# Expansion by Regions for Feynman integrals:
## from wide–angle to spacelike collinear kinematics

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# *Asymptotic expansion of Feynman integrals*

- Evaluating multi-loop Feynman integrals poses significant challenges.

- For Feynman integrals with multiple scales in the external kinematics, a natural idea is to consider the asymptotic expansion.

$$\mathcal{A} \sim \mathcal{A}_0 + \left(\frac{\Lambda_{\text{small}}}{\Lambda_{\text{large}}}\right) \mathcal{A}_1 + \left(\frac{\Lambda_{\text{small}}}{\Lambda_{\text{large}}}\right)^2 \mathcal{A}_2 + \ldots$$

- Moreover, asymptotic expansion offers insights into the intricate infrared structure of gauge theory.

- Among various techniques of doing asymptotic expansions, one usual way is the "*expansion by regions*" *(method of regions)*.

# *The expansion by regions (EbR)*

***Statement:*** $\exists$ "Regions" : $R_1,\ R_2,\ \ldots,\ R_n,$ ***such that***

$$\mathcal{I} = \mathcal{I}^{(R_1)} + \mathcal{I}^{(R_2)} + \cdots + \mathcal{I}^{(R_n)}.$$

***The original integral, $\mathcal{I}$ , can be restored by summing over contributions from the regions.***

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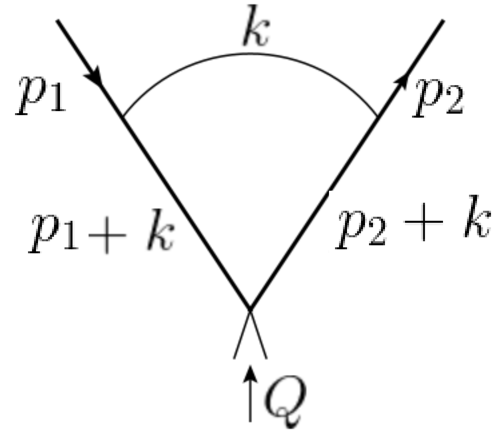
For each term on the RHS, the integrand is modified (Taylor exp.) while the integration measure is unchanged (-∞ ~ +∞).

- Dimensional regularization is taken.
- Scaleless integrals = 0. Namely, $\int d^D k (k^2)^\alpha = 0.$

# *The expansion by regions (EbR)*

Example: one-loop Sudakov form factor

(Becher, Broggio, Ferroglia 2014)




The on-shell limit kinematics

$$p_1^\mu \sim Q\,(\underset{+}{1},\ \underset{-}{\lambda},\ \underset{\perp}{\lambda^{1/2}}),\qquad p_2^\mu \sim Q\,(\underset{+}{\lambda},\ \underset{-}{1},\ \underset{\perp}{\lambda^{1/2}})$$

$$p_1^2/Q^2 \sim p_2^2/Q^2 \sim \boxed{\lambda \to 0}$$

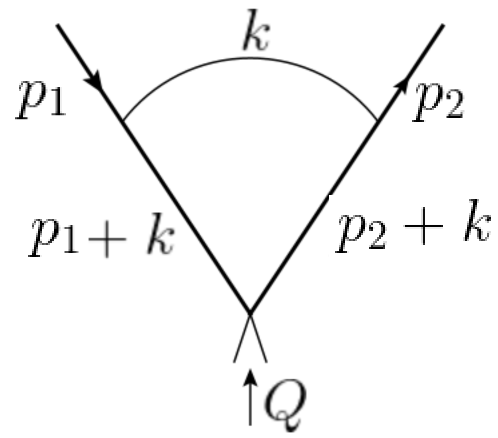
The Feynman integral

$$\mathcal{I} = \mathcal{C} \cdot \int d^D k \frac{1}{(k^2 + i0)\,((p_1 + k)^2 + i0)\,((p_2 + k)^2 + i0)}$$

can be evaluated directly, or, we can apply the EbR.

# *The expansion by regions (EbR)*

Step 1: identify **4** regions in total:




**Hard region:** $k^\mu \sim Q(1,1,1)$

**Collinear-1 region:** $k^\mu \sim Q(1,\lambda,\lambda^{1/2})$

**Collinear-2 region:** $k^\mu \sim Q(\lambda,1,\lambda^{1/2})$

**Soft region:** $k^\mu \sim Q(\lambda,\lambda,\lambda)$

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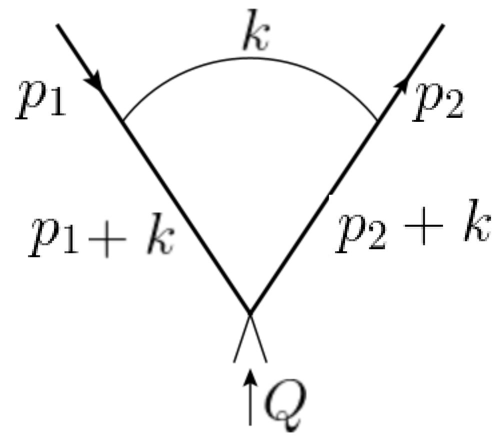


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Step 2: perform expansion around each region:

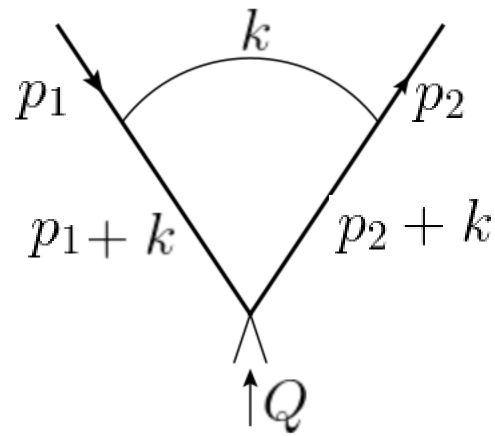
$$\mathcal{I}_H = \mathcal{C}\cdot\int d^D k \frac{1}{(k^2+i0)\,(k^2+2p_1\cdot k+i0)\,(k^2+2p_2\cdot k+i0)} + \dots$$

$$\mathcal{I}_{C_1} = \mathcal{C}\cdot\int d^D k \frac{1}{(k^2+i0)\,((p_1+k)^2+i0)\,(2p_2\cdot k+i0)} + \dots$$

$$\mathcal{I}_{C_2} = \mathcal{C}\cdot\int d^D k \frac{1}{(k^2+i0)\,(2p_1\cdot k+i0)\,((p_2+k)^2+i0)} + \dots$$

$$\mathcal{I}_S = \mathcal{C}\cdot\int d^D k \frac{1}{(k^2+i0)\,(2p_1\cdot k+p_1^2+i0)\,(2p_2\cdot k+p_2^2+i0)} + \dots$$

# *The expansion by regions (EbR)*




Step 1: ....

Step 2: ....

Step 3: sum over their contributions, and the original integral is reproduced:

$$\mathcal{I} = \mathcal{I}_H + \mathcal{I}_{C_1} + \mathcal{I}_{C_2} + \mathcal{I}_S = \frac{1}{Q^2}\left(\ln\frac{Q^2}{(-p_1^2)}\ln\frac{Q^2}{(-p_2^2)} + \frac{\pi^2}{3} + \dots\right)$$

This equality holds to **all** orders of λ!

More examples are presented in papers in the recent 20 years.

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***The original integral, $\mathcal{I}$ , can be restored by summing over contributions from the regions.***

The EbR works for all known examples so far.

However, two fundamental questions remain unanswered today:

- How to prove the EbR? (Why is this technique true?)
- How to identify the regions? (How to use this technique?)

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- How **to identify the regions**? (How to use this technique?)

**our focus today**

# *Euclidean space: region structure understood*

- Actually, all-order results have been established for Feynman integrals with Euclidean kinematics.
  - For original papers, see (Tkachov 1980s; Smirnov 1990)
  - A detailed demonstration can be found in Smirnov's book "Applied Asymptotic Expansions in Momenta and Masses"

- Each region is characterized by a certain subgraph carrying large loop momenta.

***Expansion by Regions***

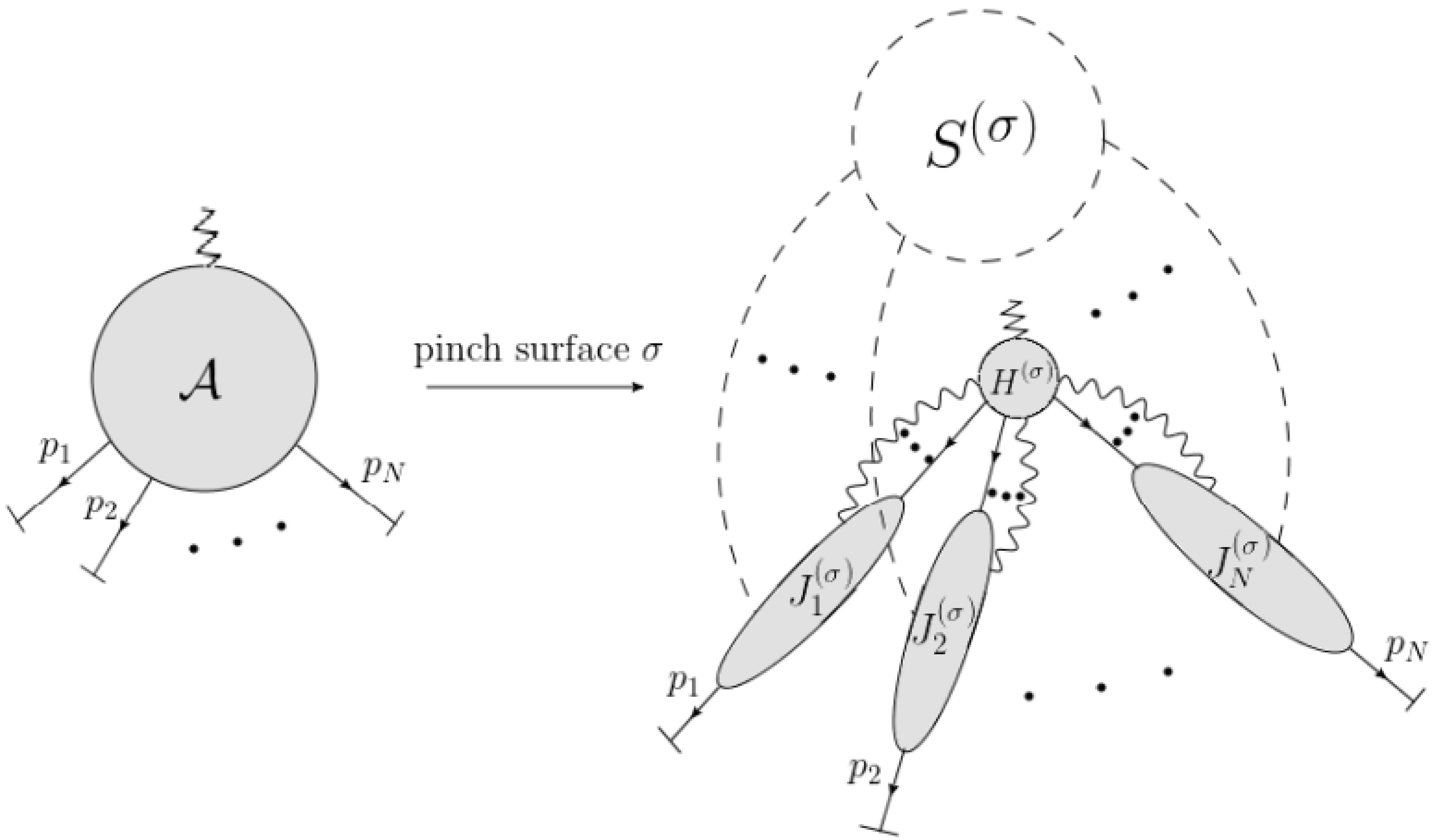
== "***Expansion by Subgraphs***".

- However, it would be highly nontrivial to extend this statement to Minkowski kinematics.

# *The expansion by regions (EbR)*

Why is such an extension nontrivial?

- ***Complicated infrared structure in Minkowski space:***

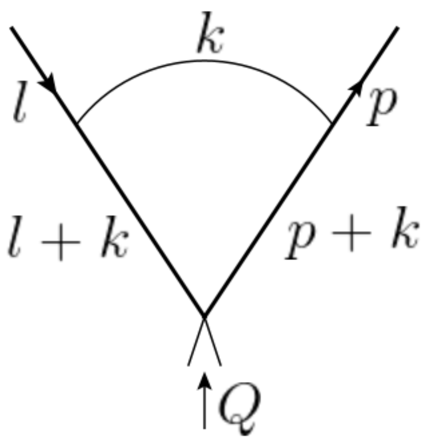


# *The expansion by regions (EbR)*

Why is such an extension nontrivial?

- ***Complicated mode structure near singularities:***

Recall that at one-loop level, we have




Hard region: $k^\mu \sim Q(1,1,1)$

Collinear-1 region: $k^\mu \sim Q(1,\lambda,\lambda^{1/2})$

Collinear-2 region: $k^\mu \sim Q(\lambda,1,\lambda^{1/2})$

Soft region: $k^\mu \sim Q(\lambda,\lambda,\lambda)$

***Question: Can other types of modes, such as***

$$(\lambda,\lambda^2,\lambda^{1/2}),\quad (\lambda^2,\lambda^2,\lambda^2),\ \cdots$$

***become relevant at higher loop orders?***

# *Progress on identifying the regions*

***Progress from 2010: a geometric approach to determine the regions systematically.*** *(Pak & Smirnov 2010; Jantzen, Smirnov, Smirnov 2012)*

- Each Feynman integral corresponds to an (N+1)-dim polytope.
- The regions corresponds to the **lower facets** (certain types of boundaries) of this polytope.
- For each lower facet, its normal vector (called "region vectors") explicitly shows the scaling of the Lee-Pomeransky parameters in the corresponding region.

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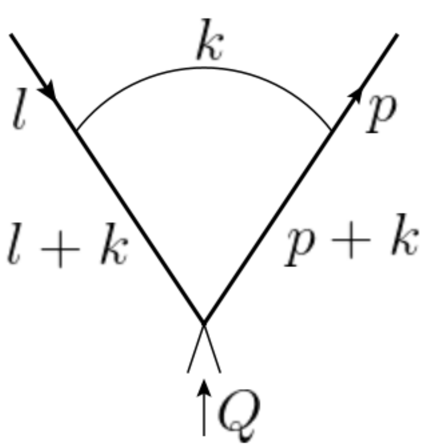



$$(0,0,0,1) \;\rightarrow\; x_1 \sim x_2 \sim x_3 \sim \lambda^0 \quad \text{(hard region)}$$
$$(-1,0,-1,1) \;\rightarrow\; x_1 \sim x_3 \sim \lambda^{-1},\; x_2 \sim \lambda^0 \quad \text{(collinear-1 region)}$$
$$(0,-1,-1,1) \;\rightarrow\; x_2 \sim x_3 \sim \lambda^{-1},\; x_1 \sim \lambda^0 \quad \text{(collinear-2 region)}$$
$$(-1,-1,-2,1) \;\rightarrow\; x_1 \sim x_2 \sim \lambda^{-1},\; x_3 \sim \lambda^{-2} \quad \text{(soft region)}$$

# *Identifying regions from polytopes*

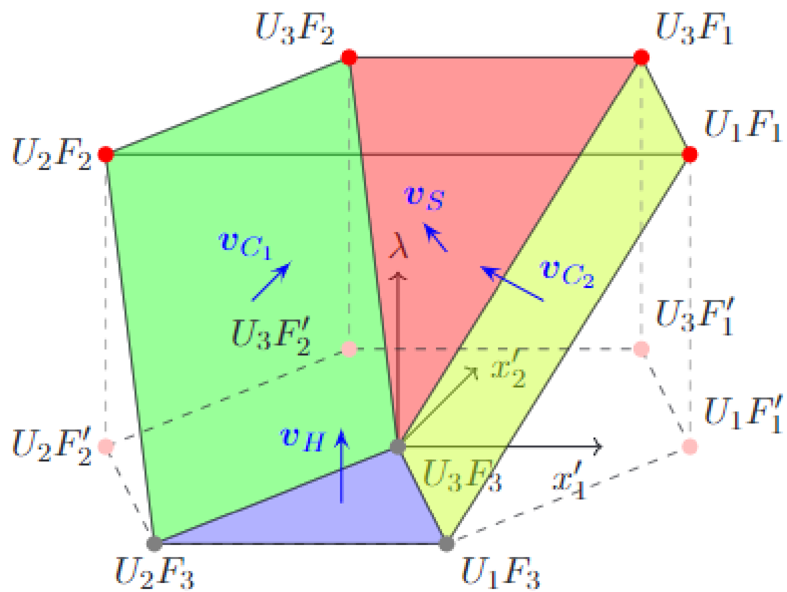
- There have been computer codes based on this approach:

  Asy2 *(Pak & A.Smirnov 2010; Jantzen, A.Smirnov, V.Smirnov 2012)*

  ASPIRE *(Ananthanarayan, Pal, Ramanan, Sarkar 2018)*

  pySecDec

  *(Heinrich, Jahn, Jones, Kerner, Langer, Magerya, Poldaru, Schlenk, Villa 2021)*



# *Identifying regions from polytopes*

However, there is still much to improve.

- All-loop-order results may not be directly available.

  Note that dim(polytope) = #(propagators)+1.

- The output of this approach describes regions in parameter space, while physically we are interested in their momentum-space interpretation.

- This geometric approach may miss some regions, which are "hidden" inside the polytope.
  - This problem has been noticed in ***(Jantzen, A.Smirnov, V.Smirnov 2012)***, and the geometric approach has been modified accordingly (Asy → Asy2).
  - However, such modifications work only for relatively simple cases.

# *More recent progress*

***Progress from 2022: all–order region analysis in both parameter & momentum space.***

*(Gardi, Herzog, Jones, Ma, Schlenk 2022; Ma 2023; Gardi, Herzog, Jones, Ma 2024)*

- Two types of regions: "**facet regions**" and "**hidden regions**".
- For a given Feynman integral, most regions are facet regions, which correspond to lower facets of the polytope.
- For certain types of asymptotic expansions, we have understood the facet region structure to all loop orders.
- For massless scattering, facet regions all feature a single and connected subgraph, exchanging an off-shell (hard or Glauber) momentum.

