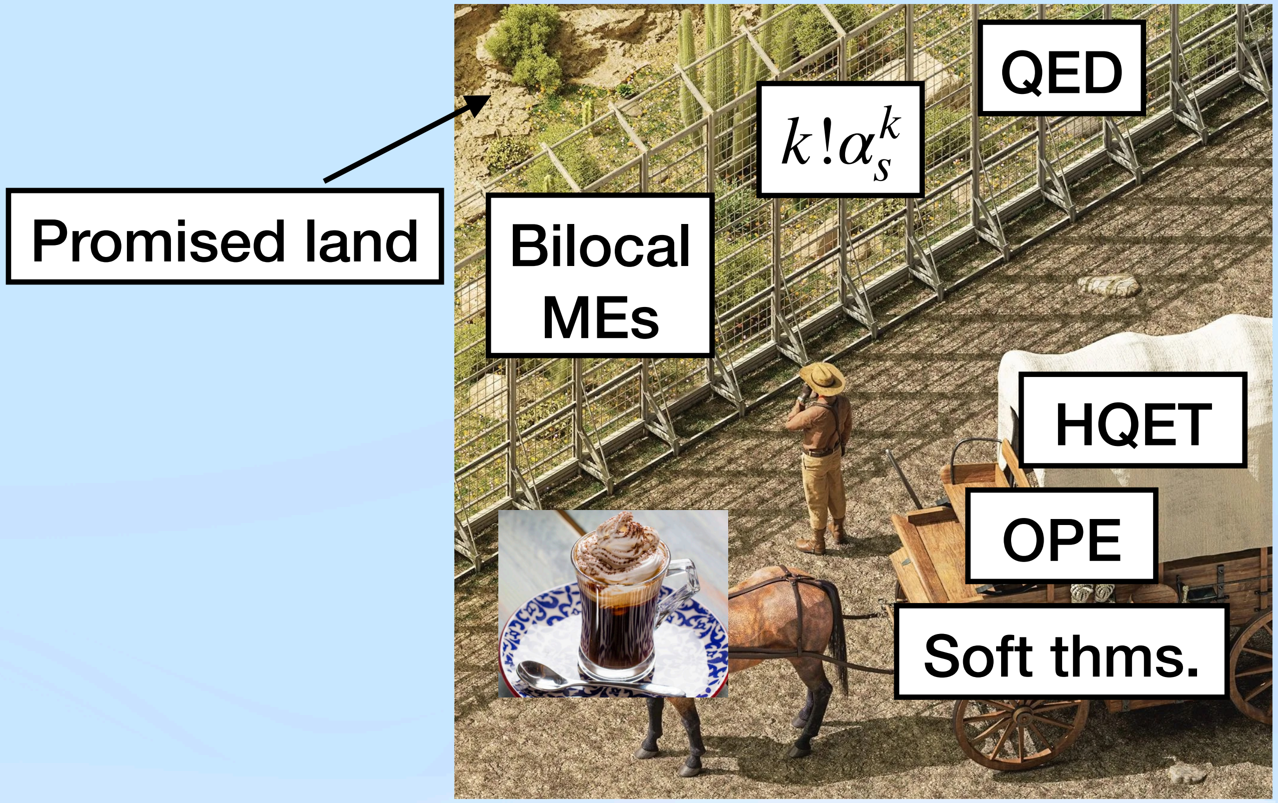

Semileptonic decays at the frontier

Vienna, March 11 2025

Jack Jenkins

TP1 Theoretical Particle Physics
Universität Siegen
P3H

CKM fits and semileptonics

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Ideally:

Dominant semileptonic modes $b \to c$ and $s \to u$ fix (λ, A),
angles fixed by $b \to u$ and γ..

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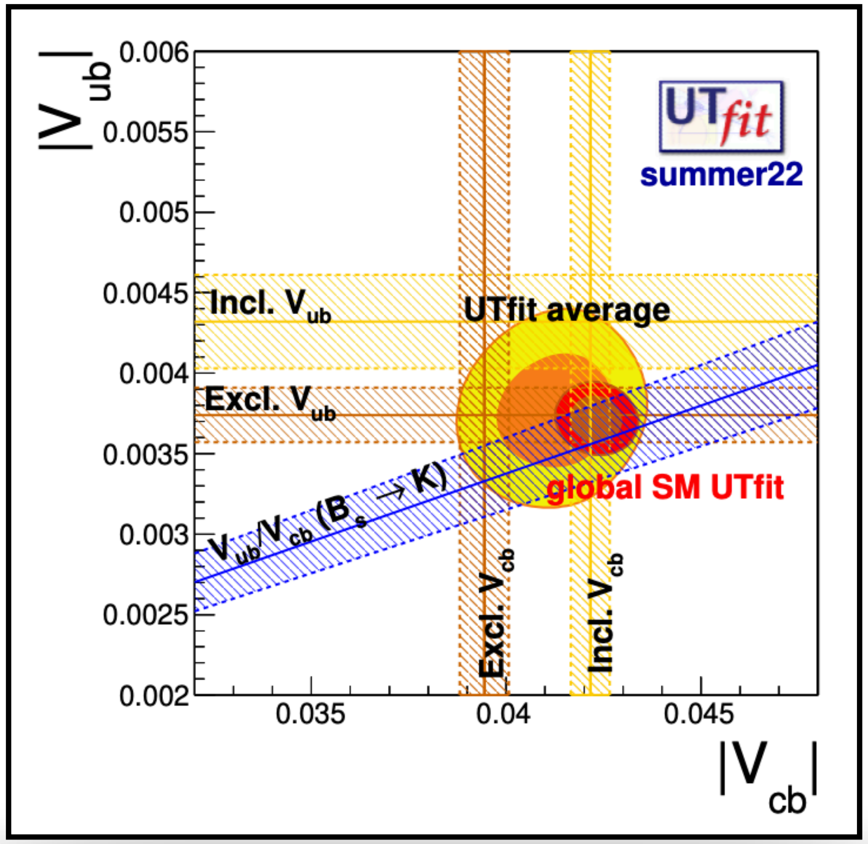
Reality:

There are 'puzzles' everywhere.. especially $|V_{cb}|$ and $|V_{ub}|$ from inclusive vs. exclusive semileptonic B decays

[2212.03894]



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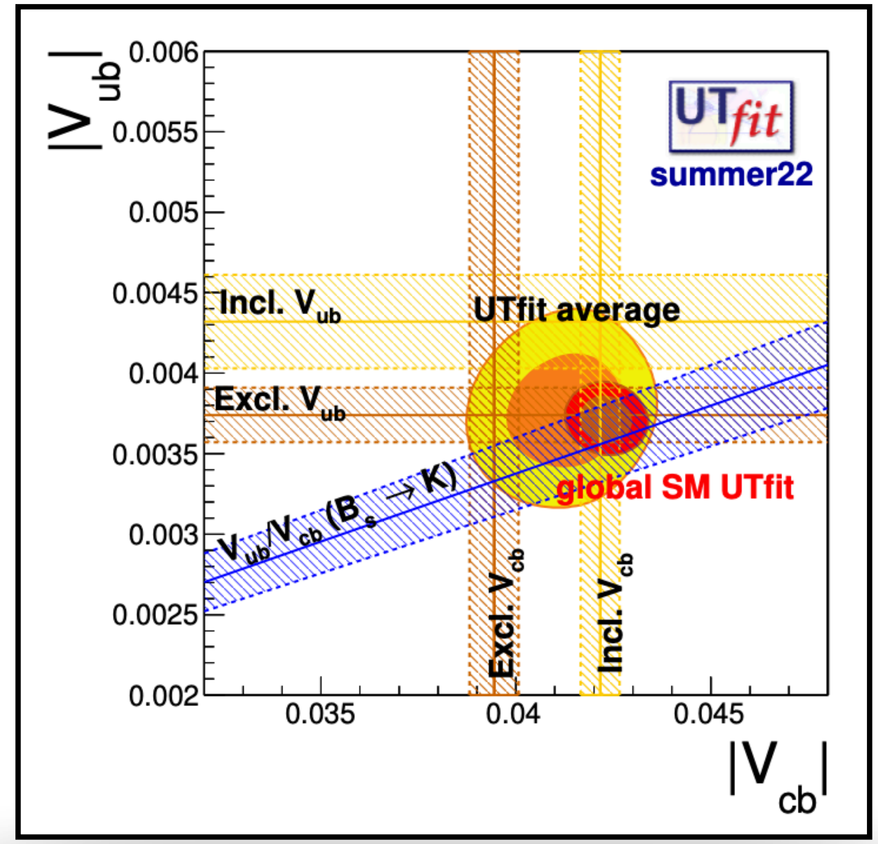

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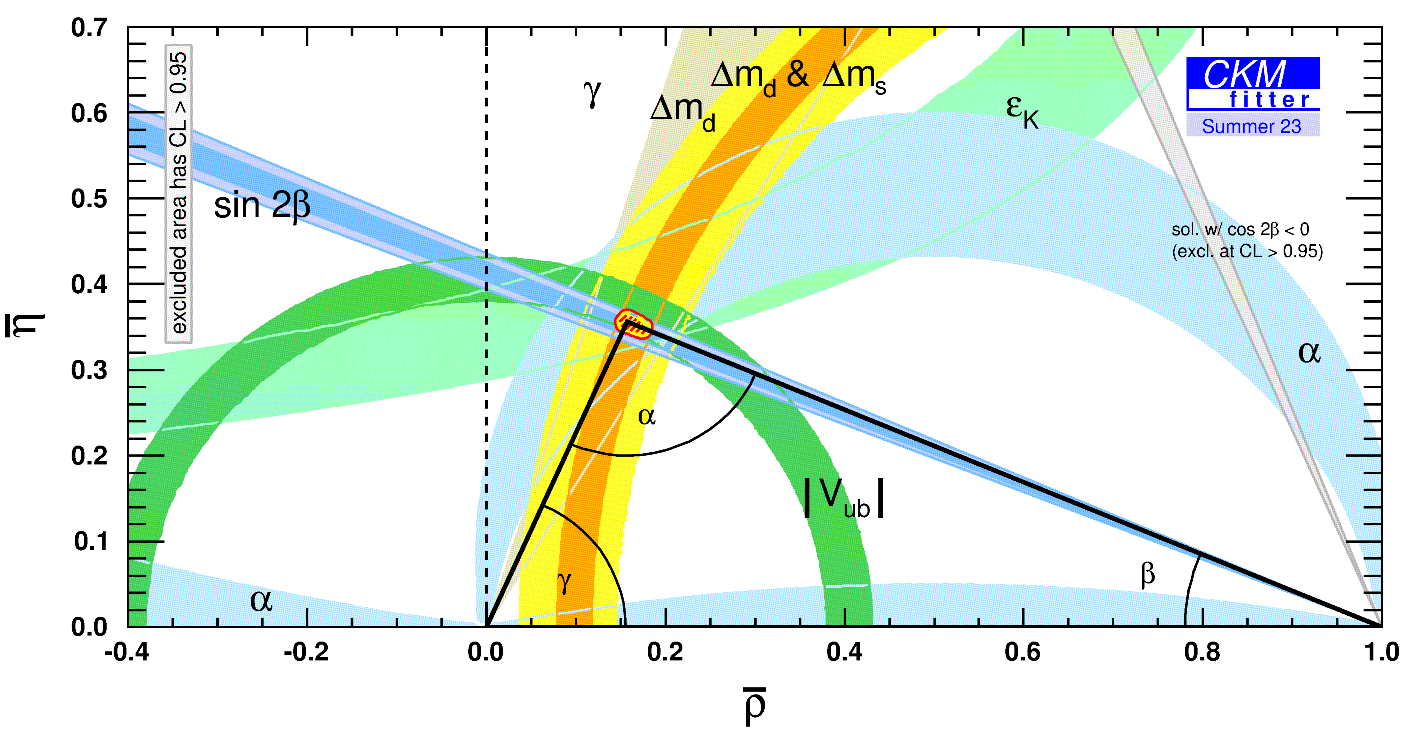

"Resolved" by including other observables assuming CKM unitarity (SM fit)

But we want to test CKM unitarity!

