\,,\,\Phi(y)]|0\rangle = \langle 0|[\Pi(x)\,,\,\Phi(y)]|0\rangle$$

- Taking $x_0 = y_0$ yields the Lehmann Sum Rule

$$\int \frac{d\mu^2}{2\pi}\rho(\mu^2) = 1$$

- Assume that a one particle state with mass m exists with a probability Z

$$\rho(s) = Z\delta(s-m^2) + \rho_{\rm cont}(s)\Theta(s-m^2+\delta m^2)$$

This implies $0 \leq Z \leq 1$

- $Z = 1$: the one-particle state exhausts the sum rule: Free field!
- $Z = 0$ means that there is no one-particle state in the Hilbert Space

Look at the perturbative calculation based on the usual Fock space:

- Yields a divergent result: "Wave Function Renormalization", Cut-Off Λ

$$Z = 1 - \frac{\alpha}{4\pi} \ln\left(\frac{\Lambda}{m}\right) + \dots$$

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- Yields a divergent result: “Wave Function Renormalization”, Cut-Off Λ

$$Z = 1 - \frac{\alpha}{4\pi} \ln\left(\frac{\Lambda}{m}\right) + \dots$$

- Compare to the case of the IR problem:
  interpret this as the start of an exponential series

$$Z = \exp\left[-\frac{\alpha}{4\pi} \ln\left(\frac{\Lambda}{m}\right)\right] = \left(\frac{\Lambda}{m}\right)^{-\alpha/(4\pi)} \to 0$$

What - at the end - is renormalization?

- We regularize by cutting off high scales
- This leaves us with a regularized theory with a finite overlap with the Fock states used for the Feynman diagrams
- We pick observables (S matrix) for which the cut off can be removed at any order in perturbation theory, so the theory has predictive power
- This can be done for renormalizable theories,
- ... but is also means that we have no clue about the physics at large scales, since our observables are by construction insensitive to high scales.

Conclusions?

- The Hilbert Space of the "full" states in QFT does not have any overlap with the Fock states used to set up perturbation theory
- This becomes manifest in the divergencies present in the Feynman diagrams
- Regularization ensures some overlap between the Fock states and the ones of the (regularized) full theory.
- In renormalizable theories we can calculate the S matrix by taking the limit $\Lambda \to \infty$ to obtain cross sections and devay rates.
- Other quantites we know form Quantum Mechanics (such as Eigenstates of the Hamiltonian or other observables) cannot be computed
- The time evolution in QFT also seems to be "pseudo unitary"..

Quantum Field Theory is really very different from Quantum Mechanics