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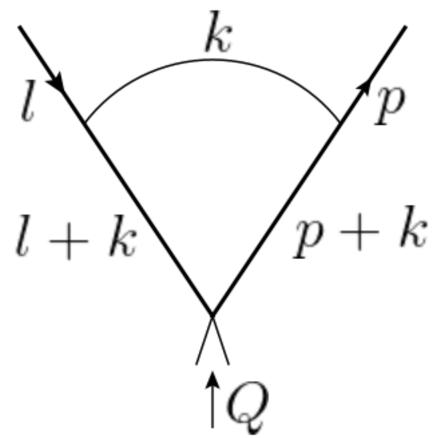
*(Gardi, Herzog, Jones, Ma, Schlenk 2022; Ma 2023; Gardi, Herzog, Jones, Ma 2024)*

- Two types of regions: "**facet regions**" and "**hidden regions**".
- Additionally, there may be hidden regions residing within the interior of the polytope.
- For each given graph, we have developed an algorithm telling whether or not any hidden regions might exist.
- For massless scattering, hidden regions all feature multiple and disconnected subgraphs, each exchanging an off-shell (hard or Glauber) momentum.

# Simplest examples

- **Facet regions**
  - The hard, collinear-1, collinear-2, and soft regions for the Sudakov form factor.

Hard region: $k^{\mu} \sim Q(1,1,1)$
Collinear-1 region: $k^{\mu} \sim Q(1,\lambda,\lambda^{1/2})$
Collinear-2 region: $k^{\mu} \sim Q(\lambda,1,\lambda^{1/2})$
Soft region: $k^{\mu} \sim Q(\lambda,\lambda,\lambda)$



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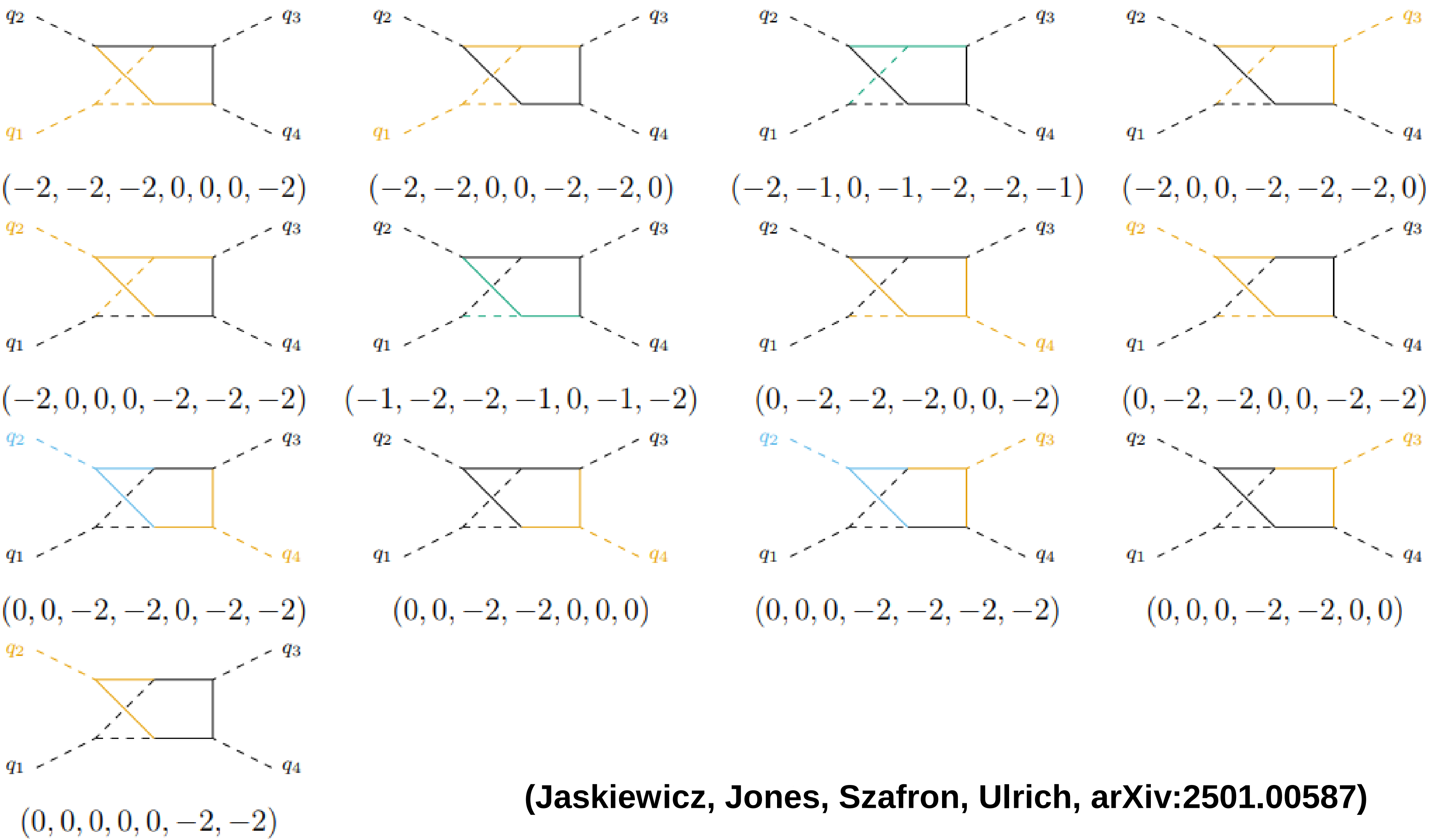
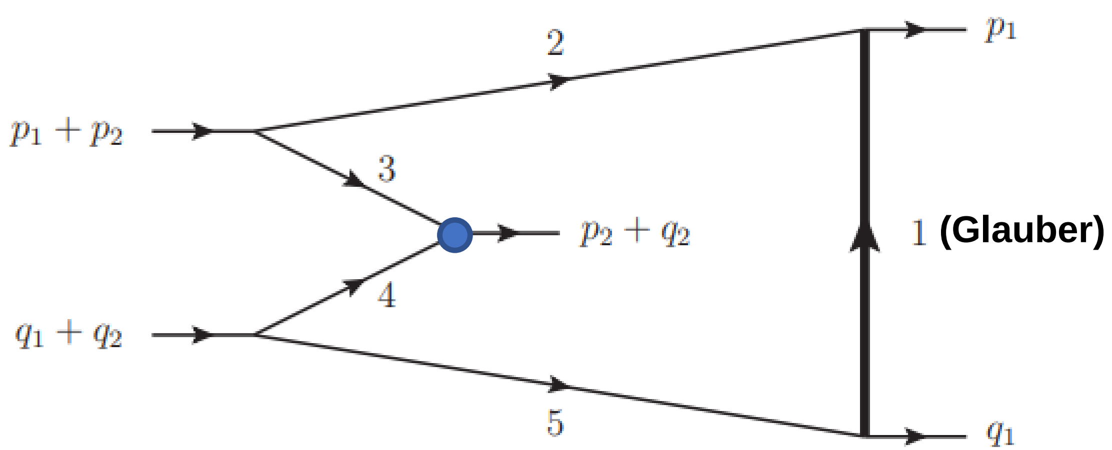



**(Jaskiewicz, Jones, Szafron, Ulrich, arXiv:2501.00587)**

# *Simplest examples*

- **Hidden regions**
  - The Glauber region in the 1-loop 5-point graph:




(Jantzen, A.Smirnov, V.Smirnov, arXiv:1206.0546)

# *Simplest examples*

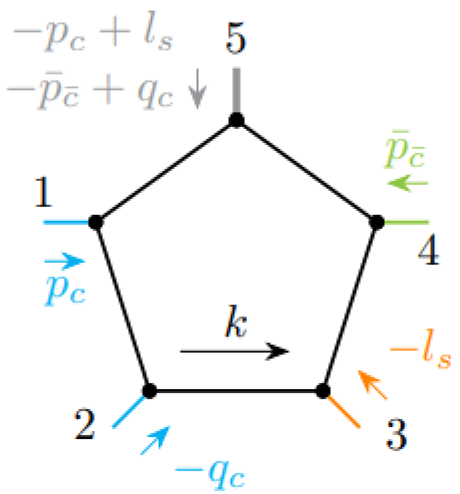
- **Hidden regions**

MITP-24-066
August 19, 2024

**Factorization restoration through Glauber gluons**

Thomas Becher[a], Patrick Hager[b], Sebastian Jaskiewicz[a], Matthias Neubert[b,c], and Dominik Schwienbacher[a]
[a] *Institut für Theoretische Physik & AEC, Universität Bern, Sidlerstrasse 5, CH-3012 Bern, Switzerland*
[b] *PRISMA+ Cluster of Excellence & MITP, Johannes Gutenberg University, 55099 Mainz, Germany*
[c] *Department of Physics, LEPP, Cornell University, Ithaca, NY 14853, U.S.A.*

We analyze the low-energy dynamics of gap-between-jets cross sections at hadron colliders, for which phase factors in the hard amplitudes spoil collinear cancellations and lead to double ("super-leading") logarithmic behavior. Based on a method-of-regions analysis, we identify three-loop contributions from perturbative active-active Glauber-gluon exchanges with the right structure to render the cross section consistent with PDF factorization below the gap veto scale. The Glauber contributions we identify are unambiguously defined without regulators beyond dimensional regularization.




$$\mathcal{F} = -\underbrace{x_1 x_3\, s_{23}}_{\lambda^{-3}} - \underbrace{x_1 x_4\, s_{51}}_{\lambda^{-3}} - \underbrace{x_3 x_5\, s_{45}}_{\lambda^{-3}} - \underbrace{x_4 x_5\, m^2}_{\lambda^{-3}} - \underbrace{x_2 x_4\, s_{34}}_{\lambda^{-2}} - \underbrace{x_2 x_5\, s_{12}}_{\lambda^{-2}} \,. \qquad (15)$$

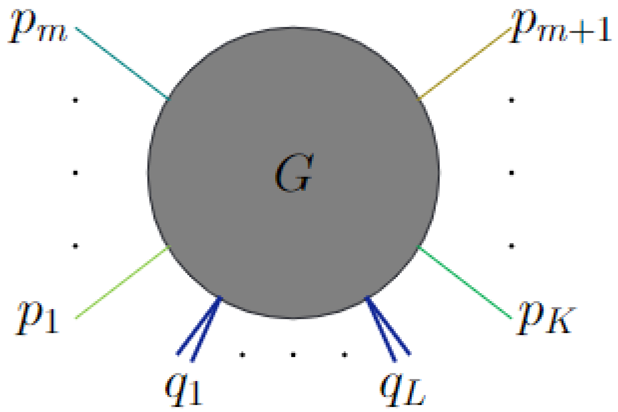
The first four terms constitute the leading power contribution, which using (8) can be factorized in the form

$$\mathcal{F} = \underbrace{\left(-q_c^- x_1 + (p_c^- - q_c^-)\, x_5\right)}_{\lambda^{-2}} \underbrace{\left(l_s^+ x_3 - \bar{p}_{\bar{c}}^+ x_4\right)}_{\lambda^{-1}} \,. \qquad (16)$$

# Wide-angle kinematics

# The "on–shell expansion"

- Let's consider the following asymptotic expansion:




massless

$$p_i^2 \sim \lambda Q^2 \quad (i = 1, \ldots, K), \quad q_j^2 \sim Q^2 \quad (j = 1, \ldots, L), \quad p_{i_1} \cdot p_{i_2} \sim Q^2 \quad (i_1 \neq i_2).$$

small virtuality — large virtuality — wide-angle scattering

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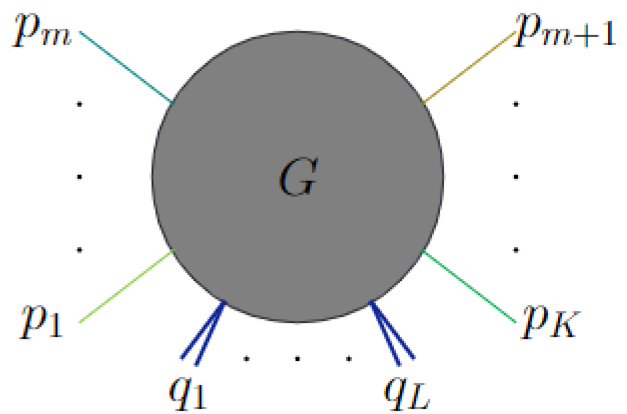


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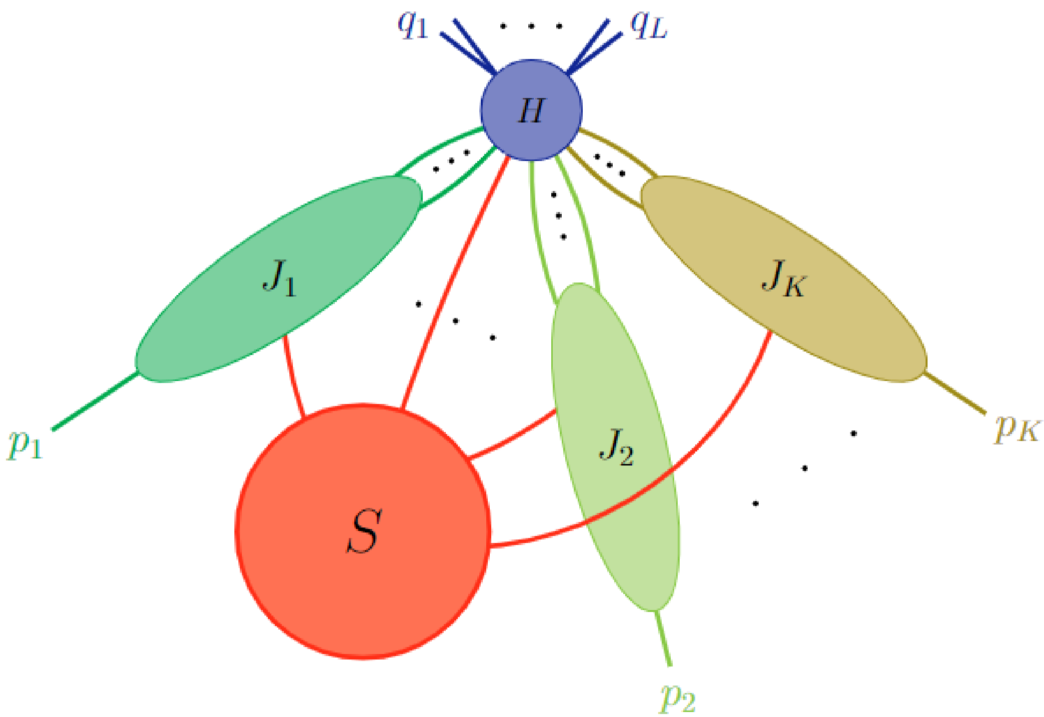
small virtuality — large virtuality — wide-angle scattering

- ***Result: the possibly relevant modes are***

$$k_H^\mu \sim Q(1,1,1), \quad k_{C_i}^\mu \sim Q(1,\lambda,\lambda^{1/2}), \quad k_S^\mu \sim Q(\lambda,\lambda,\lambda).$$

# *Facet regions in the on–shell expansion*

- More precisely, the general structure of each facet region is




$$k_H^\mu \sim Q(1,1,1),$$

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$$k_S^\mu \sim Q(\lambda,\lambda,\lambda).$$

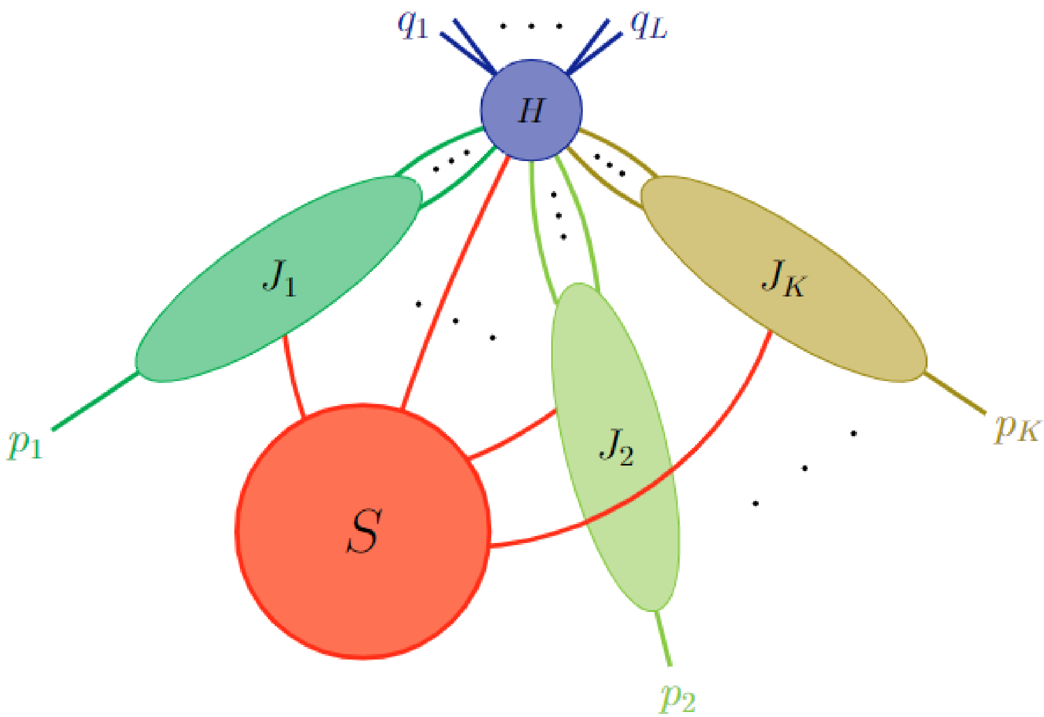
with additional requirements on the subgraphs $H$, $J$, and $S$.

This conclusion was proposed in [Gardi, Herzog, Jones, YM, Schlenk, JHEP07(2023)197],

and later proved in [YM, JHEP09(2024)197].