→ need to improve the theory for V_{ub} and V_{cb} from tree-level B decays

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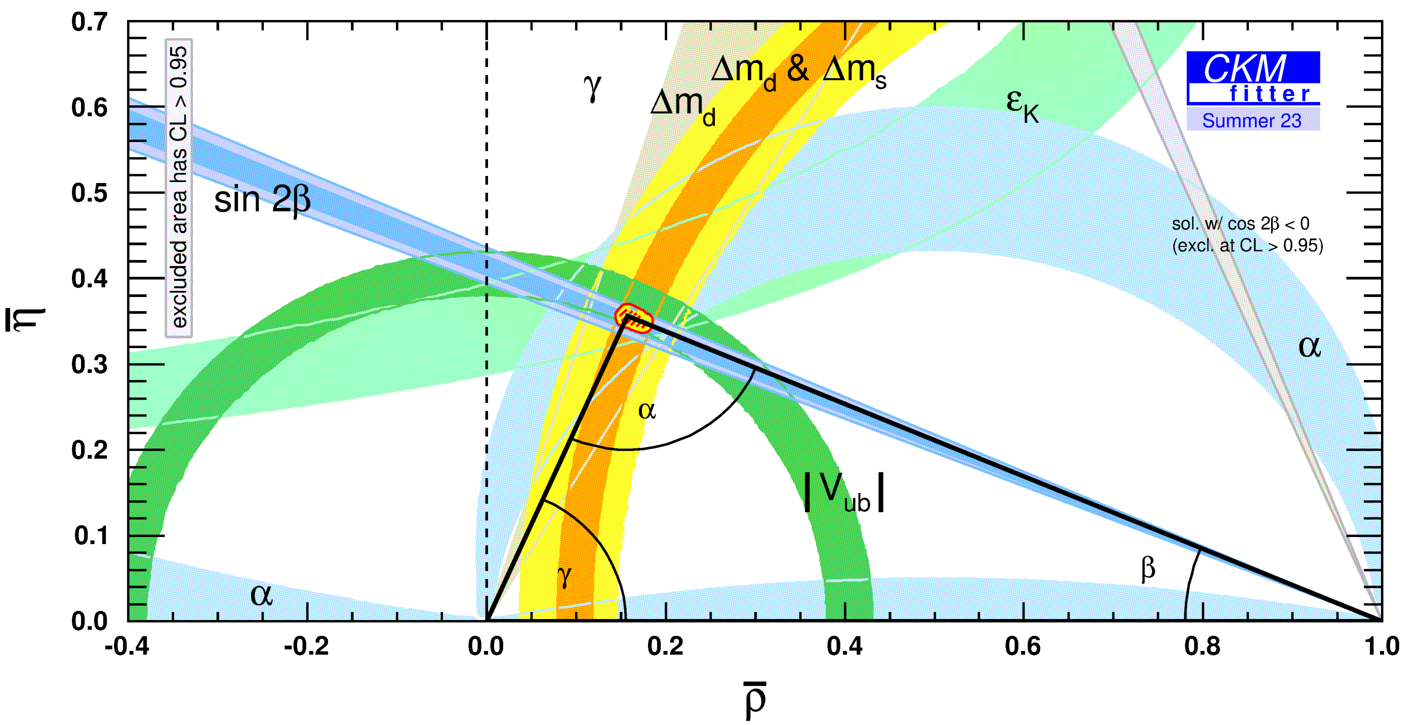

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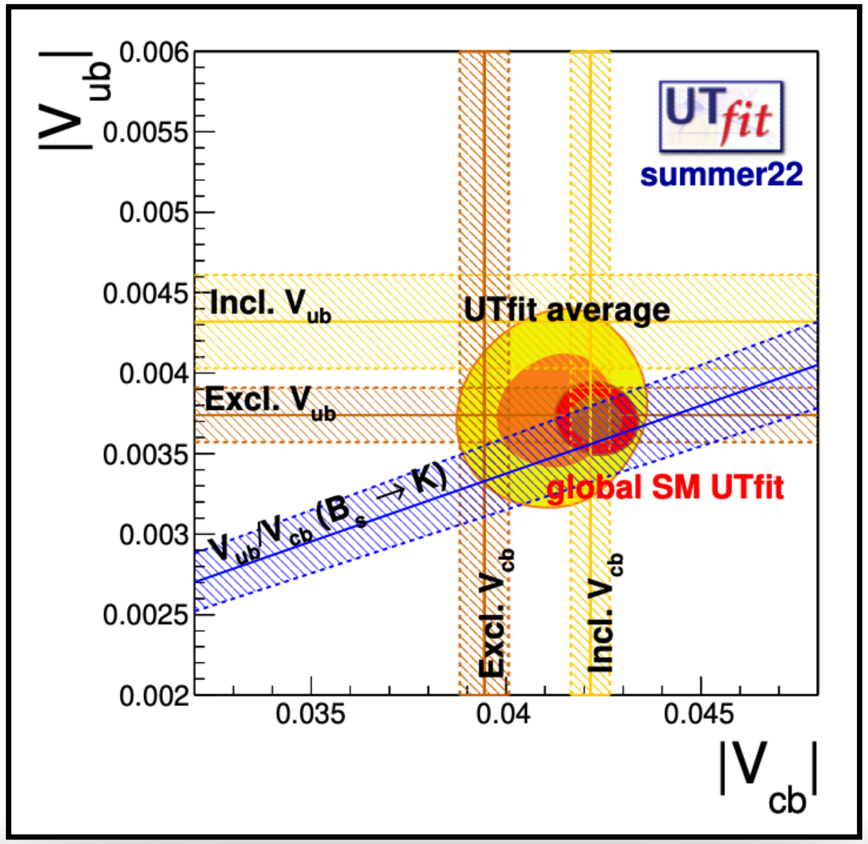


Inclusive V_{ub}/V_{cb} (?)

"Resolved" by including other observables assuming CKM unitarity (SM fit)

But we want to test CKM unitarity!

→ need to improve the theory for V_{ub} and V_{cb} from tree-level B decays

CKM fits and semileptonics

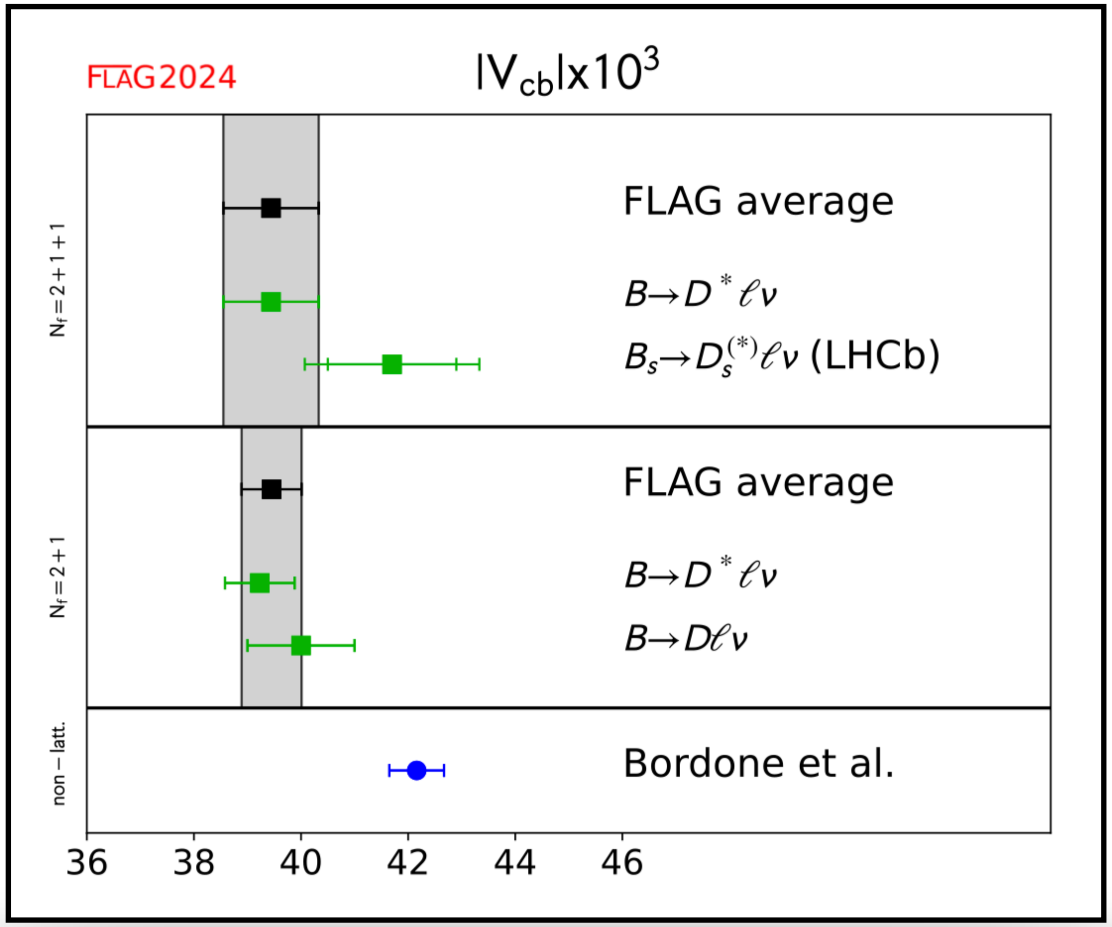
Steady progress recently, especially $|V_{cb}|$ (three loop calculations, spectrum measurements, lattice)

$$|V_{cb}|_{inc} = (42.16 \pm 0.51) \times 10^{-3}$$ Bordone, Capdevilla, Gambino [2310.20324]

