# *QCD evolution equations at four loops*

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European Research Council
Established by the European Commission

University of Vienna Seminar, Vienna, May 27, 2025

# *Work at four loops:*

- *Non-Singlet Splitting Functions at Four Loops in QCD – The Fermionic Contributions –*
  B. Kniehl, S. M., V. Velizhanin, and A. Vogt arXiv:2505.09381
- *Four-loop splitting functions in QCD – The gluon-to-gluon case –*
  G. Falcioni, F. Herzog, S. M., A. Pelloni and A. Vogt arXiv:2410.08089
- *Constraints for twist-two alien operators in QCD*
  G. Falcioni, F. Herzog, S. M., and S. Van Thurenhout arXiv:2409.02870
- *Four-loop splitting functions in QCD – The quark-to-gluon case –*
  G. Falcioni, F. Herzog, S. M., A. Pelloni and A. Vogt arXiv:2404.09701
- *Additional moments and x-space approximations of four-loop splitting functions in QCD*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt arXiv:2310.05744
- *The double fermionic contribution to the four-loop quark-to-gluon splitting function*
  G. Falcioni, F. Herzog, S. M., J. Vermaseren and A. Vogt arXiv:2310.01245
- *Four-loop splitting functions in QCD – The gluon-to-quark case –*
  G. Falcioni, F. Herzog, S. M., and A. Vogt arXiv:2307.04158
- *Four-loop splitting functions in QCD – The quark-quark case –*
  F. Herzog, G. Falcioni, S. M., and A. Vogt arXiv:2302.07593

# *Work at four loops (cont'd):*

- *Low moments of the four-loop splitting functions in QCD*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:2111.15561`
- *On quartic colour factors in splitting functions and the gluon cusp anomalous dimension*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:1805.09638`
- *Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:1707.08315`

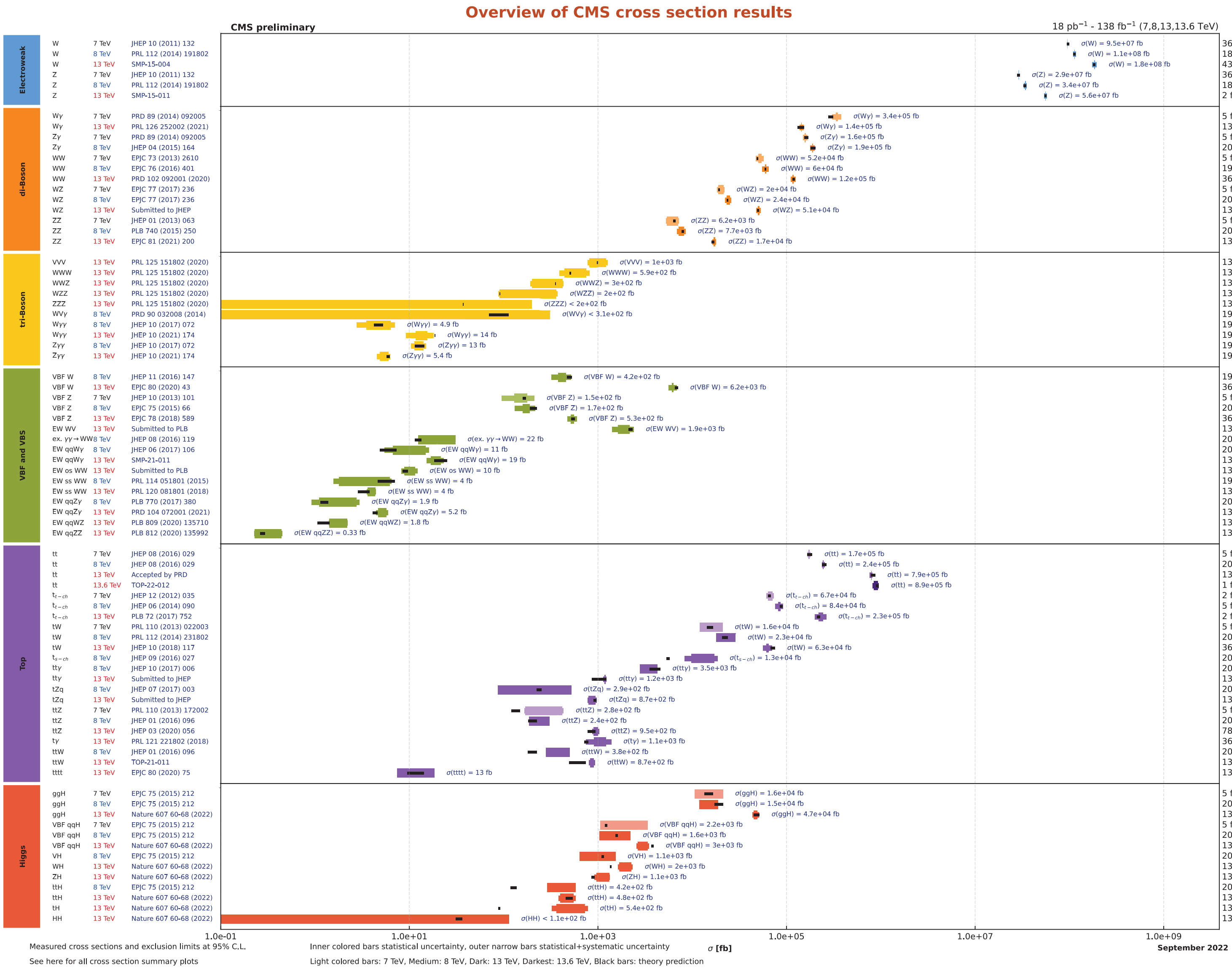
# *Past:*

- Many papers of MVV and friends . . . `2001 - ...`

# *Motivation*

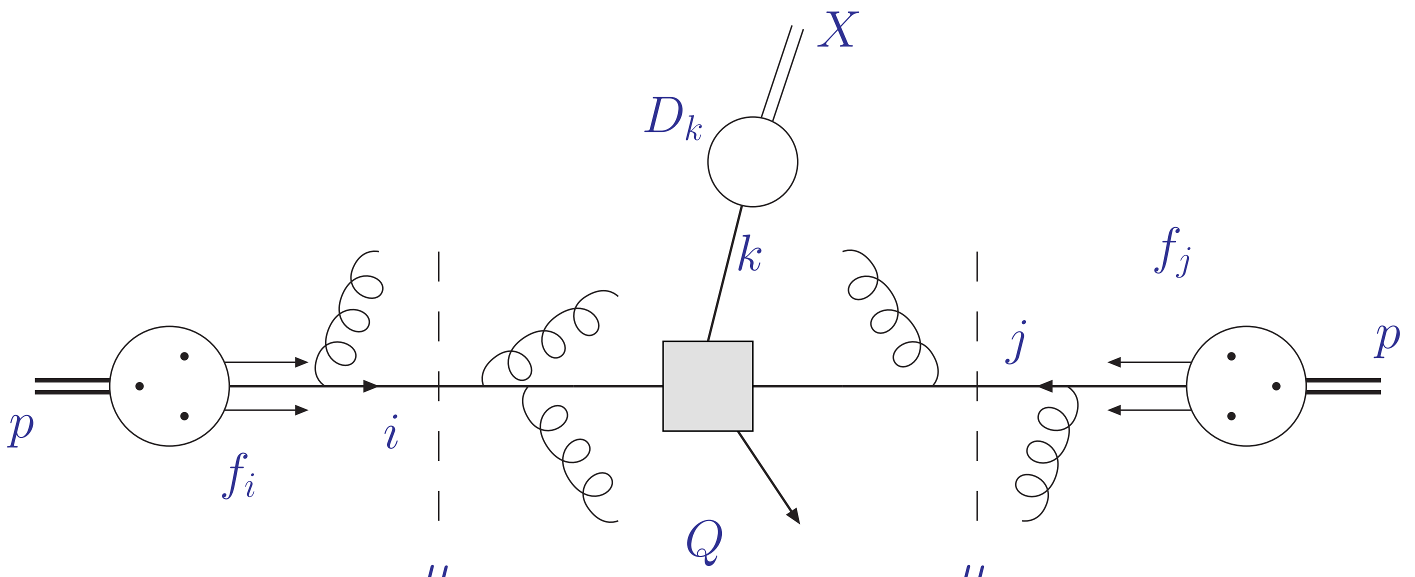
# Standard Model cross sections

- Standard Model cross sections and predictions at the LHC CMS coll. '22



# *QCD factorization*

# QCD factorization

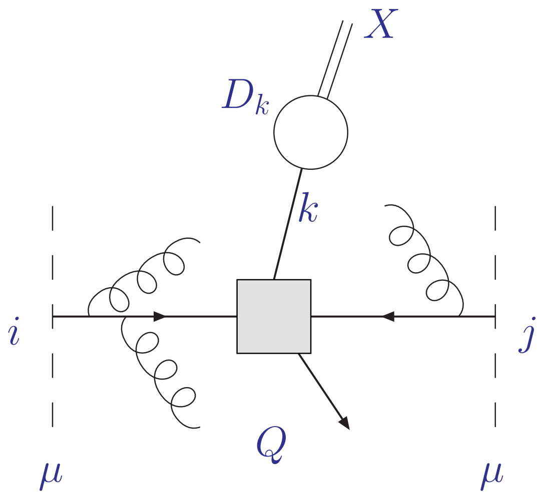



$$\sigma_{pp\to X} = \sum_{ij} f_i(\mu^2) \otimes f_j(\mu^2) \otimes \hat{\sigma}_{ij\to X}\left(\alpha_s(\mu^2), Q^2, \mu^2, m_X^2\right)$$

- Factorization at scale $\mu$
  - separation of sensitivity to dynamics from long and short distances
- Hard parton cross section $\hat{\sigma}_{ij\to X}$ calculable in perturbation theory
  - cross section $\hat{\sigma}_{ij\to k}$ for parton types $i$, $j$ and hadronic final state $X$
- Non-perturbative parameters: parton distribution functions $f_i$, strong coupling $\alpha_s$, particle masses $m_X$
  - known from global fits to exp. data, lattice computations, . . .

# *Hard scattering cross section*

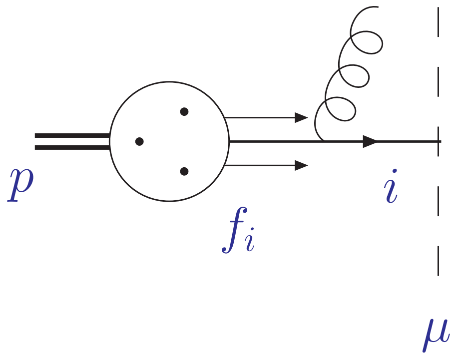
- Parton cross section $\hat{\sigma}_{ij\to k}$ calculable pertubatively in powers of $\alpha_s$
  - known to NLO, NNLO, ... ($\mathcal{O}(\mathrm{few}\%)$ theory uncertainty)

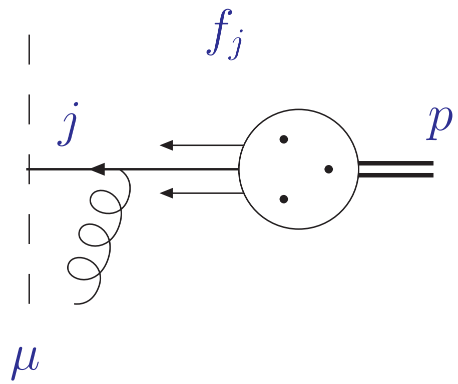



- Accuracy of perturbative predictions
  - LO (leading order) ($\mathcal{O}(50-100\%)$ unc.)
  - NLO (next-to-leading order) ($\mathcal{O}(10-30\%)$ unc.)
  - NNLO (next-to-next-to-leading order) ( $\lesssim$ $\mathcal{O}(10\%)$ unc.)
  - $\mathrm{N}^3$LO (next-to-next-to-next-to-leading order)
  - ...

# *Parton luminosity*

- Long distance dynamics due to proton structure



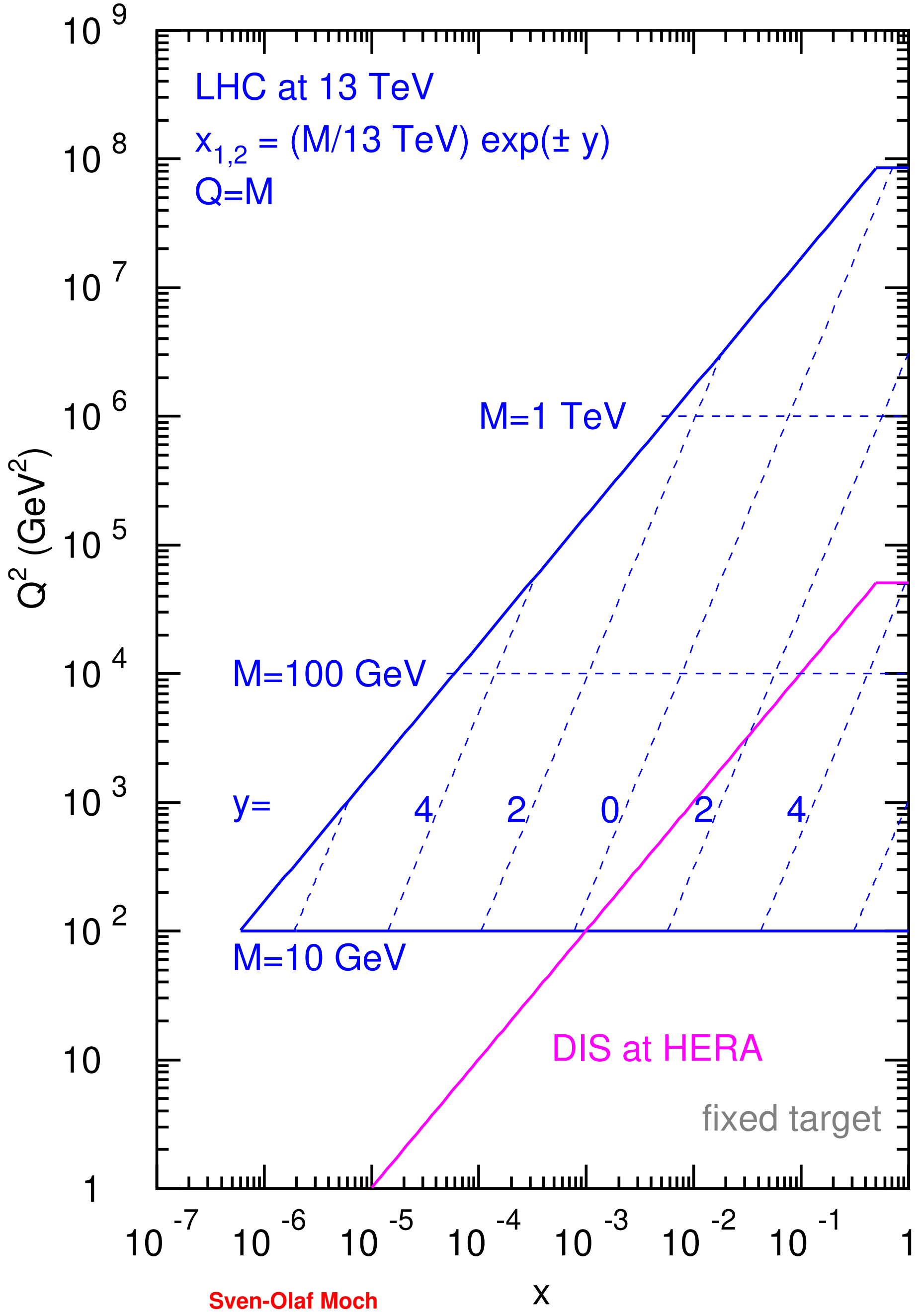




- Cross section depends on parton distributions $f_i$

$$\sigma_{pp \to X} = \sum_{ij} f_i(\mu^2) \otimes f_j(\mu^2) \otimes \left[ \ldots \right]$$

