# CP CONSERVATION IN THE STRONG INTERACTIONS

BJÖRN GARBRECHT, TUM

PARTICLE PHYSICS SEMINAR

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## Outline

- Introduction: QCD $\theta$-parameter, EFT, topology, neutron EDM
- Functional quantization and path integral contour
- Canonical quantization and $\theta$-vacua

# INTRODUCTION

## CP-odd terms in effective field theories

## Topology

# $CP$ violation in the strong interactions?

No empirical evidence—neutron electric dipole moment (EDM) strongly constrained:

$$d_n = (0.0 \pm 1.1_{\text{stat}} \pm 0.2_{\text{sys}}) \times 10^{-26}\, e\,\text{cm} \quad \text{[2020 @ PSI]}$$

QCD with massive quarks

$$\mathcal{L} \supset \frac{1}{2g^2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} + \sum_{j=1}^{N_f} \bar{\psi}_j \left( \mathrm{i}\not{D} - m_j \mathrm{e}^{\mathrm{i}\alpha_j \gamma^5} \right) \psi_j + \frac{1}{16\pi^2} \theta \operatorname{tr} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

Believed to cause a neutron electric dipole moment (EDM) $d_n \sim 10^{-15}\, e\,\text{cm} \left( \theta + \sum_j \alpha_j \right)$

[Baluni (1979); Crewther, Di Vecchia, Veneziano, Witten (1979)]

*Or does it?*

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$\propto \vec{E}^2 + \vec{B}^2$

$\gamma^5$, parity-odd

$\propto \vec{E} \cdot \vec{B}$
parity-odd

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# Effective interactions with $\theta$

$F_{\mu\nu}\tilde{F}^{\mu\nu}$ total derivative $\longrightarrow$ No effects in perturbation theory $\longrightarrow$ Use EFT

$\mathrm{SU}(N_f)_\mathrm{L} \times \mathrm{SU}(N_f)_\mathrm{R}$ global symmetry in the limit of massless quarks

Chiral $\mathrm{U}(1)_A$ symmetry of the quarks is anomalous however
$\longrightarrow$ $\mathcal{L}$ invariant under [Fujikawa (1979,80)]

| chiral trafo | | "spurion" trafo |
|---|---|---|
| $\psi \to \mathrm{e}^{\mathrm{i}\beta\gamma_5}\psi$<br>$\bar{\psi} \to \bar{\psi}\mathrm{e}^{\mathrm{i}\beta\gamma_5}$ | plus | $m_j \mathrm{e}^{\mathrm{i}\alpha_j\gamma^5} \to m_j \mathrm{e}^{\mathrm{i}(\alpha_j - 2\beta)\gamma^5}$<br>$\theta \to \theta + 2N_f\beta$ |

Spurions break the symmetries explicitly. $\longrightarrow$ Approximate symmetries
This pattern should be replicated by any effective theory.

Rephasing invariant: $\bar{\theta} = \theta + \bar{\alpha}$, where $\bar{\alpha} = \sum_{j=1}^{N_f} \alpha_j$, $\longrightarrow$ $\theta$ is an angle

# Integrating out gauge fields: Effective interactions

Topological effects described by effective 't Hooft vertex ($\Gamma_{N_f}$ some coefficient) ['t Hooft (1976,86)] [Kaplan, Sen ('24)]

$$\mathcal{L} + \frac{1}{16\pi^2}\theta\,\mathrm{tr}\,F_{\mu\nu}\tilde{F}^{\mu\nu} \to \mathcal{L} - \Gamma_{N_f}\mathrm{e}^{\mathrm{i}\xi}\prod_{j=1}^{N_f}(\bar{\psi}_j P_{\mathrm{L}}\psi_j) - \Gamma_{N_f}\mathrm{e}^{-\mathrm{i}\xi}\prod_{j=1}^{N_f}(\bar{\psi}_j P_{\mathrm{R}}\psi_j)$$

As a spurion, $\xi \to \xi + 2N_f\beta \longrightarrow$

Two options: $\xi = \theta$ (in general misaligned with masses) $\rightarrow$ $CP$ violation

$\xi = -\bar{\alpha}$ (present claim, aligned with mass terms) $\rightarrow$ no $CP$ violation

$\chi$PT at low energies

$$U = U_0\mathrm{e}^{\frac{\mathrm{i}}{f_\pi}\Phi} \qquad U_0:\ \textit{chiral condensate} \qquad \Phi = \begin{bmatrix} \pi^0 + \eta' & \sqrt{2}\,\pi^+ \\ \sqrt{2}\,\pi^- & -\pi^0 + \eta' \end{bmatrix}$$

$\xi = \theta$: $\chi$ral condensate aligned with $\theta$

$\xi = -\bar{\alpha}$: $\chi$ral condensate aligned with quark mass phases

$$\mathcal{L} = \frac{f_\pi^2}{4}\mathrm{Tr}\,\partial_\mu U\partial^\mu U^\dagger + \frac{f_\pi^2 B_0}{2}\mathrm{Tr}(M U + U^\dagger M^\dagger) + |\lambda|\mathrm{e}^{-\mathrm{i}\xi}f_\pi^4\det U + |\lambda|\mathrm{e}^{\mathrm{i}\xi}f_\pi^4\det U^\dagger \qquad M = \mathrm{diag}\{m_u\mathrm{e}^{\mathrm{i}\alpha_u}, m_d\mathrm{e}^{\mathrm{i}\alpha_d}\}$$

$$\mathcal{L}_{\text{neutron}} \supset -\frac{c_1}{f_\pi}\partial_\mu\pi^a\bar{\mathcal{N}}T^a\gamma^\mu\gamma_5\mathcal{N} \qquad CP \text{ even}$$

$$\mathcal{N} = \begin{pmatrix} p \\ n \end{pmatrix} \qquad +\frac{c_2\bar{m}}{f_\pi}(\xi + \alpha_u + \alpha_d + \alpha_s)\bar{\mathcal{N}}\pi^a T^a\mathcal{N} \qquad CP \text{ odd}$$