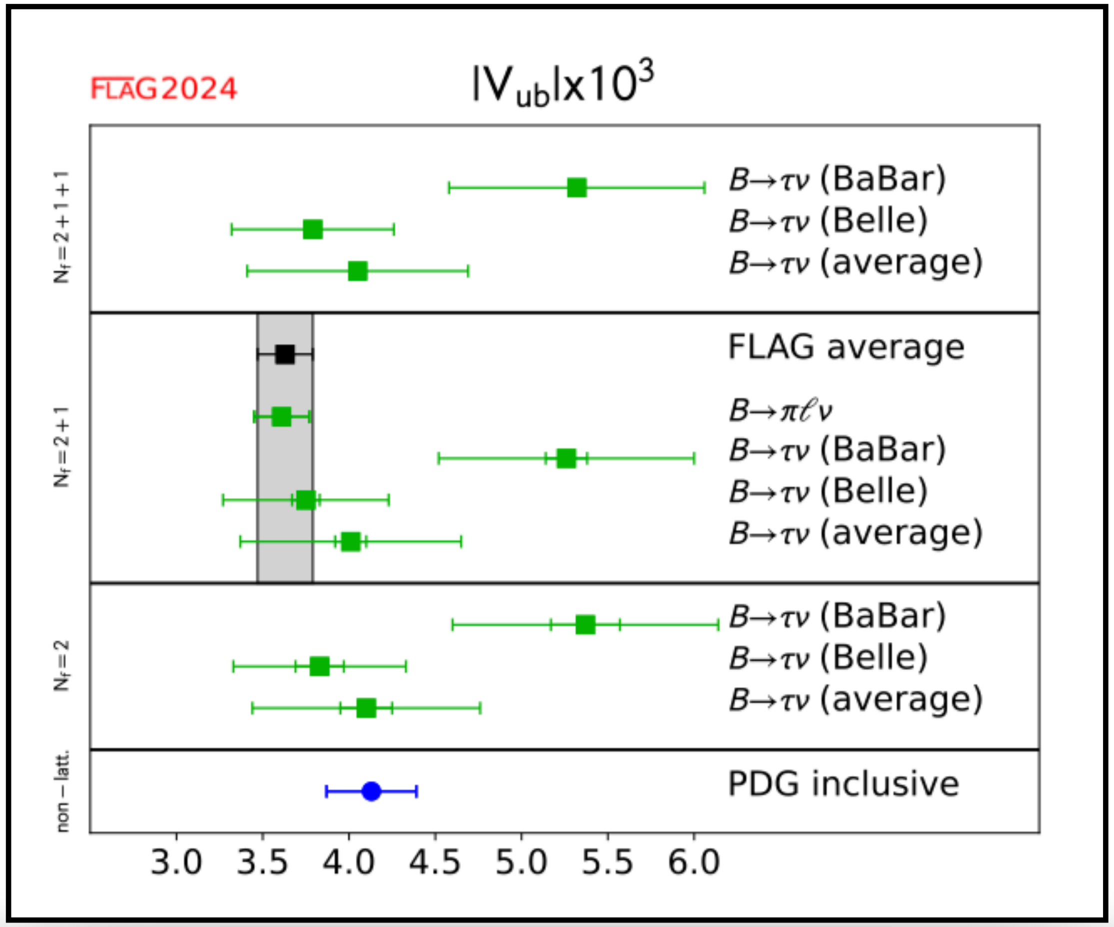
$$= (41.69 \pm 0.63) \times 10^{-3}$$ Bernlochner et. al. [2310.20324]

$$= (41.97 \pm 0.48) \times 10^{-3}$$ Finauri, Gambino [2310.20324]

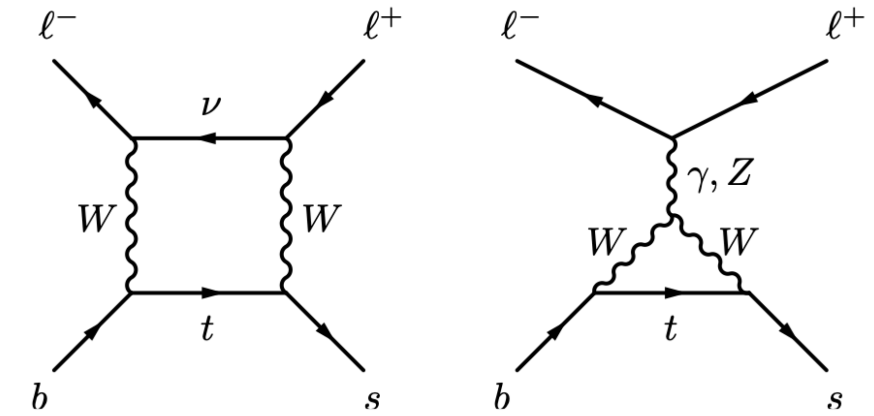
$$|V_{cb}|_{excl} = (39.46 \pm 0.53) \times 10^{-3}$$ [2411.04268]



$$|V_{ub}|_{excl} = (3.60 \pm 0.14) \times 10^{-3}$$



Neutral currents



Loop suppressed, rates are very small
(sensitive to BSM)

Huber et. al.

$$BR(B \to X_s \mu\mu)\,|_{SM} = (16.87 \pm 1.25) \times 10^{-7}$$

$$BR(K^+ \to \pi^+ \nu\bar{\nu})\,|_{SM} = (7.86 \pm 0.61) \times 10^{-11}$$

D'Ambrosio et. al. [2206.14748]

Methods largely the same as for charged currents:
form factors for exclusives, OPE for inclusive..

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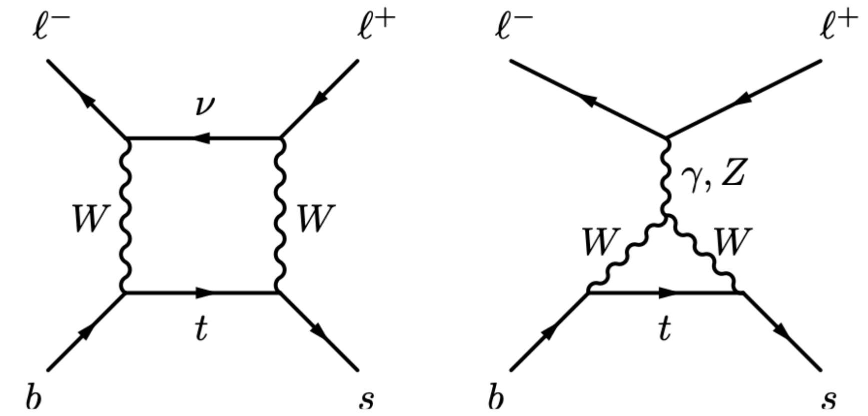
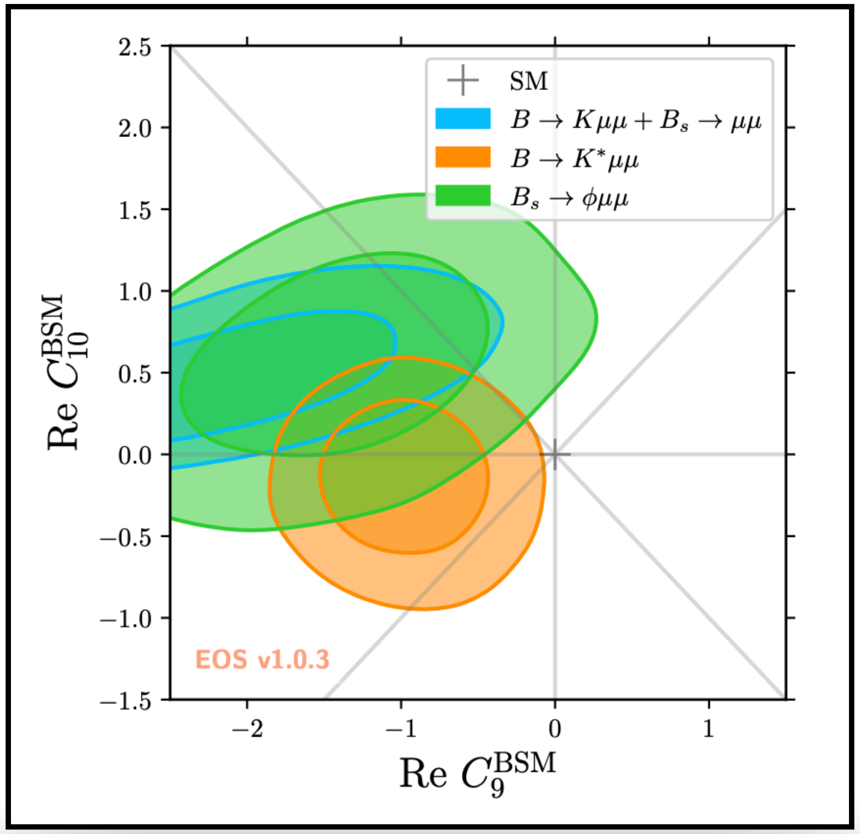


Gubernari, Reboud, van Dyk, Virto
[2305.06301]



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New: conceptual and technical challenges arise from neutral intermediate states

$$B \to X_s c\bar{c}(\to \mu\mu)$$

$$K^+ \to \pi^+(\pi^+\pi^- \to \mu^+\mu^-)$$

Especially challenging for B decays because there are many intermediate states to take into account (DD, DD^*, D^*D^*)

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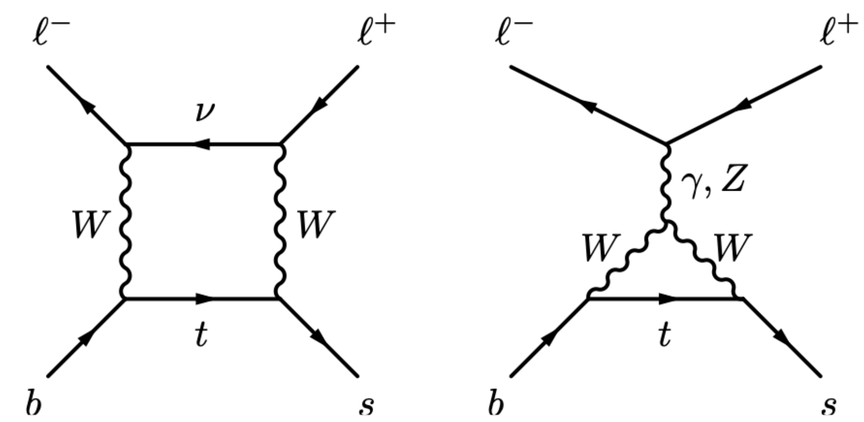