- Parton distributions known from global fits to exp. data
  - available fits accurate to NNLO
  - information on proton structure depends on kinematic coverage

# *Parton kinematics at LHC*

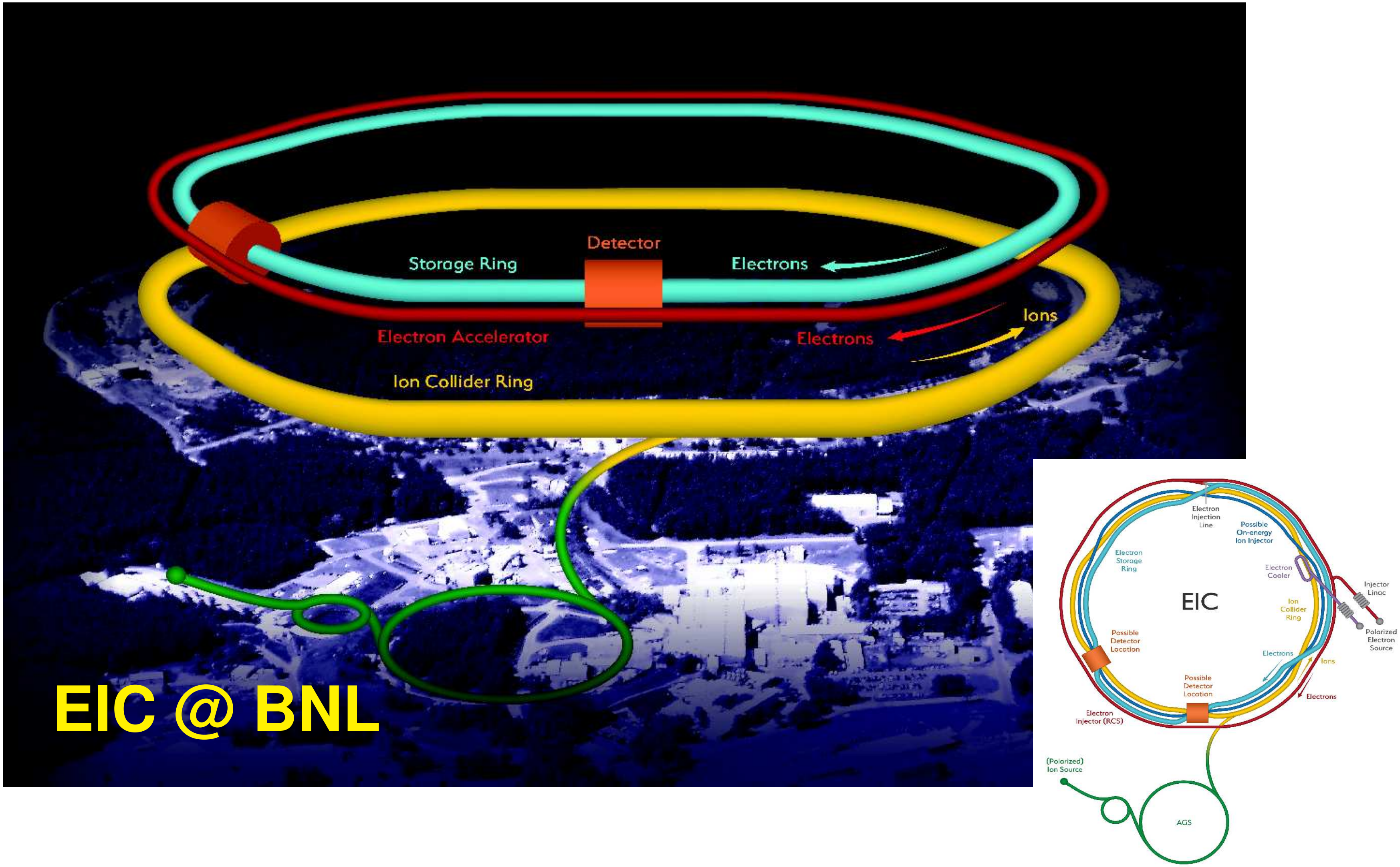



- Information on proton structure depends on kinematic coverage
  - summary of collider kinematics
- LHC run at $\sqrt{s}$ = 13 TeV
  - parton kinematics well covered by HERA and fixed target experiments
- Parton kinematics with $x_{1,2} = M/\sqrt{S}e^{\pm y}$
  - forward rapidities sensitive to small-$x$
  - high scales $Q \simeq 1$ TeV probe large $x \simeq 0.1 \ldots 0.8$

*EIC*

# *Bright future for precision hadron physics*

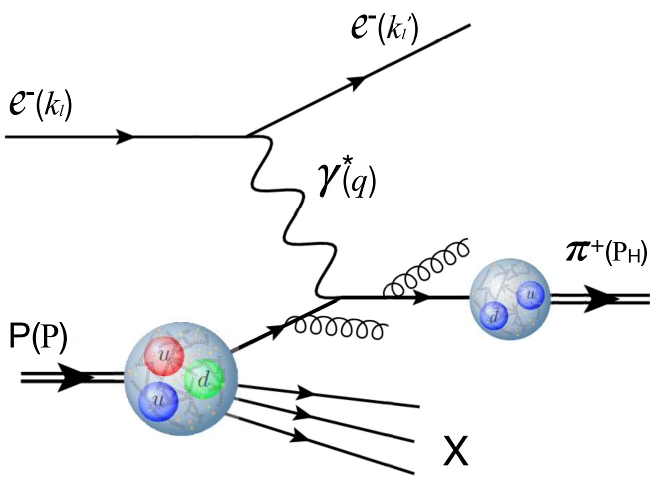
- Electron-Ion Collider
  *A machine that will unlock the secrets of the strongest force in Nature*



# *Deep-inelastic scattering*

- DIS structure functions
  - unpolarized $F_2$, $F_L$, $F_3$
  - spin dependent $g_1$

- Semi-inclusive DIS
  - production of identified hadrons in DIS
  - multiple hadron species: $\pi$, K, D, p, n, $\Lambda$, $\ldots$
  - probe of hadron structure
    in broad kinematic range

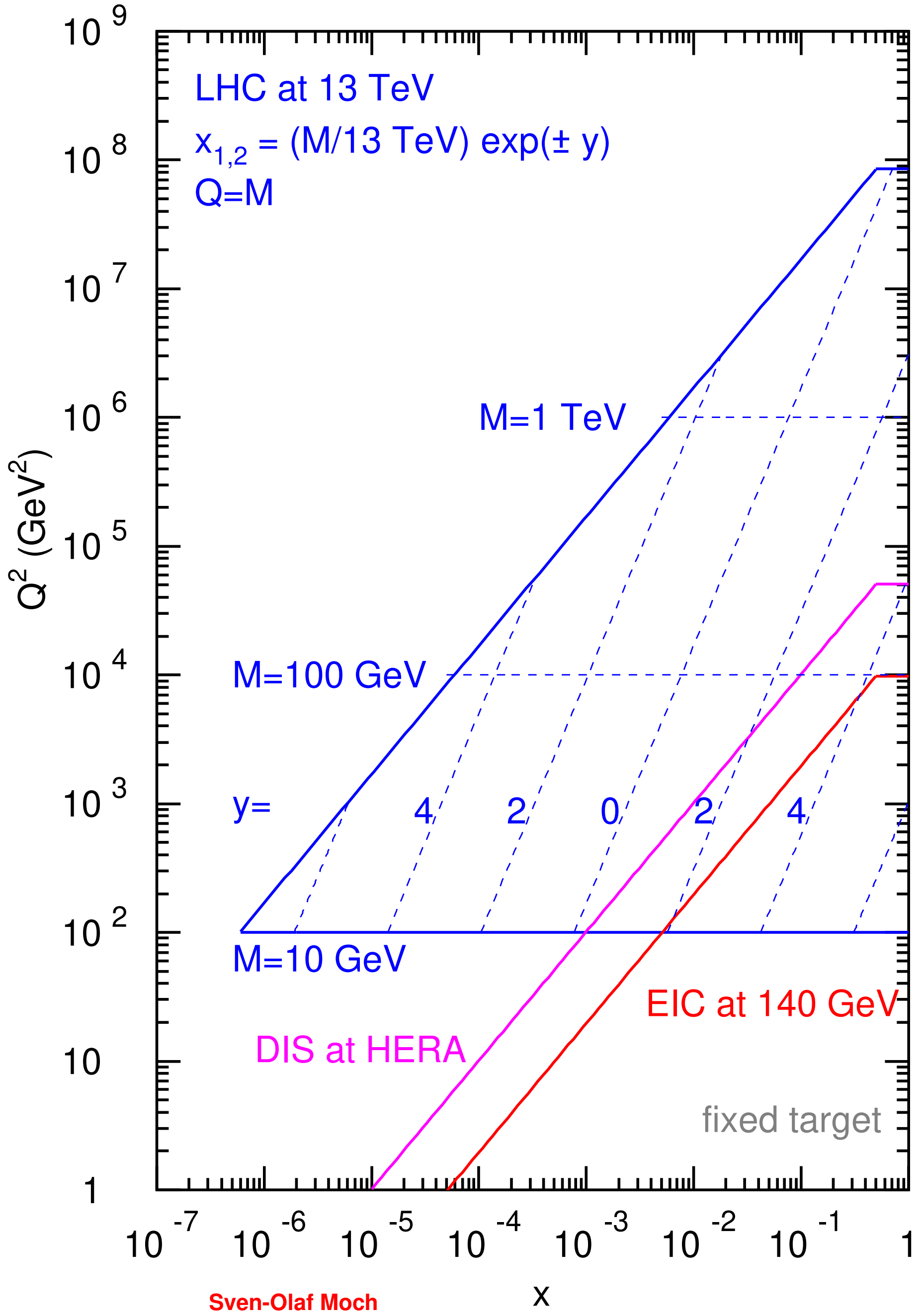



- QCD factorization at scale $\mu^2$

$$\sigma_{\gamma H \to H'} = \sum_{ij} f_{i/H}(\mu^2) \otimes \hat{\sigma}_{\gamma i \to j}\left(\alpha_s(\mu^2), Q^2, \mu^2\right) \otimes D_{H'/j}(\mu^2)$$

  - parton distribution function (PDF) $f_{i/H}(x, \mu^2)$
  - parton-to-hadron fragmentation function (FF) $D_{H'/j}(z, \mu^2)$

# Parton kinematics at EIC




- EIC run at $\sqrt{s}$ = 140 GeV
  - $ep$-collisions at EIC cover large part of phase space relevant for LHC
  - overlap with HERA and fixed target experiments

*Novel measurements at EIC*

- 3D-images of hadron in position and momentum space (including spin)
- Measurements with unprecedented precision

# *Parton evolution*

# Parton evolution

- Evolution equations for parton distributions
  - non-singlet valence PDFs $q_{\rm ns}^{\rm v} = \sum_f (q_f - \bar{q}_f)$
  - flavor asymmetries $q_{{\rm ns},ff'}^{\pm} = (q_f \pm \bar{q}_f) - (q_{f'} \pm \bar{q}_{f'})$

$$\frac{d}{d\ln\mu^2} q_{\rm ns}^{\pm,{\rm v}} = P_{\rm ns}^{\pm,{\rm v}} \otimes q_{\rm ns}^{\pm,{\rm v}}$$

  - quark-flavor singlet PDFs $q_{\rm s} = \sum_f (q_f + \bar{q}_f)$ and gluon PDF $g$
  - 2x2 matrix equation

$$\frac{d}{d\ln\mu^2} \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix} = \begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix} \otimes \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix}$$

- Splitting functions $P$ up to N$^3$LO (work in progress)

$$P_{\rm ij} = \underbrace{\alpha_s P_{\rm ij}^{(0)} + \alpha_s^2 P_{\rm ij}^{(1)} + \alpha_s^3 P_{\rm ij}^{(2)}}_{\text{NNLO: standard approximation}} + \alpha_s^4 P_{\rm ij}^{(3)} + \ldots$$

- Anomalous dimensions (Mellin transform)

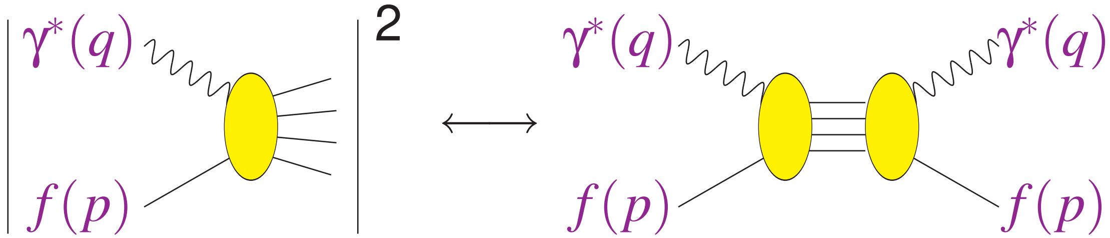
$$\gamma_{\rm ij}(N) = -\int_0^1 dx\, x^N P_{\rm ij}(x) = \alpha_s \gamma_{\rm ij}^{(0)} + \alpha_s^2 \gamma_{\rm ij}^{(1)} + \alpha_s^3 \gamma_{\rm ij}^{(2)} + \alpha_s^4 \gamma_{\rm ij}^{(3)} + \ldots$$

# *Operator product expansion (I)*

- Direct computation of physical observable
  - structures functions in deep-inelastic scattering (DIS)

## *Optical theorem*

- Total cross section related to imaginary part of Compton amplitude
  - Bjorken variable $x = Q^2/(2p \cdot q)$ and momentum transfer $Q^2 = -q^2$




- Optical theorem relates hadronic tensor $W_{\mu\nu}$ to imaginary part of Compton amplitude $T_{\mu\nu} = \mathrm{i} \int d^4z\, \mathrm{e}^{\mathrm{i}q\cdot z} \langle \mathrm{P} | \, T\left( j^\dagger_\mu(z) j_\nu(0) \right) | \mathrm{P} \rangle$

$$W_{\mu\nu} = e_{\mu\nu} \frac{1}{2x} F_L(x, Q^2) + d_{\mu\nu} \frac{1}{2x} F_2(x, Q^2) + \mathrm{i}\epsilon_{\mu\nu\alpha\beta} \frac{p^\alpha q^\beta}{p \cdot q} F_3(x, Q^2)$$

- OPE of $T_{\mu\nu}$ for short distances $z^2 \simeq 0$ in Bjorken limit $Q^2 \to \infty$, $x$ fixed
  Wilson ‘72; Christ, Hasslacher, Mueller ‘72