## Topology in four-dimensional spacetime—winding number $\Delta n$

$$U = \begin{pmatrix} a_{\mathrm{R}} + \mathrm{i}a_{\mathrm{I}} & -b_{\mathrm{R}} + \mathrm{i}b_{\mathrm{I}} \\ b_{\mathrm{R}} + \mathrm{i}b_{\mathrm{I}} & a_{\mathrm{R}} - \mathrm{i}a_{\mathrm{I}} \end{pmatrix} \in \mathrm{SU}(2) \text{ for } a_{\mathrm{R}}^2 + a_{\mathrm{I}}^2 + b_{\mathrm{R}}^2 + b_{\mathrm{I}}^2 = 1$$

$$\Rightarrow \text{Homotopy: } \mathrm{SU}(3) \supset \mathrm{SU}(2) \cong S^3 \longrightarrow \pi_3(\mathrm{SU}(2)) = \pi_3(S^3) = \mathbb{Z}$$

Theta-term/topological term is a total divergence:

gauge invariant

$$\frac{1}{4}\mathrm{tr} F_{\mu\nu}\tilde{F}_{\mu\nu} = \partial_\mu K_\mu$$

gauge dependent

$$K_\mu = \epsilon_{\mu\nu\alpha\beta}\mathrm{tr}\left[\frac{1}{2}A_\nu \partial_\alpha A_\beta + \frac{1}{3}A_\nu A_\alpha A_\beta\right]$$

### Topological quantization for pure gauge $A_\mu \to -\frac{\mathrm{i}}{g}(\partial_\mu U)U^{-1}$ at $\partial\Omega \cong S^3$

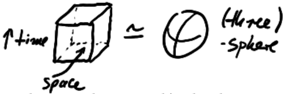
$$\Delta n = \frac{1}{16\pi^2}\int_\Omega \mathrm{d}^4 x F_{\mu\nu}\tilde{F}_{\mu\nu} = \frac{1}{4\pi^2}\oint_{\partial\Omega} \mathrm{d}^3\sigma K_\perp \in \mathbb{Z}$$

Haar measure for pure gauge

$$K_\mu = \tfrac{1}{6}\varepsilon_{\mu\nu\lambda\rho}\mathrm{tr}[(U^{-1}\partial_\nu U)(U^{-1}\partial_\lambda U)(U^{-1}\partial_\rho U)]$$

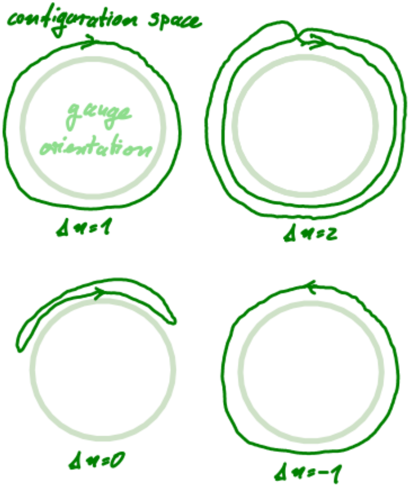
[see e.g. Coleman, "Aspects of symmetry" (1985)]

E.g. take boundary of $\Omega = \mathbb{R}^4$ as a sphere $S^3$:




Or $\Omega = T^4$(lattice), $\Omega = S^4$(Euclidean dS): $\Delta n \in \mathbb{Z}$ based on slightly more involved argument

# Topology—instantons




$\Delta n \neq 0$ implies nontrivial physical field configurations

Cf. anti-instanton: $A_\mu{}^u{}_v = -\dfrac{\sigma_{\mu\nu}{}^u{}_v x_\nu}{x^2+\rho^2}$

(extended solution to *Euclidean* EOMs)

[Belavin, Polyakov, Schwarz, Tyupkin (1975)]

Surface term decays as $1/|x|^3 \rightarrow$ surface integral does not need to vanish

Theta term contributes to the action though being a total derivative

## Topology on spatial hypersurfaces—point compactification, large gauge transformations

Consider temporal gauge $A^0 = 0$ (in view of canonical quantization)

Chern–Simons functional:

$$W[\vec{A}] = \frac{1}{4\pi^2}\varepsilon_{ijk}\int_V \mathrm{d}^3x\,\mathrm{tr}\left[\frac{1}{2}A_i\partial_j A_k - \frac{\mathrm{i}}{3}A_i A_j A_k\right] \equiv \frac{1}{4\pi^2}\int_V \mathrm{d}^3x\, K_0$$

Define $\vec{A}_U = U\vec{A}U^{-1} + \mathrm{i}\,U^{-1}\vec{\nabla}U$ (residual gauge freedom in temporal gauge)

With extra constraint $U(\vec{x}) \to$ const. on $\partial V$ (periodic on $T^3$)
$\to$ Point compactification, homotopy $V \cong S^3$ ($V \cong T^3$)

space

$\vec{A} = iU^{-1}\vec{\nabla}U$
& $U(\vec{x}) = \text{const.}$
on boundary

(three)
-sphere

$U^{(\nu)}$: equivalence classes of "large" ($\nu \neq 0$) gauge transformation on spacelike ($\tau = \text{const.}$) hypersurface $V \simeq S^3$ ($V \simeq T^3$) with $W[\vec{A}_{U^{(\nu)}}] - W[\vec{A}] = \nu \in \mathbb{Z}$

**However: Extra constaint not enforced by $A^0 = 0$ —would require extra provisions**

# FUNCTIONAL QUANTIZATION

Take time to infinity before summing over topological sectors

# Euclidean path integral & topology

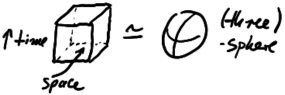
Topological term $F\tilde{F}$ total derivative—how can it contribute?
Does interference of sectors have a material effect?