# Some remarks

1. This picture is natural




$$k_H^\mu \sim Q(1,1,1),$$

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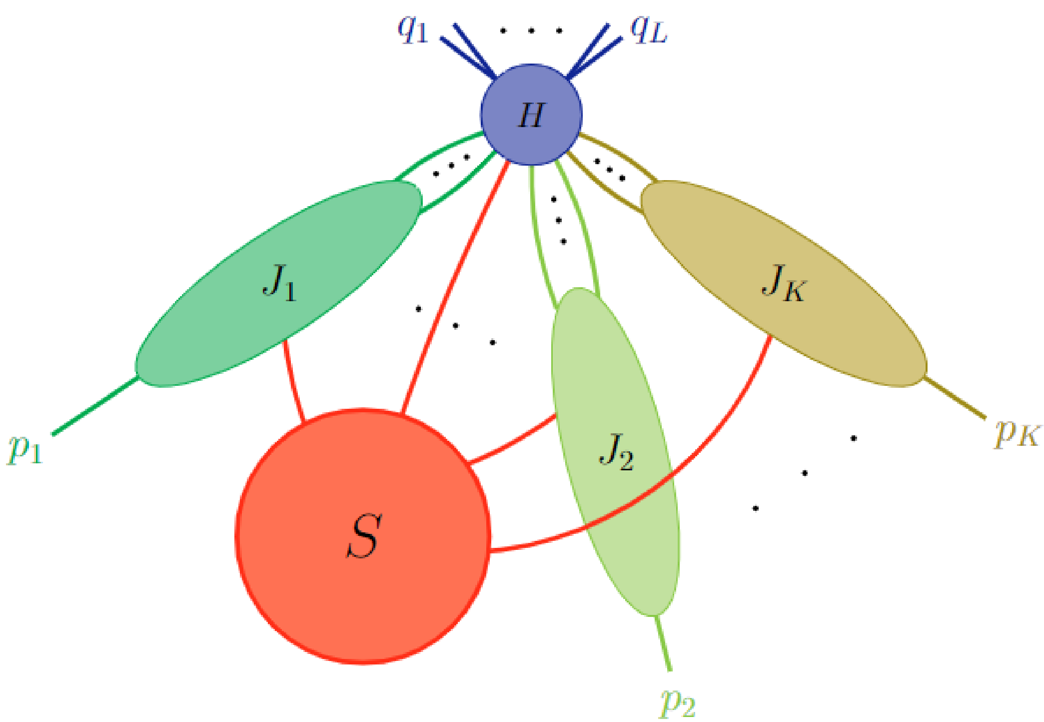
because it describes neighborhoods of the singularities of the integrand.

# *Some remarks*

2. This picture is highly nontrivial

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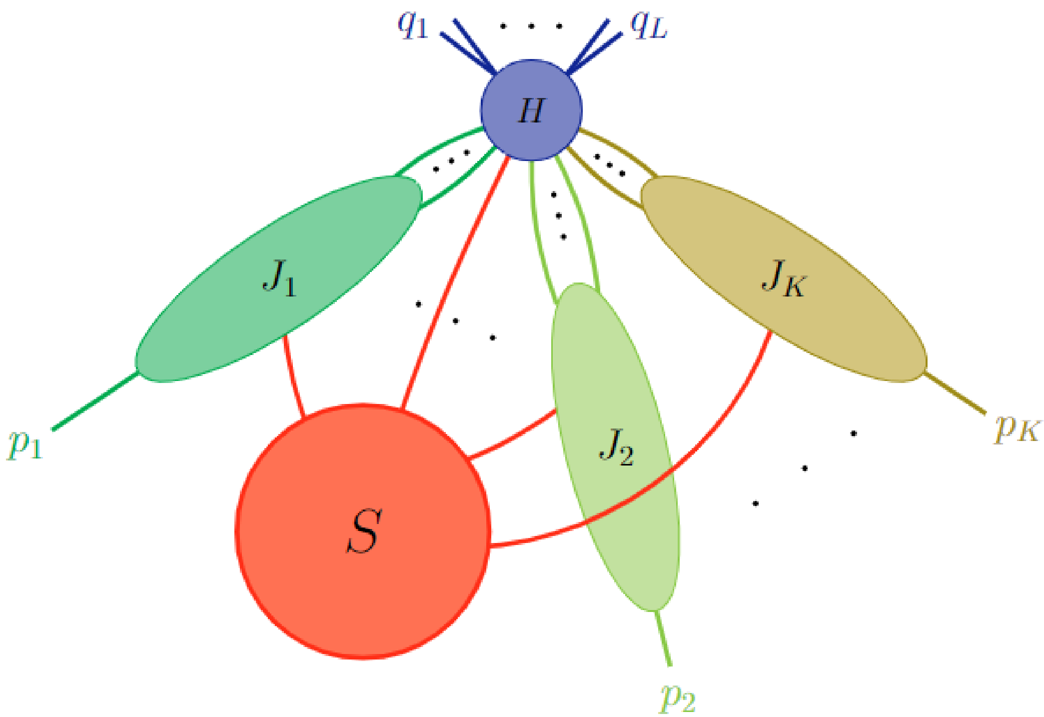
because the small parameter λ is unique.

→ ***This further validates*** **SCET$_\text{I}$ !**

YM, JHEP09(2024)197

# Some remarks

3. This picture is independent of the spin or spacetime dimension.

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Meanwhile, regulators may affect the list of regions.

YM, JHEP09(2024)197

# *Hidden regions in the on–shell expansion*

- Most of the regions are facet regions:
  - most graphs have only facet regions;
  - for the remaining graphs, hidden regions are very few compared with the facet regions.

- For the 2->2 massless wide-angle scattering graphs,
  - one loop: no graphs with hidden regions;
  - two loops: ≈100 graphs, none with hidden regions;
  - three loops: ≈1000 graphs, 10 of them have hidden regions.

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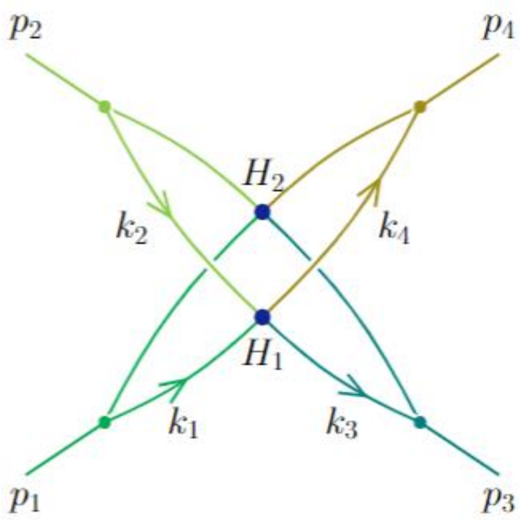
same origin

only 1 for each graph;
"Landshoff scattering"

# *Hidden regions in the on–shell expansion*

- The "*Landshoff scattering*":




- In scalar theory, from straightforward power counting, above is ***the only*** region that contributes to the leading asymptotic behavior. So this region must be included.
- This region ***cannot*** be detected by any computer codes.

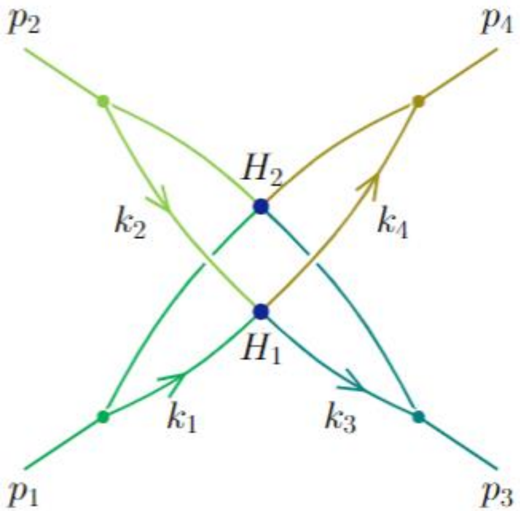
Gardi, Herzog, Jones, YM, JHEP08(2024)127

# *Power counting details*

- To see why this region is leading:

$$k_i^\mu = Q\left(\xi_i v_i^\mu + \lambda\kappa_i \bar{v}_i^\mu + \sqrt{\lambda}\tau_i u_i^\mu + \sqrt{\lambda}\nu_i n^\mu\right), \qquad i = 1,2,3,4.$$

(Botts & Sterman, 1989)




$$\xi_2 = \xi_1 - \frac{1}{2}\sqrt{\lambda}\cos^2(\theta)\left(\tan\left(\frac{\theta}{2}\right)\Delta\tau - \cot\left(\frac{\theta}{2}\right)\Sigma\tau\right) + \lambda(\kappa_2 - \kappa_1),$$

$$\xi_3 = \xi_1 + \frac{1}{2}\sqrt{\lambda}\tan\left(\frac{\theta}{2}\right)\Delta\tau + \lambda(\kappa_2 - \kappa_4),$$

$$\xi_4 = \xi_1 - \frac{1}{2}\sqrt{\lambda}\cot\left(\frac{\theta}{2}\right)\Sigma\tau + \lambda(\kappa_2 - \kappa_3).$$

- With this parameterization, $\int d^D k_1 d^D k_2 d^D k_3 = Q^{3D}\int \prod_{i=1}^{3} d\xi_i d\kappa_i d\tau_i d\nu_i$

- Under change of variables $\{\xi_2, \xi_3\} \to \{\kappa_4, \tau_4\}$,

$$\det\left(\frac{\partial(\xi_2,\xi_3)}{\partial(\kappa_4,\tau_4)}\right) = \lambda^{3/2}\cos(\theta)\cot(\theta).$$

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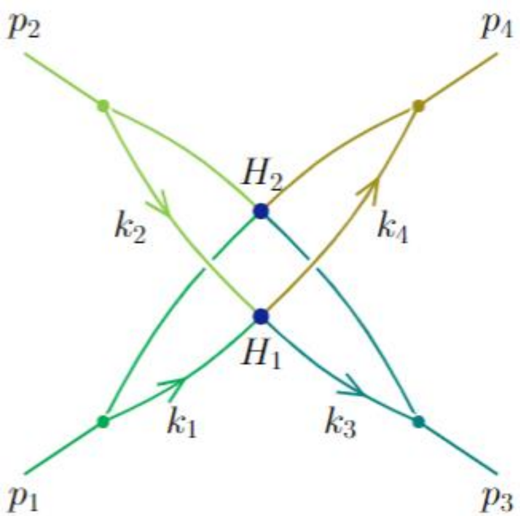



$$\int \prod_{i=1}^{3} d\xi_i\ d\kappa_i d\tau_i d\nu_i = \mathcal{C}\cdot\int_0^1 d\xi_1 \underbrace{\left(\int\prod_{i=1}^{3}(\lambda d\kappa_i)(\lambda^{\frac{1}{2}}d\tau_i)(\lambda^{\frac{1}{2}}d\nu_i)^{1-2\epsilon}\right)}_{\lambda^{6-3\epsilon}}$$

$$\cdot\int d\kappa_4 d\tau_4\ \underbrace{\det\left(\frac{\partial(\xi_2,\xi_3)}{\partial(\kappa_4,\tau_4)}\right)}_{\lambda^{3/2}}.$$

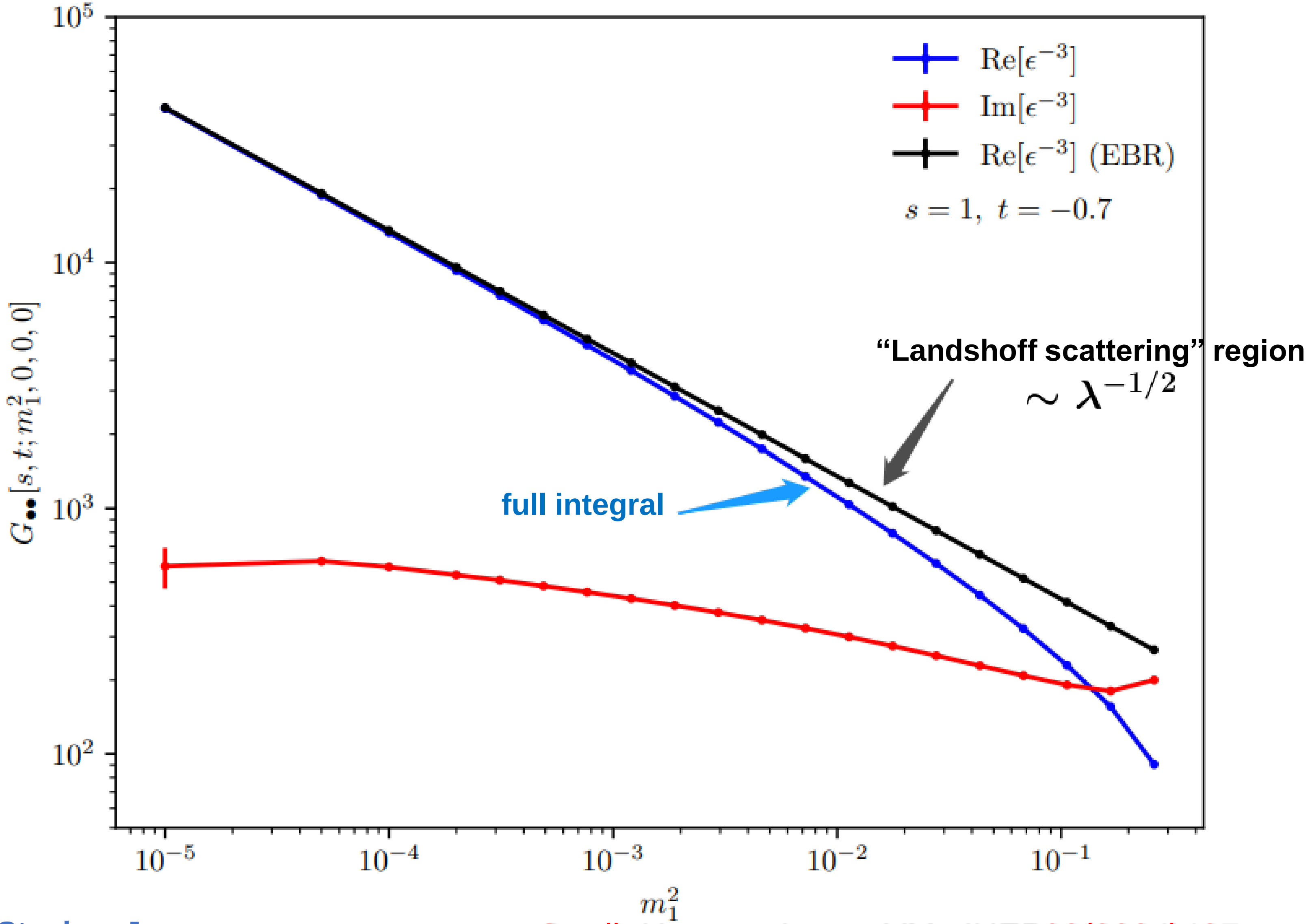
- Power counting result:

$$\mathcal{I}\sim\lambda^\mu, \quad \mu = -\frac{1}{2} - 3\epsilon.$$

- Meanwhile, **μ ≥ 0** for all the other regions.