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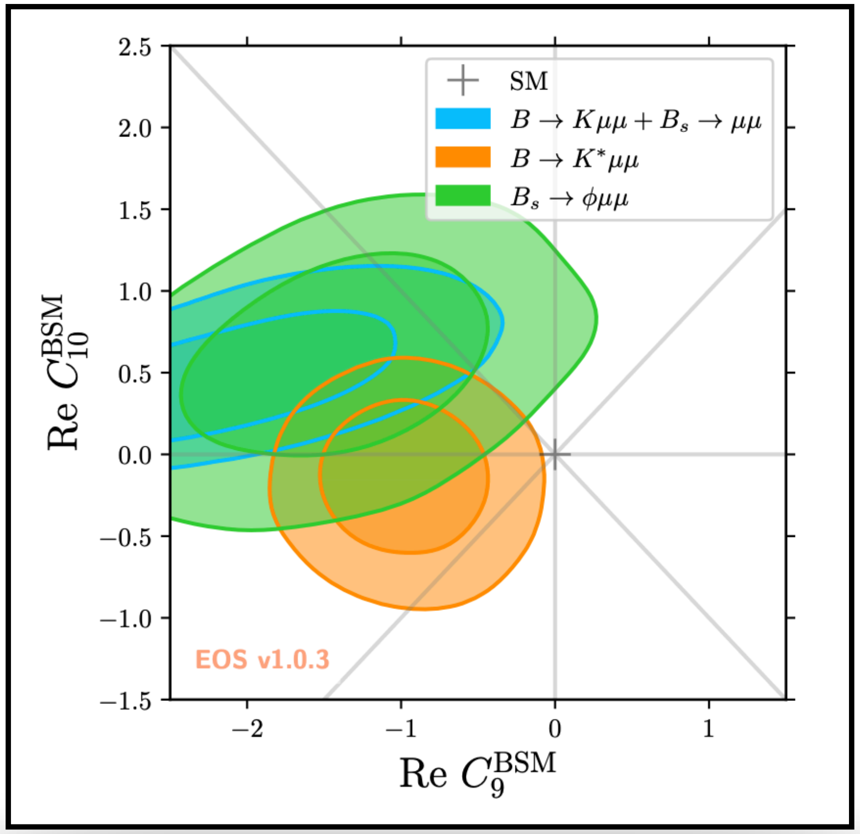

For the <u>inclusive</u> mode $B \to X_s \mu^+\mu^-$, virtual effects can be calculated in QCD, supplemented with <u>inclusive</u> hadronic inputs (spectral functions)

Huber et. al. [1908.07507]

Outline

- Inclusive B-decays
  - Heavy quark expansion, Phenomenology of Rare Decays, Schemes for heavy quark masses and HQET Wilson coeffs.
- Chiral dynamics
  - $K \to \pi\nu\nu$
  - $B \to \pi\ell\nu$

Inclusive B Decays

Huber, Hurth, Lunghi, JJ, Qin, Vos
[2404.03517]

Charged currents

$$(\bar{c}_L \gamma_\mu b_L)(\bar{\ell}_L \gamma^\mu \nu_L)$$

$$\mathscr{L}_{b\to c} = -\frac{4G_F}{\sqrt{2}} V_{cb} C_{V-A}(\mu) Q_{V-A}$$

$b \to c$ current is conserved in QCD

$$C_{V-A}(\mu) = 1 + \frac{\alpha(\mu)}{2\pi}\left(\ln\frac{\mu^2}{M_Z^2} + \frac{11}{6}\right) \simeq 1.005$$

Scale dependence from QED logs, but no new operators appear (chiral limit $m_b \ll M_W$)

Bigi et. al. [2309.02849]

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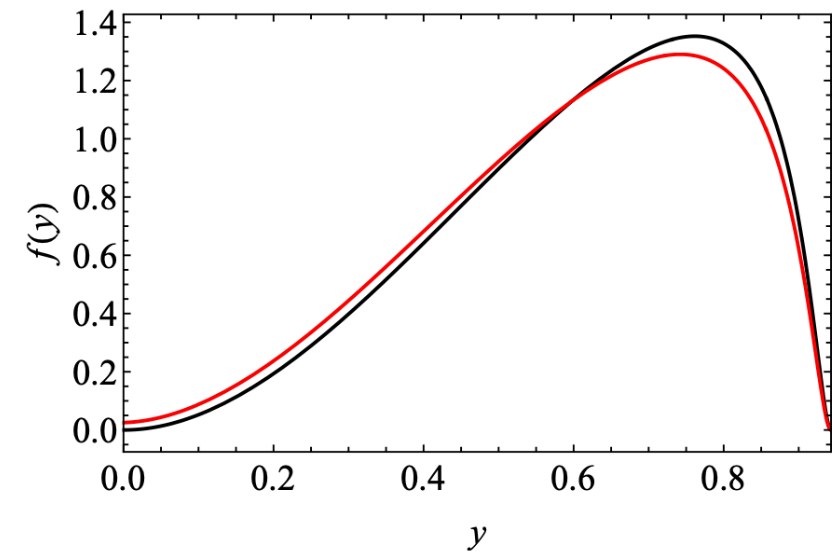
Leading power ($m_b \gg \Lambda$)

$$\rho = m_c^2/m_b^2 \qquad y = 2E_\ell/m_b$$

$$\frac{d\Gamma}{dy} = \frac{G_F^2 m_b^5}{192\pi^2}|V_{cb}|^2\left[f_0(y,\rho) + \frac{\alpha_s}{\pi}f_1(y,\rho) + \left(\frac{\alpha_s}{\pi}\right)^2 f_2(y,\rho) + \frac{\alpha_e}{\pi}f_{em}(y,\rho)\right]$$



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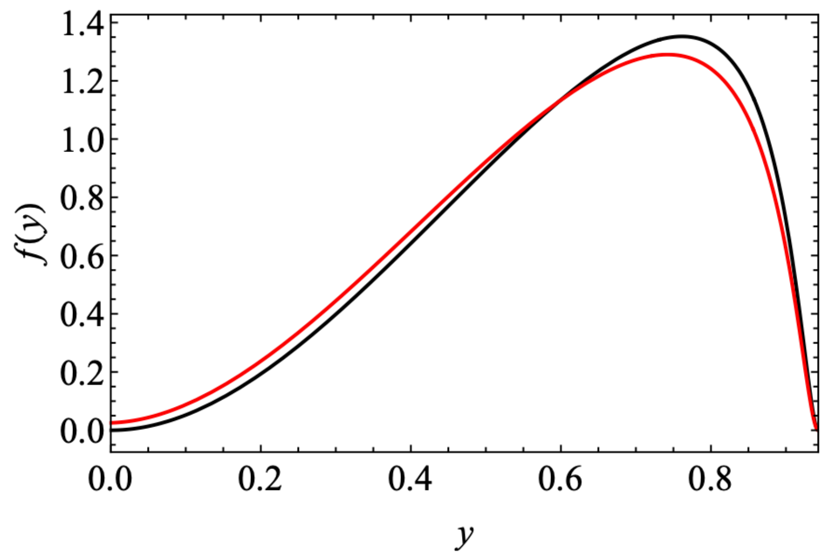


Total rate (integral over y) at N3LO

Rather sensitive to scheme for heavy quark mass (m_b^5)

Fael, Schönwald, Steinhauser [2011.13654]

(adapted, no power corrections)

Kinetic scheme:

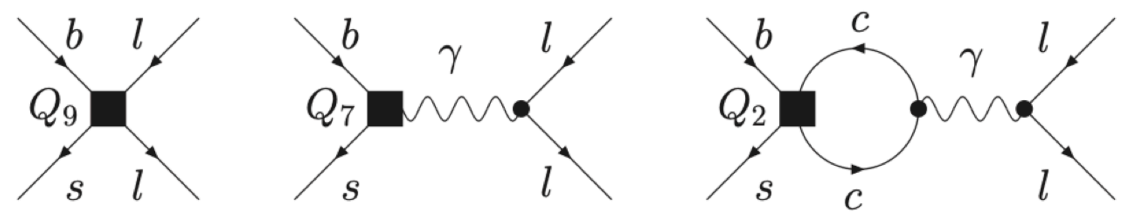
$$\Gamma_{c\ell\nu} \sim 1 - 0.1162_{\alpha_s} - 0.0350_{\alpha_s^2} - 0.0097_{\alpha_s^3}$$

$$\Gamma_{u\ell\nu} \sim 1 - 0.020_{\alpha_s} - 0.012_{\alpha_s^2} + 0.017_{\alpha_s^3}$$

Fael, Vienna 09'24 (prelim.)

Neutral currents

Semileptonic operators mix with the nonleptonic operators at order α



Since the lowest order amplitude is order α, the running is an $O(1)$ relative effect (!)

Interplay between QCD and QED logarithms ($\mu \gg m_b$)

$$\alpha_s \ll 1 \quad \alpha_e/\alpha_s \ll 1$$
$$\alpha_s \ln(\mu/\mu_0) \sim 1$$

$$Q_1 = (\bar{s}_L \gamma_\mu T^a c_L)(\bar{c}_L \gamma^\mu T^a b_L)$$
$$Q_2 = (\bar{s}_L \gamma_\mu c_L)(\bar{c}_L \gamma^\mu b_L)$$

$$Q_3 = (\bar{s}_L \gamma_\mu b_L) \textstyle\sum_q (\bar{q} \gamma^\mu q)$$
$$Q_4 = (\bar{s}_L \gamma_\mu T^a b_L) \textstyle\sum_q (\bar{q} \gamma^\mu T^a q)$$
$$Q_5 = (\bar{s}_L \gamma_{\alpha\beta\delta} b_L) \textstyle\sum_q (\bar{q} \gamma^{\alpha\beta\delta} q)$$
$$Q_6 = (\bar{s}_L \gamma_{\alpha\beta\delta} b_L) \textstyle\sum_q (\bar{q} \gamma^{\alpha\beta\delta} T^a q)$$

$$Q_{3Q} = (\bar{s}_L \gamma_\mu b_L) \textstyle\sum_q e_q (\bar{q} \gamma^\mu q)$$
$$Q_{4Q} = (\bar{s}_L \gamma_\mu T^a b_L) \textstyle\sum_q e_q (\bar{q} \gamma^\mu T^a q)$$
$$Q_{5Q} = (\bar{s}_L \gamma_{\alpha\beta\delta} b_L) \textstyle\sum_q e_q (\bar{q} \gamma^{\alpha\beta\delta} q)$$
$$Q_{6Q} = (\bar{s}_L \gamma_{\alpha\beta\delta} b_L) \textstyle\sum_q e_q (\bar{q} \gamma^{\alpha\beta\delta} T^a q)$$
$$Q_b = \dots$$

$$Q_9 = (\bar{s}_L \gamma_\mu b_L) \textstyle\sum_\ell (\bar{\ell} \gamma^\mu \ell)$$
$$Q_{10} = (\bar{s}_L \gamma_\mu b_L) \textstyle\sum_\ell (\bar{\ell} \gamma^\mu \gamma^5 \ell)$$

Organize perturbation theory around solution to 13x13 ADM at LL

Huber, Lunghi, Misiak, Wyler [0512066]