Recall: Euclidean path integral projects on ground state

$$\lim_{T\to\infty} \frac{\mathrm{e}^{-HT}}{\mathrm{e}^{-E_0 T}} \qquad \text{or} \qquad \lim_{T\to\infty} \frac{\mathrm{e}^{-\mathrm{i}HT(1-\mathrm{i}\varepsilon)}}{\mathrm{e}^{-\mathrm{i}E_0 T(1-\mathrm{i}\varepsilon)}}$$

$H$: Hamiltonian
$E_0$: ground state energy

$\longrightarrow$ Consider $\Omega = \mathbb{R}^4$ (or different *spatial* topologies)




Finite action $\longrightarrow$ pure gauge at infinity
$\longrightarrow$ Topological quantization $\longrightarrow$ Phases $\mathrm{e}^{\mathrm{i}\Delta n\theta}$

No reason for topological quantization in finite $\Omega \subset \mathbb{R}^4$

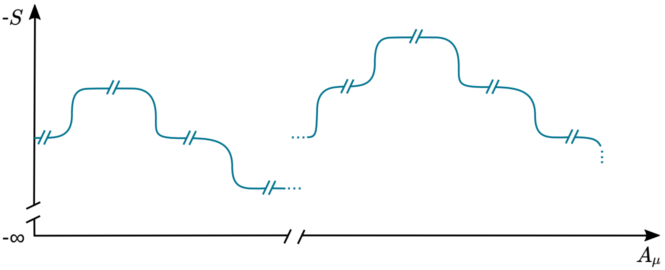
Must take $T \to \infty$ before summing over sectors:

$$Z = \lim_{N\to\infty} \sum_{\Delta n=-N}^{N} \lim_{VT\to\infty} \int_{\Delta n} \mathcal{D}\phi\, \mathrm{e}^{-S_{\mathrm{E}}[\phi]}$$

## More technically: Integration contour from Lefschetz thimbles

Parametrization of the path integral through steepest descent contours about classical saddle points $\longrightarrow$ Contour integration on Lefschetz thimbles

$$\frac{\partial\phi(x;u)}{\partial u} = \overline{\frac{\delta S_\mathrm{E}[\phi(x;u)]}{\delta\phi(x;u)}} \Longrightarrow -\frac{\partial \mathrm{Re} S_\mathrm{E}[\phi(x;u)]}{\partial u} \leq 0 \ \text{ and } \ \frac{\partial \mathrm{Im} S_\mathrm{E}[\phi(x;u)]}{\partial u} = 0$$




Each thimble emerges from a critical point and corresponds to one $\Delta n \in \mathbb{Z}$

Keeping $VT$ finite while summing over different $\Delta n$ ($\Leftrightarrow$ different boundary conditions, infinite distance in field space) does *not* correspond to a nonsingular deformation of the Cauchy contour

**Integration contour sweeps over full thimbles first, i.e. $VT \to \infty$ before sum over $\Delta n$**

## So is it $\xi = -\bar{\alpha}$ or $\xi = \theta$?

- Take $\langle F(x)\tilde{F}(x)\rangle$ as measure for $CP$ violation
- Each element in the sequence over $N$ vanishes (not so when limits ordered the other way around):

$$\boxed{\langle F(x)\tilde{F}(x)\rangle = \lim_{\substack{N\to\infty \\ N\in\mathbb{N}}} \lim_{VT\to\infty} \frac{\sum_{\Delta n=-N}^{N} \frac{\Delta n}{VT} Z_{\Delta n}}{\sum_{\Delta n=-N}^{N} Z_{\Delta n}} = 0 \qquad CP \text{ conserved}}$$

- Index theorem: $\Delta n$ is the difference of the numbers of right and left chiral zero modes
- Left/right chiral quasi-zero modes in spectral representation of fermion correlation regulated by $1/(m\,\mathrm{e}^{\mp\mathrm{i}\alpha})$

$$S(x,x') \underset{\boxed{\Delta n = -1}}{\uparrow}{=} \frac{\hat{\psi}_{0\mathrm{L}}(x)\hat{\psi}^\dagger_{0\mathrm{L}}(x')}{m\mathrm{e}^{-\mathrm{i}\alpha}} + \sum\!\!\!\!\!\!\int_{\lambda^{\mathrm{E}}\neq 0} \frac{\hat{\psi}_\lambda(x)\hat{\psi}^\dagger_\lambda(x')}{\lambda}$$

- Contributions from discrete modes to correlation function vanish for $VT\to\infty$ $\to$ Quark correlations remain aligned with quark mass after interference of $\Delta n$-sectors

$$\to \boxed{\xi = -\bar{\alpha}}$$

- Order of limits matters because series is not positive definite due to phases $\mathrm{e}^{\mathrm{i}\Delta n\theta}$, not absolutely summable

# Fermion correlations and instantons

Dilute instanton gas (DIGA) picture (to determine phase of 't Hooft vertex—not quantitatively accurate for actual QCD)

Leading contribution to two-point function (no instantons)

$$\langle\psi(x)\bar\psi(x')\rangle = \mathrm{i}S_{0\mathrm{inst}}(x,x')$$

$$\mathrm{i}S_{0\mathrm{inst}}(x,x') = (-\gamma^\mu\partial_\mu + \mathrm{i}m_i\mathrm{e}^{-\mathrm{i}\alpha_i\gamma^5})\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}p(x-x')}}{p^2-m_i^2+\mathrm{i}\epsilon}$$

## Green's function in $n$-instanton, $\bar n$-anti-instanton background (DIGA)

$$\mathrm{i}S_{n,\bar n}(x,x') \approx \mathrm{i}S_{0\mathrm{inst}}(x,x') + \sum_{\bar\nu=1}^{\bar n}\frac{\hat\psi_{0\mathrm{L}}(x-x_{0,\bar\nu})\hat\psi^\dagger_{0\mathrm{L}}(x'-x_{0,\bar\nu})}{m\mathrm{e}^{-\mathrm{i}\alpha}} + \sum_{\nu=1}^{n}\frac{\hat\psi_{0\mathrm{R}}(x-x_{0,\nu})\hat\psi^\dagger_{0\mathrm{R}}(x'-x_{0,\nu})}{m\mathrm{e}^{\mathrm{i}\alpha}}$$

$\hat\psi_{0\mathrm{L,R}}$: 't Hooft zero modes

Alignment of $\alpha$ in Lagrangian mass and instanton-induced $\chi$SB $\longrightarrow$ No $CP$ violation here

## Sum/interference over DIGA configurations

$$\boxed{\begin{aligned}\langle\psi(x)\bar\psi(x')\rangle &= \lim_{\substack{N\to\infty\\ N\in\mathbb{N}}}\lim_{VT\to\infty}\frac{\sum_{\Delta n=-N}^{N}\langle\psi(x)\bar\psi(x')\rangle_{\Delta n}}{\sum_{\Delta n=-N}^{N} Z_{\Delta n}}\\ &= \mathrm{i}S_{0\mathrm{inst}}(x,x') + \mathrm{i}\kappa\bar h(x,x')m^{-1}\mathrm{e}^{-\mathrm{i}\alpha\gamma^5}\\ &\longrightarrow \xi = -\alpha \text{ (alignment)}\end{aligned}}$$