Gardi, Herzog, Jones, YM, JHEP08(2024)127

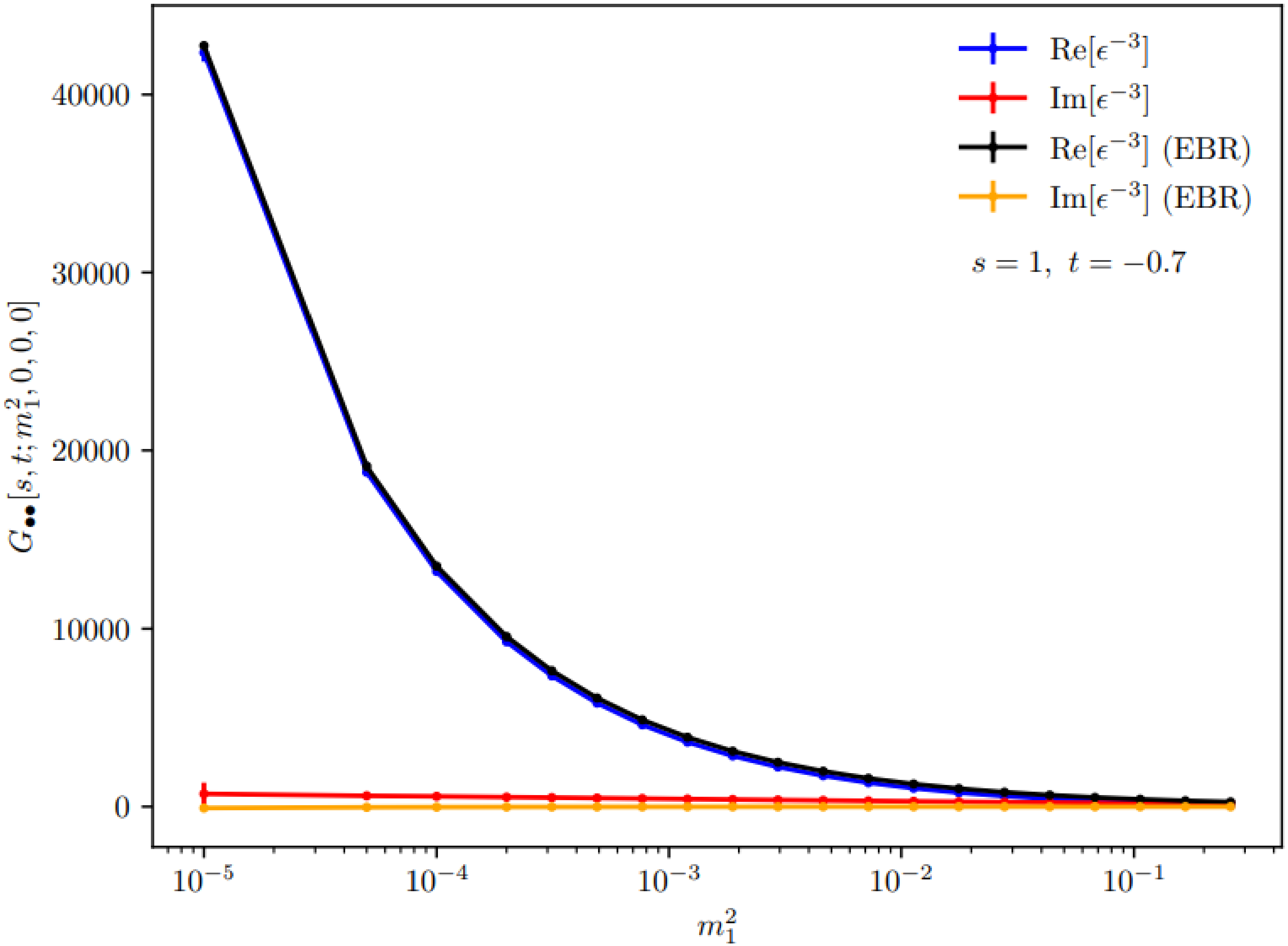
# *Numerical evidences*




Graph by Stephen Jones

Gardi, Herzog, Jones, YM, JHEP08(2024)127

# Numerical evidences

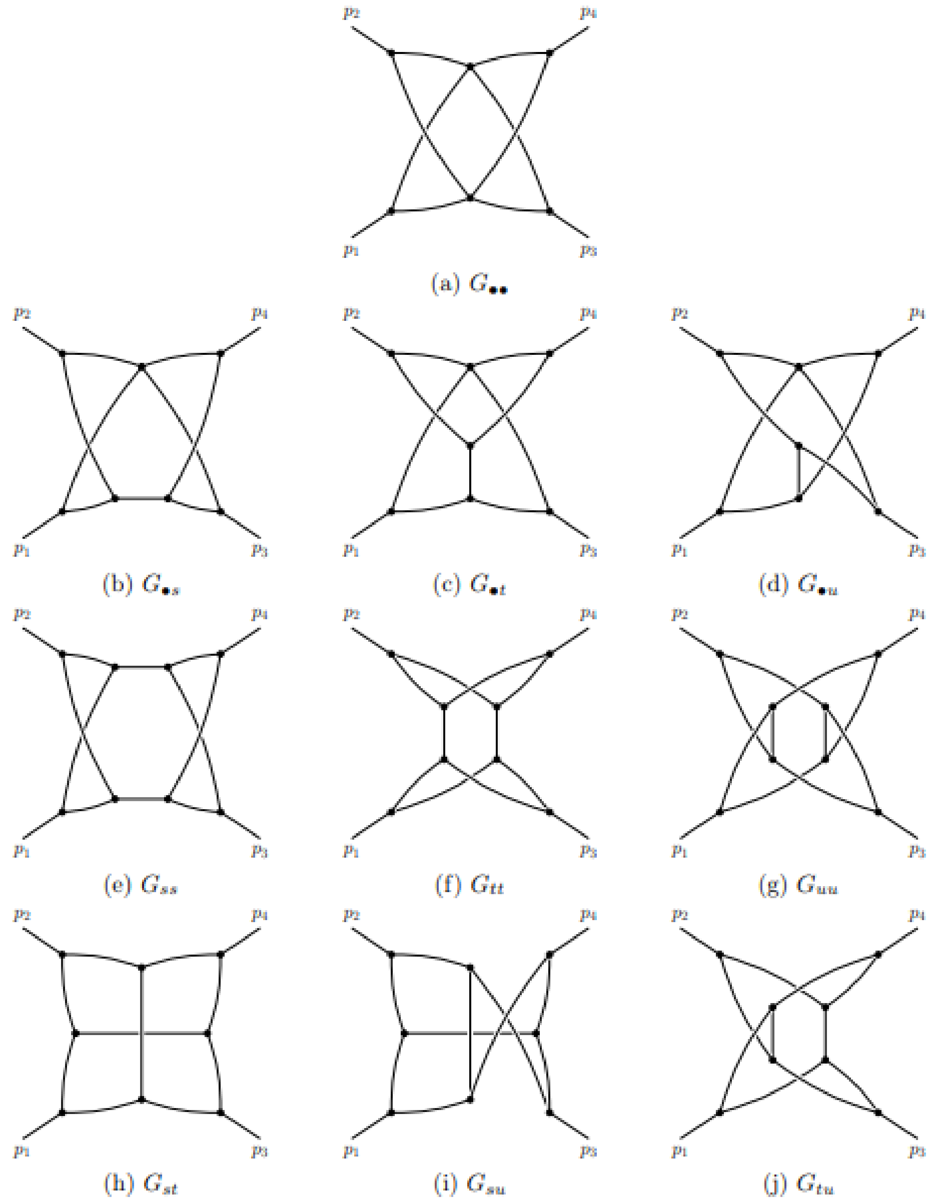



Graph by Stephen Jones

Gardi, Herzog, Jones, YM, JHEP08(2024)127

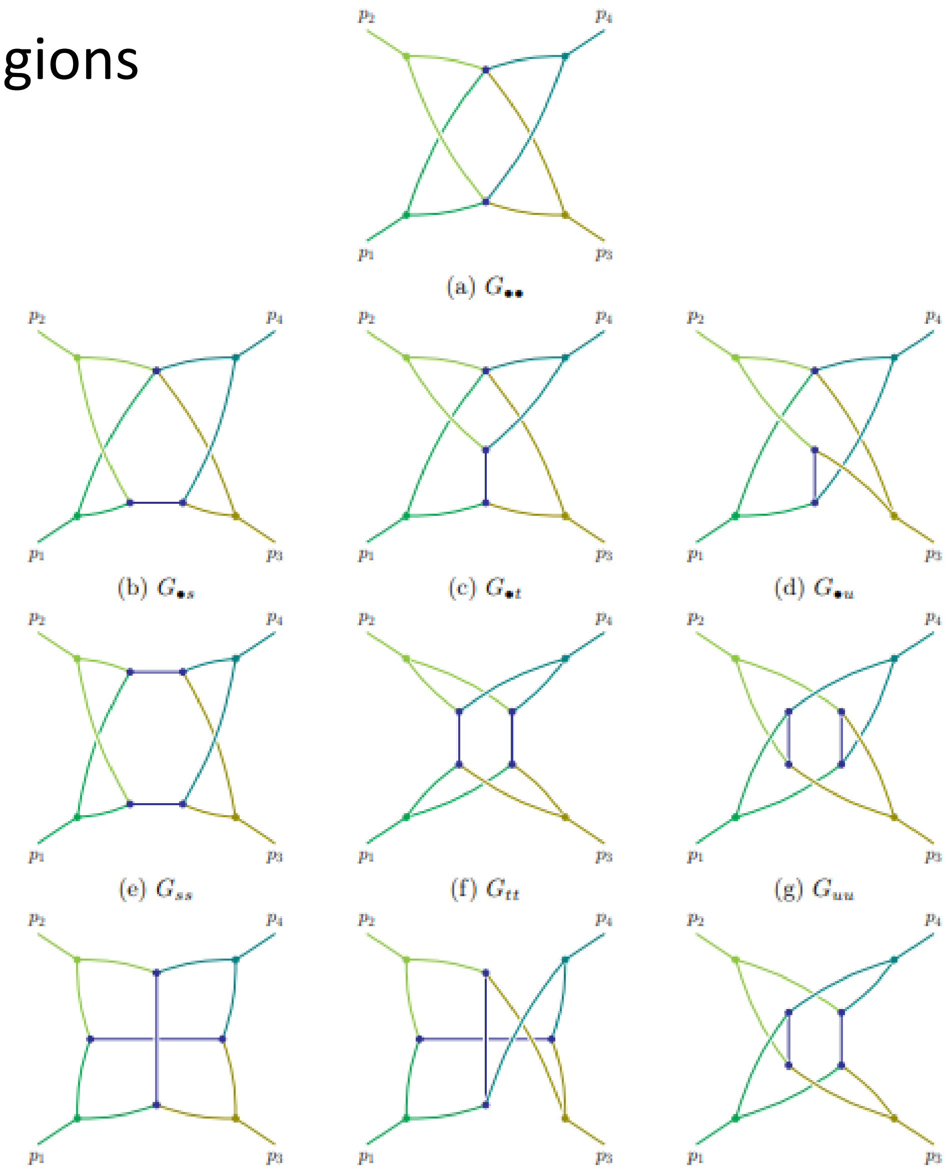
# *All the 3–loop graphs with hidden regions*

- All the 3-loop graphs



# *All the 3–loop graphs with hidden regions*

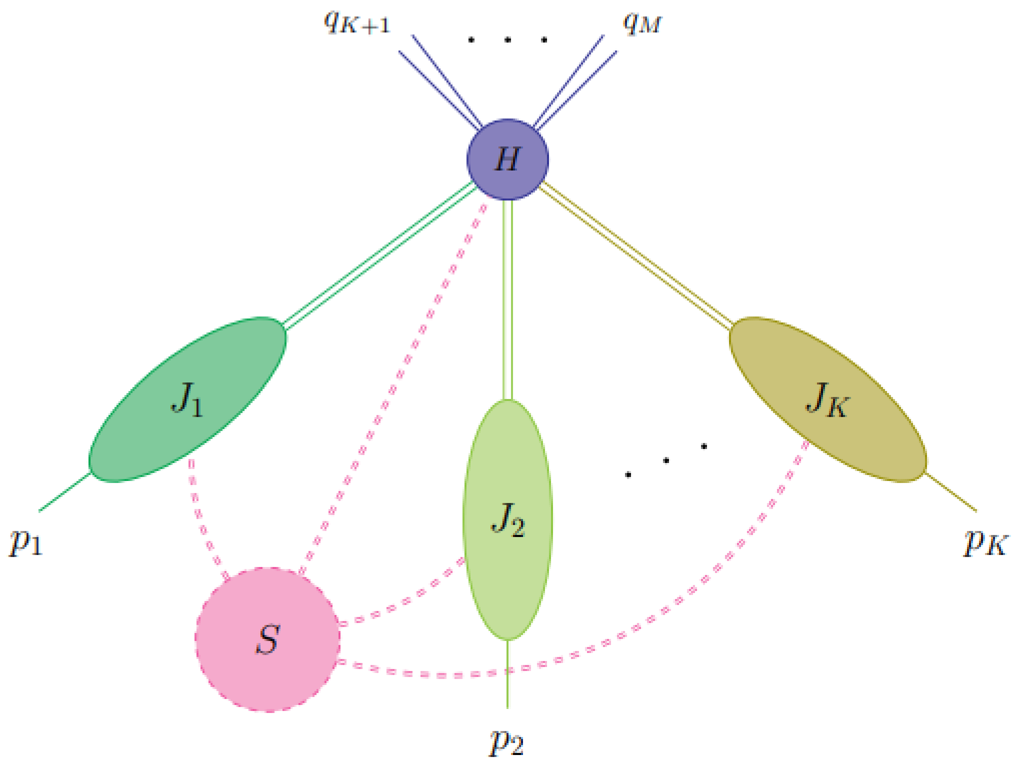
- Corresponding hidden regions



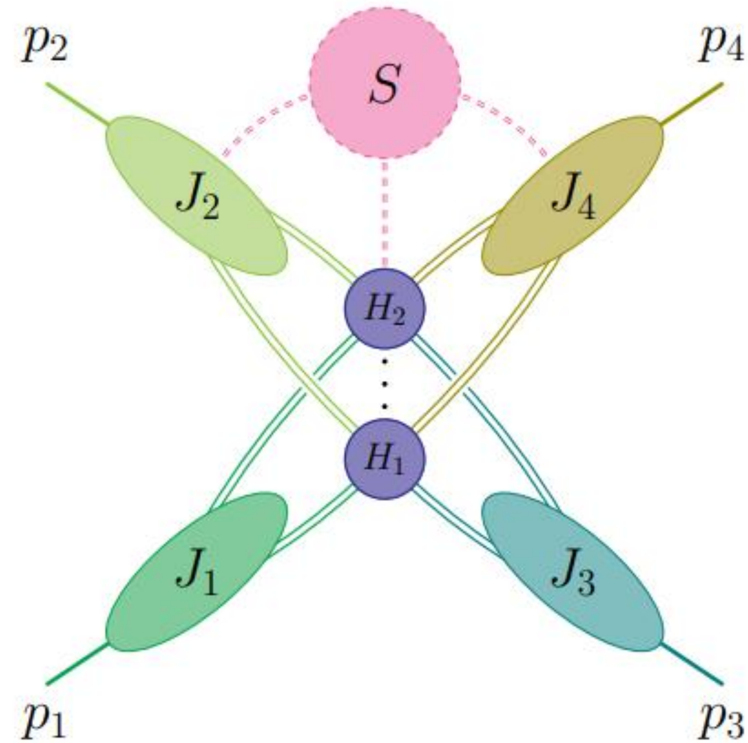

(a) $G_{\bullet\bullet}$ (b) $G_{\bullet s}$ (c) $G_{\bullet t}$ (d) $G_{\bullet u}$ (e) $G_{ss}$ (f) $G_{tt}$ (g) $G_{uu}$ (h) $G_{st}$ (i) $G_{su}$ (j) $G_{tu}$

# *Regions in the on–shell expansion*

***Conjecture:***




**facet region**

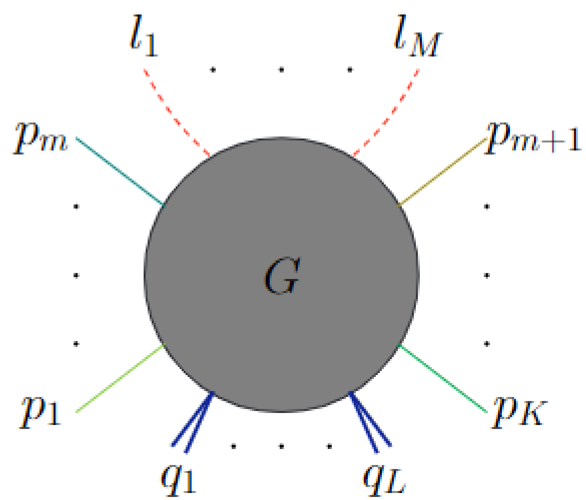



**hidden region**

Gardi, Herzog, Jones, YM, JHEP08(2024)127

# *The "soft expansion"*

- Including some soft external momenta




**massless**

**exactly on-shell** **large virtuality** **exactly on-shell**

$$p_i^2 = 0 \quad (i = 1, \ldots, K), \quad q_j^2 \sim Q^2 \quad (j = 1, \ldots, L), \quad l_k^2 = 0 \quad (k = 1, \ldots, M),$$

$$p_{i_1} \cdot p_{i_2} \sim Q^2 \quad (i_1 \neq i_2), \quad p_i \cdot l_k \sim q_j \cdot l_k \sim \lambda Q^2, \quad l_{k_1} \cdot l_{k_2} \sim \lambda^2 Q^2 \quad (k_1 \neq k_2).$$

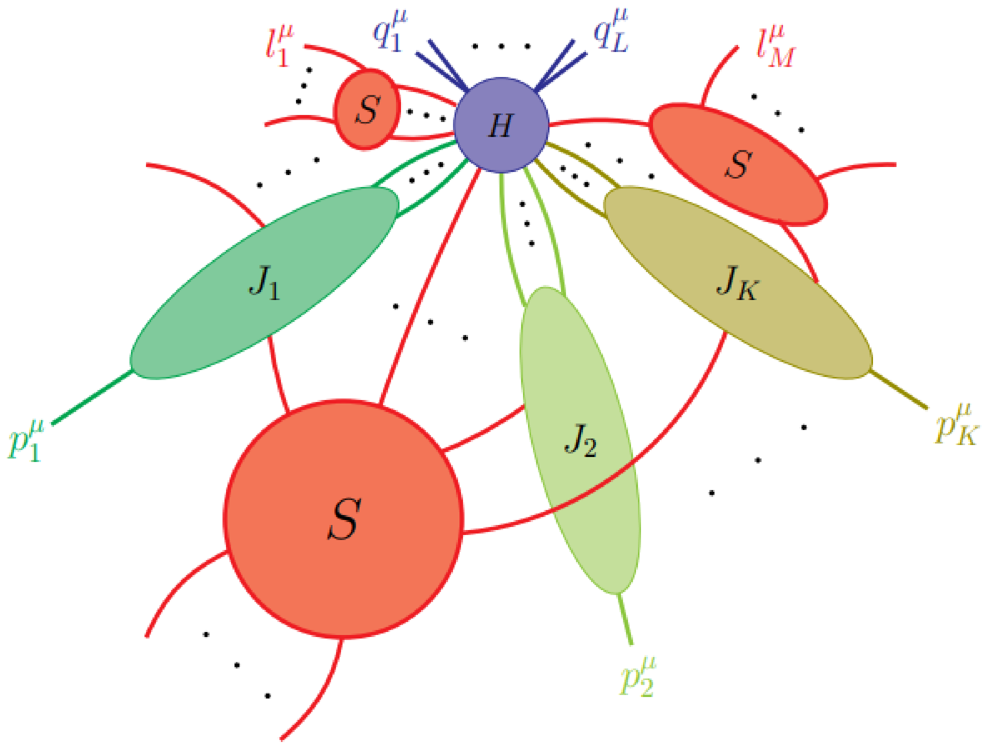
**wide-angle scattering** **soft momenta**

YM, JHEP09(2024)197

# *Regions in the soft expansion*

- ***Result: the possibly relevant modes are:***

$$k_H^\mu \sim Q(1,1,1), \quad k_{C_i}^\mu \sim Q(1,\lambda,\lambda^{1/2}), \quad k_S^\mu \sim Q(\lambda,\lambda,\lambda).$$

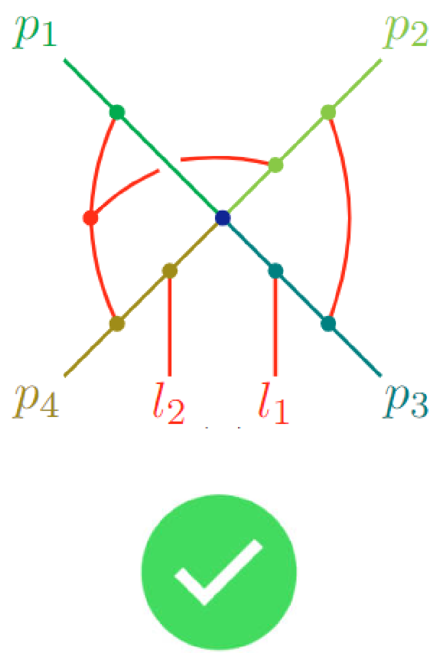


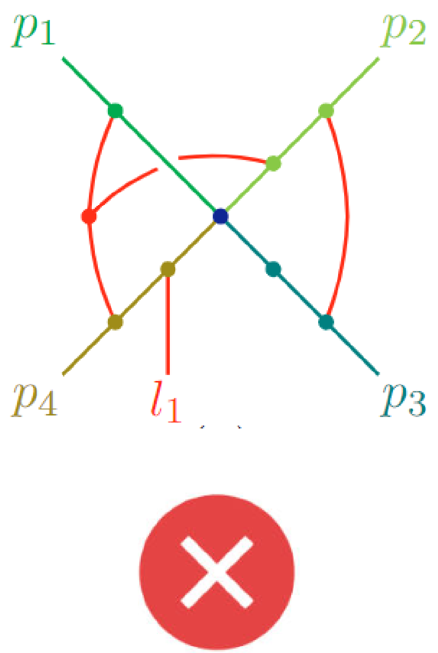

- Interesting feature: additional requirements for the subgraphs.