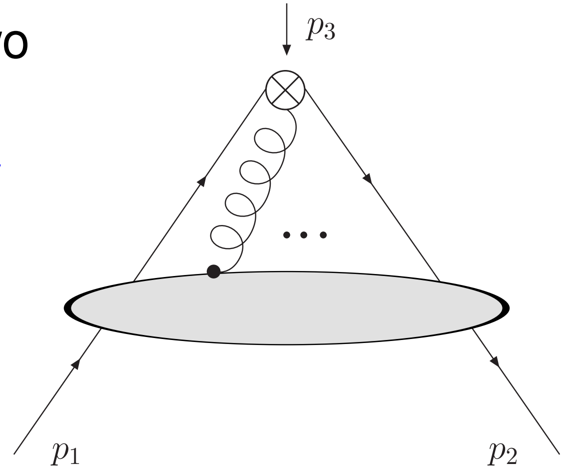
# *Operator product expansion (II)*

- OPE for parton states gives coefficient functions in Mellin space $C^N_{a,i}$

$$T_{\mu\nu,\mathrm{k}} = \sum_{N,\mathrm{j}} \left(\frac{1}{2x}\right)^N \left[ e_{\mu\nu}\, C^N_{L,\mathrm{j}}\left(\frac{Q^2}{\mu^2},\alpha_s\right) + d_{\mu\nu}\, C^N_{2,\mathrm{j}}\left(\frac{Q^2}{\mu^2},\alpha_s\right) + \mathrm{i}\epsilon_{\mu\nu\alpha\beta}\frac{p^\alpha q^\beta}{p\cdot q} C^N_{3,\mathrm{j}}\left(\frac{Q^2}{\mu^2},\alpha_s\right)\right] A^N_{\mathrm{jk}}\left(\mu^2\right) + \text{ higher twists}$$

- Operator matrix elements $A^N_{\mathrm{ij}} = \langle j|O^N_{\mathrm{i}}|j\rangle$ in parton state
- Anomalous dimensions $\gamma_{\mathrm{ij}}(N)$ from collinear singularities of Compton amplitude $T_{\mu\nu}$ after mass factorization
  - established computational approach through four loops
    one loop Buras '80; two loops Kazakov, Kotikov '90; S.M., Vermaseren '99; three loops S.M., Vermaseren, Vogt '04; four loops Davies, Vogt, Ruijl, Ueda, Vermaseren '17; S.M., Ruijl, Ueda, Vermaseren, Vogt to appear
- Versatile calculation method
  - photon-DIS $\longrightarrow \gamma_{\mathrm{qq}}\,, \gamma_{\mathrm{qg}}$
  - Higgs (scalar)-DIS $\longrightarrow \gamma_{\mathrm{gq}}\,, \gamma_{\mathrm{gg}}$
  - graviton-DIS $\longrightarrow \Delta\gamma_{\mathrm{ij}}$ (polarized quantities) S.M., Vermaseren, Vogt '14

# *Operator matrix elements*




- Scalar singlet operators of spin-$N$ and twist two from contraction with light-like vector $\Delta_\mu$
  - quarks $\psi$, field strenghth $F^{\mu;a} = \Delta_\nu F^{\mu\nu;a}$
  - $N$ covariant derivatives $D = \Delta_\mu D^\mu$

$$\mathcal{O}_{\rm q}^{(N)} = \frac{1}{2}\text{Tr}\left[\overline{\psi}\not\Delta\, D^{N-1}\psi\right]$$

$$\mathcal{O}_{\rm g}^{(N)} = \frac{1}{2}\text{Tr}\left[F_\nu\, D^{N-2}F^\nu\right]$$

- Direct computation of OMEs $A_{\rm ij}^N = \langle j|O_{\rm i}^N|j\rangle$ in parton state
  - anomalous dimensions $\gamma_{\rm ij}(N)$ from renormalization of operators
- Physical operators mix under renormalization with alien operators
  Dixon, Taylor '74; Kluberg-Stern, Zuber '75; Joglekar, Lee '76

# *Workflow*

- Zero-momentum transfer through operator gives 2-point functions
- Feynman diagrams generation with **Qgraf** Nogueira '91
- Four-loop IBP reduction with **Forcer** Ruijl, Ueda, Vermaseren '17
- Symbolic manipulations with **Form** Vermaseren '00; Kuipers, Ueda, Vermaseren, Vollinga '12 and **TForm** Tentyukov, Vermaseren '07

## *Quark pure-singlet splitting function* $P_{\rm qq} = P_{\rm ns}^{+} + P_{\rm ps}$

$$\begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix}$$

# Moments of pure-singlet splitting function

- Moments $N = 2, \dots 20$ for pure-singlet anomalous dimension $\gamma_{\rm ps}^{(3)}(N)$

$$
\begin{aligned}
\gamma_{\rm ps}^{(3)}(N\!=\!2) &= -691.5937093\, n_f + 84.77398149\, n_f^{\,2} + 4.466956849\, n_f^{\,3}\,, \\
\gamma_{\rm ps}^{(3)}(N\!=\!4) &= -109.3302335\, n_f + 8.776885259\, n_f^{\,2} + 0.306077137\, n_f^{\,3}\,, \\
&\dots \\
\gamma_{\rm ps}^{(3)}(N\!=\!20) &= -0.442681568\, n_f + 0.805745333\, n_f^{\,2} - 0.020918264\, n_f^{\,3}\,.
\end{aligned}
$$

- Results $N \leq 8$ agree with inclusive DIS S.M., Ruijl, Ueda, Vermaseren, Vogt '21 (also for $N = 10$ and $N = 12$)
- Quartic color terms $d_R^{abcd}d_R^{abcd}$ agree with S.M., Ruijl, Ueda, Vermaseren, Vogt '18
- Large-$n_f$ parts agree with all-$N$ results Davies, Vogt, Ruijl, Ueda, Vermaseren '17;
- $\zeta_4$ terms in $\gamma_{\rm ps}^{(3)}(N)$ agree with Davies, Vogt '17 based on no-$\pi^2$ theorem Jamin, Miravitllas '18; Baikov, Chetyrkin '18
- Checked by $n_f^2$ terms at all-$N$ Gehrmann, von Manteuffel, Sotnikov, Yang '23

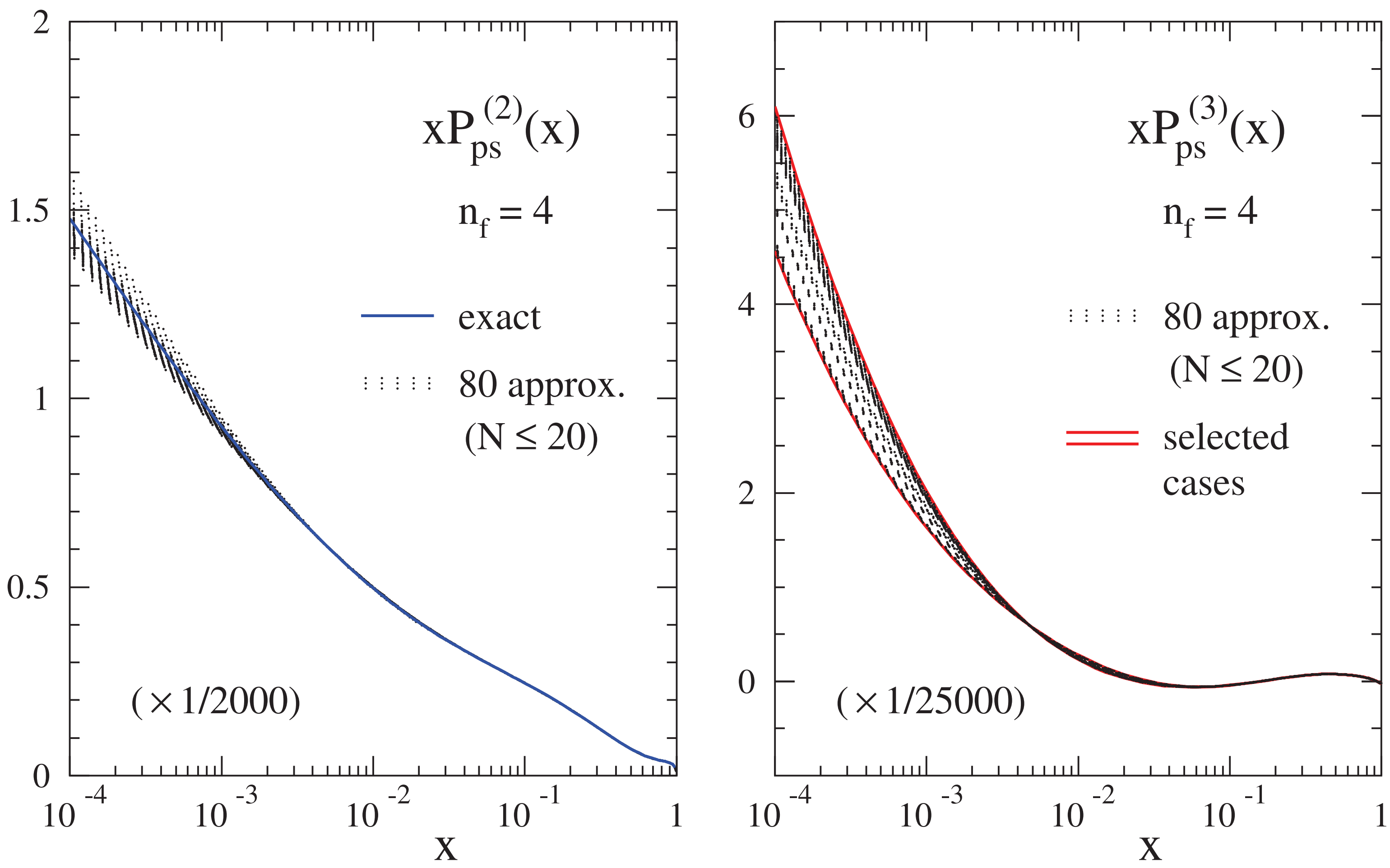
## Outlook

- Higher moments $N = 22, \dots$ to be published

# Approximations in $x$-space

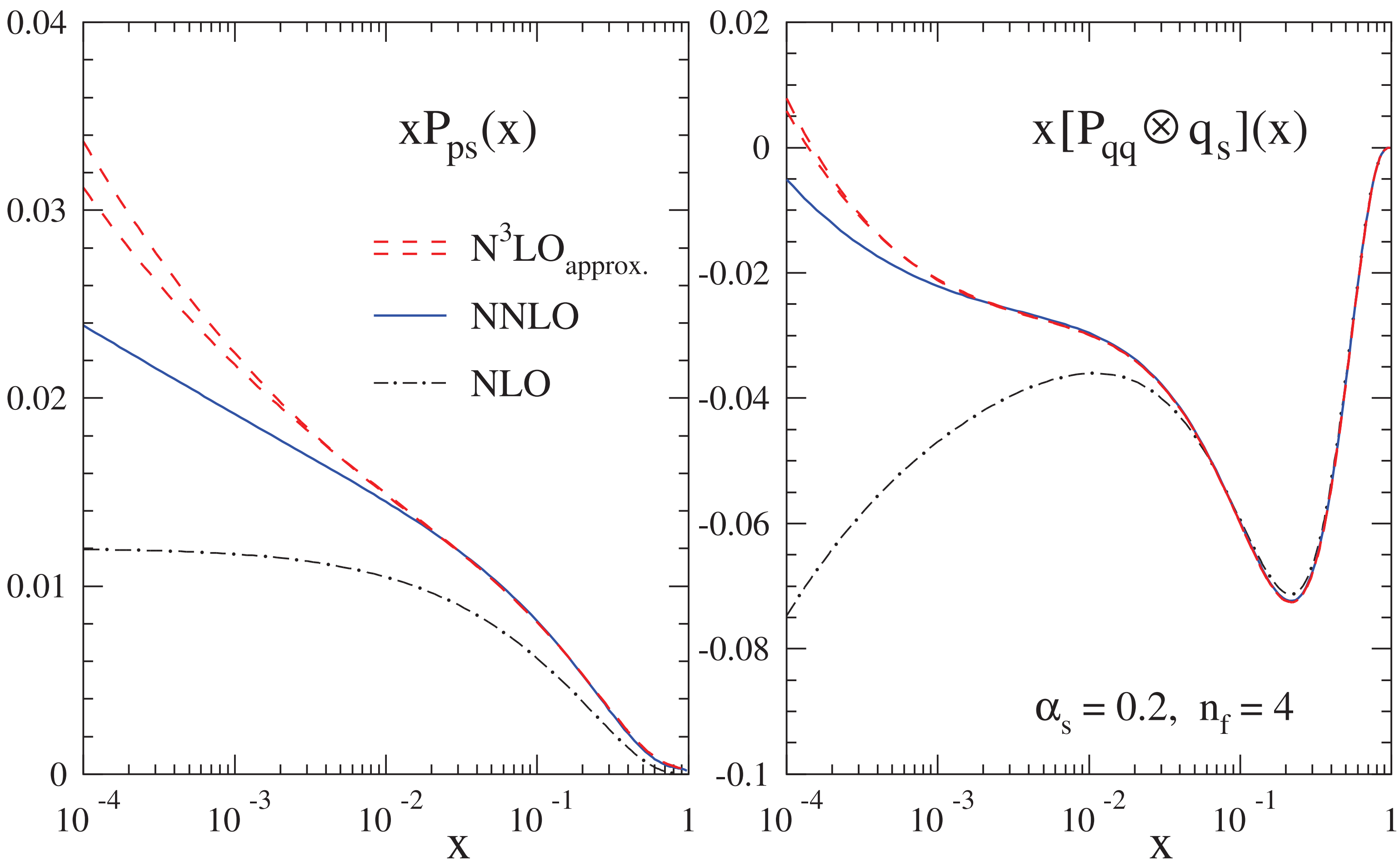
- Large- and small-$x$ information about four-loop splitting function $P_{\rm ps}^{(3)}(x)$
  - leading logarithm $(\ln^2 x)/x$ — Catani, Hautmann '94
  - sub-dominant logarithms $\ln^k x$ with $k = 6, 5, 4$ — Davies, Kom, S.M., Vogt '22
  - leading large-$x$ terms $(1-x)^j \ln^k(1-x)$ with $j \geq 1$ and $k \leq 4$ with $k = 4, 3$ known — Soar, S.M., Vermaseren, Vogt '09
- Approximation of four-loop splitting function $P_{\rm ps}^{(3)}(x)$ with suitable ansatz
  - unknown leading small-$x$ terms: $(\ln x)/x$, $1/x$
  - unknown sub-dominant logarithms: $\ln^k x$ with $k = 3, 2, 1$
  - two remaining large-$x$ terms $(1-x)\ln^k(1-x)$ with $k = 2, 1$
  - different two-parameter polynomials together one function (dilogarithm $\mathrm{Li}_2(x)$ or $\ln^k(1-x)$ with $k = 2, 1$, suppressed as $x \to 1$)
- Approximations for phenomenology with fixed $n_f = 3, 4, 5$
  - easy-to-use
  - no correlations between different $n_f$ dependenct terms accounted for

# *Pure-singlet splitting function*




- Approximations to pure-singlet splitting function $P_{\rm ps}^{(n)}(x)$ at $n_f = 4$ with 80 trial functions
  - left: three-loops ($n = 2$) with comparison to known result
  - right: four-loops ($n = 3$) with remaining uncertainty