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# CANONICAL QUANTIZATION

Done consistently without extra gauge constraint, point compactification

Properly normalizable physical states

# Theta vacuum, standard story

Take $A^0 = 0$, *assume* in addition:

For $|\vec{x}| \to \infty$: $\vec{A}(\vec{x}) = \mathrm{i}\, U^{-1}(\vec{x}) \vec{\nabla}\, U(\vec{x})$ and $U(\vec{x}) \to \text{const.}$

But why? [cf. Jackiw (1980)]

Consider initial and final states, taking $x_4 \to \pm\infty$

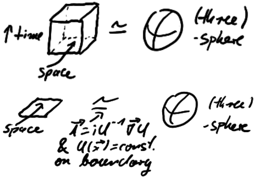
$\to$ *Ansatz*: Construct from pure gauge configurations on these surfaces, with

$$n_{\pm\infty} = \frac{1}{4\pi^2} \int\limits_{x^4 = \pm\infty} \mathrm{d}^3\sigma K_\perp \in \mathbb{Z}$$

Chern–Simons number

**not** gauge invariant

point compactification




Prevacua: $n_{-\infty} \to |n\rangle$, $n_{\infty} \to \langle n|$ (field eigenstates)

Gauge invariant (up to phase) state $|\theta\rangle = \sum_n \mathrm{e}^{-\mathrm{i}n\theta} |n\rangle$ [Callan, Dashen, Gross (1976); Jackiw, Rebbi (1976); Jackiw (1980)]

⚠ States not normalizable in the proper sense: $\langle \theta^{(i)} | \theta^{(j)} \rangle = \sum_n \delta(\theta^{(i)} - \theta^{(j)} + 2\pi n)$

[cf. e.g. Callan, Dashen, Gross (1976); issue taken by Okubo, Marshak (1992)]

Without ado, this contradicts 1st Dirac–von Neumann axiom of quantum mechanics.

[Dirac 1932, von Neumann 1932]

# Theta

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In this section, I shall remain with the formalism as developed thus far, and study further the action of the gauge symmetry. We have remarked already on the invariance of the quantized theory, when $A^0 = 0$, under transformations which in infinitesimal form are described by Eq. (31), and in finite form by

$$\mathbf{A} \to U^{-1}\mathbf{A}U - \frac{1}{g}U^{-1}\nabla U\,. \tag{36a}$$

Here $U$ is a $2\times 2$ unitary, $c$-number SU(2) matrix, depending on position, but not on time. We shall make a very important hypothesis concerning the physically admissible finite transformations. While some plausible arguments can be given in support of this hypothesis (see below) in the end we must recognize it as an assumption, without which the subsequent development cannot be made. We shall assume that the allowed gauge transformation matrices $U$ tend to a definite limit as $r$ passes to infinity.

$$\lim_{r\to\infty} U(\mathbf{r}) = U_\infty\,. \tag{36b}$$

[Jackiw, *Introduction to the Yang–Mills quantum theory* (1980)]

# Theta vacuum, standard story

Take $A^0 = 0$, *assume* in addition:

For $|\vec{x}| \to \infty$: $\vec{A}(\vec{x}) = \mathrm{i}\, U^{-1}(\vec{x}) \vec{\nabla}\, U(\vec{x})$ and $U(\vec{x}) \to \text{const.}$

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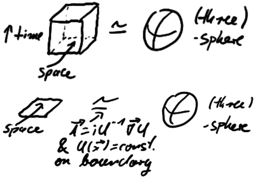


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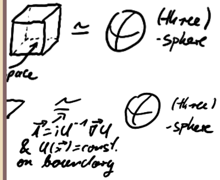


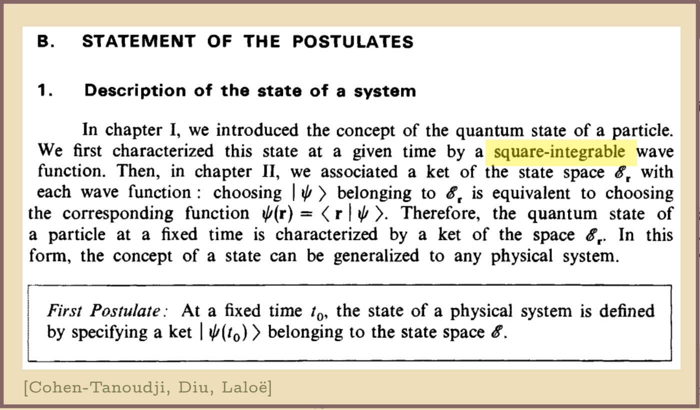

B. STATEMENT OF THE POSTULATES

1. Description of the state of a system

In chapter I, we introduced the concept of the quantum state of a particle. We first characterized this state at a given time by a square-integrable wave function. Then, in chapter II, we associated a ket of the state space $\mathscr{E}_r$ with each wave function: choosing $|\psi\rangle$ belonging to $\mathscr{E}_r$ is equivalent to choosing the corresponding function $\psi(\mathbf{r}) = \langle \mathbf{r} | \psi \rangle$. Therefore, the quantum state of a particle at a fixed time is characterized by a ket of the space $\mathscr{E}_r$. In this form, the concept of a state can be generalized to any physical system.

*First Postulate*: At a fixed time $t_0$, the state of a physical system is defined by specifying a ket $|\psi(t_0)\rangle$ belonging to the state space $\mathscr{E}$.