YM, JHEP09(2024)197

# *Regions in the soft expansion*

- The interactions between the soft subgraph and the jets follow the "information-delivery" picture.
- Some external soft information must be <u>delivered</u> to each jet.
- Any soft component **adjacent to ≥3 jets** can be a <u>messenger</u>.
- Example:



# Spacelike collinear kinematics

# *Spacelike collinear external momenta*

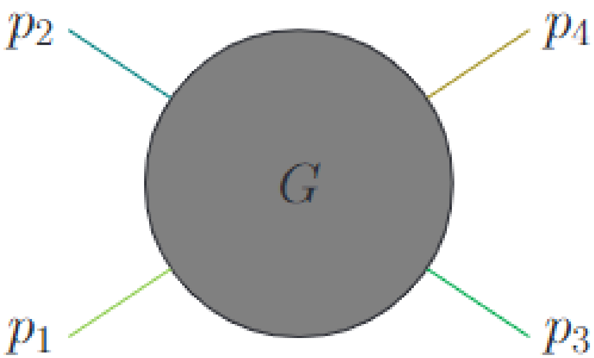
Given two collinear momenta $p_1$ and $p_2$, we call them

- timelike collinear: if they are both in the initial or final state;
- spacelike collinear: if one is initial and the other is final.

Feynman graphs with spacelike collinear external momenta possibly feature Glauber singularities, which are responsible for factorization breaking. ***(Catani, de Florian & Rodrigo 2011; Forshaw, Seymour & Siodmok 2012; ...)***

Note that Glauber singularities are absent in wide-angle kinematics: one can always deform the momentum-space integration contour to avoid them. ***(Collins & Sterman 1981)***

# The "*Regge–limit expansion*"




$$(p_1 + p_2)^2 = s,$$
$$(p_1 + p_3)^2 = t,$$
$$(p_1 + p_4)^2 = u,$$

kinematic limit:

$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0,$$
$$|t| \ll s \sim |u|,$$

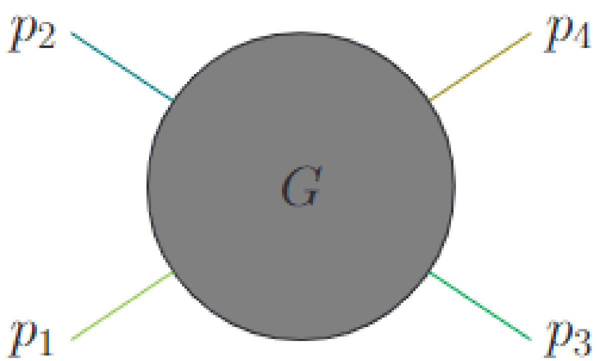
Gardi, Herzog, Jones, YM, JHEP08(2024)127

# The “Regge–limit expansion”

**Modes in facet regions:**

H, C1, C2,

1- and 2-loop

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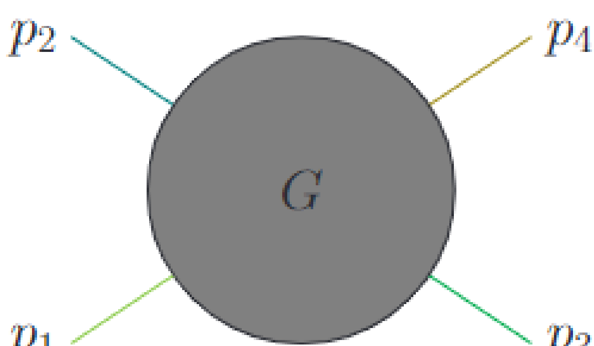
Gardi, Herzog, Jones, YM, JHEP08(2024)127

# *The "Regge–limit expansion"*

**Modes in facet regions:**

H, C1, C2, — 1- and 2-loop

…, S, S·C1, $C1^2$, S·C2, $C2^2$, — 3- and 4-loop

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Gardi, Herzog, Jones, YM, JHEP08(2024)127

# *The "Regge–limit expansion"*

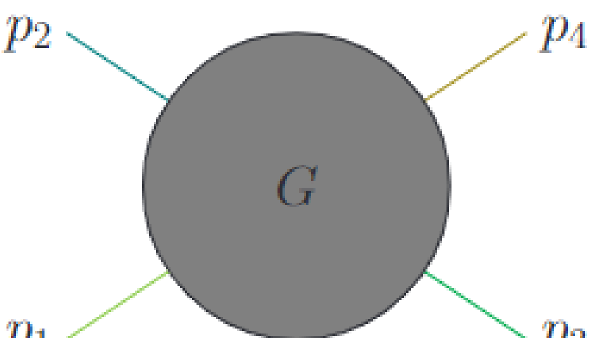
**Modes in facet regions:**

H, $C_1$, $C_2$, — 1- and 2-loop

…, S, $S\cdot C_1$, $C_1^2$, $S\cdot C_2$, $C_2^2$, — 3- and 4-loop

…, $S^2\cdot C_1$, $C_1^3$, $S^2\cdot C_2$, $C_2^3$, — 5- and 6-loop

… — … "cascade of modes"

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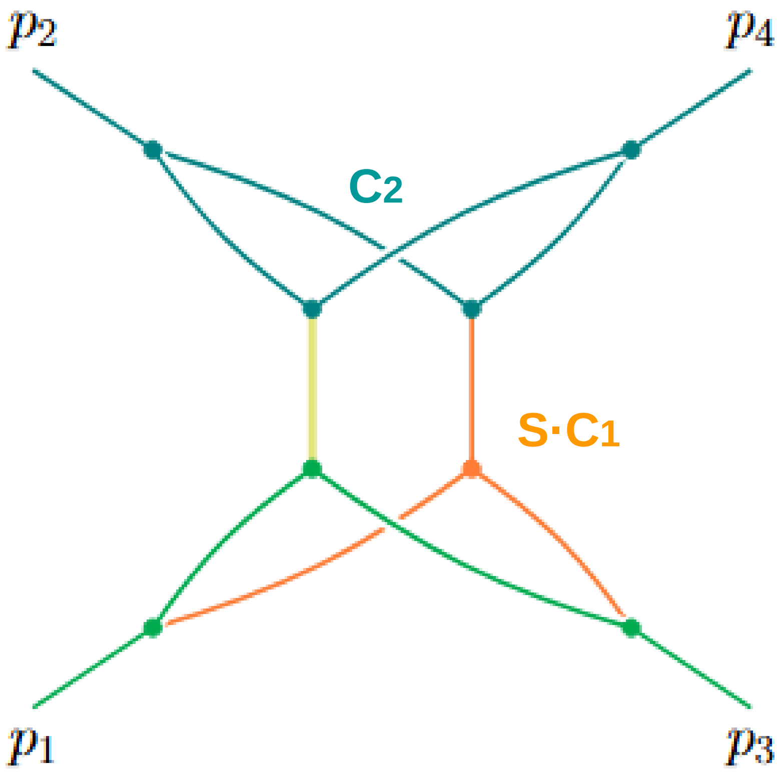
Gardi, Herzog, Jones, YM, JHEP08(2024)127

# *The "Regge–limit expansion"*

**Example:**

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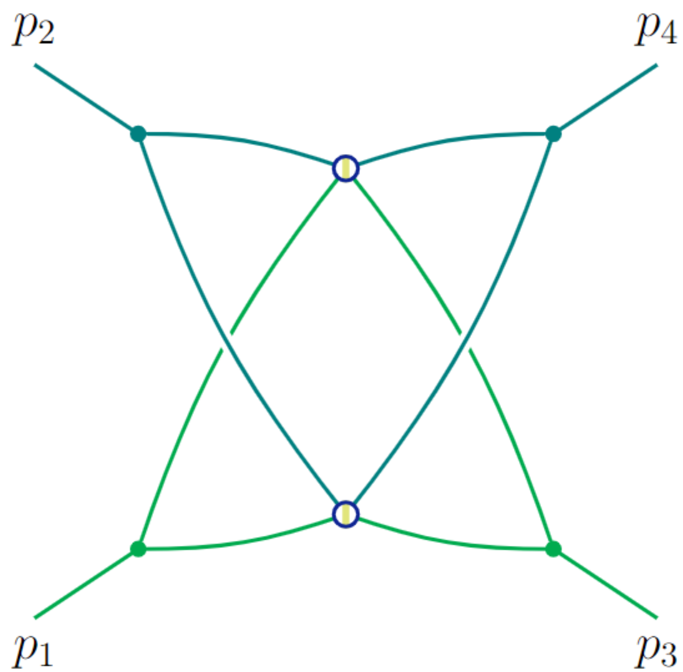
# *The "Regge–limit expansion"*

**Modes in hidden regions:**

... + *Glauber*

$(\lambda, \lambda, \lambda^{1/2})$

Starting from 3 loops:




Gardi, Herzog, Jones, YM, JHEP08(2024)127

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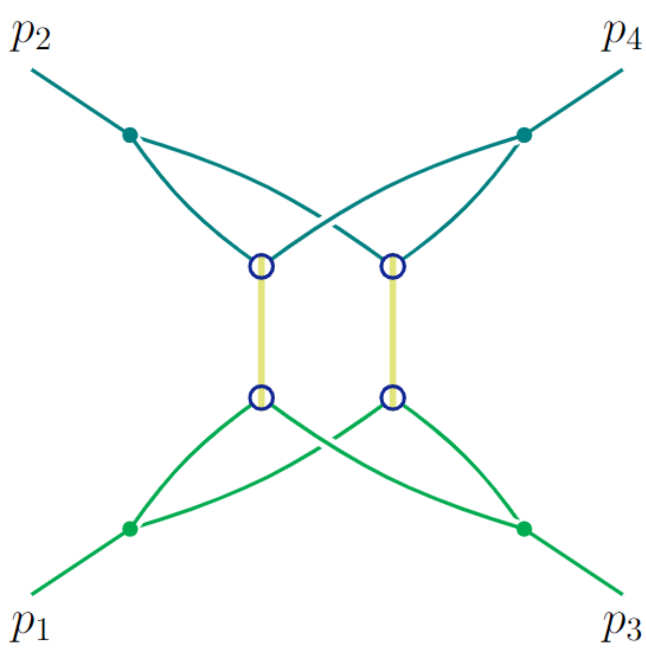


Gardi, Herzog, Jones, YM, JHEP08(2024)127

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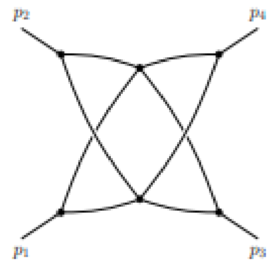
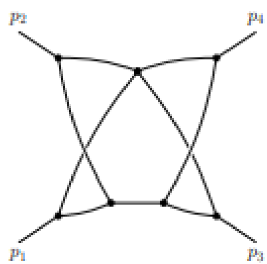
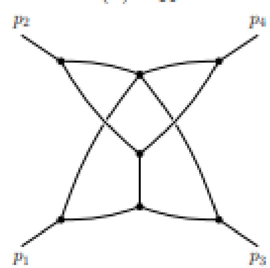


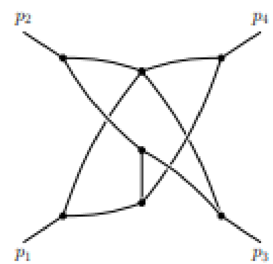

(a) $G_{\bullet\bullet}$



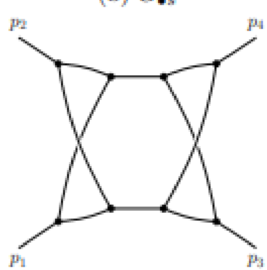

(b) $G_{\bullet s}$



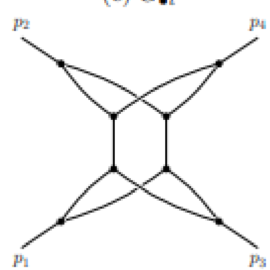

(c) $G_{\bullet t}$



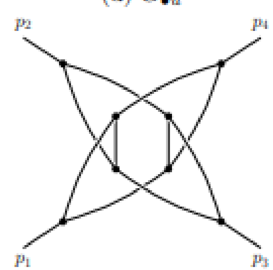

(d) $G_{\bullet u}$



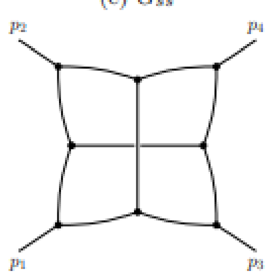

(e) $G_{ss}$



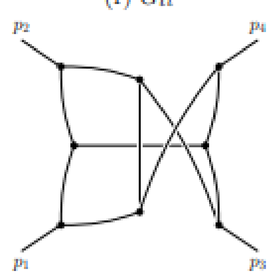

(f) $G_{tt}$



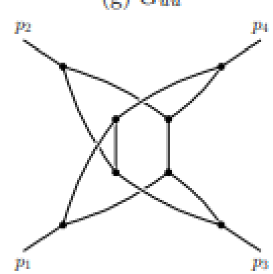

(g) $G_{uu}$




(h) $G_{st}$

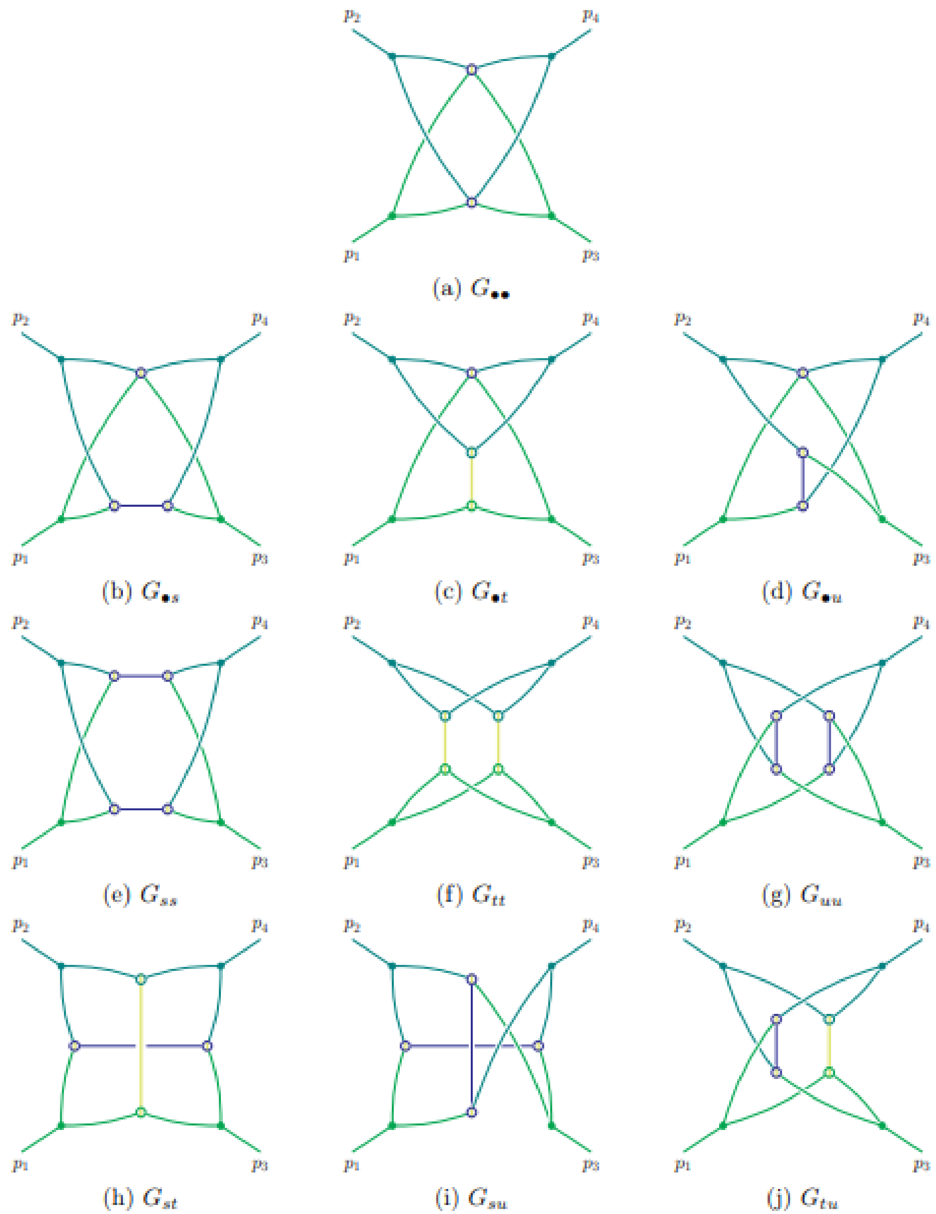



(i) $G_{su}$




(j) $G_{tu}$

# The “Regge–limit expansion”



(a) $G_{\bullet\bullet}$
(b) $G_{\bullet s}$
(c) $G_{\bullet t}$
(d) $G_{\bullet u}$
(e) $G_{ss}$
(f) $G_{tt}$
(g) $G_{uu}$
(h) $G_{st}$
(i) $G_{su}$
(j) $G_{tu}$