Neutral currents

Angular analysis sensitive to different combinations of Wilson coefficients

$$\frac{d^2\Gamma_{sll}}{dq^2 dz} = \frac{3}{8}\left[(1+z^2)H_T(q^2) + 2zH_A(q^2) + 2(1-z^2)H_L(q^2)\right]$$

$$\frac{d\Gamma}{dq^2} = H_T + H_L \qquad \frac{dA_{FB}}{dq^2} = \frac{3}{4}H_A$$

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Structure functions of local OPE

$$h_I^{ij} = 1 - \frac{\alpha_s C_F}{2\pi}\omega_I^{ij} + \frac{1}{m_b^2}\chi_I^{ij} + \dots$$

Simplified formulae at the scale $\mu \sim m_b$

$$H_T(q^2) = 2\Gamma_0 m_b^3(1-s)^2 s\left[(C_9^2 + C_{10}^2)h_T^{99}(s) + \frac{4}{s^2}C_7^2 h_T^{77}(s) + \frac{4}{s}C_7C_9h_T^{79}(s)\right] + H_T^{brem}(q^2) \qquad s = q^2/m_b^2$$

$$H_A(q^2) = -4\Gamma_0 m_b^3(1-s)^2 s\left[C_9C_{10}h_A^{90}(s) + \frac{2}{s}C_7C_{10}h_A^{70}(s)\right] + H_A^{brem}(q^2)$$

$$H_L(q^2) = \Gamma_0 m_b^3(1-s)^2\left[(C_9^2 + C_{10})^2 h_L^{99}(s) + 4C_7^2 h_L^{77}(s) + 4C_7C_9h_L^{79}(s)\right] + H_L^{brem}(q^2)$$

Nonlocal (some re-expand into “effective” local terms)

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Normalization

$$\Gamma_0 = \frac{G_F^2}{48\pi^3}\frac{\alpha^2}{16\pi^2}|V_{tb}V_{ts}|^2$$

$$\Gamma_0^{c\ell\nu} = \frac{G_F^2}{192\pi^3}|V_{cb}|^2 \qquad \Gamma_0^{u\ell\nu} = \frac{G_F^2}{192\pi^3}|V_{ub}|^2$$

$$\frac{|V_{tb}V_{ts}|^2}{|V_{cb}|} \sim 1$$

$$\frac{|V_{tb}V_{ts}|^2}{|V_{ub}|^2} \sim \frac{|V_{cb}|^2}{|V_{ub}|^2}$$

Heavy quark expansion

Matching of QCD → bHQET

"Light" QCD charm, but don't neglect the mass

$$\mathscr{L}_{QCD}^{N_\ell+1} \to \bar{b}_v iv\cdot Db_v + \sum_{i=1}^{N_l} \bar{q}_i(i\not{D} - m_i)q_i + \mathscr{L}_{YM}$$
$$+\frac{1}{m_b^2}\left[\bar{b}_v(iD_\perp)^2 b_v + C_G(\mu)\bar{b}_v(i\sigma_{\mu\nu})[iD_\perp^\mu, iD_\perp^\mu]b_v\right] + O(1/m_b^3)$$

Dual expansion in $\alpha_s(\mu)$ and $1/m_b(\mu)$

Matching coefficients at $\mu = m_b$ (MS)

Grozin, Marquard, Piclum, Steinhauser [0707.1388]

$$C_G(m_b) = 1 + 0.1492_{\alpha_s} + 0.0676_{\alpha_s^2} + 0.0497_{\alpha_s^3}$$

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Leading power (in QCD)

$\langle \bar{B}(p)|\bar{b}\gamma_\mu b|\bar{B}(p)\rangle = 2M_B p_\mu$ **exactly** (CVC)

Define HQET matrix elements of physical states

$$\langle \dots \rangle = \frac{1}{2M_B}\langle \bar{B}|\dots|\bar{B}\rangle$$

$$\mu_\pi^2(\mu) = -\langle \bar{b}_v(iD^\perp)^2 b_v\rangle$$

$$\mu_G^2(\mu) = \langle \bar{b}_v(iD_\mu^\perp)(iD_\nu^\perp)(-i\sigma^{\mu\nu})b_v\rangle$$

$$\rho_{LS}^3(\mu) = \langle \bar{b}_v(iD_\mu^\perp)(iv\cdot D)(iD^{\perp\mu})b_v\rangle$$

$$\rho_D^3(\mu) = \langle \bar{b}_v(iD_\mu^\perp)(iv\cdot D)(iD_\nu^\perp)(-i\sigma^{\mu\nu})b_v\rangle$$

$$f_q^a(\mu) = \langle \bar{B}^a|(\bar{b}_v\gamma_\mu q)(\bar{q}\gamma^\mu b_v)|\bar{B}^a\rangle$$

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$$+\frac{1}{m_b^2}\left[\bar{b}_v (iD_\perp)^2 b_v + C_G(\mu)\bar{b}_v (i\sigma_{\mu\nu})[iD_\perp^\mu, iD_\perp^\mu] b_v\right] + O(1/m_b^3)$$

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Matching coefficients at $\mu = m_b$ (MS)

Grozin, Marquard, Piclum, Steinhauser [0707.1388]

$$C_G(m_b) = 1 + 0.1492_{\alpha_s} + 0.0676_{\alpha_s^2} + 0.0497_{\alpha_s^3}$$

Leading power (in QCD)

$\langle \bar{B}(p) | \bar{b}\gamma_\mu b | \bar{B}(p)\rangle = 2M_B p_\mu$ **exactly** (CVC)

Define HQET matrix elements of physical states

$$\langle \ldots \rangle = \frac{1}{2M_B}\langle \bar{B} | \ldots | \bar{B}\rangle$$

$$\mu_\pi^2(\mu) = -\langle \bar{b}_v (iD^\perp)^2 b_v\rangle$$

$$\mu_G^2(\mu) = \langle \bar{b}_v (iD_\mu^\perp)(iD_\nu^\perp)(-i\sigma^{\mu\nu}) b_v\rangle$$

$$\rho_{LS}^3(\mu) = \langle \bar{b}_v (iD_\mu^\perp)(iv\cdot D)(iD^{\perp\mu}) b_v\rangle$$

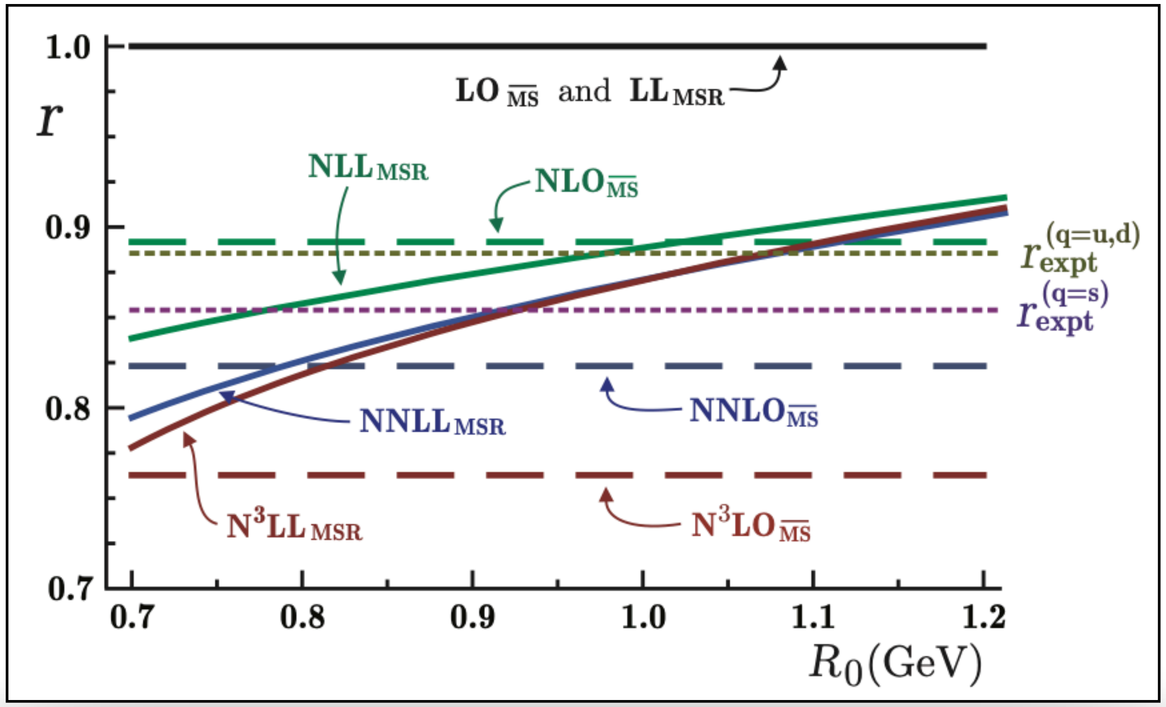
$$\rho_D^3(\mu) = \langle \bar{b}_v (iD_\mu^\perp)(iv\cdot D)(iD_\nu^\perp)(-i\sigma^{\mu\nu}) b_v\rangle$$

$$f_q^a(\mu) = \langle \bar{B}^a | (\bar{b}_v \gamma_\mu q)(\bar{q}\gamma^\mu b_v) | \bar{B}^a\rangle$$

HQET charm quark ($N_l = 3$)

$$R = \frac{M_{B*}^2 - M_B^2}{M_{D*}^2 - M_D^2} = \frac{C_G(m_b)}{C_G(m_c)} + O(1/m_{b,c})$$

Hoang, Jain, Scimemi, Stewart [0908.3189]



Heavy quark expansion

Power corrections (even up to $1/m_b^3$) can be extracted from the distribution of semileptonic B (in principle even D) decays

Finauri, Gambino [2310.20324]

m_b^{kin}	$\overline{m}_c(2\,\mathrm{GeV})$	μ_π^2	$\mu_G^2(m_b)$	$\rho_D^3(m_b)$	ρ_{LS}^3	$\mathrm{BR}_{c\ell\nu}$	$10^3\lvert V_{cb}\rvert$
4.573	1.090	0.454	0.288	0.176	−0.113	10.63	41.97
0.012	0.010	0.043	0.049	0.019	0.090	0.15	0.48
1	0.380	-0.219	0.557	-0.013	-0.172	-0.063	-0.428
	1	0.005	-0.235	-0.051	0.083	0.030	0.071
		1	-0.083	0.537	0.241	0.140	0.335
			1	-0.247	0.010	0.007	-0.253
				1	-0.023	0.023	0.140
					1	-0.011	0.060
						1	0.696
							1

Bernlochner et. al. [2205.10274]

	$\lvert V_{cb}\rvert \times 10^3$	m_b^{kin}	$\overline{m}_c$	μ_G^2	μ_π^2	ρ_D^3
Value	41.69	4.56	1.09	0.37	0.43	0.12
Uncertainty	0.59	0.02	0.01	0.07	0.24	0.20