[Cohen-Tanoudji, Diu, Laloë]

Gauge invariant (u … 976); … kiw (1980)]

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...pace

$\simeq$ (three) -sphere

$\vec{A} = iU^{-1}\vec{\nabla}U$ & $U(\vec{x})=$const. on boundary

$\simeq$ (three) -sphere

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[Dirac 1932, von Neumann 1932]

# Canonical quantization of the gauge field

Minkowski spacetime, temporal gauge $A^0 = 0$, no sources $\longrightarrow$

$$\begin{aligned} g\vec{E}^a &= -\partial/\partial t\,\vec{A}^a \\ g\vec{B}^a &= \vec{\nabla}\times\vec{A}^a - 1/2\,f^{abc}\vec{A}^a\times\vec{A}^b \end{aligned}$$

Canonical momentum conjugate to $\vec{A}^a$:

$$g\vec{\Pi}^a = -\vec{E}^a + \frac{g^2}{8\pi^2}\theta\vec{B}^a$$

- No constraints on $\partial V$ accounted for $\longrightarrow$ $\Psi[\vec{A}]$ must be defined for $U(\vec{x}) \neq \text{const.}$ on $\partial V$
- Residual gauge dofs.: Throw out unphysical states (leading to gauge dependence) "First quantize, then constrain"

The corresponding operator must observe the commutation relations:

$$[A^{a,i}(\vec{x}), \Pi^{b,j}(\vec{x}\,')] = \mathrm{i}\delta^{ij}\delta^{ab}\delta^3(\vec{x}-\vec{x}\,')\,, \quad [\Pi^{a,i}(\vec{x}), \Pi^{b,j}(\vec{x}\,')] = 0\,, \quad \text{take e.g. } \vec{\Pi}^a = \frac{\delta}{\mathrm{i}\delta\vec{A}^a}$$

Hamiltonian density:

$$\mathcal{H} = \frac{1}{2}\left((\vec{E}^a)^2 + (\vec{B}^a)^2\right) = \frac{1}{2}\left(\left(g\frac{\delta}{\mathrm{i}\delta\vec{A}^a} - \frac{g^2}{8\pi^2}\theta\vec{B}^a\right)^2 + (\vec{B}^a)^2\right)$$

[e.g. Jackiw (1980)]

## Wave functional in gauge theory (temporal gauge $A_0 = 0$)

Since $[U^{(n)}, H] = 0$, can find states $\Psi_{\theta^{(i)}}[\vec{A}_{(U^{(1)})^n}] = \mathrm{e}^{\mathrm{i}n\theta^{(i)}}\Psi_{\theta^{(i)}}[\vec{A}]$

Wave functionals not properly normalizable because of summation over gauge redundancies:

$$\int \mathcal{D}\vec{A}\,\Psi^{(a)*}_{\theta^{(i)}}[\vec{A}]\Psi^{(b)}_{\theta^{(j)}}[\vec{A}] = \sum_{\nu=-\infty}^{\infty}\int_{0\le W[\vec{A}]<1}\mathcal{D}\vec{A}\,\mathrm{e}^{-\mathrm{i}(\theta^{(i)}-\theta^{(j)})(W[\vec{A}]+\nu)}\psi^{(a)*}_{\theta^{(i)}}[\vec{A}]\psi^{(b)}_{\theta^{(j)}}[\vec{A}]$$
$$=2\pi\delta(\theta^{(i)}-\theta^{(j)})\int_{0\le W[\vec{A}]<1}\mathcal{D}\vec{A}\,\mathrm{e}^{-\mathrm{i}(\theta^{(i)}-\theta^{(j)})W[\vec{A}]}\psi^{(a)*}_{\theta^{(i)}}[\vec{A}]\psi^{(b)}_{\theta^{(j)}}[\vec{A}]$$
$$=2\pi\delta(\theta^{(i)}-\theta^{(j)})\delta_{ab} \qquad \text{Bloch theorem}$$

*How about gauge fixing if we must sum over dedundancies?*

*What about gauge transformations with $U(\vec{x}) \neq \mathrm{const.}$ on $\partial V$?*

*What about the first postulate?*

Cf. $T^4$/lattice/finite $T$:

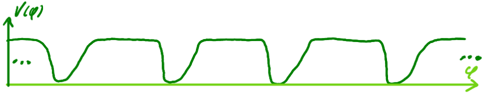
$$Z = \sum_a \int \mathcal{D}\vec{A}\,\Psi^{(a)*}_{\theta^{(i)}}[\vec{A}]\mathrm{e}^{-\beta H}\Psi^{(b)}_{\theta^{(i)}}[\vec{A}]$$

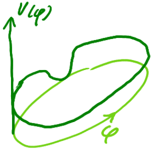
Not properly normalizable either

## Crystal or circle?

Functionals $\Psi_\theta(\vec{A})$ with above periodicity properties can be compared with Bloch states

Bloch states live on a crystal:
$\vec{A}_{U_4^{(1)}}$ is a *different* site than $\vec{A}$




In contrast: In gauge theory $\vec{A}_{U_4^{(1)}}$ is a *redundant* description of the configuration $\vec{A}$—corresponding to $\varphi \to \varphi + 2\pi n$ on a circle

On a crystal: Bloch states do not correpsond to normalized wave functions, these are rather wave packets made up of Bloch states. Packets, however, not translation (gauge) invariant

On a circle: Truncation of the inner product according to a single period leads to properly normalizable states, corresponding here to *gauge fixing* $\vec{A} \in \mathcal{A}$ so that each physical configuration appears one time and one time only:

$$\int_{\mathcal{A}} \underbrace{\mathcal{D}\vec{A}\, f_{\mathcal{A}}[\vec{A}]}_{\substack{\text{gauge invariant} \\ \text{under change of } \mathcal{A}}} \quad \Psi^{(a)*}_{\theta^{(i)}}[\vec{A}]\, \Psi^{(b)}_{\theta^{(j)}}[\vec{A}]$$

Note: Under gauge fixed inner product, $\Psi^{(a)}_{\theta^{(i)}}$, $\Psi^{(b)}_{\theta^{(j)}}$ no longer orthogonal for $\theta^{(i)} \neq \theta^{(j)}$

## Form of the wave functional & Constraining the Hilbert space

Require: Gauge invariance & $\frac{\delta}{\mathrm{i}\delta\vec{A}(\vec{x})}$ should remain Hermitian under restricted inner product