# *Main conclusions*

***1. The regions corresponding to a given graph can be predicted from the infrared picture!***

***– on–shell expansion: hard, collinear, soft.***

***– soft expansion: hard, collinear, soft.***

***– Regge limit: hard, collinear, soft, Glauber, (collinear)$^2$, soft •collinear, ...***

***with the mode interactions following certain patterns.***

***2. Landshoff scattering in the wide–angle kinematics***

↕ "*spacelike collinearization*"

***Glauber singularities in the spacelike collinear limit***

# The "Landshoff–Glauber correspondence"

***2→3 scattering in the spacelike collinear limit***

***(1+2→3+4+5, with $p_2 // p_3$):***

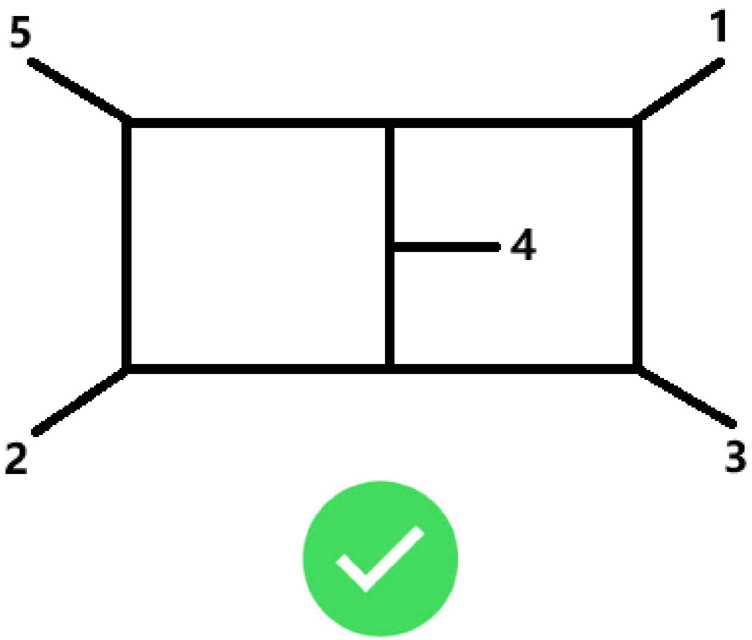
in collaboration with W.Chen, E.Gardi, R.Ma, Y.Zhang, Z.Zhu

# The "Landshoff–Glauber correspondence"

***2→3 scattering in the spacelike collinear limit***

***(1+2→3+4+5, with $p_2//p_3$):***

***Factorization breaking effects emerge from two loop level***

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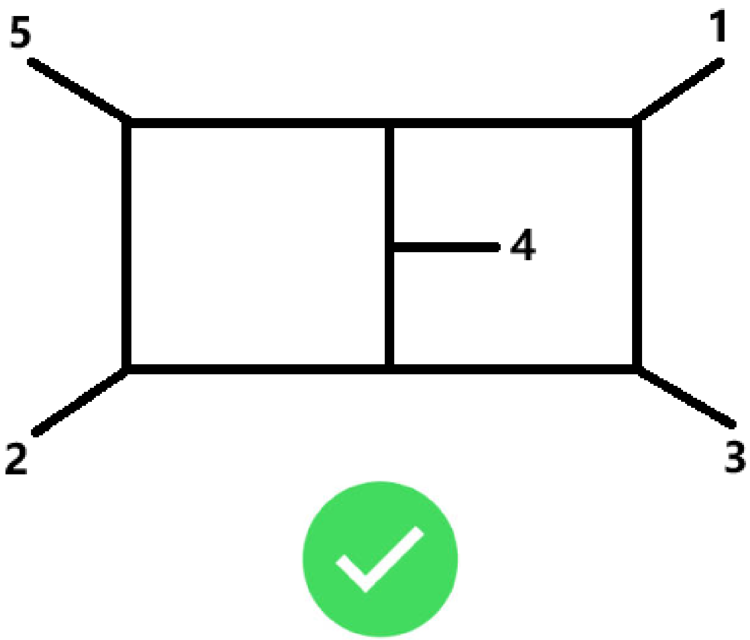


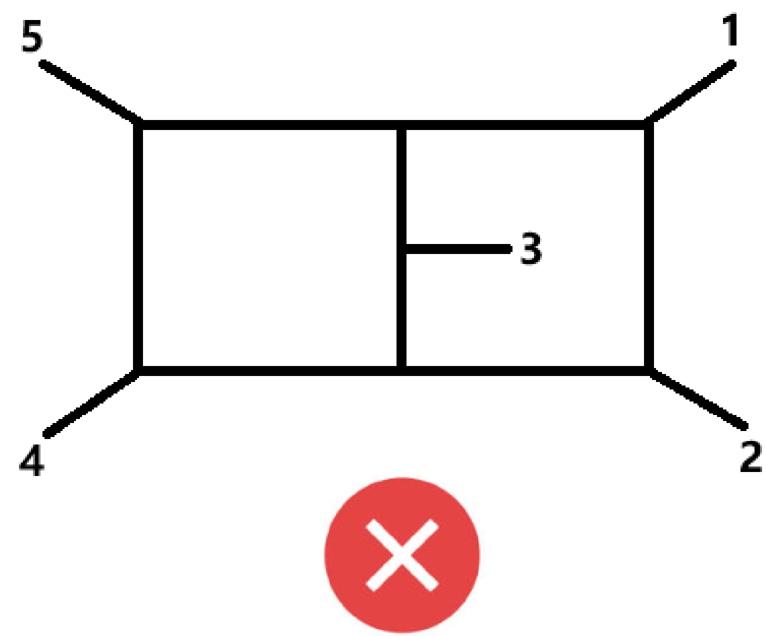


in collaboration with W.Chen, E.Gardi, R.Ma, Y.Zhang, Z.Zhu

# *Conclusions and outlook*

The following related topics can be investigated further.

1. Inclusion of massive propagators.
2. Generalize to phase-space integrals.
3. SCET, Glauber-SCET, SCET gravity, etc.
4. Local infrared subtractions.
5. Can one even justify the EbR with the help of our results?
6. Landau analysis of singularities.
7. Mathematical studies of convex/tropical geometry, etc.

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…

THANK YOU!

# Backup slides

# *The expansion by regions (EbR)*

- These two questions are related.
  1. How to prove the EbR? (Why is this technique true?)
  2. How to identify the regions? (How to use this technique?)

Recall the statement

$\exists$ "Regions" : $R_1, \; R_2, \; \ldots, \; R_n,$ such that

$$\mathcal{I} = \mathcal{I}^{(R_1)} + \mathcal{I}^{(R_2)} + \cdots + \mathcal{I}^{(R_n)}.$$

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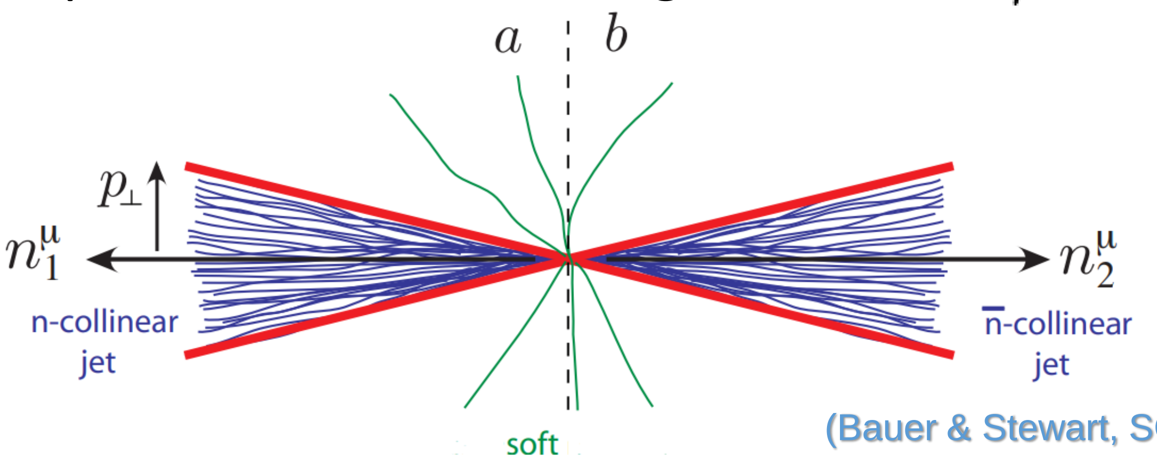
- Moreover, Jantzen showed that the EbR works to all orders, provided that the regions satisfy certain requirements.

*(Jantzen 2011)*

# *The expansion by regions (EbR)*

- Usually, regions are determined based on heuristic examples or experience.

- ***Soft–Collinear Effective Theory (SCET):*** an effective theory describing the interactions of soft and collinear degrees of freedom in the presence of a hard interaction.
- For example, the SCET describing $e^+e^- \to \gamma^* \to \mathbf{dijets}$




(Bauer & Stewart, SCET Lecture Notes 2013)

involves the hard mode (integrated out), the collinear modes, and the soft mode.

# *The expansion by regions (EbR)*

- Usually, regions are determined based on heuristic examples or experience.

- ***Soft–Collinear Effective Theory (SCET):*** an effective theory describing the interactions of soft and collinear degrees of freedom in the presence of a hard interaction.
- The **SCETI Lagrangian** (leading order):

$$
\begin{aligned}
\mathcal{L} &= \sum_n (\mathcal{L}_{n\xi} + \mathcal{L}_{ng}) + \mathcal{L}_{\text{soft}} \\
&= \sum_n \Bigg( e^{-ix\cdot\mathcal{P}}\overline{\xi}_n \left( in\cdot D + i\not{D}_{n\perp}\frac{1}{i\overline{n}\cdot D_n}\not{D}_{n\perp}\right)\frac{\not{\overline{n}}}{2}\xi_n \\
&\quad + \frac{1}{2g^2}\mathrm{Tr}\{[i\mathcal{D}^\mu, i\mathcal{D}_\mu]^2\} + \tau\mathrm{Tr}\{[i\mathcal{D}_{\mathrm{s}}^\mu, A_{n\mu}]^2\} + 2\mathrm{Tr}\{b_n[i\mathcal{D}_{\mathrm{s}}^\mu, [i\mathcal{D}_\mu, c_n]]\}\Bigg) \\
&\quad + \overline{\psi}_{\mathrm{s}} i\not{D}_{\mathrm{s}}\psi_{\mathrm{s}} - \frac{1}{2}\mathrm{Tr}\{G_{\mathrm{s}}^{\mu\nu}G_{\mathrm{s},\mu\nu}\} + \tau_{\mathrm{s}}\mathrm{Tr}\{(i\partial_\mu A_{\mathrm{s}}^\mu)^2\} + 2\mathrm{Tr}\{b_{\mathrm{s}} i\partial_\mu i\mathcal{D}_{\mathrm{s}}^\mu c_{\mathrm{s}}\}\,.
\end{aligned}
$$

# *Parametric representation*

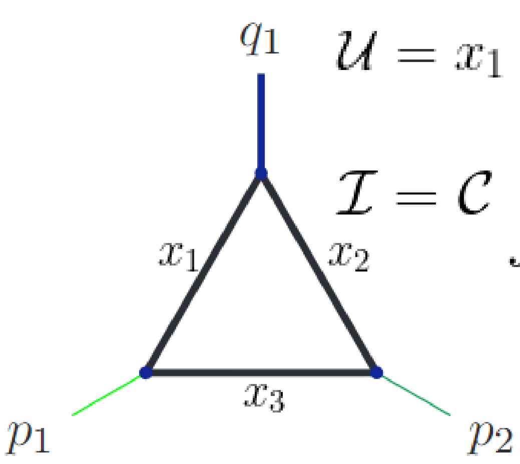
The Lee-Pomeransky representation (Lee & Pomeransky 2013)

$$\mathcal{I}(G) = \frac{\Gamma(D/2)}{\Gamma((L+1)D/2-\nu)\prod_{e\in G}\Gamma(\nu_e)} \int_0^\infty \left( \prod_{e\in G} dx_e x_e^{\nu_e - 1} \right) \left( \mathcal{P}\left(\boldsymbol{x},\boldsymbol{s}\right) \right)^{-D/2},$$

$$\mathcal{P}(\boldsymbol{x},\boldsymbol{s}) \equiv \mathcal{U}(\boldsymbol{x}) + \mathcal{F}(\boldsymbol{x},\boldsymbol{s}),$$

$$\mathcal{U}(\boldsymbol{x}) = \sum_{T^1} \prod_{e \notin T^1} x_e, \qquad \mathcal{F}(\boldsymbol{x},\boldsymbol{s}) = -\sum_{T^2} s_{T^2} \prod_{e\notin T^2} x_e + \mathcal{U}(\boldsymbol{x}) \sum_e m_e^2 x_e \ .$$

**spanning trees** **spanning 2-trees**




$$\mathcal{U} = x_1 + x_2 + x_3, \quad \mathcal{F} = (-p_1^2)x_1x_3 + (-p_2^2)x_2x_3 + (-q_1^2)x_1x_2$$

$$\mathcal{I} = \mathcal{C}\int_0^\infty dx_1 dx_2 dx_3 x_1^{\nu_1 - 1} x_2^{\nu_2-1} x_3^{\nu_3 - 1}$$

$$\cdot \left(x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2\right)^{-D/2}$$

# *Parametric representation*

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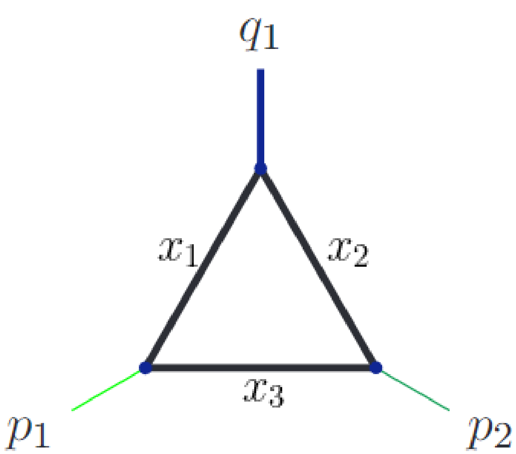
**spanning trees** **spanning 2-trees**




Each region → a certain scaling of the $\boldsymbol{x}$

$$\textbf{Hard region}: \ \boldsymbol{x_1, x_2, x_3 \sim \lambda^0}$$

$$\textbf{Collinear region to } \boldsymbol{p_1}: \boldsymbol{x_1, x_3 \sim \lambda^{-1}}, \ \boldsymbol{x_2 \sim \lambda^0}$$

$$\textbf{Collinear region to } \boldsymbol{p_2}: \boldsymbol{x_1 \sim \lambda^0, x_2, x_3 \sim \lambda^{-1}}$$

$$\textbf{Soft region}: \boldsymbol{x_1, x_2 \sim \lambda^{-1}}, \ \boldsymbol{x_3 \sim \lambda^{-2}}$$

# *Regions in different representations*

- Momentum space:

$$\text{Hard region: } k^\mu \sim Q(1,1,1)$$
$$\text{Collinear-1 region: } k^\mu \sim Q(1,\lambda,\lambda^{1/2})$$
$$\text{Collinear-2 region: } k^\mu \sim Q(\lambda,1,\lambda^{1/2})$$
$$\text{Soft region: } k^\mu \sim Q(\lambda,\lambda,\lambda)$$

- Parameter space:

$$\text{Hard region}: \ x_1, x_2, x_3 \sim \lambda^0$$
$$\text{Collinear region to } p_1: x_1, x_3 \sim \lambda^{-1},\ x_2 \sim \lambda^0$$
$$\text{Collinear region to } p_2: x_1 \sim \lambda^0, x_2, x_3 \sim \lambda^{-1}$$
$$\text{Soft region}: x_1, x_2 \sim \lambda^{-1},\ x_3 \sim \lambda^{-2}$$

- Relation between the scalings:

$$x_e \sim (D_e)^{-1}$$

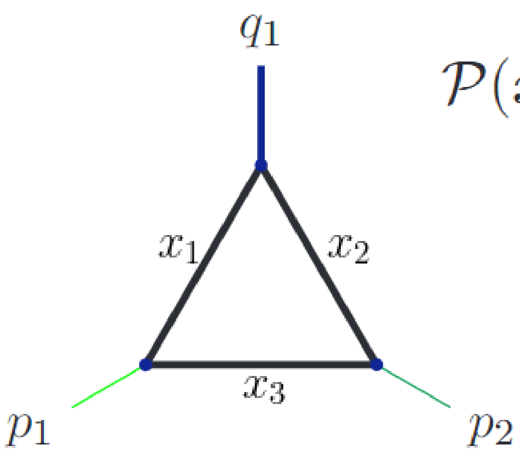
# *Identifying regions from Newton polytopes*

- Given the Lee-Pomeransky polynomial,

$$\mathcal{P}(\boldsymbol{x};\boldsymbol{s}) = \mathcal{U}(\boldsymbol{x}) + \mathcal{F}(\boldsymbol{x};\boldsymbol{s}),$$

take the **exponents** of each term:

$$s x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n} \to (v_1, v_2, \ldots, v_n; a) \quad \text{if} \quad s \sim \lambda^a Q^2$$




$$\mathcal{P}(\boldsymbol{x},\boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

**(1,0,0;0)** **(0,1,0;0)** **(0,0,1;0)** **(1,0,1;1)** **(0,1,1;1)** **(1,1,0;0)**

Construct a Newton polytope, defined as the convex hull of the points.