Minimal subtraction

Schemes are defined by counterterms for the fields, masses and couplings in renormalizable QFT, or an EFT with symmetry-preserving regulators

Mass-dependent schemes are defined to all orders by specifying certain conditions that correlation functions should fulfill order by order (eg: textbook pole scheme for massive leptons, also kinetic scheme)

$$m_b^{os} = m_b^{kin}(\mu_k;\mu) + \bar{\Lambda}(\mu_k;\mu) - \frac{\mu_\pi^{kin}(\mu_k,\mu)^2}{2m_b^{kin}} + O(1/m_b^3)$$

The μ_G^2 term doesn't even show up, because it is an 'irrelevant' operator and we have to take $m_Q \to 0$ to compute these matrix elements

Perturbative analogue of the all-orders formula (very schematic)

$$M_B = \langle \mathscr{L}_{QCD} \rangle = \langle \mathscr{L}_{HQET} \rangle$$

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Mass-independent "R" schemes can be required to be 'renormalon free' and there is a huge freedom on what to subtract in addition to the asymptotic part of a series for an observable

$$M_B = m_b(R(\mu),\mu) + \delta M_b(R(\mu),\mu)$$

$$M_B[\alpha_s(\mu)] = m_b(\mu) \sum_n c_n \alpha_s^n(\mu)$$
$$+ \int \phi(\gamma)\, d\gamma\, \exp[\gamma/\alpha_s(\mu)]$$

observable

Includes all asymptotics

Minimal: take $R(\mu)$ from pole-MS relation at fixed order, with $R(\mu) = \mu$

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$$M_{B*}^2 - M_B^2 = C_G(R)\mu_G^2(R) + \delta(\Delta M_B^2)$$

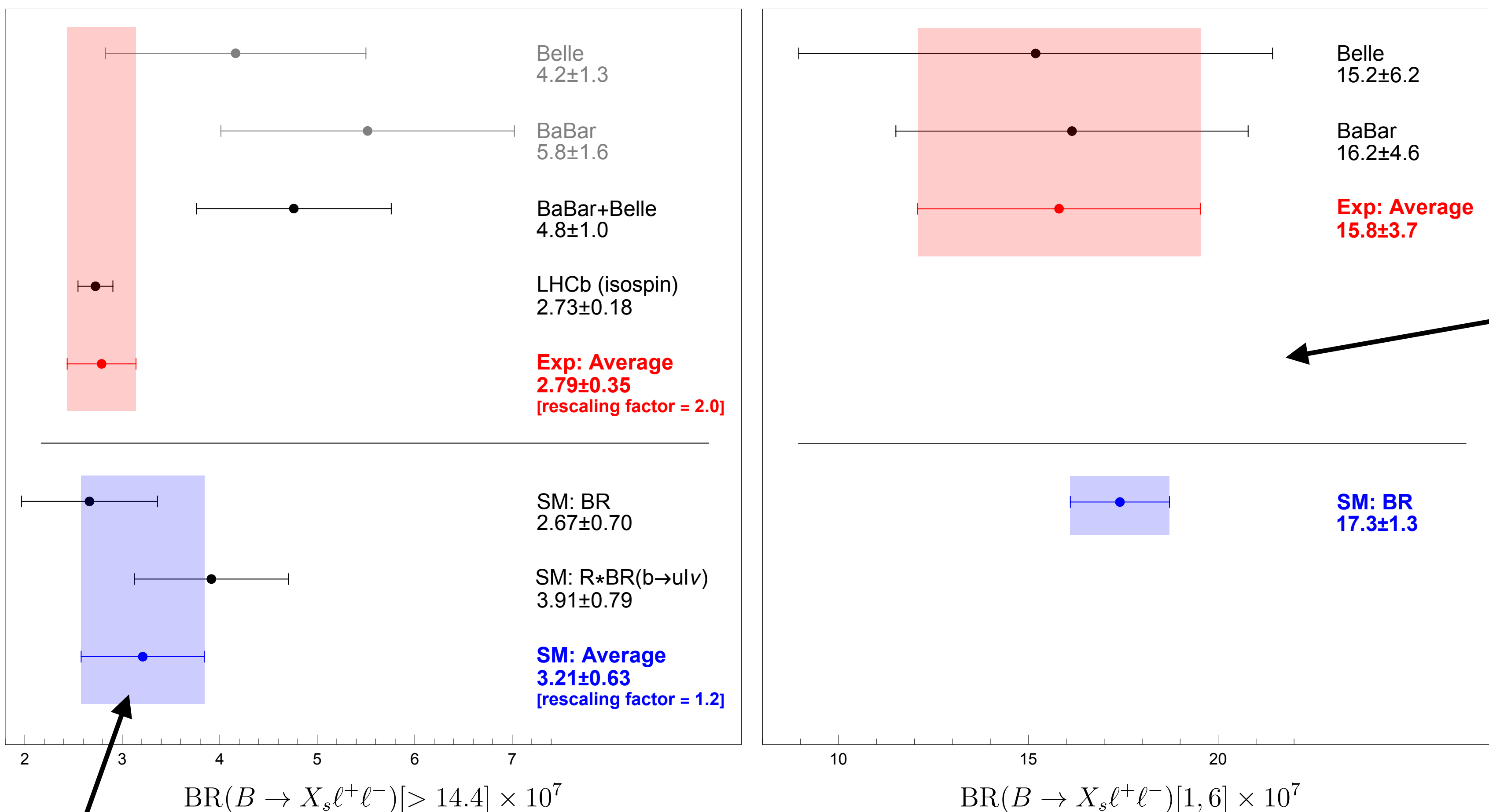
For semileptonics: need renormalon-free HQ masses and HQET matrix elements

Minimal (?): take $R(\mu)$ from pole-MS magnetic moment relation at fixed order, with $R(\mu) = \mu$

Phenomenology

Branching ratios above / below narrow resonances

Hurth, Huber, Lunghi, JJ, Qin, Vos [2404.03517]



No LHCb yet

Theory mature at low-q^2, power corrections are small

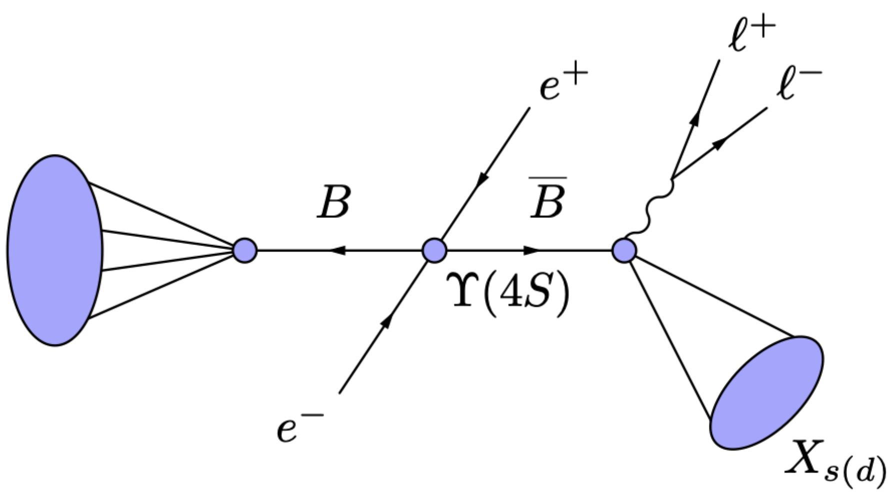
Effects of power corrections are large at high-q^2, even after normalizing to $B \to X_u$

$$\mathscr{B}[>14.4] = (3.05 - 5.87\lambda_2^{eff} + 8.09\rho_1) \times 10^{-7}$$

$$\mathscr{R}[>14.4] = (24.90 + 2.49\lambda_2^{eff} + 10.72\rho_1) \times 10^{-4}$$

B-Tagging

B factories

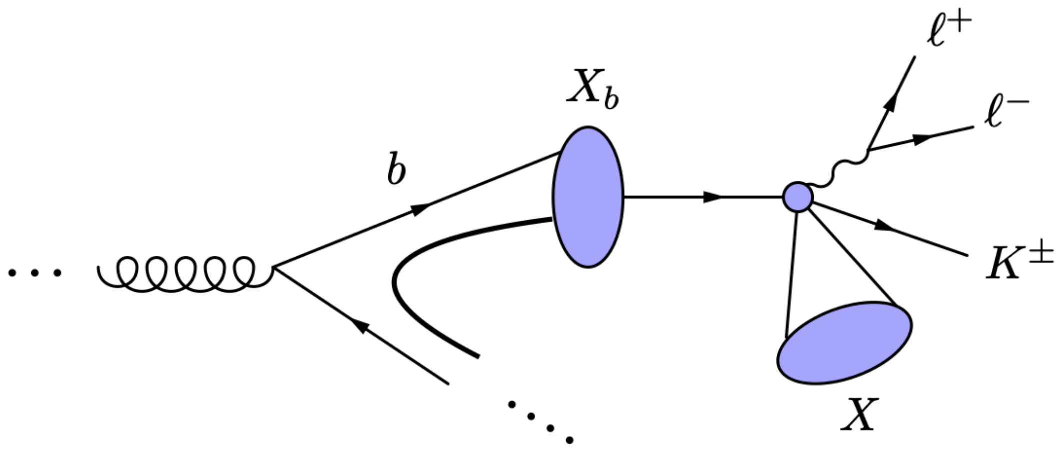


Reconstruct $\bar{B}$ momentum from tagging recoil B
(Low efficiency, gain in systematics)

BaBar and Belle used sum over exclusive modes
(including neutrals $\pi^0 \to \gamma\gamma$)

Belle [0208029]	$\mathscr{B}$	65 M $B\bar{B}$ pairs
Belle [0503044]	$\mathscr{B}$	152 M
Belle [1402.7134]	$\mathscr{A}_{FB}$	772 M
BaBar [0404006]	$\mathscr{B}$	89 M
BaBar [1312.5364]	$\mathscr{B}$	471 M

LHCb (?)