$\Longrightarrow \Psi^{(a)}[\vec{A}] \overset{(*)}{=} \Psi^{(a)}[\vec{A}]_{\mathrm{g.i.}} \exp(\mathrm{i}\varphi[\vec{A}])$ valid for *all* $U(\vec{x})$ (also nonconstant on boundary)

gauge invariant

independent of state $(a)$

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$\longrightarrow$ "Diagonalize" $H$:

$$\begin{aligned}
\Psi'[\vec{A}] &= \mathrm{e}^{-\mathrm{i}\theta W[\vec{A}]}\Psi[\vec{A}]\,,\\
H' &= \mathrm{e}^{-\mathrm{i}\theta W[\vec{A}]} H \mathrm{e}^{\mathrm{i}\theta W[\vec{A}]} = \frac{1}{2}\int \mathrm{d}^3x\,\mathrm{tr}\left[-g^2\frac{\delta^2}{\delta\vec{A}^2} + \vec{B}^2\right]\\
&= -\frac{g^2}{2}\sum\!\!\!\!\!\!\int_\sigma \frac{\delta^2}{\delta\vec{A}^2(\sigma)} + \frac{1}{2}\int \mathrm{d}^3x\,\mathrm{tr}\,\vec{B}^2\,,\ \sigma\in\{\sigma_{\text{gauge}},\sigma_{\|}\}
\end{aligned}$$

$$\frac{\delta}{\delta\vec{A}(\vec{x})}W[\vec{A}] = \frac{g}{8\pi^2}\vec{B}(\vec{x})$$

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**Only trivial one-dimensional representations of SU(2)**

$$\Psi[\vec{A}_U] = \mathrm{e}^{\mathrm{i}\varphi[\vec{A}_U]}\Psi[\vec{A}]\ \text{(eigenstate of } U\text{)},\qquad U_3 = U_2\, U_1$$

$$\mathrm{e}^{\mathrm{i}\varphi[\vec{A}_{U_3}]} = \mathrm{e}^{\mathrm{i}\varphi[\vec{A}_{U_2}]}\mathrm{e}^{\mathrm{i}\varphi[\vec{A}_{U_1}]} \Rightarrow \mathrm{e}^{\mathrm{i}\varphi[\vec{A}_{U_2U_1}]} - \mathrm{e}^{\mathrm{i}\varphi[\vec{A}_{U_1U_2}]} = 0$$

$\Rightarrow \Psi'[\vec{A}]$ is gauge invariant $(**)$

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Throw states not satisfying $(*,**)$ out of the Hilbert space $\to CP$ conserved

## Gauß' law in the constrained Hilbert space

For $\Omega(\vec{x})$ an infinitesimal generator of gauge transformations

$\longrightarrow$ Noether charge:

$$Q(\Omega) = \frac{1}{g}\int \mathrm{d}^3x\,\mathrm{tr}\left[\Pi^i(D^i\Omega)\right] = \int_V \mathrm{d}^3x\,\mathrm{tr}\left[\left(-E^i + \frac{g^2}{8\pi^2}\theta B^i\right) D^i\Omega\right]$$

$$= \int \mathrm{d}^3x\,\mathrm{tr}\left[\Omega D^i\left(E^i - \frac{g^2}{8\pi^2}\theta B^i\right)\right] + \int_{\partial V} \mathrm{d}a^i\,\mathrm{tr}\left[\Omega\left(-E^i + \frac{g^2}{8\pi^2}\theta B^i\right)\right]$$

For $\Omega(\vec{x}) = 0$ when $\vec{x} \in \partial V$ and since $\Psi'$ is gauge invariant

$\rightarrow$ Gauß' law: $\vec{D}\cdot\vec{E}\,\Psi'[\vec{A}] = 0$

Usually, the argument is made the other way around: Impose Gauß' law to throw states out of the Hilbert space [Jackiw (1980)] (automatic in Dirac formalism)

Since $[Q(\Omega), W[\vec{A}]] = 0$ for $\Omega(\vec{x}) = 0$ when $\vec{x} \in \partial V$ this also holds when $\Psi'[\vec{A}] \to \mathrm{e}^{\mathrm{i}\tilde{\theta}\,W[\vec{A}]}\Psi'[\vec{A}]$, so imposing Gauß' law does not fix $\tilde{\theta}$, does not tell us about large gauge transformations

## Nondiagonal basis

Redefining derivatives w.r.t. $\vec{A}$ as

$$\vec{D}_{\vec{A}} \Psi[\vec{A}] = \mathrm{i} \left( \frac{\delta}{\mathrm{i}\delta \vec{A}} - \theta \frac{g}{8\pi^2} \vec{B} \right) \Psi[\vec{A}]$$

corresponds to a canonical transformation of the momentum operator.

Induces translation as

$$T[\Delta \vec{A}]\, \Psi[\vec{A}] = \mathrm{e}^{-\mathrm{i}\theta\left( W[\vec{A}+\Delta\vec{A}] - W[\vec{A}]\right)} \Psi[\vec{A} + \Delta \vec{A}]$$

gauge invariant under shift with $-\mathrm{i}\delta/\delta\vec{A}$

For a shift $\Delta \vec{A}_{\text{gauge}}$ corresponding to a *general* gauge transformation:

$$T[\Delta \vec{A}_{\text{gauge}}]\, \Psi[\vec{A}] = \Psi[\vec{A}] \quad \text{if} \quad \Psi[\vec{A}] = \mathrm{e}^{\mathrm{i}\theta\, W[\vec{A}]} \Psi_{\text{g.i.}}[\vec{A}]$$

$\theta$ in $\Psi_\theta$ is pinned to $\theta$ in $H$ so that $CP$ is conserved

## Conclusion

There is no $CP$ violation in QCD.

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Flaws in the standard calculation and resolutions:

- **Taking $T \to \infty$ after summing over sectors corresponds to an inequivalent deformation of the integration contour**

  Maintain Cauchy contour and order of limits

- **No point compactification/topology in temporal gauge w/o extra constraint $U(\vec{x}) \underset{|\vec{x}|\to\infty}{\longrightarrow}$ const.**

  Define $\Psi$ for *all* temporal gauges

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> These two points are the only way in which we differ from the standard lore.