**Regions <-> the lower facets of this Newton polytope.**

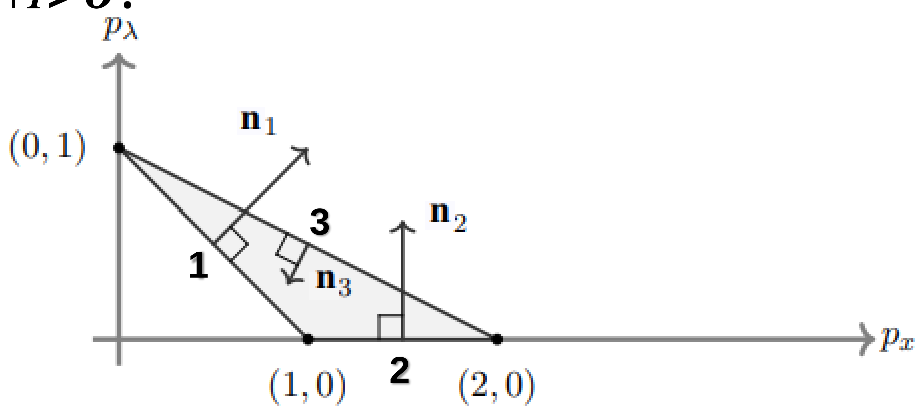
(Entries of the vector normal to a lower facet are precisely the scalings of $x_1$, $x_2$, ....)

# *Identifying regions from Newton polytopes*

**Regions <-> the lower facets of this Newton polytope**

Given a graph with $N$ propagators, the Newton polytope $\triangle$ is $N$+1 dimensional.

- **Facets:** the $N$-dimensional boundaries of $\triangle$.
- **Lower facets:** those facets whose inward-pointing normal vectors $\boldsymbol{v}$ satisfy $v_{N+1}>0$.




$1,2 \in F^+$

$3 \notin F^+$

- The vector $\boldsymbol{v}$ is usually referred to as the **region vector**, and its entries show the scaling of $\boldsymbol{x}$.

# *Identifying regions from Newton polytopes*

Back to our example:

Each region (hard, collinear-1, collinear-2, soft) corresponds to a specific facet containing certain points.

$$\mathcal{P}(\boldsymbol{x},\boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

**(1,0,0;0)** **(0,1,0;0)** **(0,0,1;0)** **(1,0,1;1)** **(0,1,1;1)** **(1,1,0;0)**

These points are in the hard facet, with $\boldsymbol{v}_h$ **= (0,0,0;1)**.

In comparison,

$$\textbf{Hard region}: \; \boldsymbol{x_1, x_2, x_3 \sim \lambda^0}$$

# *Identifying regions from Newton polytopes*

Back to our example:

Each region (hard, collinear-1, collinear-2, soft) corresponds to a specific facet containing certain points.

$$\mathcal{P}(\boldsymbol{x},\boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

**(1,0,0;0)** **(0,0,1;0)** **(1,0,1;1)** **(1,1,0;0)**

These points are in the collinear-1 facet, with $\boldsymbol{v}_{C1}$ **= (-1,0,-1;1)**.

$$\textbf{Collinear region to } \boldsymbol{p_1}: \; \boldsymbol{x_1, x_3 \sim \lambda^{-1}, \; x_2 \sim \lambda^0}$$

# *Identifying regions from Newton polytopes*

Back to our example:

Each region (hard, collinear-1, collinear-2, soft) corresponds to a specific facet containing certain points.

$$\mathcal{P}(\boldsymbol{x},\boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

**(0,0,1;0)** **(1,1,0;0)**

**(0,1,0;0)** **(0,1,1;1)**

These points are in the collinear-2 facet, with $\boldsymbol{v}_{C2}$ **= (0,-1,-1;1)**.

$$\textbf{Collinear region to } p_2: \; x_1 \sim \lambda^0, \; x_2, x_3 \sim \lambda^{-1}$$

# *Identifying regions from Newton polytopes*

Back to our example:

Each region (hard, collinear-1, collinear-2, soft) corresponds to a specific facet containing certain points.

$$\mathcal{P}(\boldsymbol{x},\boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

$x_3 \rightarrow$ **(0,0,1;0)**, $p_1^2 x_1 x_3 \rightarrow$ **(1,0,1;1)**, $p_2^2 x_2 x_3 \rightarrow$ **(0,1,1;1)**, $q_1^2 x_1 x_2 \rightarrow$ **(1,1,0;0)**

These points are on the soft facet, with $\boldsymbol{v_S}$ **= (-1,-1,-2;1)**.

$$\textbf{Soft region}: \; x_1, x_2 \sim \lambda^{-1}, \; x_3 \sim \lambda^{-2}$$

# Infrared structures of wide-angle scattering

- **The Landau equations**

$$\alpha_e l_e^2(k,p,q) = 0 \qquad \forall e \in G$$
$$\frac{\partial}{\partial k_a}\mathcal{D}\left(k,p,q;\alpha\right) = 0 \qquad \forall a \in \{1,\ldots,L\}\,.$$

**are necessary conditions for infrared singularity. The solutions of the Landau equations are called pinch surfaces.**

- **The pinch surfaces of hard processes has been studied in detail in the past decades.**

- **Motivation: it looks that the infrared regions are in one-to-one correspondence with the pinch surfaces!**

# *Regions in the on–shell expansion*

E.Gardi, F.Herzog, S.Jones, YM, J.Schlenk, JHEP07(2023)197

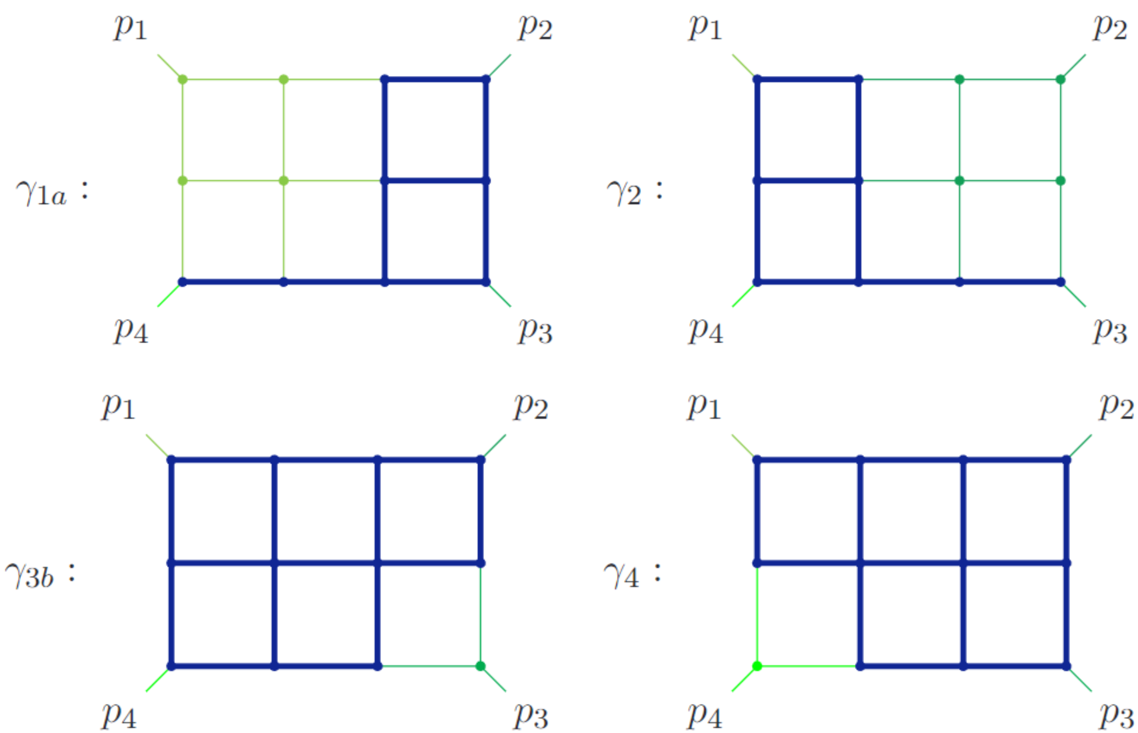
- **Each solution of the Landau equations corresponds to a region, provided that some requirements of H, J, and S are satisfied.**

- *Requirement of $H$: all the internal propagators of $H_{\text{red}}$, which is the reduced form of $H$, are off-shell.*
- *Requirement of $J$: all the internal propagators of $\widetilde{J}_{i,\text{red}}$, which is the reduced form of the contracted graph $\widetilde{J}_i$, carry exactly the momentum $p_i^\mu$.*
- *Requirement of $S$: every connected component of $S$ must connect at least two different jet subgraphs $J_i$ and $J_j$.*

# *A graph–finding algorithm*

- Based on this conclusion, we can construct a **graph-finding algorithm** to unveil all the regions.
- A fishnet example

**Step 1: constructing the "primitive jets":**

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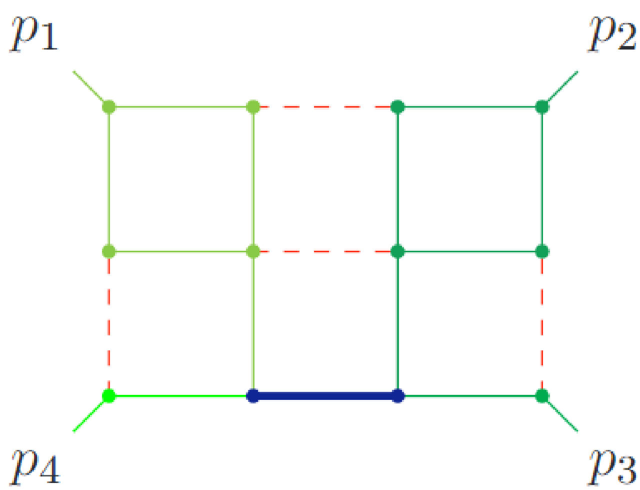
# *A graph–finding algorithm*

- Based on this conclusion, we can construct a **graph-finding algorithm** to unveil all the regions.
- A fishnet example

**Step 2: overlaying the "primitive jets":**




**Step 3: removing pathological configurations.**

This algorithm does not involve constructing Newton polytopes, and can be much faster.

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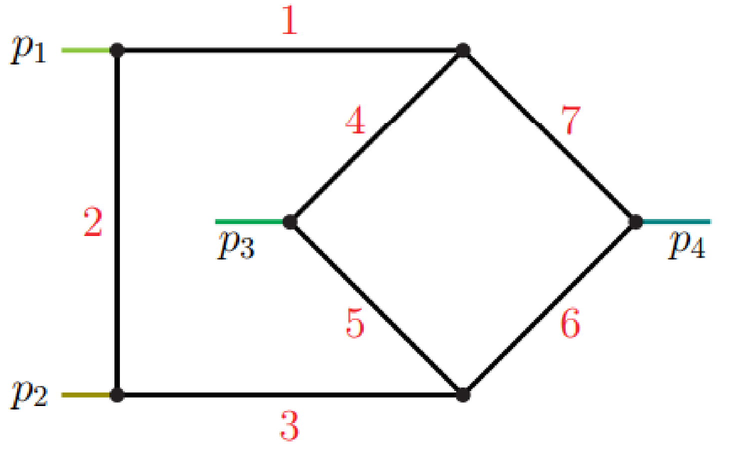
# *Landau analysis of cancellations*

- Each region (except the hard region) must correspond to an infrared singularity, satisfying the Landau equations:

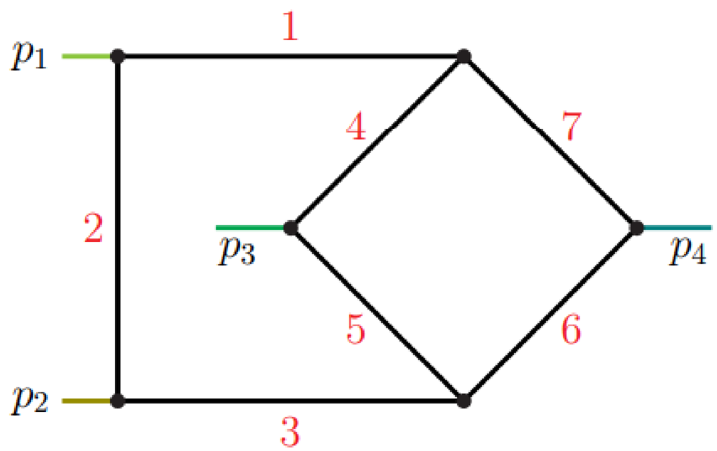
$$\mathcal{F}(\boldsymbol{\alpha}; \boldsymbol{s}) = 0,$$
$$\forall i, \quad \alpha_i = 0 \ \text{ or } \ \partial\mathcal{F}/\partial\alpha_i = 0.$$

- Therefore, $\mathcal{F}$ having both positive and negative terms does not necessarily imply a region, because the Landau equation above may not be satisfied.
- For example,



# *Landau analysis of cancellations*

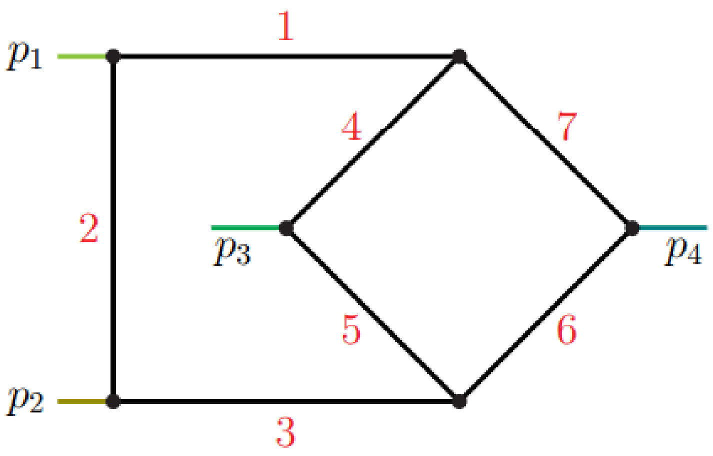
- For example,




$$
\begin{aligned}
\mathcal{F}(\boldsymbol{\alpha};\boldsymbol{s}) = \quad & (-p_1^2)\left[\alpha_1\alpha_2(\alpha_4+\alpha_5+\alpha_6+\alpha_7)+\alpha_2\alpha_4\alpha_7\right] \\
& +(-p_2^2)\left[\alpha_2\alpha_3(\alpha_4+\alpha_5+\alpha_6+\alpha_7)+\alpha_2\alpha_5\alpha_6\right] \\
& +(-p_3^2)\left[\alpha_4\alpha_5(\alpha_1+\alpha_2+\alpha_3+\alpha_6+\alpha_7)+\alpha_1\alpha_5\alpha_7+\alpha_3\alpha_4\alpha_6\right] \\
& +(-p_4^2)\left[\alpha_6\alpha_7(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5)+\alpha_1\alpha_4\alpha_6+\alpha_3\alpha_5\alpha_7\right] \\
& +(-q_{12}^2)\left[\alpha_1\alpha_3(\alpha_4+\alpha_5+\alpha_6+\alpha_7)+\alpha_3\alpha_4\alpha_7+\alpha_1\alpha_5\alpha_6\right] \\
& +(-q_{13}^2)\alpha_2\alpha_5\alpha_7+(-q_{14}^2)\alpha_2\alpha_4\alpha_6.
\end{aligned}
$$

# *Landau analysis of cancellations*

- For example,




One can check that any possible cancellation within $\mathscr{F}$ is not compatible with the Landau equations.

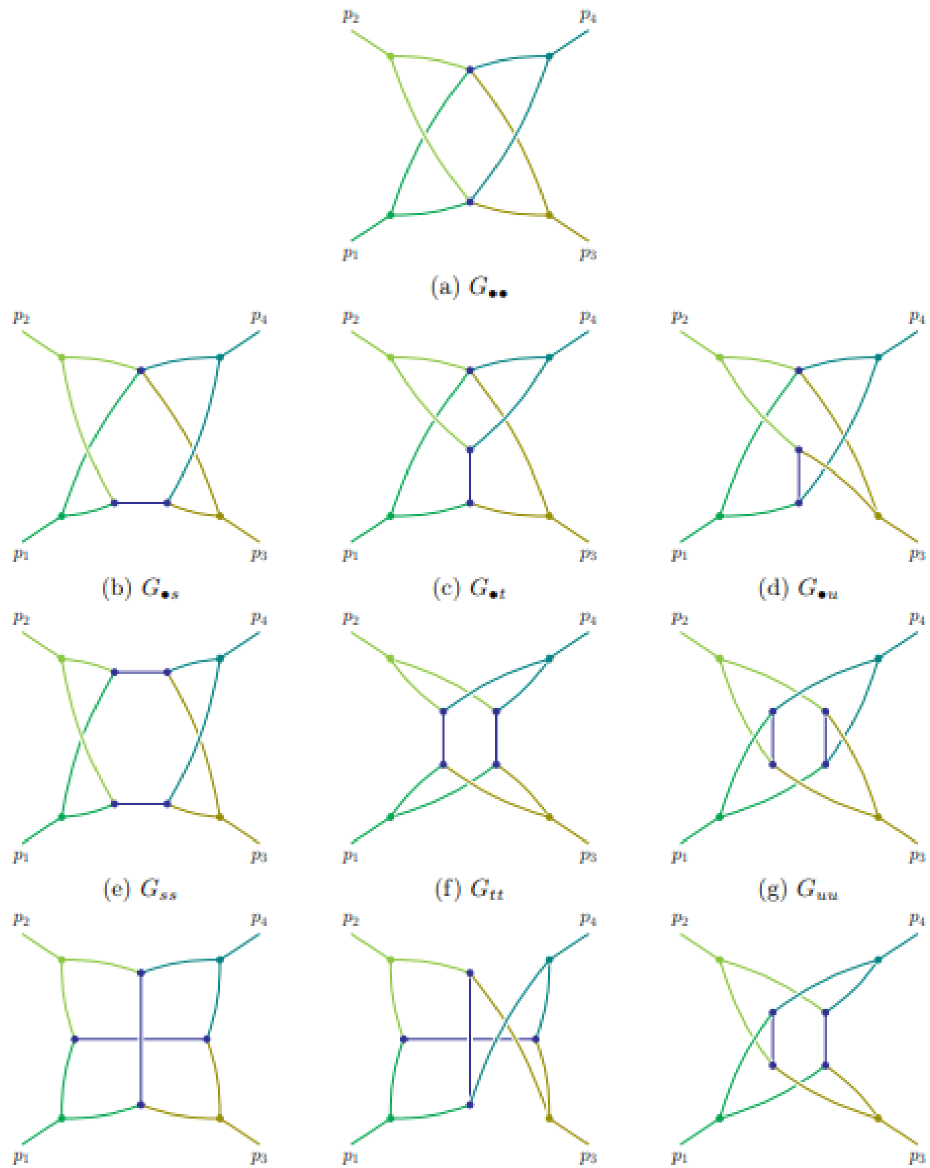
- Therefore, all the regions are from the lower facets of the Newton polytope.