# *Pure-singlet splitting function*




- Left: results for $P_{\rm ps}(x)$ up to N$^3$LO; $\alpha_s \;=\; 0.2$ fixed, $n_f \;=\; 4$
- Right: contribution to evolution kernel $d\ln q_{\rm s}/d\ln \mu_f^2$ up to N$^3$LO for typical quark-singlet shape

$$xq_{\,\rm s}(x,\mu_0^2) \;=\; 0.6\,x^{\,-0.3}(1-x)^{3.5}\,(1+5.0\,x^{\,0.8})$$

# *Scale stability of evolution (I)*

- PDF evolution
  - splitting functions enter PDF evolution via convolution

$$\frac{d}{d \ln \mu^2} \, f_{\rm i}(x) \quad = \quad \sum_{\rm j} \int_x^1 \frac{dz}{z} \, P_{\rm ij}(z) \, f_{\rm j}\left(\frac{x}{z}\right)$$

- Interplay between $P(z \sim x \to 0)$ and $f(\frac{x}{z} \to 1)$
  - $P(\mathbf{z} \sim \mathbf{x} \to \mathbf{0})$ has largest uncertainty
  - $f(\frac{\mathbf{x}}{\mathbf{z}} \to \mathbf{1})$ is suppressed
- Model singlet PDFs

$$x{\rm q}_{\,\rm s}(x,\mu_0^2) \quad = \quad 0.6\, x^{\,-0.3}(\mathbf{1-x})^{\mathbf{3.5}}\,(1+5.0\,x^{\,0.8})$$

$$x{\rm g}(x,\mu_0^2) \quad = \quad 1.6\, x^{\,-0.3}(\mathbf{1-x})^{\mathbf{4.5}}\,(1-0.6\,x^{\,0.3})$$

- Residual small-$x$ uncertainty in four-loop splitting functions at $x \sim \mathcal{O}(10^{-4})$ affects PDFs only at $x \sim \mathcal{O}(10^{-5})$
  - edge of LHC parton kinematics (low scales, forward region)
  - $x \sim 10^{-5}$ corresponds to $y \sim 4$ and $Q \sim 10$ GeV

# *Gluon-to-quark splitting function* $P_{\rm qg}$

$$\begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix}$$

# *Moments of gluon-to-quark splitting function*

- Moments $N = 2, \ldots 20$ for gluon-to-quark anomalous dimension $\gamma_{\rm qg}^{(3)}(N)$

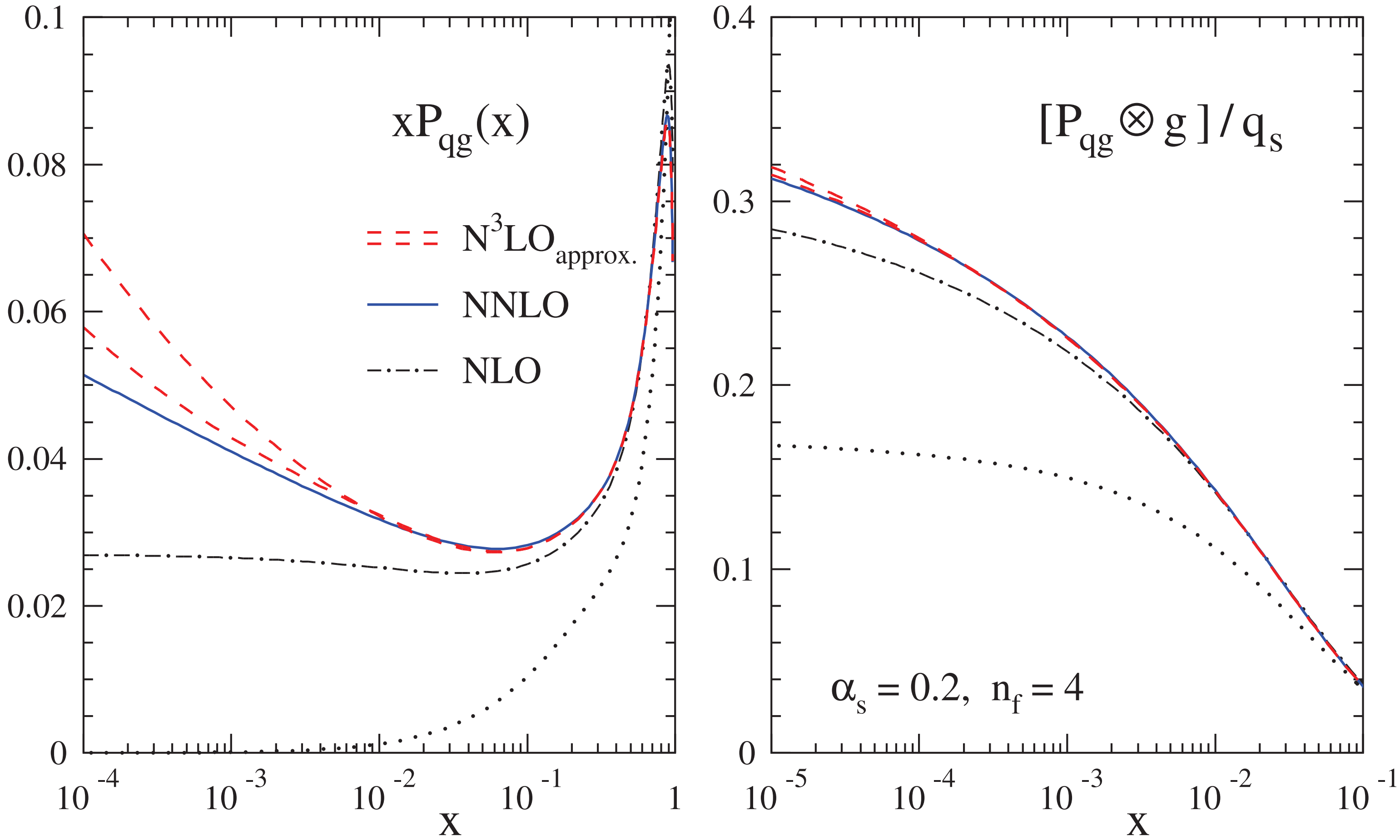
$$\gamma_{\rm qg}^{(3)}(N\!=\!2) \quad = \quad -654.4627782\, n_f + 245.6106197\, n_f^2 - 0.924990969\, n_f^3 \,,$$
$$\gamma_{\rm qg}^{(3)}(N\!=\!4) \quad = \quad 290.3110686\, n_f - 76.51672403\, n_f^2 - 4.911625629\, n_f^3 \,,$$
$$\gamma_{\rm qg}^{(3)}(N\!=\!6) \quad = \quad 335.8008046\, n_f - 124.5710225\, n_f^2 - 4.193871425\, n_f^3 \,,$$
$$\gamma_{\rm qg}^{(3)}(N\!=\!8) \quad = \quad 294.5876830\, n_f - 135.3767647\, n_f^2 - 3.609775642\, n_f^3 \,,$$
$$\ldots$$
$$\gamma_{\rm qg}^{(3)}(N\!=\!20) \quad = \quad 52.24329555\, n_f - 109.3424891\, n_f^2 - 2.153153725\, n_f^3 \,.$$

- Approximation of four-loop splitting function $P_{\rm qg}^{(3)}(x)$ again with known large- and small-$x$ information and suitable ansatz

## *Outlook*

- Higher moments $N = 22, \ldots$ to be published

# *Gluon-to-quark splitting function*




- Left: results for $P_{\rm qg}(x)$ up to N$^3$LO; $\alpha_s \;=\; 0.2$ fixed, $n_f \;=4$
- Right: contribution to evolution kernel $d\ln g/d\ln\mu_f^2$ up to N$^3$LO for typical gluon shape

$$xg(x,\mu_0^2) \;=\; 1.6\,x^{\,-0.3}(1-x)^{4.5}\;\left(1-0.6\,x^{\,0.3}\right)$$

## *Quark-to-gluon splitting function* $P_{\rm gq}$

$$\begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix}$$

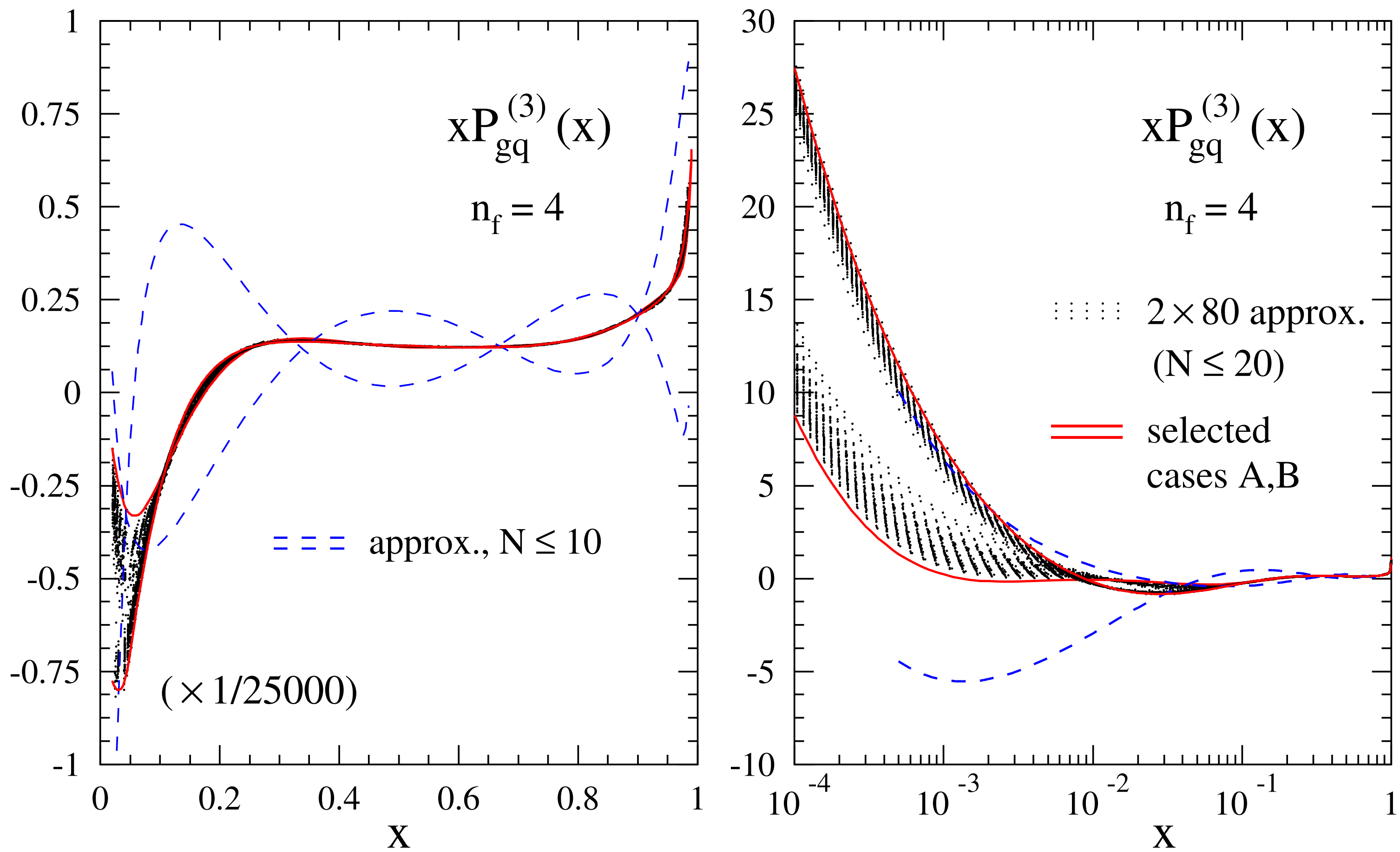
# *Moments of quark-to-gluon splitting function*

- Moments for quark-to-gluon anomalous dimension $\gamma_{\rm gq}^{(3)}(N)$
  - moments $N = 2, \ldots 10$ — S.M., Ruijl, Ueda, Vermaseren, Vogt '23
  - moments $N = 12, \ldots 20$ — Falcioni, Herzog, S.M., Pelloni, Vogt '24

$$
\begin{aligned}
\gamma_{\rm gq}^{(3)}(N\!=\!2) &= -16663.2255 + 4439.14375\, n_f - 202.555479\, n_f^2 - 6.37539072\, n_f^3 \,, \\
\gamma_{\rm gq}^{(3)}(N\!=\!4) &= -6565.73145 + 1291.06746\, n_f - 16.1461902\, n_f^2 - 0.83976340\, n_f^3 \,, \\
\gamma_{\rm gq}^{(3)}(N\!=\!6) &= -3937.47937 + 679.718506\, n_f - 1.37207753\, n_f^2 - 0.13979433\, n_f^3 \,, \\
\gamma_{\rm gq}^{(3)}(N\!=\!8) &= -2803.64411 + 436.393057\, n_f + 1.81494624\, n_f^2 + 0.07358858\, n_f^3 \,, \\
&\ldots \\
\gamma_{\rm gq}^{(3)}(N\!=\!20) &= -1054.26140 + 105.497994\, n_f + 2.39223577\, n_f^2 + 0.19938504\, n_f^3 \,.
\end{aligned}
$$

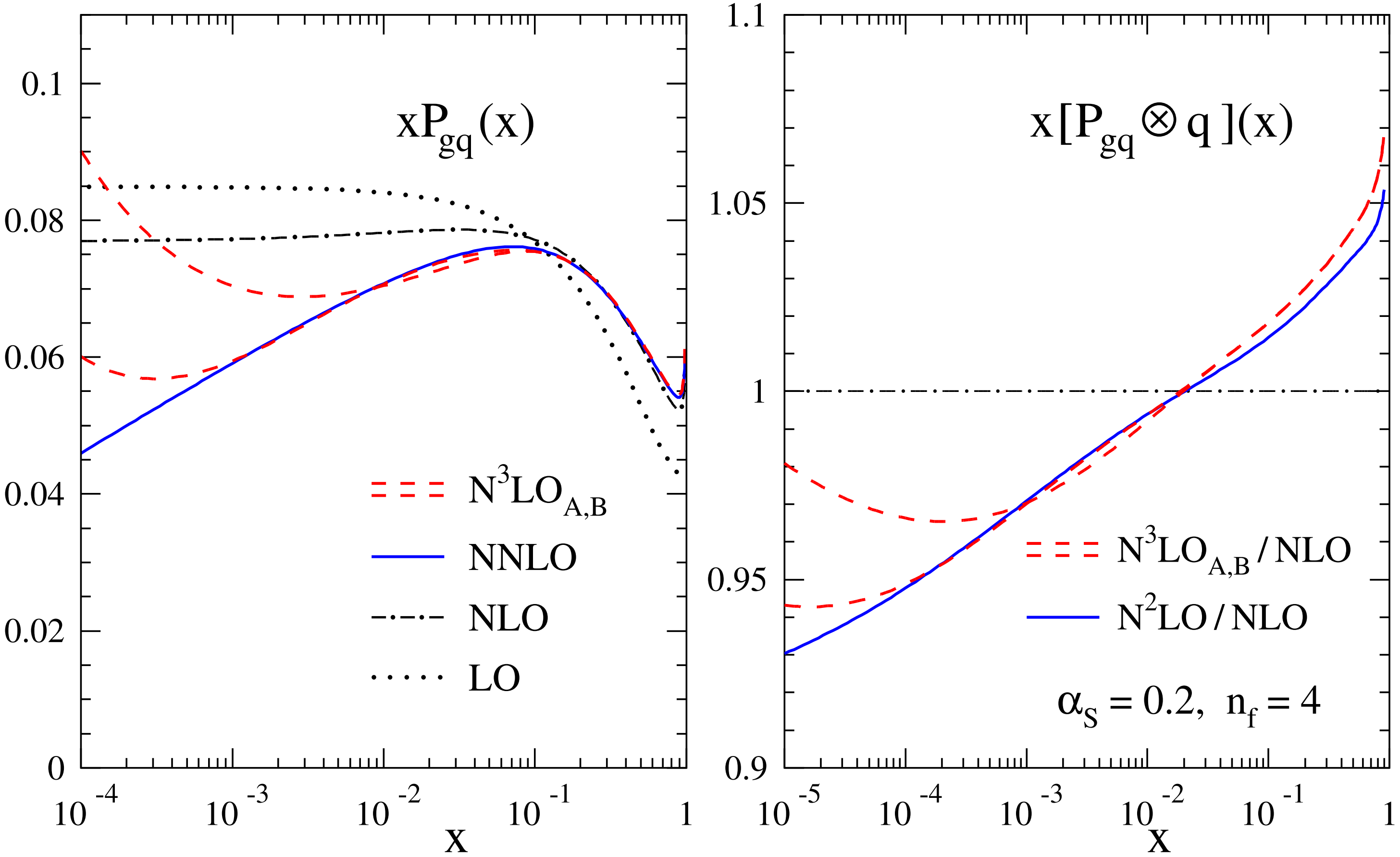
- Approximation of four-loop splitting function $P_{\rm gq}^{(3)}(x)$ again with known large- and small-$x$ information and suitable ansatz