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**THANK YOU!**

# Effective chiral Lagrangian ($\chi$PT)

$$U = U_0 \mathrm{e}^{\frac{\mathrm{i}}{f_\pi}\Phi} \qquad U_0: \textit{ chiral condensate}$$

$$\Phi = \begin{bmatrix} \pi^0 + \eta' & \sqrt{2}\,\pi^+ \\ \sqrt{2}\,\pi^- & -\pi^0 + \eta' \end{bmatrix}$$

Chiral Lagrangian (lowest order terms) inherits "spurious" symmetries:

$$\mathcal{L} = \frac{f_\pi^2}{4}\mathrm{Tr}\,\partial_\mu U \partial^\mu U^\dagger + \frac{f_\pi^2 B_0}{2}\mathrm{Tr}(M\,U + U^\dagger M^\dagger) + |\lambda|\mathrm{e}^{-\mathrm{i}\xi} f_\pi^4 \det U + |\lambda|\mathrm{e}^{\mathrm{i}\xi} f_\pi^4 \det U^\dagger$$
$$+\mathrm{i}\bar{N}\not{\partial}N - \left(m_N \bar{N}\tilde{U}P_\mathrm{L}N + \mathrm{i}c\bar{N}\tilde{U}^\dagger\not{\partial}P_\mathrm{L}\tilde{U}N + d\bar{N}\tilde{M}^\dagger P_\mathrm{L}N + e\bar{N}\tilde{U}\tilde{M}\tilde{U}P_\mathrm{L}N + \mathrm{h.c.}\right)$$

$$M = \mathrm{diag}\{m_u \mathrm{e}^{\mathrm{i}\alpha_u}, m_d \mathrm{e}^{\mathrm{i}\alpha_d}, m_s \mathrm{e}^{\mathrm{i}\alpha_s}\}$$

$\tilde{M}$, $\tilde{U}$ reduced to subspace $(u, d)$

nucleon doublet $N = \begin{pmatrix} p \\ n \end{pmatrix}$

Effective interaction $\propto \det U$ cannot be quantitatively reliably handled in $\chi$PT but yet represents pattern of broken axial symmetry.

# Neutron electric dipole moment

$$\mathrm{i}\mathcal{M} = -2\mathrm{i}D(q^2)\varepsilon^*_\mu(\vec{q})\bar{u}_{s'}(\vec{p}')\frac{\mathrm{i}}{4}[\gamma^\mu,\gamma^\nu]\,q_\nu \mathrm{i}\gamma_5 u_s(\vec{p})$$

$$\rightarrow \mathcal{L}_{\text{eff}} = D(0)\bar{n}\,\underbrace{F_{\mu\nu}\frac{\mathrm{i}}{4}[\gamma^\mu,\gamma^\nu]\mathrm{i}\gamma_5}_{\subset \vec{S}\cdot\vec{E}}\,n$$


- $\chi$PT value: $d_n = 3.2 \times 10^{-16}(\xi + \bar{\alpha})\,e\,\mathrm{cm}$
- Experimental bound: $|d_n| < 1.8 \times 10^{-26}\,e\,\mathrm{cm}$ (90% c.l.) [nEDM/PSI (2020)]
- Calculations e.g. of neutron EDM implicitly *assume* $\xi = \theta$
  [e.g. Baluni (1979); Crewther, Di Vecchia, Veneziano, Witten (1979)]
- However $\xi = -\bar{\alpha}$ also perfectly valid by arguments used to this end
- Another signature—weaker bounds: $\eta' \to \pi\pi$

# Fermion correlations

The effective vertex generates the following correlation functions at tree level:

$$\langle\prod_{j=1}^{N_f}\psi_j(x_j)\bar{\psi}_j(x_j')\rangle_{\text{inst}}=\left(\mathrm{e}^{-\mathrm{i}\xi}\prod_{j=1}^{N_f}P_{\mathrm{L}j}+\mathrm{e}^{\mathrm{i}\xi}\prod_{j=1}^{N_f}P_{\mathrm{R}j}\right)\bar{H}(x_1,\ldots,x_1',\ldots)$$

Goal: Compute correlation function and compare with EFT answer above to fix $\xi$

Cf. leading contribution to two-point function

$$\begin{aligned}\langle\psi_i(x)\psi_j(x')\rangle =&\mathrm{i}S_{0\text{inst }ij}(x,x')\\ \mathrm{i}S_{0\text{inst }ij}(x,x') =&(-\gamma^\mu\partial_\mu+\mathrm{i}m_i\mathrm{e}^{-\mathrm{i}\alpha_i\gamma^5})\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}p(x-x')}\frac{\delta_{ij}}{p^2-m_i^2+\mathrm{i}\epsilon}\end{aligned}$$

So $\xi=\theta/\xi=-\bar{\alpha}$ implies $CP$-violation/no $CP$-violation

Only one explicit calculation based on dilute instanton gas (DIGA) finding $\xi=\theta$
['t Hooft (1986)]

## Fermion correlations

- Obtain correlation functions from Green's functions in fixed background of instantons and anti-instantons
- Interfere all instanton configurations
  - First, within one topological sector
  - Then over the different sectors

DIGA to dermine $CP$ phase of 't Hooft vertex—not quantitatively accurate for actual QCD

**Green's function in $n$-instanton, $\bar{n}$-anti-instanton background (DIGA)**

$$\mathrm{i}S_{n,\bar{n}}(x,x') \approx \mathrm{i}S_{0\mathrm{inst}}(x,x') + \sum_{\bar{\nu}=1}^{\bar{n}} \frac{\hat{\psi}_{0\mathrm{L}}(x-x_{0,\bar{\nu}})\hat{\psi}^{\dagger}_{0\mathrm{L}}(x'-x_{0,\bar{\nu}})}{m\mathrm{e}^{-\mathrm{i}\alpha}} + \sum_{\nu=1}^{n} \frac{\hat{\psi}_{0\mathrm{R}}(x-x_{0,\nu})\hat{\psi}^{\dagger}_{0\mathrm{R}}(x'-x_{0,\nu})}{m\mathrm{e}^{\mathrm{i}\alpha}}$$

$\hat{\psi}_{0\mathrm{L,R}}$: 't Hooft zero modes

Comments:

- For small masses, zero modes dominate close to the cores of instantons, far away from instantons the solution goes to the zero-instanton configuration [Diakonov, Petrov (1986)]
- Alignment of phase $\alpha$ between Lagrangian mass and instanton-induced $\chi$SB $\longrightarrow$ No indication of $CP$ violation here

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cf.

$$\mathrm{i}S_{0\mathrm{inst}}(x,x') = (-\gamma^{\mu}\partal_{\mu} + \mathrm{i}m\mathrm{e}^{-\mathrm{i}\alpha\gamma^5}) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}p(x-x')} \frac{1}{p^2 - m^2 + \mathrm{i}\epsilon}$$

C…

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# Interferences within the topological sectors