Sum over exclusive modes, isospin re-weighting
$B^{0,+} \to K^+(n\pi^\pm)\mu^+\mu^-$ (avoid neutrals)

Koppenburg [CERN-THESIS-2002-010]

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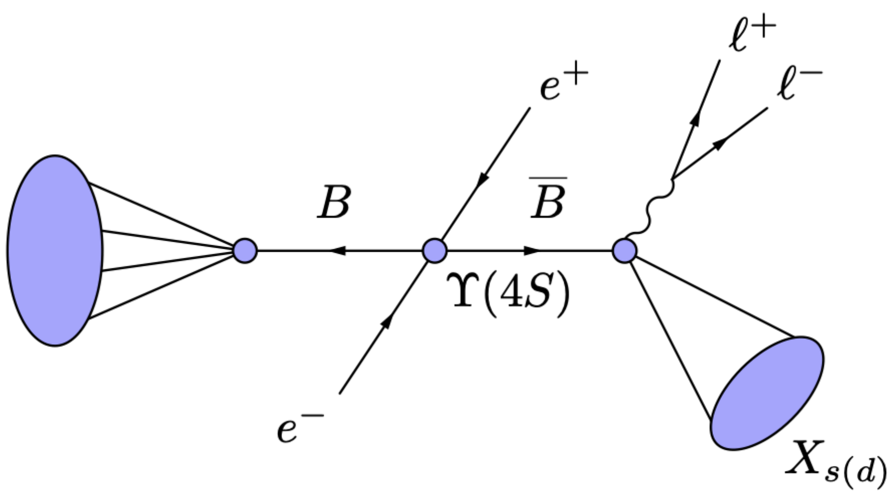

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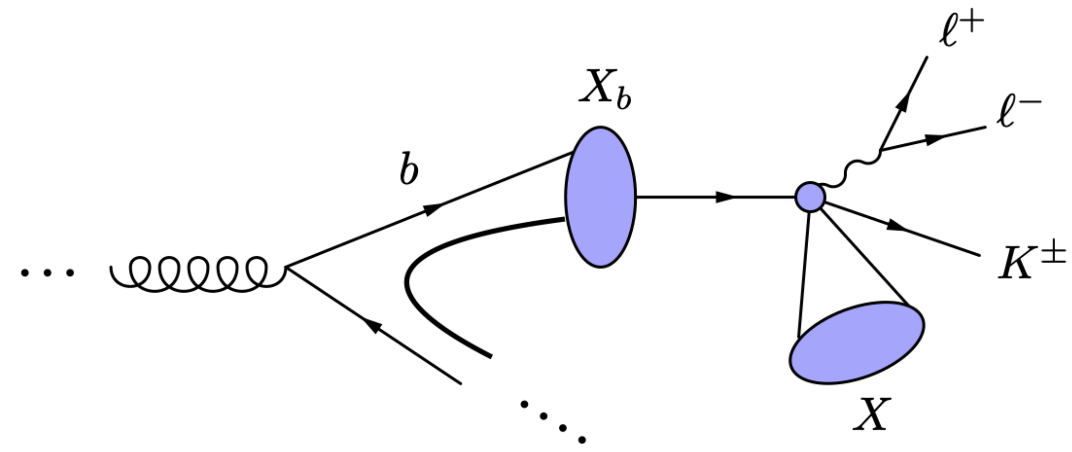

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Isospin extrapolation, semi-inclusive strategy
$X_b \to K^+\mu^+\mu^-X$ (vertex 3 charged particles)

Amhis, Owen [2106.15943]

Separately measure and subtract $\bar{B}_s$ and Λ_b contaminations to X_b using an additional K or p

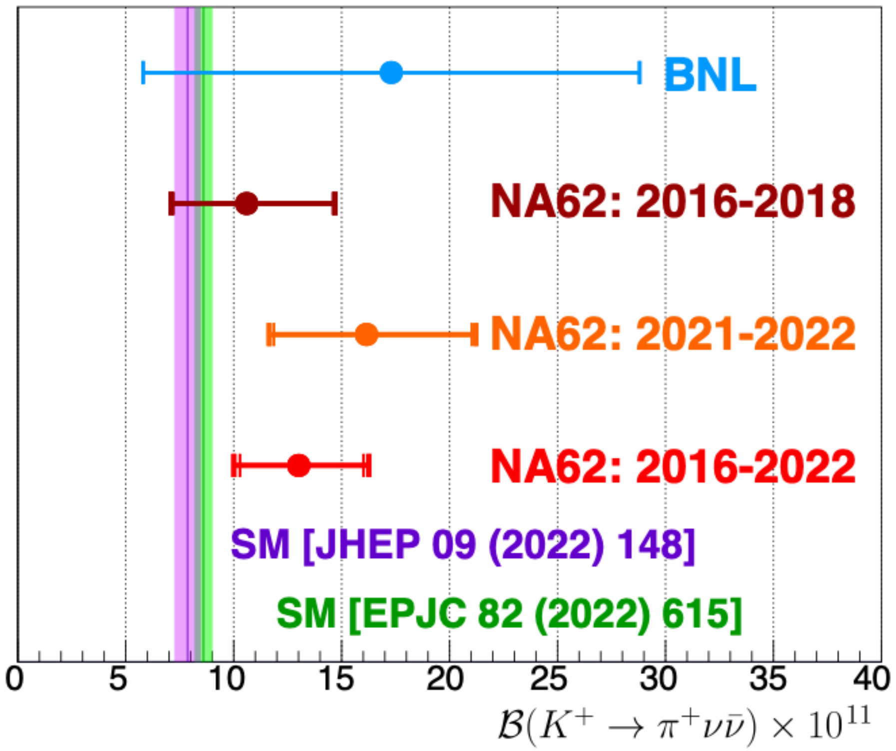
$$\bar{B}_s: \quad X_b \to K^+K^-\mu^+\mu^-X$$

$$\Lambda_b: \quad X_b \to pK^-\mu^+\mu^-X$$

Chiral dynamics: Rare Kaon Decays

Anshika Bansal, JJ, Daniel Winney
[preliminary]

Motivation: NA62 update



$$BR(K^+ \to \pi^+ \nu\bar{\nu})\Big|_{exp} = 13.0(^{+3.0}_{-2.7})_{stat}(^{+1.3}_{-1.2})_{syst}$$

NA62 [2412.12015]

$$BR(K^+ \to \pi^+ \nu\bar{\nu})\Big|_{SM} = (7.73 \pm 0.16_{pert} \pm 0.25_{non-pert} \pm 0.54_{par}) \times 10^{-11}$$

Brod, Gorbahn, Stamou [2105.02868]

Four frontiers for precision in $K^+ \to \pi^+ \nu\bar{\nu}$:

- Experiment (still statistically limited)
- Progress on $|V_{cb}|$ in B sector: top quark contribution is proportional to $|V_{ts}V_{td}|^2 \sim |V_{cb}|^4$
- $V_{ts}^* V_{td} X_t(m_t)$ at higher order in perturbative QCD
- Intrinsic hadronic uncertainties (local and nonlocal FFs)

Scale separation

$Q_\nu = (\bar{d}_L \gamma_\mu s_L)(\bar{\nu}_L \gamma^\mu \nu_L)$

Dominant contribution from Q_ν sensitive to large top quark mass, known at NLO QCD and NLO EW

Brod, Gorbahn, Stamou [1009.0947]

RGE invariant below the weak scale (CVC)

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$Q_2 = (\bar{d}_L \gamma_\mu T^a c_L)(\bar{c}_L \gamma^\mu T^a s_L) - (\bar{d}_L \gamma_\mu T^a u_L)(\bar{u}_L \gamma^\mu T^a s_L)$

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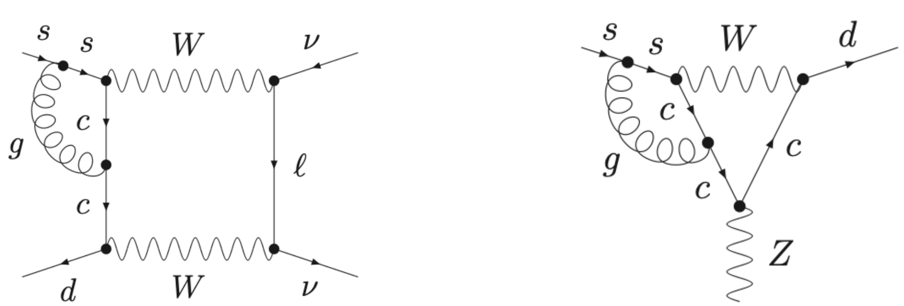
$Q_3 = (\bar{d}_L \gamma_\mu s_L) \sum_q \bar{q} \gamma^\mu q$

$\ldots$



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Charm mass cannot be neglected at the electroweak scale, due to interplay of GIM and CKM suppression of the charm / top contributions

$$V_{cs}^* V_{cd} \sim \lambda, \quad V_{ts}^* V_{td} \sim \lambda^5$$

Resummation of $x_c^2 \alpha_s^n (\alpha_s \ln x_c)^k$ corrections to all orders in k and for $n = 0,1$

Buras, Gorbahn, Haisch, Nierste [0603079]

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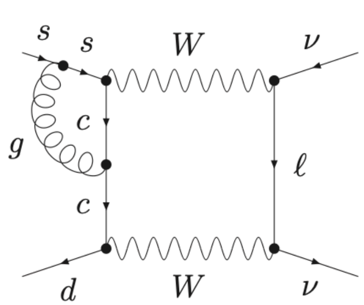

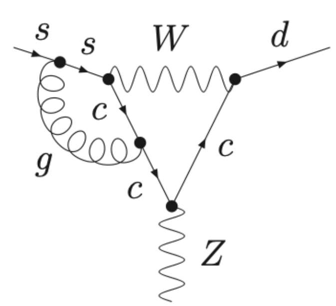

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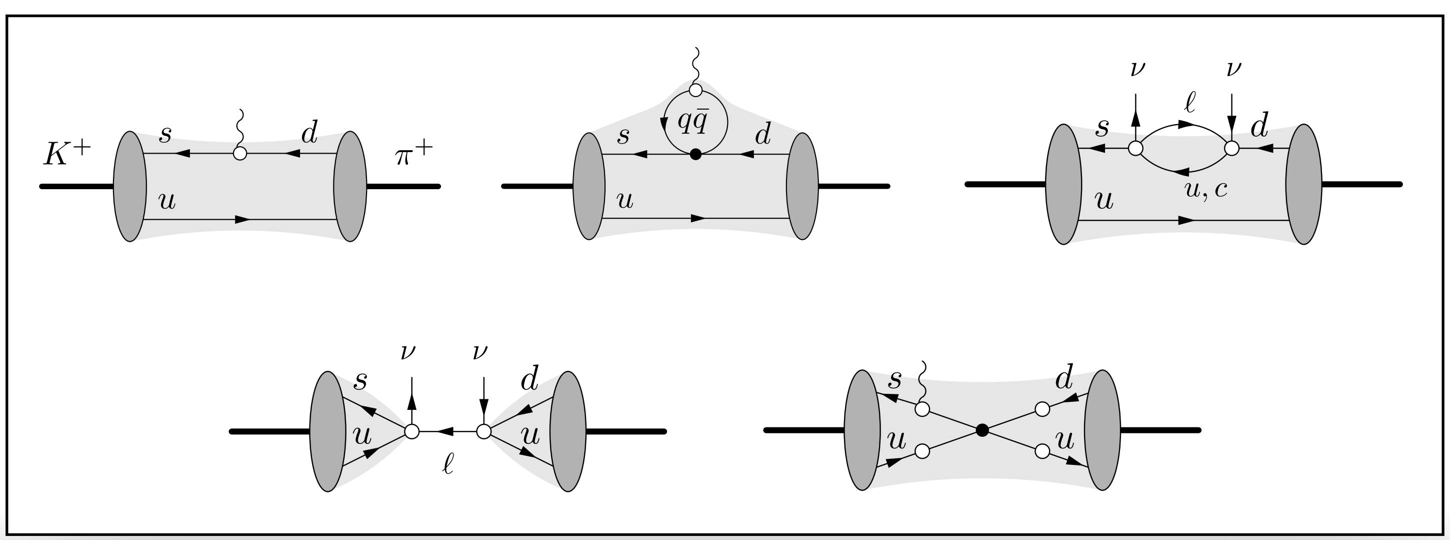


Nonlocal operators / matrix elements from factorization

Actual values of these FFs (??)