- Actually, as one can check in this way, most cases where $\mathscr{F}$ is indefinite does not have regions due to cancellations.

# All the 3–loop graphs with hidden regions

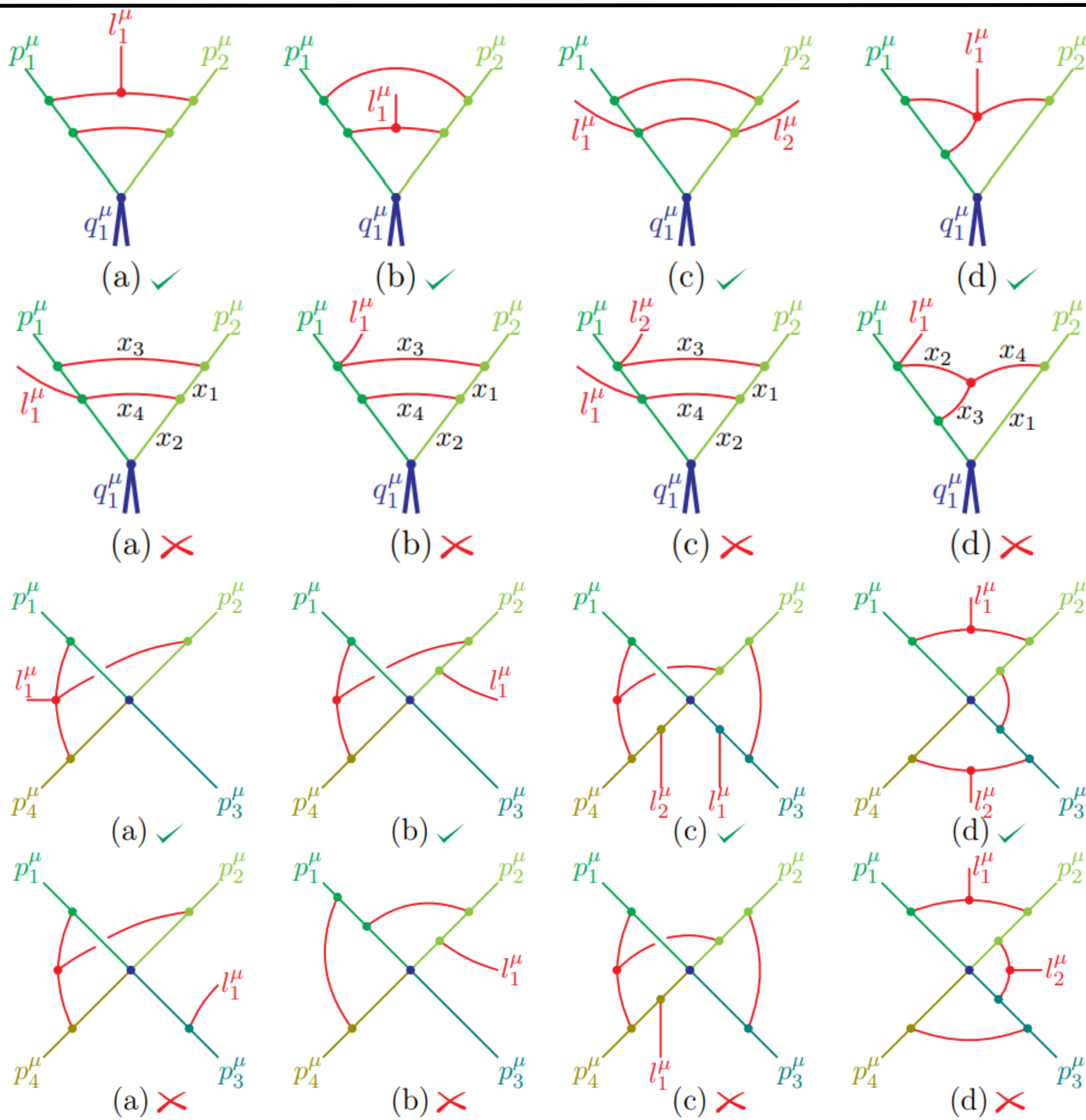
***To identify these regions systematically:***

*Dissect the original polytope into several distinct sectors, such that these regions, which are hidden inside the original polytope, appear as lower facets of the new sub-polytopes.*




(a) $G_{\bullet\bullet}$ (b) $G_{\bullet s}$ (c) $G_{\bullet t}$ (d) $G_{\bullet u}$ (e) $G_{ss}$ (f) $G_{tt}$ (g) $G_{uu}$ (h) $G_{st}$ (i) $G_{su}$ (j) $G_{tu}$

# *Regions in the soft expansion*

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# *Regions in the soft expansion*

- This study may also go beyond QCD.
- For example, some rules for the "Soft-Collinear Gravity" coincide with what we have found:

> graviton attached to a purely soft vertex.
>
> The above argument generalises to the following all-order statement: *In soft loop-corrections to the soft theorem, contrary to the tree-level case, the emitted soft graviton must always attach to a purely-soft vertex, and never directly to any of the energetic particle lines.* The reason is that soft-collinear interactions involve the soft field at the multipole-expanded point $x_-^\mu$ to any order in the $\lambda$-expansion. Hence, if the emitted graviton couples directly to an energetic line, one can always route its momentum such that the entire loop integral will depend only on $n_{i-}k\, n_{i+}^\mu/2$ of a single collinear direction, $i$, and no soft invariant can be formed to provide a scale to the loop diagram.
>
> Continuing with two soft loops, whenever the diagram contains a second purely

(Beneke, Hager, Szafron, "*Soft-Collinear Gravity and Soft Theorems*")

See also

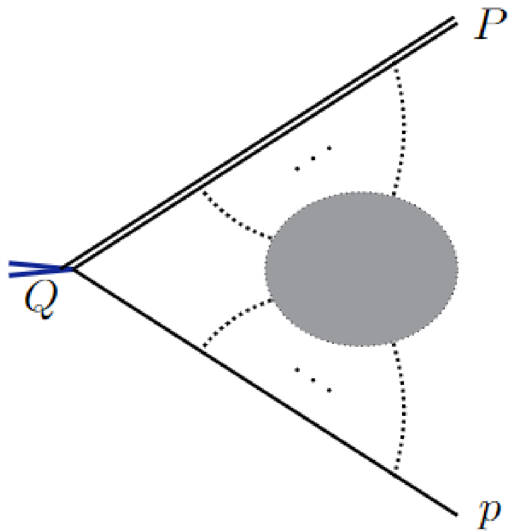
(Beneke, Hager, Szafron, 2021)

(Beneke, Hager, Schwienbacher, 2022)

(Beneke, Hager, Sanfilippo, 2023) et al.

# *The "mass expansion"*

- The heavy-to-light decay process:




$$P^2 = M^2 \sim Q^2, \quad p^2 = m^2 \sim \lambda Q^2, \quad P \cdot p \sim Q^2.$$

**large mass** **small mass**

- In addition to the hard, collinear, and soft modes, more complicated modes can be present.

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# *Regions in the mass expansion*

| | |
|---|---|
| • More modes are included: | Starting from |
| hard mode $Q(1,1,1)$, | 1 loop |
| collinear mode $Q(1,\lambda,\lambda^{1/2})$, | 1 loop |
| soft mode $Q(\lambda,\lambda,\lambda)$, | 2 loops |
| soft·collinear mode $Q(\lambda,\lambda^2,\lambda^{3/2})$, | 3 loops |
| soft² mode $Q(\lambda^2,\lambda^2,\lambda^2)$, …. | 4 loops |
| semihard mode $Q(\lambda^{1/2},\lambda^{1/2},\lambda^{1/2})$, | 2 loops |
| semihard·collinear, semihard·soft, …., | 3 loops, nonplanar |
| semicollinear mode $Q(1,\lambda^{1/2},\lambda^{1/4})$, | 3 loops, nonplanar |
| semihard·semicollinear, …. | 4 loops, nonplanar |

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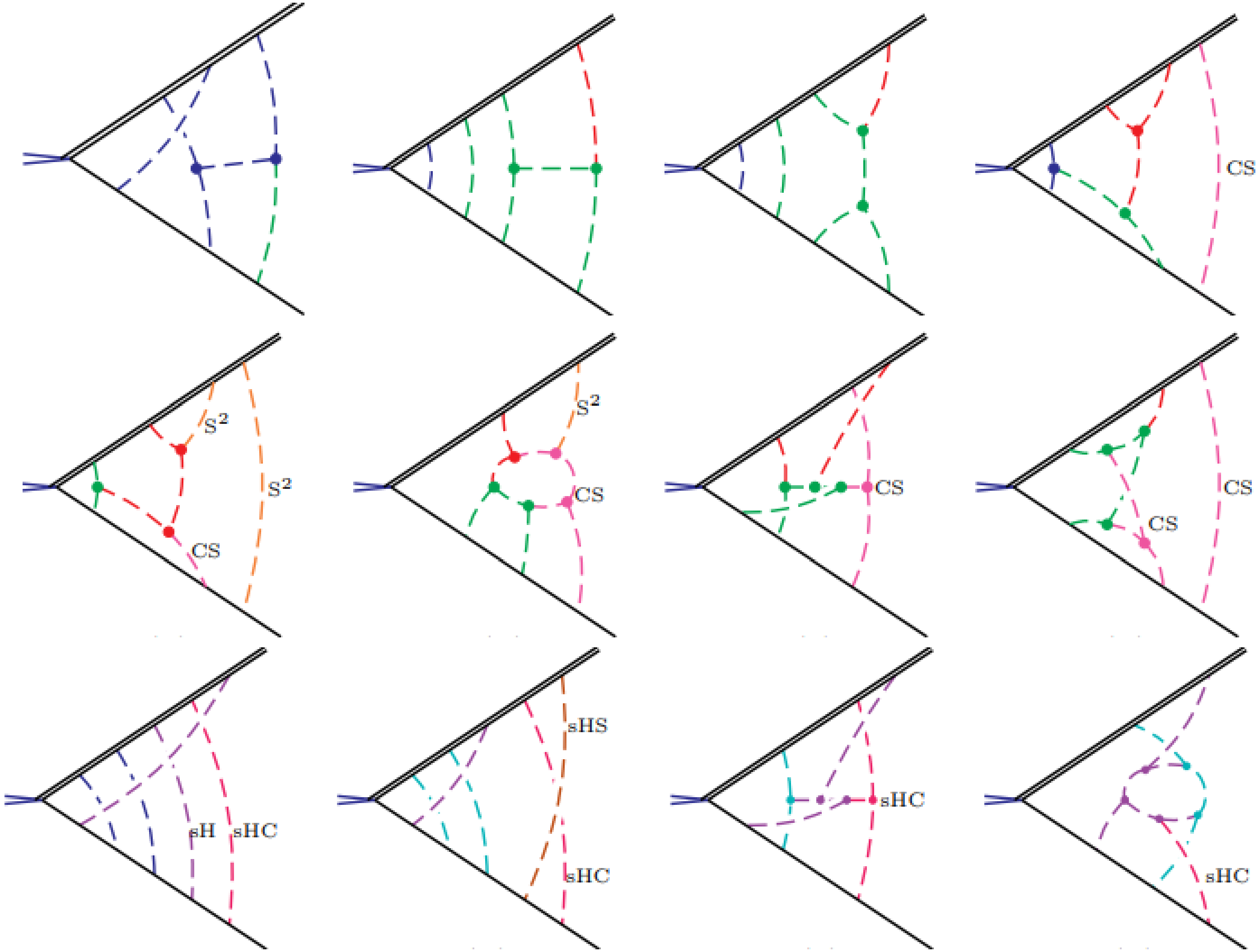
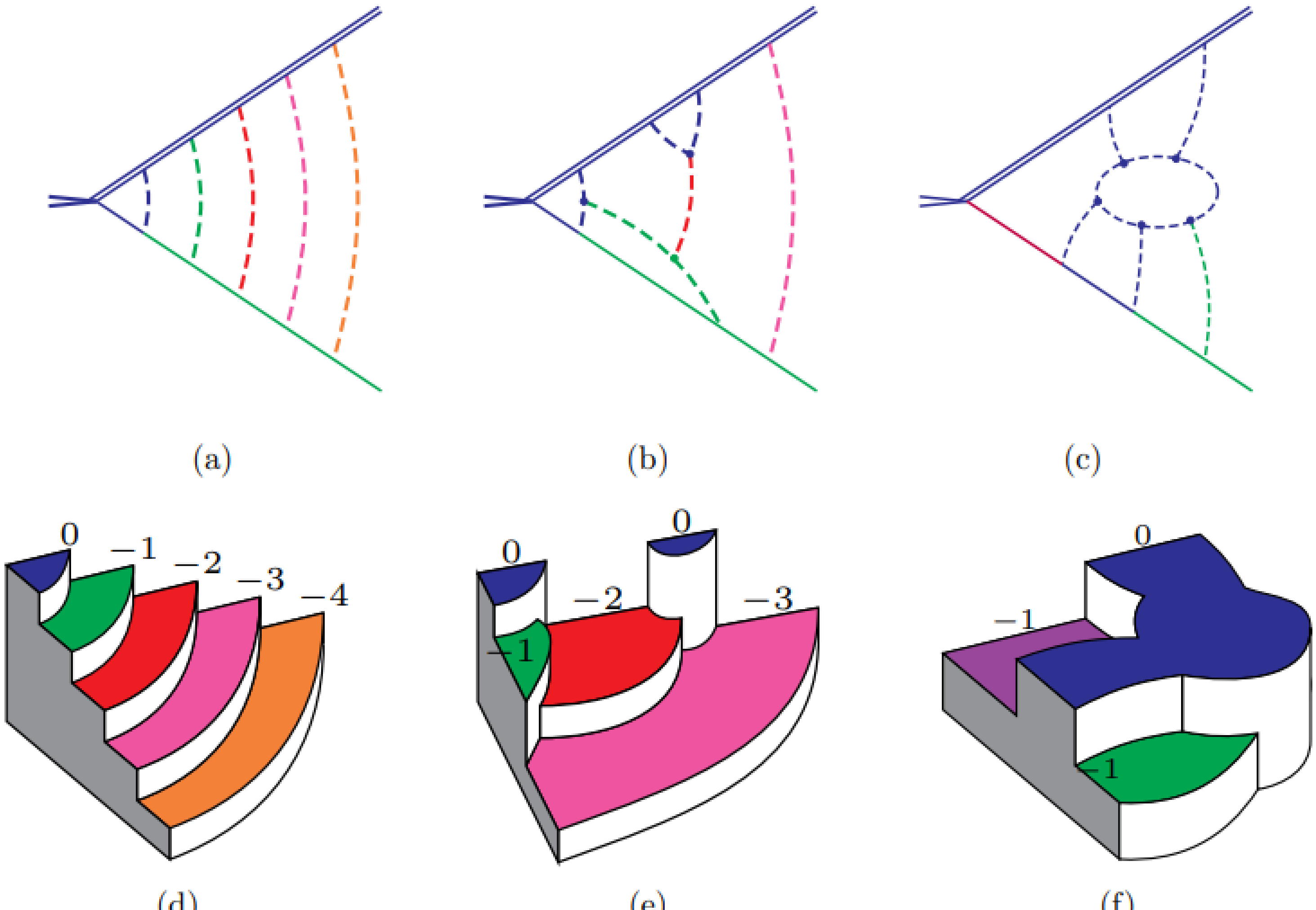
# *Regions in the mass expansion*

- Examples



# *A formalism for planar graphs*

- For planar graphs, each region can be depicted as a "***terrace***".

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# *A formalism for planar graphs*

- For planar graphs, each region can be depicted as a "***terrace***".

(d) (e) (f)

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# Main conclusion 1

The regions corresponding to a given graph can be predicted from the infrared picture!

– on–shell expansion: hard, collinear, soft.

– soft expansion: hard, collinear, soft.

– Regge limit: hard, collinear, soft, Glauber, (collinear)$^2$, soft •collinear, ...

with the mode interactions following certain patterns.

Above shows the "expansion–by–subgraphs" prescription in each given external kinematics.

Can we unify these results?

(a prescription for generic asymptotic expansions?)

# *To unify these prescriptions*

- **Understand the mode structure**
  - **Wide-angle kinematics**
    - **- The mode structure depends on the virtualities of the external momenta.**
    - **- There is a finite number of modes in general.**
  - **Spacelike-collinear kinematics**
    - **- The mode structure can be obtained from above + “spacelike collinearization”.**
    - **- When there are multiple collinear directions, there are infinite modes in general ---“cascade of modes”.**

- **Understand the mode interactions**
  - **How do the mode subgraphs connect to each other?**
  - **Any further requirements of these subgraphs? (necessary and sufficient condition for a region)**

- **Develop a graph-finding algorithm to obtain the regions directly from the graphs**

# *Local infrared subtractions*

- Aim: construct counterterms removing both IR and UV singularities at the level of **integrand**.

- We need the "hard-collinear" and "soft-collinear" approximations that are exactly used for the method of regions.

- Main differences: ① no hard region. ② more nested approx.

- Recent progresses at two loops:

- 2-loop 2→2 wide-angle scattering (Anastasiou & Sterman 2018)
- 2-loop $e^+e^- \to W, Z, \gamma^*$ (Anastasiou, Haindl, Sterman, Yang, Zeng 2020)
- 2-loop $q\bar{q} \to W, Z, \gamma^*$ (Anastasiou & Sterman 2022)
- 2-loop $gg \to h \cdots h$ (Anastasiou, Karlen, Sterman, Venkata 2023)