Within a topological sector, interfere/sum/integrate over

- all instanton/anti-instanton numbers $n + \bar{n}$ with $\Delta n = n - \bar{n}$ fixed
- locations of all instantons/anti-instantons
- remaining collective coordinates

$\longrightarrow$ Dilute instanton gas approximation (skip technicalities)

Can also obtain coincident fermion correlations using the index theorem and anomalous current only

# Correlation function for fixed $\Delta n$

$$\langle\psi(x)\bar\psi(x')\rangle_{\Delta n} = \sum_{\substack{\bar n, n\geq 0\\ n-\bar n=\Delta n}} \frac{1}{\bar n! n!}\left[\bar h(x,x')\left(\frac{\bar n}{m e^{-i\alpha}}P_L + \frac{n}{m e^{i\alpha}}P_R\right)(VT)^{\bar n+n-1} + iS_{0\mathrm{inst}}(x,x')(VT)^{\bar n+n}\right] \times (i\kappa)^{\bar n+n}(-1)^{n+\bar n}e^{i\Delta n(\alpha+\theta)}$$

$$= \left[\left(e^{i\alpha}I_{\Delta n+1}(2i\kappa VT)P_L + e^{-i\alpha}I_{\Delta n-1}(2i\kappa VT)P_R\right)\frac{i\kappa}{m}\bar h(x,x') + I_{\Delta n}(2i\kappa VT)iS_{0\mathrm{inst}}(x,x')\right] \times (-1)^{\Delta n}e^{i\Delta n(\alpha+\theta)}$$

Instantons per spacetime volume: $i\kappa \propto e^{-S_E}$

$\chi$SB rank-two spinor-tensor from integrating quark zero-modes over the locations of the instantons: $\bar h(x,x')$

Modified Bessel function: $I_\nu(x)$

**Sum is dominated by particular value of $n \approx \bar n$:** [Diakonov, Petrov (1986)]

$$\langle n\rangle = \frac{\sum_{n=0}^{\infty} n\frac{(\kappa VT)^n}{n!}}{\sum_{n=0}^{\infty}\frac{(\kappa VT)^n}{n!}} = \kappa VT\,, \quad \frac{\sqrt{\langle(n-\langle n\rangle)^2\rangle}}{\langle n\rangle} = \frac{1}{\sqrt{\kappa VT}}\,, \quad \text{cf. } \lim_{x\to\infty}\frac{I_{\Delta n}(ix\,e^{-i0^+})}{I_{\Delta n'}(ix\,e^{-i0^+})} = 1$$

$\longrightarrow$ No relative $CP$ phase between mass and instanton induced breaking of $\chi$ral symmetry—alignment in infinite-volume limit

Correspondingly, partition function for fixed $\Delta n$: [cf. Leutwyler, Smilga (1992)]

$$Z_{\Delta n} = I_{\Delta n}(2\mathrm{i}\kappa VT)\,(-1)^{\Delta n}\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$$

**Note:** The topological phase $\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$ multiplies $\langle\psi(x)\bar{\psi}(x')\rangle_{\Delta n}$ and $Z_{\Delta n}$ entirely—not just the contributions induced by instantons.

Other correlation functions ($n$ point, stress-energy, for some observer,...) are calculated from the Feynman diagram with the Green's function in the $n$ instanton, $\bar{n}$ anti-instanton background.
Then it remains to average over $n$, $\bar{n}$, locations and remaining collective coordinates.

There is no $CP$ violation/misalignment of phases to this end. It remains to consider the interference between the topological sectors.

## Interferences among topological sectors (are immaterial)

Topological quantization $\leftrightarrow$ Interference between sectors for $VT \to \infty$

### Fermion correlator

$$\langle \psi(x)\bar{\psi}(x')\rangle = \lim_{\substack{N\to\infty\\ N\in\mathbb{N}}} \lim_{VT\to\infty} \frac{\sum_{\Delta n=-N}^{N} \langle \psi(x)\bar{\psi}(x')\rangle_{\Delta n}}{\sum_{\Delta n=-N}^{N} Z_{\Delta n}}$$

$$= \mathrm{i}S_{0\mathrm{inst}}(x,x') + \mathrm{i}\kappa \bar{h}(x,x') m^{-1} \mathrm{e}^{-\mathrm{i}\alpha\gamma^5} \quad \text{(same as for fixed } \Delta n\text{)}$$

Recall: $\mathrm{i}S_{0\mathrm{inst}}(x,x') = (-\gamma^\mu\partial_\mu + \mathrm{i}m\mathrm{e}^{-\mathrm{i}\alpha\gamma^5}) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}p(x-x')} \frac{1}{p^2 - m^2 + \mathrm{i}\epsilon}$

$\longrightarrow$ No relative $CP$-phase between mass and instanton term

$\longrightarrow \xi = -\alpha$

$\longrightarrow$ $CP$ is conserved

# Limits ordered the other way around

First sum over all $\Delta n$ as well:

$$\sum_{\bar{n},n\geq 0} \frac{1}{\bar{n}!n!} \left[ \bar{h}(x,x')(\bar{n}\, m^{-1} e^{i\alpha} P_{\rm L} + n\, m^{-1} e^{-i\alpha} P_{\rm R}) \,(VT)^{\bar{n}+n-1} + iS_{0\rm inst}(x,x')\,(VT)^{\bar{n}+n} \right] \times (-mi\kappa)^{\bar{n}+n} e^{i\Delta n(\alpha+\theta)}$$

$$= \left[ -\left(e^{-i\theta} P_{\rm L} + e^{i\theta} P_{\rm R}\right) \frac{i\kappa}{m} \bar{h}(x,x') + iS_{0\rm inst}(x,x') \right] e^{-2i\kappa\, VT \cos(\alpha+\theta)}$$

$$Z \to \sum_{n,\bar{n}} \frac{1}{n!\bar{n}!} (-i\kappa\, VT)^{\bar{n}+n} e^{-i(\bar{n}-n)(\alpha+\theta)} = e^{-2i\kappa\, VT \cos(\alpha+\theta)}$$

Then, $VT \to \infty$ trivial as $VT$-dependence cancels

$\longrightarrow$ Relative $CP$ phase leading to $CP$-violating observables

However: Changing the order does not correspond to a nonsingular integration contour.