# *Quark-to-gluon splitting function (I)*




- Approximations for $P_{\rm gq}^{(3)}(x)$ based on moments $N \leq 10$ vs. $N \leq 20$
  - clear improvements at large-$x$ (left) and small-$x$ (right)

# Quark-to-gluon splitting function (II)




- Left: results for $P_{gq}(x)$ up to N$^3$LO; $\alpha_s = 0.2$ fixed, $n_f = 4$
- Right: contribution to evolution kernel $d \ln q_s / d \ln \mu_f^2$ up to N$^3$LO for typical quark-singlet shape

$$xq_s(x, \mu_0^2) = 0.6\, x^{-0.3}(1-x)^{3.5}\,(1 + 5.0\, x^{0.8})$$

# *Gluon-gluon splitting function* $P_{\rm gg}$

$$\begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix}$$

# *Moments of gluon-gluon splitting function*

- Moments for gluon–gluon anomalous dimension $\gamma_{\rm gg}^{(3)}(N)$
  - moments $N = 2, \ldots 10$ — S.M., Ruijl, Ueda, Vermaseren, Vogt '23
  - Moments $N = 12, \ldots 20$ — Falcioni, Herzog, S.M., Pelloni, Vogt '24

$$
\begin{aligned}
\gamma_{\rm gg}^{(3)}(N=2) &= 654.462778\, n_f - 245.610620\, n_f^2 + 0.92499097\, n_f^3 \,, \\
\gamma_{\rm gg}^{(3)}(N=4) &= 39876.1233 - 10103.4511\, n_f + 437.098848\, n_f^2 + 12.9555655\, n_f^3 \,, \\
\gamma_{\rm gg}^{(3)}(N=6) &= 53563.8435 - 14339.1310\, n_f + 652.777331\, n_f^2 + 16.6541037\, n_f^3 \,, \\
\gamma_{\rm gg}^{(3)}(N=8) &= 62279.7438 - 17150.6968\, n_f + 785.880613\, n_f^2 + 18.9331031\, n_f^3 \,, \\
&\ldots \\
\gamma_{\rm gg}^{(3)}(N=20) &= 90499.2530 - 26132.2983\, n_f + 1178.50283\, n_f^2 + 25.6433278\, n_f^3 \,.
\end{aligned}
$$

- Known large- and small-$x$ limits and suitable ansatz approximate $P_{\rm gg}^{(3)}(x)$

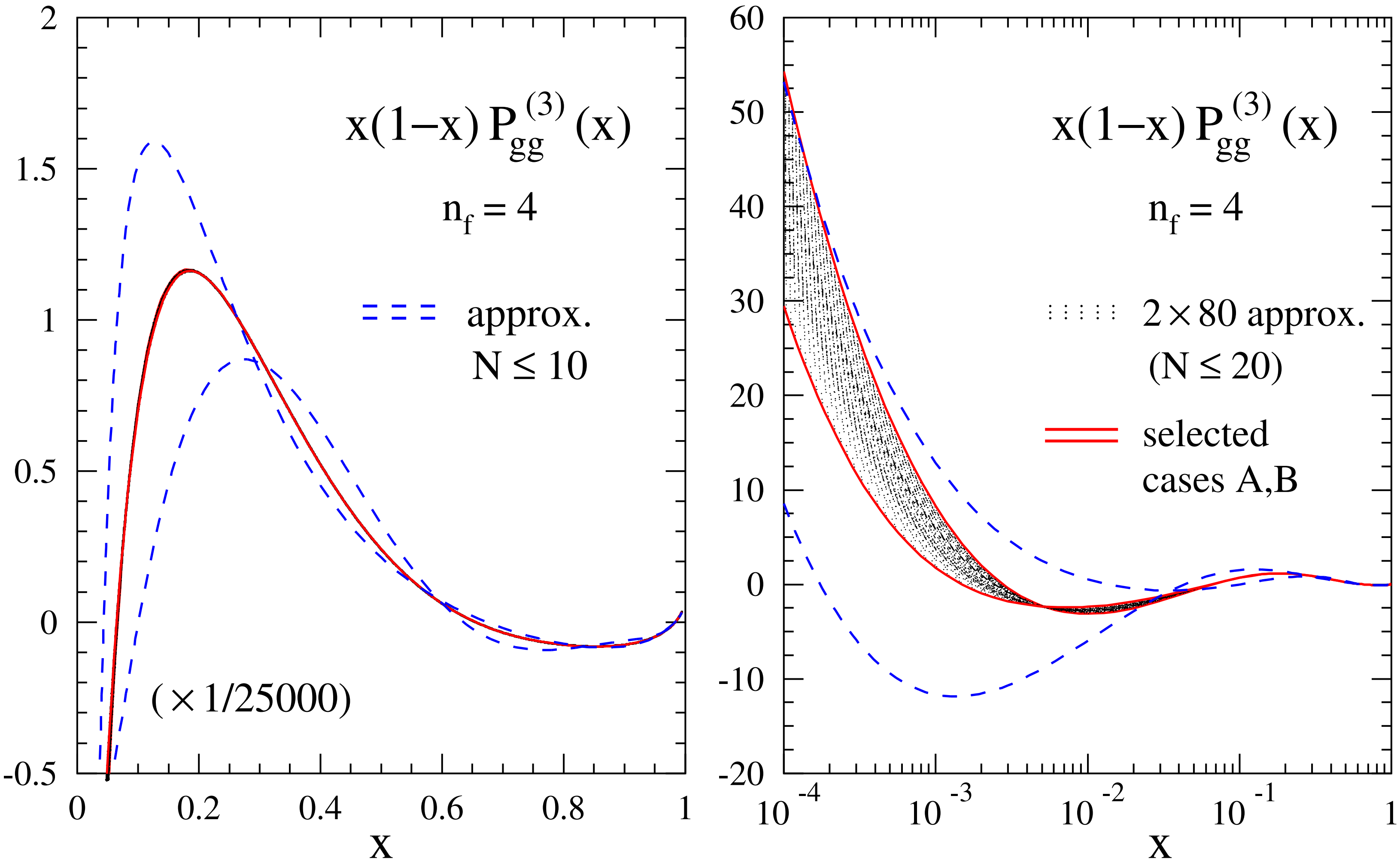
## *Applications*

- Comparison to other approximations for $P_{\rm gg}^{(3)}$

  McGowan, Cridge, Harland-Lang, Thorne '22; NNPDF collaboration '24
- Benchmark N$^3$LO evolution

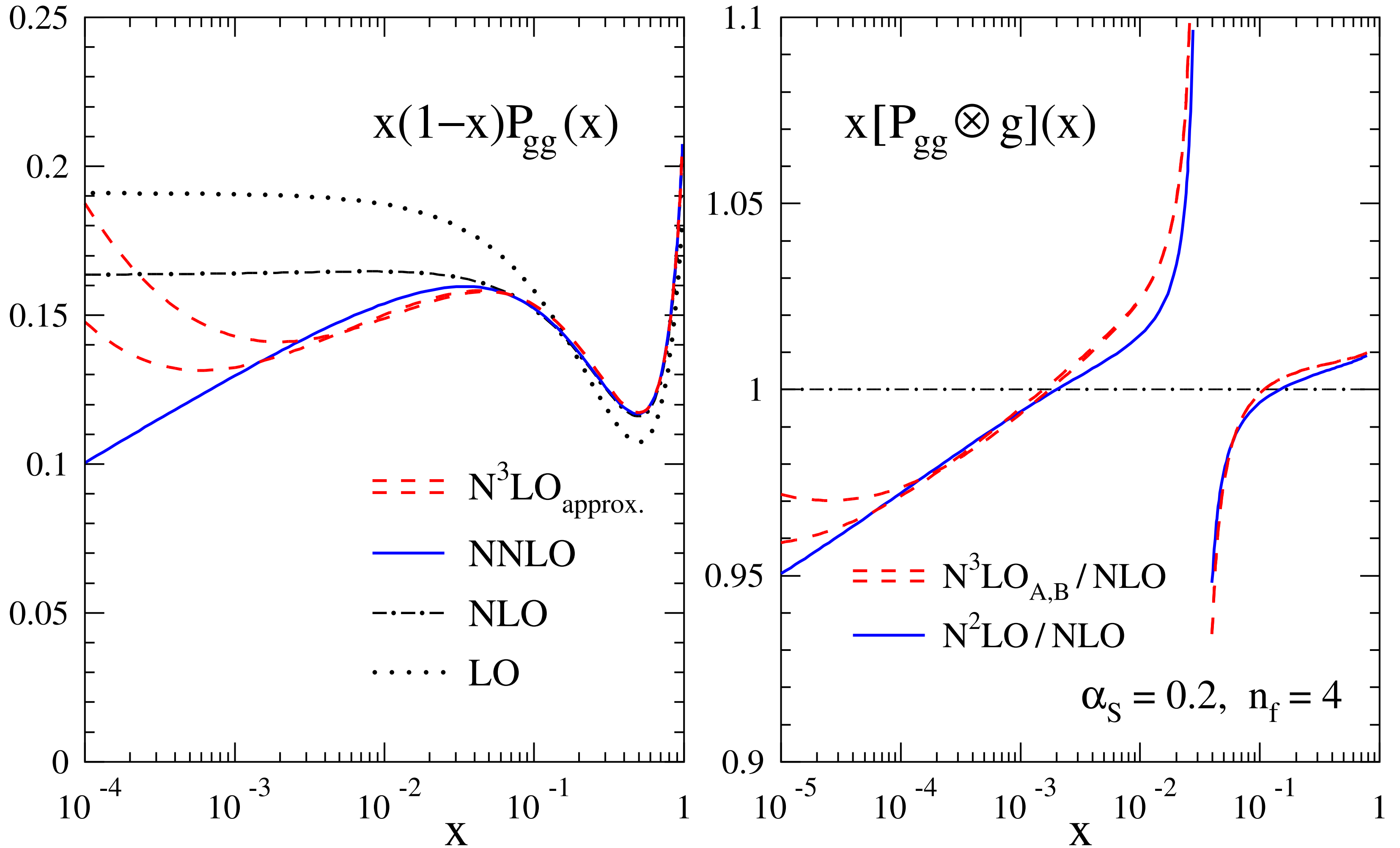
  Cooper-Sarkar, Cridge, Harland-Lang, Hekhorn, Huston, Magni, S.M., Thorne '24

# *Gluon-gluon splitting function (I)*




- Approximations for $P_{\rm gg}^{(3)}(x)$ based on moments $N \leq 10$ vs. $N \leq 20$
  - clear improvements at large-$x$ (left) and small-$x$ (right)

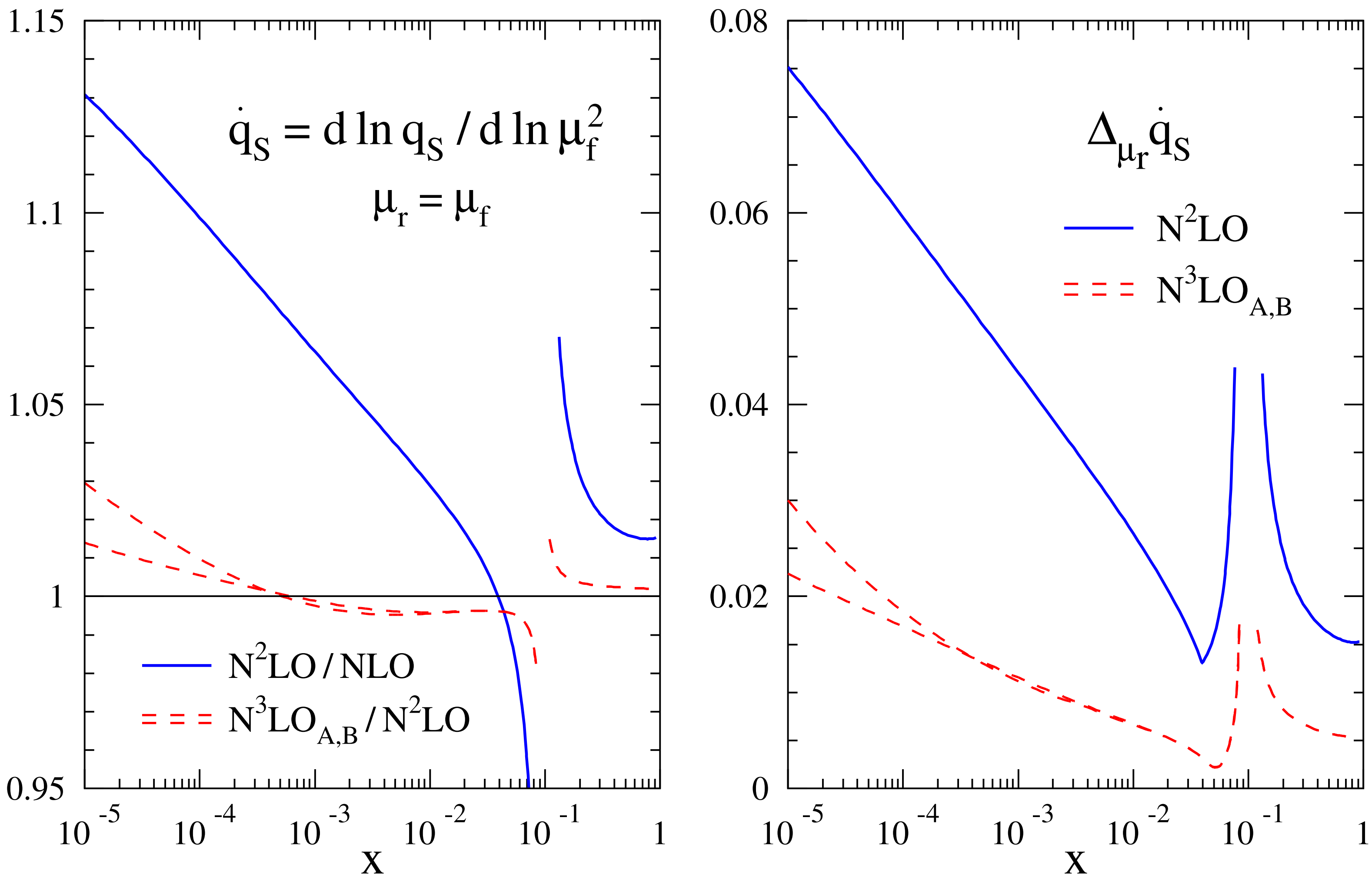
# *Gluon-gluon splitting function (II)*




- Left: results for $P_{gg}(x)$ up to N$^3$LO; $\alpha_s \;=\; 0.2$ fixed, $n_f \;=\; 4$
- Right: contribution to evolution kernel $d \ln g / d \ln \mu_f^2$ up to N$^3$LO for typical gluon shape

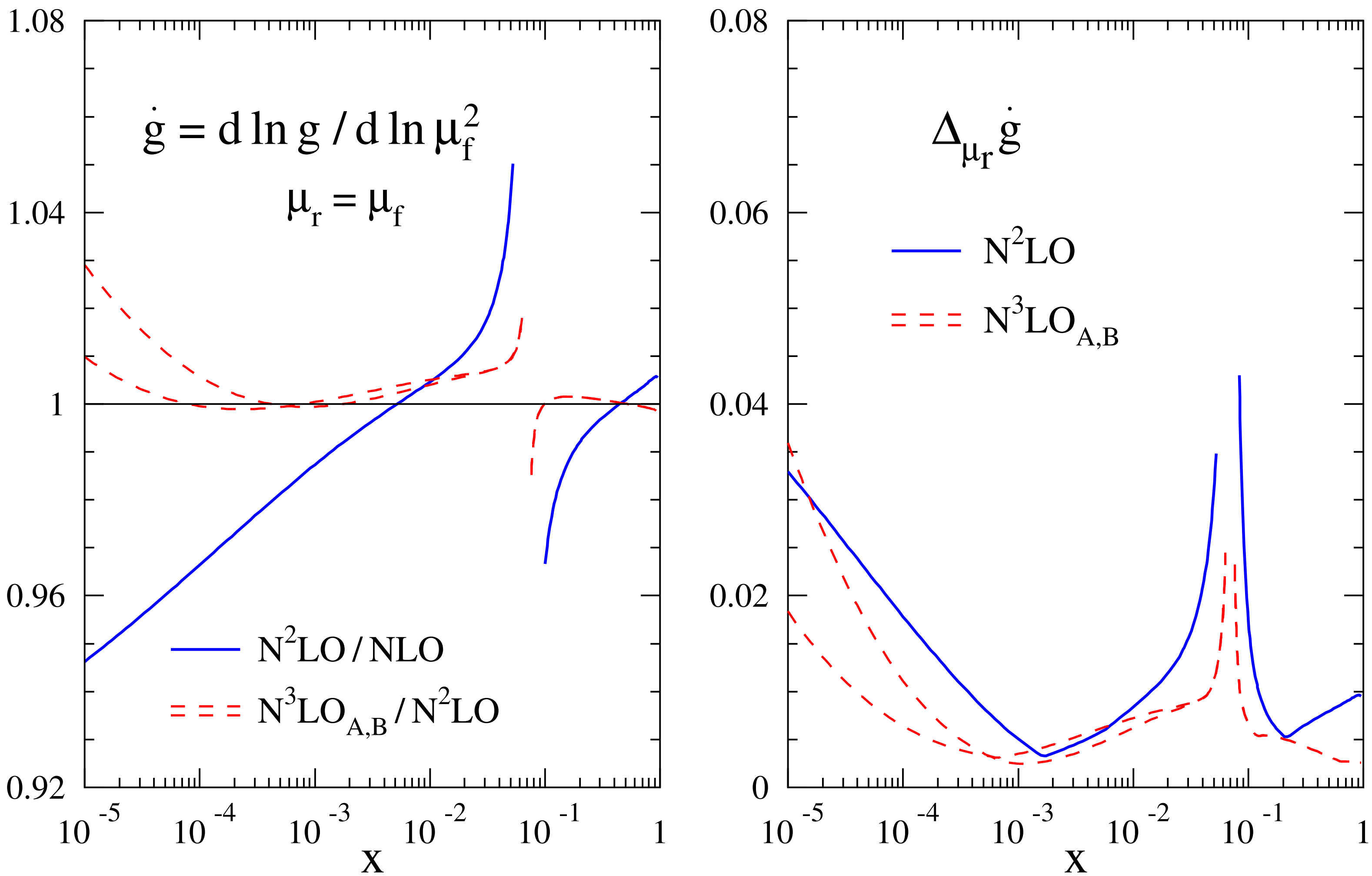
$$xg(x,\mu_0^2) \;=\; 1.6\, x^{\,-0.3}(1-x)^{4.5}\;\left(1 - 0.6\, x^{\,0.3}\right)$$

# *Scale stability of evolution (II)*




- Relative NNLO and $N^3$LO corrections to scale derivative of the quark PDF $\mathrm{q_s}$ for $\alpha_s = 0.2$ fixed, $n_f = 4$
- Renormalization scale dependence of evolution kernel $d \ln \mathrm{q_s} / d \ln \mu_r^2$

# *Scale stability of evolution (III)*




- Relative NNLO and N$^3$LO corrections to scale derivative of the quark PDF $\mathrm{g}$ for $\alpha_s = 0.2$ fixed, $n_f = 4$
- Renormalization scale dependence of evolution kernel $d \ln \mathrm{g} / d \ln \mu_r^2$

## *Renormalization*

# Quantum Chromodynamics

- QCD Lagrangian

$$\mathcal{L}_{\text{QCD}} = \underbrace{-\frac{1}{4} F_a^{\mu\nu} F^a_{\mu\nu} + \sum_{f=1}^{n_f} \overline{\psi}^f (i\not{D} - m_f)\psi^f}_{\text{classical part}} \underbrace{- \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 - \overline{c}^a \partial^\mu D^{ab}_\mu c^b}_{\text{gauge fixing}}$$

  - field strength tensor $F^a_{\mu\nu}$, matter fields $\psi_f, \overline{\psi}_f$ and ghost fields $c^a, \overline{c}^a$
  - covariant derivative $D^{ac}_\mu = \partial_\mu \delta^{ac} + g_s f^{abc} A^c \mu$

- Twist-two spin-$N$ gauge-invariant operators

$$\mathcal{O}^{(N)}_{\rm q}(x) = \frac{1}{2} \text{Tr}\left[\overline{\psi}(x) \not{\Delta}\, D^{N-1} \psi(x)\right]$$

$$\mathcal{O}^{(N)}_{\rm g}(x) = \frac{1}{2} \text{Tr}\left[F_\nu(x)\, D^{N-2} F^\nu(x)\right]$$

  - lightlike vector $\Delta_\mu$ and short-hand notations

$F^{\mu;a} = \Delta_\nu\, F^{\mu\nu;a}$, $A^a = \Delta_\mu A^{\mu;a}$, $D = \Delta_\mu\, D^\mu$, $\partial = \Delta_\mu \partial^\mu$

# *Renormalization (I)*

## *Operator renormalization*

- Physical operators $\mathcal{O}_{\rm q}^{(N)}$, $\mathcal{O}_{\rm g}^{(N)}$ mix under renormalization with alien operators Dixon, Taylor '74; Kluberg-Stern, Zuber '75; Joglekar, Lee '76
- Equation-of-motions (EOM) operators with generic local function $\mathcal{G}^a$ of gauge field and its derivatives Falcioni, Herzog '22

$$\mathcal{O}_{\text{EOM}}^{(N)} = \left(D \cdot F^a + g_s \overline{\psi} T^a \not{\Delta} \psi\right) \mathcal{G}^a(A^a, \partial A^a, \partial^2 A^a, \ldots)$$

- (Generalized) BRST-exact operators
  - ghost alien operators $\mathcal{O}_c^{(N)}$ generated by gBRST transformation acting on suitable ancestor operator Falcioni, Herzog '22
- Expansion of EOM and ghost operators in class I, II, . . .

$$\mathcal{O}_{\text{EOM}}^{(N)} = \mathcal{O}_{\text{EOM}}^{(N),I} + \mathcal{O}_{\text{EOM}}^{(N),II} + \mathcal{O}_{\text{EOM}}^{(N),III} + \mathcal{O}_{\text{EOM}}^{(N),IV} + \ldots$$

$$\mathcal{O}_c^{(N)} = \mathcal{O}_c^{(N),I} + \mathcal{O}_c^{(N),II} + \mathcal{O}_c^{(N),III} + \mathcal{O}_c^{(N),IV} + \ldots$$

# Renormalization (II)

- EOM operators

$$\mathcal{O}^{(N),I}_{\text{EOM}} = \eta(N)\, \left(D\cdot F^a + g_s\overline{\psi}\,\Delta\!\!\!/\,T^a\psi\right)\,\left(\partial^{\,N-2}A^a\right)$$

$$\mathcal{O}^{(N),II}_{\text{EOM}} = g_s\,\left(D\cdot F^a + g_s\overline{\psi}\,\Delta\!\!\!/\,T^a\psi\right)\sum_{\substack{i+j\\=N-3}} C^{abc}_{ij}(\partial^i A^b)(\partial^j A^c)$$

$$\mathcal{O}^{(N),III}_{\text{EOM}} = g_s^2\,\left(D\cdot F^a + g_s\overline{\psi}\Delta\!\!\!/ T^a\psi\right)\sum_{\substack{i+j+k\\=N-4}} C^{abcd}_{ijk}(\partial^i A^b)(\partial^j A^c)(\partial^k A^d)$$

$$\mathcal{O}^{(N),IV}_{\text{EOM}} = g_s^3\,\left(D\cdot F^a + g_s\overline{\psi}\Delta\!\!\!/ T^a\psi\right)\sum_{\substack{i+j+k+l\\=N-5}} C^{abcde}_{ijkl}(\partial^i A^b)(\partial^j A^c)(\partial^k A^d)(\partial^l A^e)$$

  - coefficients $C^{abc}_{ij}$, $C^{abcd}_{ijk}$, ...
  - product of independent colour tensors and EOM operator couplings

$$C^{abc}_{ij} = f^{abc}\kappa_{ij}$$

$$C^{abcd}_{ijk} = (f\;f)^{abcd}\kappa^{(1)}_{ijk} + d_4^{abcd}\kappa^{(2)}_{ijk} + \text{'quartic Casimirs'}$$

$$C^{abcde}_{ijkl} = (f\;f\;f)^{abcde}\kappa^{(1)}_{ijkl} + \text{'quartic Casimirs'}$$

# Renormalization (III)

- Ghost alien operators

$$\begin{aligned}
\mathcal{O}_c^{(N),I} &= -\eta(N)(\partial\overline{c}^a)(\partial^{N-1}c^a) \\
\mathcal{O}_c^{(N),II} &= -g_s \sum_{\substack{i+j\\=N-3}} \widetilde{C}_{ij}^{abc}(\partial\overline{c}^a)(\partial^i A^b)(\partial^{j+1}c^c) \\
\mathcal{O}_c^{(N),III} &= -g_s^2 \sum_{\substack{i+j+k\\=N-4}} \widetilde{C}_{ijk}^{astu}(\partial\overline{c}^a)(\partial^i A^s)(\partial^j A^t)(\partial^{k+1}c^u) \\
\mathcal{O}_c^{(N),IV} &= -g_s^3 \sum_{\substack{i+j+k+l\\=N-5}} \widetilde{C}_{ijkl}^{abcde}(\partial\overline{c}^a)(\partial^i A^b)(\partial^j A^c)(\partial^k A^d)(\partial^{l+1}c^e)
\end{aligned}$$

  - coefficients $\widetilde{C}_{ij}^{abc}$, $\widetilde{C}_{ijk}^{abcd}$, $\ldots$
  - product of independent colour tensors and ghost operator couplings

$$\begin{aligned}
\widetilde{C}_{ij}^{abc} &= f^{abc}\eta_{ij} \\
\widetilde{C}_{ijk}^{abcd} &= (f\,f)^{abcd}\eta_{ijk}^{(1)} + d_4^{abcd}\eta_{ijk}^{(2)} + \text{'quartic Casimirs'} \\
\widetilde{C}_{ijkl}^{abcde} &= (f\,f\,f)^{abcde}\eta_{ijkl}^{(1)} + \text{'quartic Casimirs'}
\end{aligned}$$

# *Renormalization (IV)*

## *Challenge*

- Determination of $N$-dependent couplings $\eta$, $\kappa_{ij}$, $\eta_{ij}$, $\kappa^{(1)}_{ijk}$, $\eta^{(1)}_{ijk}$, ...

## *Bootstrap solution*

- Structure of couplings of alien operators with $n+1$ gluons related to ones with $n$ gluons
- All-$N$ structure of couplings fixed by small set of constants (explicit computation for some fixed $N$ values)

## *Gauge invariance*

- Complete Lagrangian with twist-two physical and alien operators

$$\widetilde{\mathcal{L}} \quad = \quad \mathcal{L}_{\mathsf{QCD}} + \mathcal{O}_{\mathrm{q}} + \mathcal{O}_{\mathrm{g}} + \mathcal{O}^{(N)}_{\mathsf{EOM}} + \mathcal{O}^{(N)}_{c}$$

  - invariance of Lagrangian under generalized gauge transformation

$$A^a_\mu \to A^a_\mu + \delta_\omega A^a_\mu + \delta^\Delta_\omega A^a_\mu$$

  - parameter of gauge transformation $\omega^a$
  - “standard” QCD gauge transformation $\delta_\omega A^a_\mu = D^{ab}_\mu \omega^b(x)$
  - generalized transformation $\delta^\Delta_\omega A^a_\mu$ for operators

# Renormalization (V)

- Generalized transformation $\delta_\omega^\Delta A_\mu^a$ for operators

$$\delta_\omega^\Delta A_\mu^a = -\Delta_\mu \Bigg[ \eta(N)\, \partial^{N-1} \omega^a + g_s \sum_{\substack{i+j\\=N-3}} \widetilde{C}_{ij}^{a a_1 a_2} \left(\partial^i A^{a_1}\right) \left(\partial^{j+1} \omega^{a_2}\right)$$

$$+ g_s^2 \sum_{\substack{i+j+k\\=N-4}} \widetilde{C}_{ijk}^{a a_1 a_2 a_3} \left(\partial^i A^{a_1}\right) \left(\partial^j A^{a_2}\right) \left(\partial^{k+1} \omega^{a_3}\right)$$

$$+ g_s^3 \sum_{\substack{i+j+k+l\\=N-5}} \widetilde{C}_{ijkl}^{a a_1 a_2 a_3 a_4} \left(\partial^i A^{a_1}\right) \left(\partial^j A^{a_2}\right) \left(\partial^k A^{a_3}\right) \left(\partial^{l+1} \omega^{a_4}\right) + \mathcal{O}(g_s^4) \Bigg]$$

- Generalized gauge transformation promoted to (anti-)BRST symmetry

$$\omega^a \to c^a \qquad \omega^a \to \overline{c}^a$$

## Upshot

- Algebraic approach for derivation of a complete set of operators
- Relations between $N$-dependent couplings $\kappa_{ij}$, $\eta_{ij}$, $\kappa_{ijk}^{(1)}$, $\eta_{ijk}^{(1)}$, . . .
- Multiplicative renormalization of operators $\mathcal{O}_{\rm i}^{\rm ren}(x) = Z_{\rm ij}\, \mathcal{O}_{\rm j}^{\rm bare}(x)$

# *Renormalization (VI)*

## *Class I*

- Coupling $\eta(N)$
  - counter term $Z_{\mathrm{g}\,c}$ for mixing of physical operators into aliens
  - computation at one-loop Dixon, Taylor '74; Hamberg, van Neerven '91

$$Z_{\mathrm{g}\,c} = -\frac{a_s}{\epsilon}\frac{C_A}{N(N-1)} + \mathcal{O}(a_s^2),$$

  - one-loop result $\eta(N) = \dfrac{-1}{N(N-1)}$

## *Class II*

- Couplings $\kappa_{ij}$, $\eta_{ij}$
  - class II operators obey following relations

$$\kappa_{ij} + \kappa_{ji} = 0 \qquad \text{[anti-symmetry of } f\text{]}$$

$$\eta_{ij} = 2\kappa_{ij} + \eta(N)\binom{i+j+1}{i} \qquad \text{[gBRST]}$$

$$\eta_{ij} + \sum_{s=0}^{i}(-1)^{s+j}\binom{s+j}{j}\eta_{(i-s)(j+s)} = 0 \qquad \text{[anti-gBRST]}$$

# Renormalization (VII)

## Conjugation relation

- Double summation of anti-gBRST relation leads to

$$\eta_{ij} = \sum_{t=0}^{i} \binom{t+j}{j} \sum_{s=0}^{i-t} (-1)^s \binom{s+j+t}{j+t} \eta_{(i-t-s)(j+t+s)} .$$

- Conjugation relation with great predictive power
  - examples known anomalous dimensions of twist-two operators in non-forward kinematics S.M., VanThurenhout '21; VanThurenhout '23
- Systematic solution for $\eta_{ij}$ at one loop possible with ansatz

$$\eta_{ij} = \eta(N) \left[ c_1 \binom{i+j+1}{i} + c_2 \binom{i+j+1}{j} + c_3 (-1)^j \right]$$

  - solution from conjugation relation: $c_3 = -c_1$ and $c_1 + c_2 = 1$
  - constant $c_1 = 1/4$ determined from counter term at some fixed $N$
- Conjugation relations solved with creative telescoping Zeilberger '91
  - symbolic summation with **Mathematica** packages **Sigma** Schneider '04 and **EvaluateMultiSums** Schneider '13

# Renormalization (VIII)

## Class III

- Couplings $\kappa^{(1)}_{ijk}$, $\eta^{(1)}_{ijk}$ of class III operators obey following relations

$$\kappa^{(1)}_{ijk} + \kappa^{(1)}_{ikj} = 0 \qquad \text{[anti-symmetry of } f\text{]}$$

$$\kappa^{(1)}_{ijk} + \kappa^{(1)}_{jki} + \kappa^{(1)}_{kij} = 0 \qquad \text{[Jacobi identity]}$$

$$\eta^{(1)}_{ijk} = 2\kappa_{i(j+k+1)} \binom{j+k+1}{j} + 2[\kappa^{(1)}_{ijk} + \kappa^{(1)}_{kji}] \qquad \text{[gBRST]}$$

$$\eta^{(1)}_{ijk} = \sum_{m=0}^{i} \sum_{n=0}^{j} \frac{(m+n+k)!}{m!n!k!} (-1)^{m+n+k} \eta^{(1)}_{(j-n)(i-m)(k+m+n)} \qquad \text{[anti-gBRST]}$$

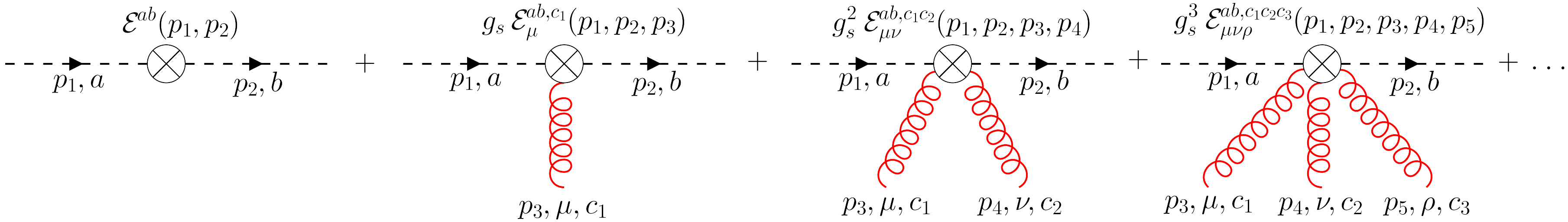
- Solution for $\eta^{(1)}_{ijk}$ at one loop from conjugation relation

$$\begin{aligned}\eta^{(1)}_{ijk} = -\frac{\eta(N)}{24}\Bigg\{ & 5(-1)^{i+j+1}\binom{i+j+1}{i} + (-1)^{i+k}\binom{i+k+1}{k} + 2(-1)^{j+k+1}\binom{j+k+1}{j} \\ & + \binom{i+k+1}{i}\left[(-1)^{i+k} + 4\binom{N-2}{j+1}\right] + \binom{j+k+1}{k}\left[5(-1)^{j+k+1} - 3\binom{N-2}{i} \right. \\ & \left. + \binom{N-2}{i+1}\right] + \binom{i+j+1}{j}\left[4(-1)^{i+j} - 15\binom{N-2}{k} - 5\binom{N-2}{k+1}\right]\Bigg\}.\end{aligned}$$

# *Renormalization (IX)*

## *Feynman rules*

- Derivation of Feynman rules with known couplings $\kappa_{ij}$, $\eta_{ij}$, $\kappa^{(1)}_{ijk}$, $\eta^{(1)}_{ijk}$, ...
  - ghost alien operator




$$\varepsilon^{ab} = \frac{1+(-1)^N}{2} i^N \eta(N) \delta^{ab} (\Delta \cdot p_1)^N$$

$$\varepsilon^{ab,c_1}_{\mu} = \frac{1+(-1)^N}{2} i^{N-1} \Delta_\mu\, f^{ac_1b} \sum_{\substack{i+j\\=N-3}} \eta_{ij} (\Delta \cdot p_1)(\Delta \cdot p_3)^i (\Delta \cdot p_2)^{j+1}$$

- Agreement with direct computation of counter terms up to three loops

  Gehrmann, von Manteuffel, Yang '23

- Agreement with Feynman rules at two loops Hamberg, van Neerven '91; Matiounine, Smith, van Neerven '98; Blümlein, Marquard, Schneider, Schönwald '22

- New results for alien operators with five and six partons

# *All-$N$ results*

# *Analytic reconstruction (I)*

- Sufficiently many Mellin moments allow for reconstruction of analytic all-$N$ expressions through solution of Diophantine equations

  Lenstra, Lenstra, Lovász '82

- Harmonic sums define basis in space of functions for $\gamma_{\rm ij}(N)$

$$S_{\pm m_1,\ldots,m_k}(N) = \sum_{i=1}^{N} \frac{(\pm 1)^i}{i^{m_1}} S_{m_2,\ldots,m_k}(i)$$

  - at weight $w$ there are $2 \cdot 3^{w-1}$ harmonic sums
  - $l$-loop $\gamma_{\rm ij}^{(l-1)}(N)$ contains harmonic sums up to weight $2l - 1$
    $\longrightarrow$ numbers grow quickly: $2, 18, 162, 1458$ sums for $l = 1, 2, 3, 4$

- Some applications in QCD
  - three-loop non-singlet transversity $\gamma_{\rm tr}^{(2)}$ Velizhanin'12
  - three-loop polarized $\Delta\gamma_{\rm ij}^{(2)}$ S.M., Vermaseren, Vogt '14
  - four-loop non-singlet $\gamma_{\rm ns}^{(3)\pm}$ (large-$n_c$) S.M., Vogt, Ruijl, Ueda, Vermaseren '17
  - four-loop non-singlet DIS $C_{\rm ns}^{(4)}$ (large-$n_f$)

    Basdew-Sharma, Pelloni, Herzog, Vogt '22
  - ...

# *Analytic reconstruction (II)*

## *Conformal symmetry and integrability*

- Gribov-Lipatov reciprocity relation (RR)
  - diagonal splitting functions $P_{\rm ii}^{(0)}(x)$ invariant under mapping $x \rightarrow \frac{1}{x}$
- RR realized for universal $\gamma_{\rm u}(N)$ in $N = 4$ SYM theory
  - uniform transcendentality sums with $w = 2l - 1$ only at $l$-loops
- RR in $N$-space for QCD implies $\gamma(N) = \gamma_{\rm u}\left(N + \gamma(N) - \beta(\alpha_s)\right)$
- RR constraints for $\gamma_{\rm u}$ reduce number to $2^{w-1}$ sums at weight $w$ for $\gamma_{\rm u}$
  - $2^{w+1} - 1$ objects with denominators $1/(N+1)$ added ($255$ at $w = 7$ )

## *Example*

- Large-$n_c$ limit of $\gamma_{\rm ns}^{(3)\pm}$ only needs harmonic sums with positive index
  - weight $w$ RR sums given by Fibonacci number $F(w)$
  - total number of unknowns (including powers $1/(N+1)$) amount to $F(w+4) - 2$ ($87$ at $w = 7$)
- Additional $46$ constraints from large-$x$/small-$x$ ($N \rightarrow \infty / N \rightarrow 0$) limit
- Solution becomes feasible with $18$ Mellin moments for $\gamma_{\rm ns}^{(3)\pm}$

# *Analytic reconstruction (III)*

- Mellin moments suffice to determine all-$N$ result for parts of $\gamma_{\rm ps}^{(3)}(N)$
  - harmonic sums and Riemann $\zeta_n$ terms up to total weight $w = 7$
- Terms proportional to $\zeta_5$ are particularly simple
  - $N$-dependent terms respect RR
  - RR implies invariance under mapping $N \to -N-1$
- Combinations of denominators $\eta = \frac{1}{N} - \frac{1}{N+1}$ and $\nu = \frac{1}{N-1} - \frac{1}{N+2}$

$$
\begin{aligned}
\gamma_{\rm ps}^{(3)}(N)\Big|_{\zeta_5} \;=\;& 160\, n_f\, C_F^3 \left(9\,\eta + 6\,\eta^2 - 4\,\nu\right) + 80/3\, n_f\, C_A\, C_F^2 \left(-9\,\eta - 6\,\eta^2 + 4\,\nu\right) \\
&+40/9\, n_f\, C_A^2\, C_F \left(-1 - 214\,\eta - 144\,\eta^2 + 104\,\nu\right) \\
&+320/3\, n_f\, \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(-1 + 56\,\eta + 36\,\eta^2 - 16\,\nu\right)
\end{aligned}
$$

- Inverse Mellin transformation generates additional terms with $\zeta_n$
  - $\zeta_n$ in $N$-space $\neq$ $\zeta_n$ in $x$-space

# *Non-singlet splitting functions* $P_{\rm ns}^{\pm,\rm v}$

# *Small-x behavior (I)*

## *The small $x$-limit: $x \to 0$*

- Structure of non-singlet splitting functions $P^{\pm}_{\rm ns}$ at small $x$
  - double-logarithmic contributions with very large coefficients
  - resummation for $P^{+}_{\rm ns}$ to leading logarithmic (LL) accuracy in Mellin-$N$ space

Kirschner, Lipatov '83

$$\gamma^{+}_{\rm ns,LL}(N,\alpha_s) \;=\; -\frac{N}{2}\left\{1-\left(1-\frac{2\alpha_s C_F}{\pi N^2}\right)^{1/2}\right\}$$

- Large-$n_c$ limit with intriguing structure

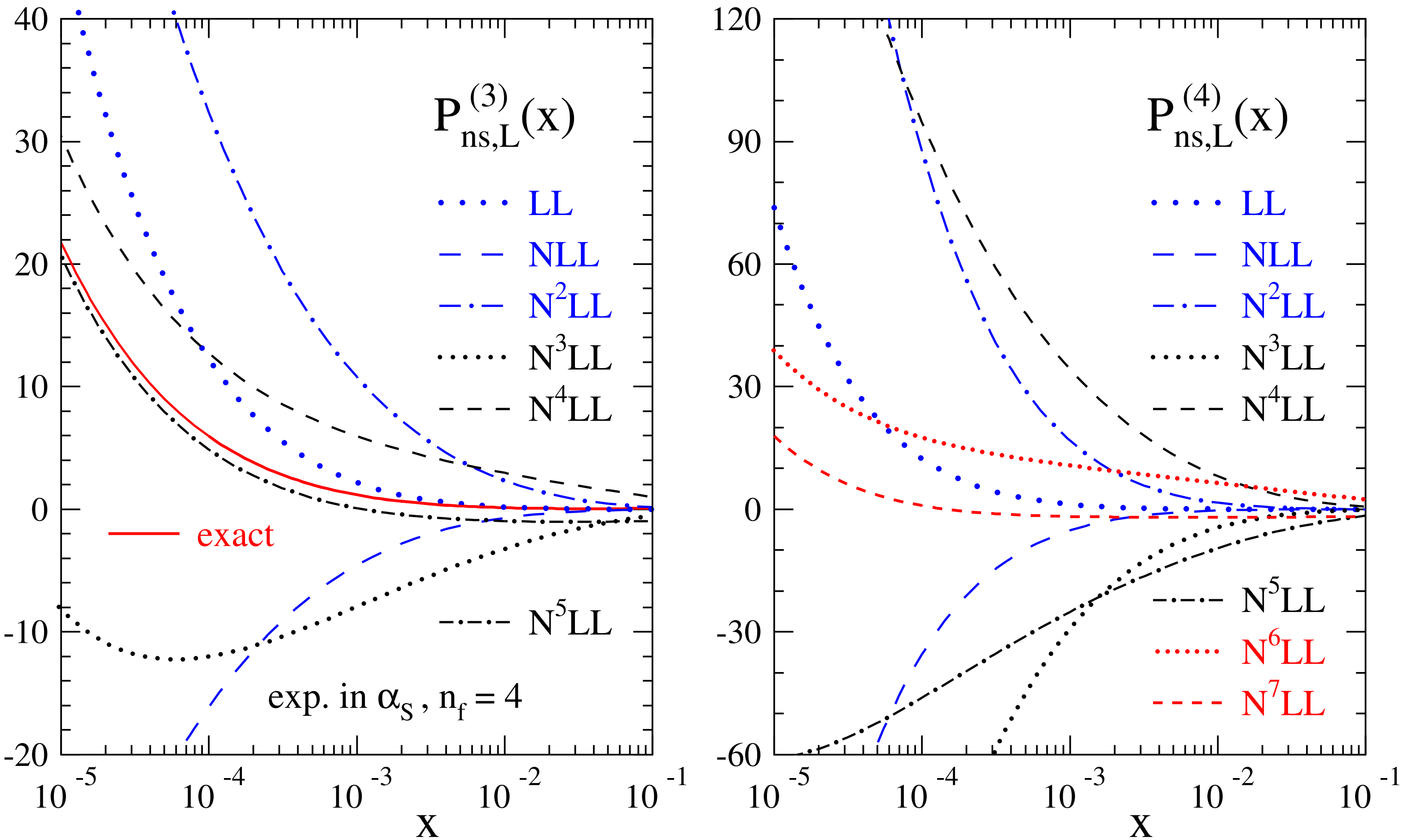
Velizhanin '14

$$\gamma^{+}_{\rm ns}(N,\alpha_s)\,\Big(N+\gamma^{+}_{\rm ns}(N,\alpha_s)-\beta(\alpha_s)/\alpha_s\Big) \;=\; O(1)$$

  - Laurent expansion about $N=0$
- Exploit structure of the (unfactorized) structure functions in dimensional regularization
- Resummation in terms of modified Bessel functions to N$^7$LL accuracy

Davies, Kom, S.M., Vogt '22

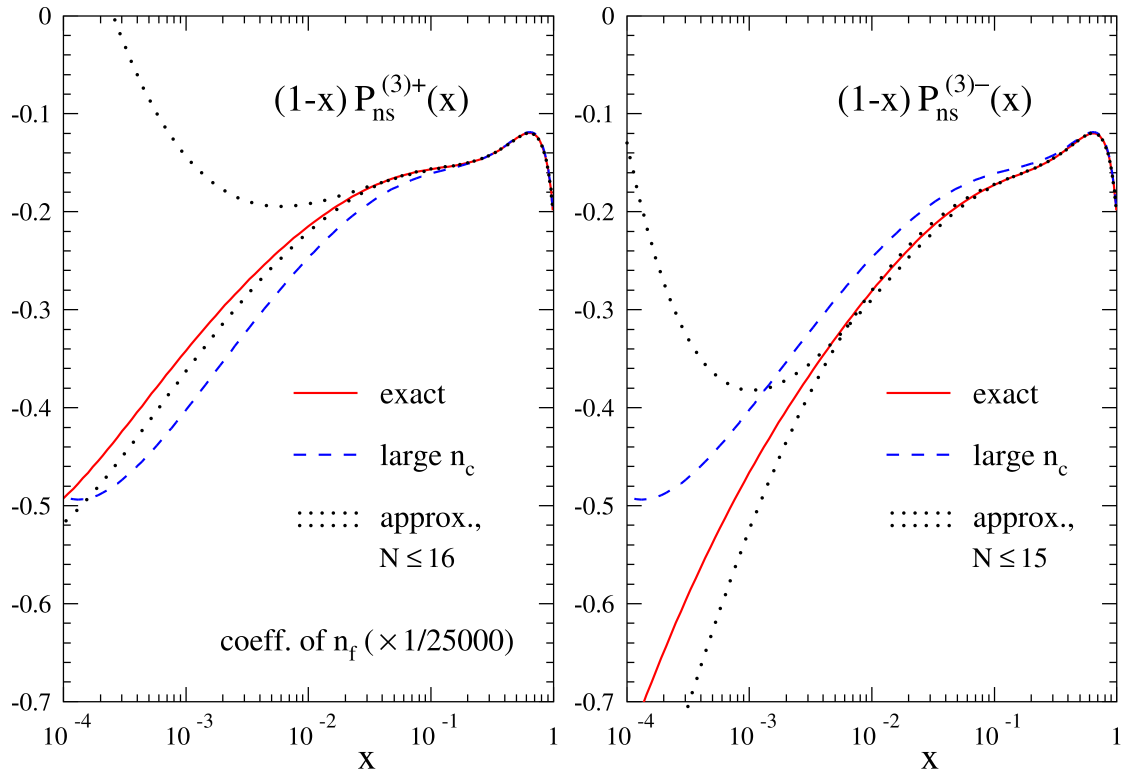
# *Small-x behavior (II)*




- Splitting functions $P_{\rm ns}^{(3),+}$ (left) and $P_{\rm ns}^{(4),+}$ (right) Davies, Kom, S.M., Vogt '22
  - small-$x$ approximations to the four-flavour splitting functions $P_{\rm ns,\,L}^{(n)}(x)$ in the large-$n_c$ limit
  - predictions up to N$^7$LL

# *Four-loop non-singlet splitting functions (I)*

- Four-loop $P_{\rm ns}^{(3)\pm}(x)$ ($n_f^1$ terms)
- Comparison to large-$n_c$ limit and uncertainty bands from approximations

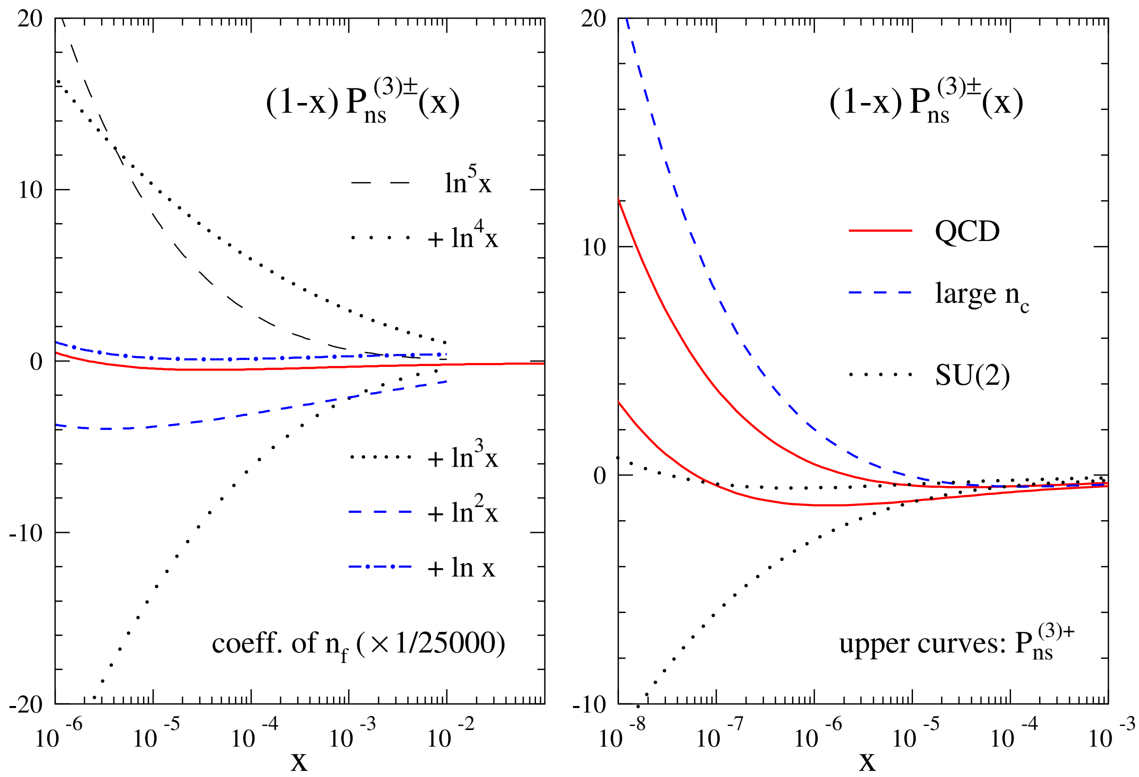



## *Analytic results*

- Contributions to non-singlet splitting functions
  - $n_f$-terms ($n_f^3$ Gracey '94; $n_f^2$ Davies, Vogt, Ruijl, Ueda, Vermaseren '16)
  - leading $n_c$ terms S.M., Vogt, Ruijl, Ueda, Vermaseren '17
  - $n_f C_F^3$ terms Gehrmann, von Manteuffel, Sotnikov, Yang '23
  - all $n_f$ terms Kniehl, S.M., Velizhanin, Vogt '25

    analytic reconstruction based on moments $N \leq 24$ for $n_f d_{FF}^{(4)}$ and $N \leq 32$ for $n_f C_F^2 C_A$

# Four-loop non-singlet splitting functions (II)




- $n_f^1$ terms of $P_{\rm ns}^{(3)\pm}(x)$
  - left: exact result and small-$x$ logarithms: $\ln^k x$ with $k = 5, 4, 3, 2, 1$
  - right: illustration of large-$n_c$ approximation with QCD $n_c = 3$ and $SU(2)$ gauge theory

# *Splitting functions at large $x$*

## *The large $x$-limit: $x \to 1$*

- Structure of diagonal splitting functions $P_{\rm ii}$ (for $\rm i = q, g$) at large $x$

$$P_{ii}^{(n-1)}(x) \;=\; \frac{A_n^{\rm i}}{(1-x)_+} + B_n^{\rm i}\,\delta(1-x) + \ldots$$

- Cusp anomalous dimension $A_n^{\rm i}$
  - known from $1/\epsilon^2$-poles of QCD form factor
- Four loop results in QCD

Large-$n_c$ (Henn, Lee, Smirnov, Smirnov, Steinhauser ‘16; S. M., Ruijl, Ueda, Vermaseren, Vogt ‘17); $n_f$ (Grozin ‘18; Henn, Peraro, Stahlhofen, Wasser ‘19); $n_f^2$ (Davies, Ruijl, Ueda, Vermaseren, Vogt ‘16; Lee, Smirnov, Smirnov, Steinhauser ‘17); $n_f^3$ (Gracey ‘94; Beneke, Braun, ‘95);
quartic colour (Lee, Smirnov, Smirnov, Steinhauser ‘19; Henn, Korchemsky, Mistlberger ‘19)

- Virtual anomalous dimension $B_n^{\rm i}$
  - parts related to $1/\epsilon$-poles of QCD form factor

# Quark virtual anomalous dimension

- Four loop result (up to one unknown $b^{\mathrm{q}}_{4,FA}$) Kniehl, S.M., Velizhanin, Vogt '25

$$B^{\mathrm{q}}_4 =$$

$$C_F^4\left(\frac{4873}{24} - 450\zeta_2 - \frac{684}{5}\zeta_2^2 - \frac{16888}{35}\zeta_2^3 + 2004\zeta_3 - 120\zeta_3\zeta_2 + \frac{128}{5}\zeta_3\zeta_2^2 - 1152\zeta_3^2 - 2520\zeta_5 - 384\zeta_5\zeta_2 + 5880\zeta_7\right)$$

$$+C_F C_A^3\left(-\frac{371201}{648} - \frac{1}{24}b^{\mathrm{q}}_{4,FA} + \frac{4582}{3}\zeta_2 - \frac{22388}{135}\zeta_2^2 + \frac{48368}{315}\zeta_2^3 - \frac{153670}{81}\zeta_3 + \frac{472}{3}\zeta_3\zeta_2 + \frac{16}{5}\zeta_3\zeta_2^2 + 528\zeta_3^2\right.$$

$$\left.+\frac{11372}{9}\zeta_5 + 504\zeta_5\zeta_2 - 2870\zeta_7\right) + C_F^2 C_A^2\left(\frac{29639}{36} - \frac{46771}{27}\zeta_2 - \frac{24340}{27}\zeta_2^2 - \frac{21988}{35}\zeta_2^3 + \frac{129662}{27}\zeta_3 + \frac{2096}{9}\zeta_3\zeta_2\right.$$

$$\left.-\frac{64}{5}\zeta_3\zeta_2^2 - \frac{7102}{3}\zeta_3^2 + \frac{5354}{9}\zeta_5 - 2104\zeta_5\zeta_2 + 8610\zeta_7\right) + C_F^3 C_A\left(-\frac{2085}{4} + 1167\zeta_2 + \frac{4334}{5}\zeta_2^2 + \frac{317188}{315}\zeta_2^3\right.$$

$$\left.-3260\zeta_3 - \frac{1988}{3}\zeta_3\zeta_2 + \frac{256}{5}\zeta_3\zeta_2^2 + 3220\zeta_3^2 - 976\zeta_5 + 2064\zeta_5\zeta_2 - 10920\zeta_7\right) + \frac{d_F^{abcd}d_A^{abcd}}{n_F}\left(b^{\mathrm{q}}_{4,FA}\right)$$

$$+n_f C_F^3\left(32 + 162\zeta_2 - \frac{408}{5}\zeta_2^2 - \frac{51472}{315}\zeta_2^3 - 308\zeta_3 - \frac{256}{3}\zeta_3\zeta_2 + 224\zeta_3^2 + 912\zeta_5\right)$$

$$+n_f C_F^2 C_A\left(-\frac{7751}{54} - \frac{3892}{27}\zeta_2 + \frac{55708}{135}\zeta_2^2 + \frac{2808}{35}\zeta_2^3 - \frac{15400}{27}\zeta_3 + \frac{2672}{9}\zeta_3\zeta_2 - \frac{1232}{3}\zeta_3^2 - \frac{7432}{9}\zeta_5\right)$$

$$+n_f C_F C_A^2\left(\frac{20027}{108} - \frac{41092}{81}\zeta_2 + \frac{2468}{45}\zeta_2^2 - \frac{4472}{135}\zeta_2^3 + \frac{9554}{27}\zeta_3 - \frac{580}{3}\zeta_3\zeta_2 + \frac{416}{3}\zeta_3^2 + \frac{1130}{9}\zeta_5\right)$$

$$+n_f \frac{d_F^{abcd}d_F^{abcd}}{n_F}\left(-192 + \frac{1888}{3}\zeta_2 - \frac{704}{15}\zeta_2^2 + \frac{2048}{45}\zeta_2^3 - \frac{992}{3}\zeta_3 + 64\zeta_3\zeta_2 + 256\zeta_3^2 - 1120\zeta_5\right)$$

$$+n_f^2 C_F^2\left(-\frac{188}{27} + \frac{1244}{27}\zeta_2 - \frac{4208}{135}\zeta_2^2 + \frac{56}{27}\zeta_3 - \frac{160}{9}\zeta_3\zeta_2 + \frac{368}{9}\zeta_5\right) + n_f^2 C_F C_A\left(-\frac{193}{54} + \frac{3170}{81}\zeta_2 - \frac{32}{9}\zeta_2^2\right.$$

$$\left.-\frac{320}{9}\zeta_3 + \frac{80}{3}\zeta_3\zeta_2 - \frac{88}{9}\zeta_5\right) + n_f^3 C_F\left(-\frac{131}{81} + \frac{32}{81}\zeta_2 - \frac{64}{135}\zeta_2^2 + \frac{304}{81}\zeta_3\right)$$

# Summary

- Experimental precision of $\lesssim 1\%$ motivates computations at higher order in perturbative QCD
  - theoretical predictions at NNLO in QCD nowadays standard
- Push for theory results at N$^3$LO (and even N$^4$LO)
  - evolutions equations expected to achieve percent-level
  - massive use of computer algebra
- Four-loop splitting functions approximated from moments $N = 2, \ldots 20$
  - residual uncertainties negligible in wide kinematic range of $x$ probed at current and future colliders
  - $P_{\rm qq} = P_{\rm ns}^{+} + P_{\rm ps}$, $P_{\rm qg}$ $P_{\rm gq}$ and $P_{\rm gg}$ all done
- More all-$N$ results to come
- Novel structural insights into QCD from integrabilty and conformal symmetry
  - Key parts of QCD inherited from $N = 4$ Super Yang-Mills theory
  - Conformal symmetry in parts of QCD evolution equations