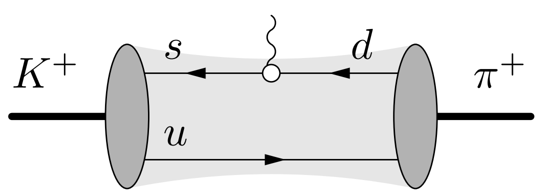
Local form factors

Charged currents:

$$\langle \pi^+(k) \,|\, \bar{u}\gamma_\mu s \,|\, K^0(p) \rangle = f_+^{K\to\pi}(q^2)(p+k)_\mu + f_-^{K\to\pi}(q^2) q_\mu$$

Neutral currents:

$$\langle \pi^+(k) \,|\, \bar{d}\gamma_\mu s \,|\, K^+(p) \rangle = f_+^{K\to\pi}(q^2)(p+k)_\mu + f_-^{K\to\pi}(q^2) q_\mu$$



Local vector form factors from V-A currents in SM (also V+A for FCNCs, hadronic current is the same)

Universal to charged-current and neutral-current $K \to \pi^+$ transitions up to isospin breaking corrections ($K^+ \to \pi^0$ complicated by $\pi^0 - \eta$ mixing LECs)

$$\frac{f_+^{K^+\pi^+}(0)}{f_+^{K^0\pi^+}(0)} = 1.0015 \pm 0.0007 \qquad \frac{\lambda_+^{K^+\pi^+}(0)}{\lambda_+^{K^0\pi^+}(0)} = 0.9986 \pm 0.0002$$

Mescia, Smith [0705.2025]

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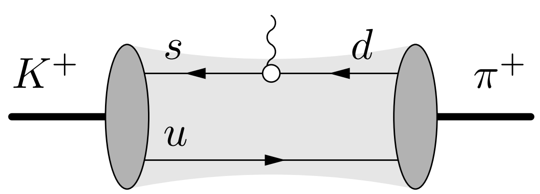

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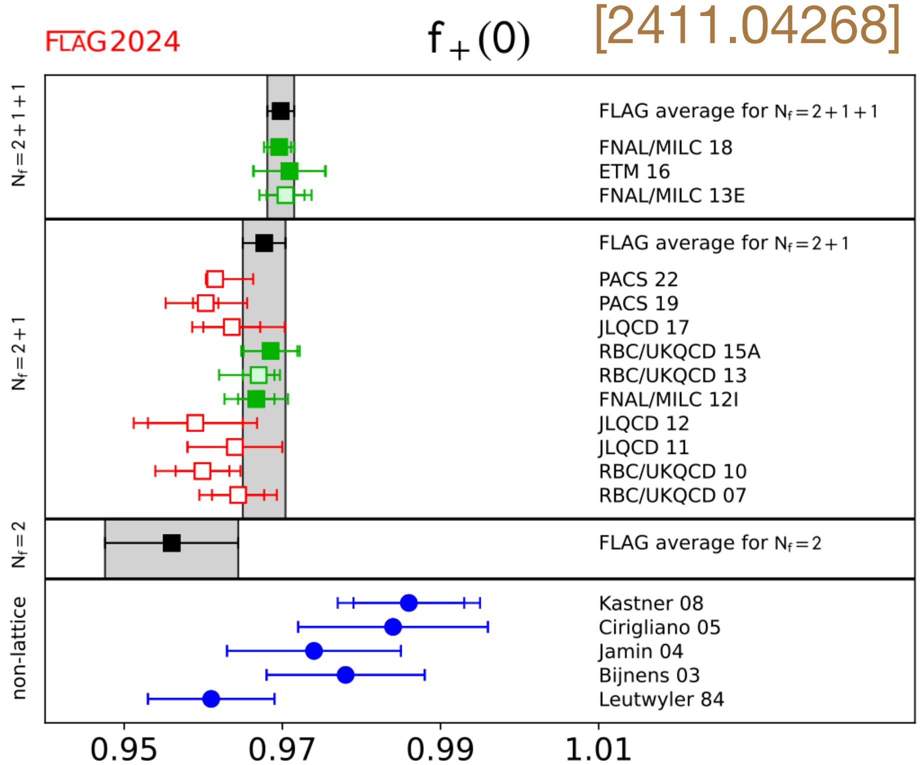
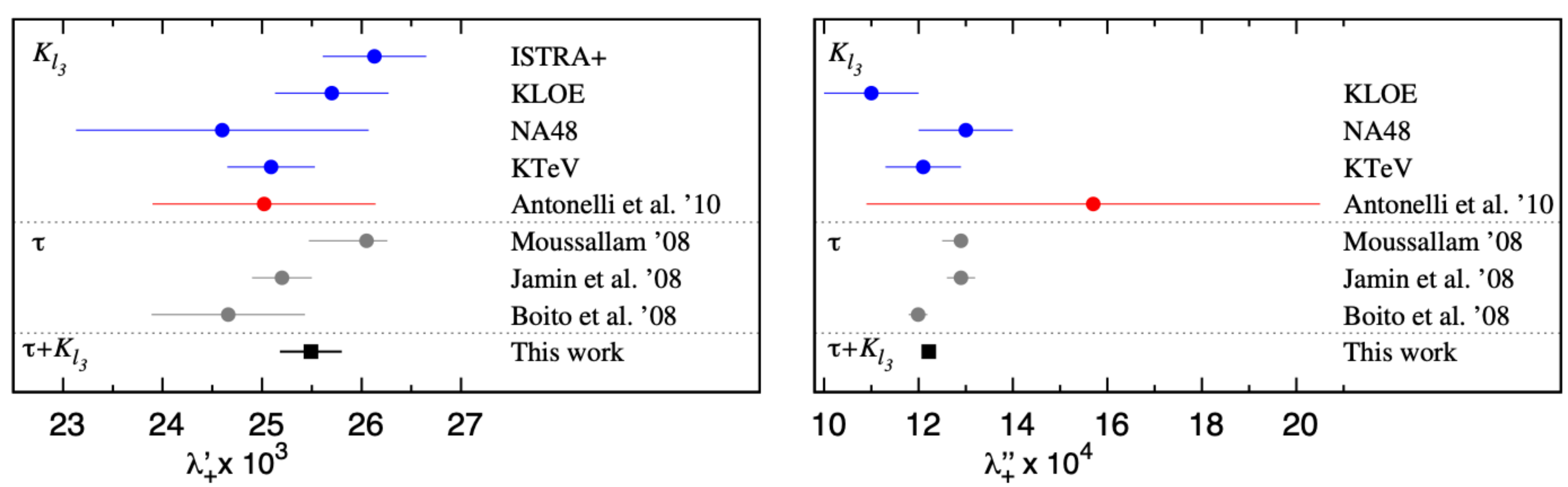
Mescia, Smith [0705.2025]

Normalization: LQCD

[2411.04268]



Slope parameters from phenomenology:
$K\to\pi\ell\nu$ and $\tau\to K\pi\bar{\nu}_\tau$ (analyticity)



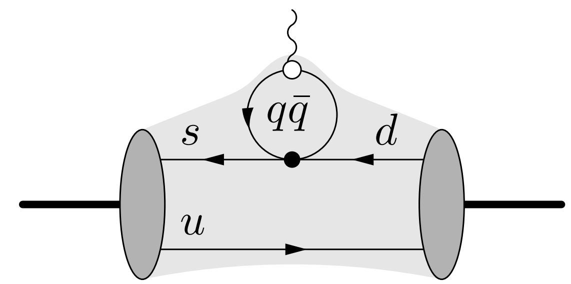
Boito, Escribano, Jamin [1007.1858]

Nonlocal form factors

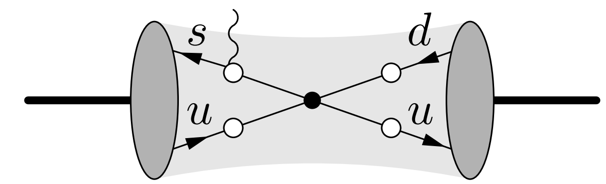
Electromagnetic form factor dominates $K^+ \to \pi^+ \ell^+ \ell^-$
and can be extracted from the spectrum up to a phase

$$\int d^4x\, e^{-iqx} \langle \pi^+ \,|\, TQ(0) J^\mu_\gamma(x) \,|\, K^+ \rangle = (q^\mu p \cdot q - p^\mu q^2) F_\gamma^{K^+\pi^+}(q^2)$$

$$Q(0) = C_1(\mu) Q_1 + C_2(\mu) Q_2$$



Weak neutral-current form factor in $K^+ \to \pi^+ \nu \bar{\nu}$

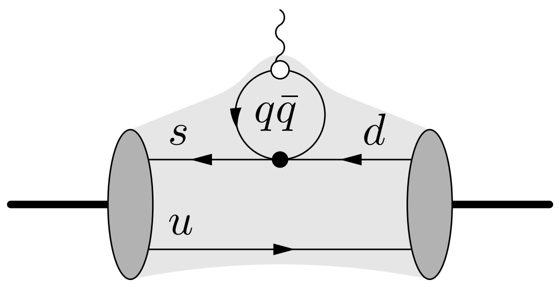


$$\int d^4x\, e^{-iqx} \langle \pi^+ \,|\, TQ(0) J^\mu_Z(x) \,|\, K^+ \rangle$$

$$= (q^\mu p \cdot q - p^\mu q^2) F_{Z\|}^{K^+\pi^+}(q^2) + q^\mu F_{Z\perp}^{K^+\pi^+}(q^2)$$

no contribution
to rate ($m_\nu = 0$)

Nonlocal form factors



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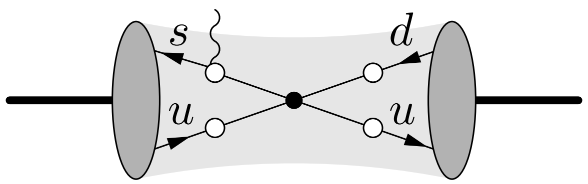

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no contribution to rate ($m_\nu = 0$)

Weak and electromagnetic charges are not aligned

$$J^\gamma_\mu = \frac{2}{3}(\bar{u}\gamma_\mu u + \bar{c}\gamma_\mu c) - \frac{1}{3}(\bar{d}\gamma_\mu d + \bar{s}\gamma_\mu \bar{s})$$

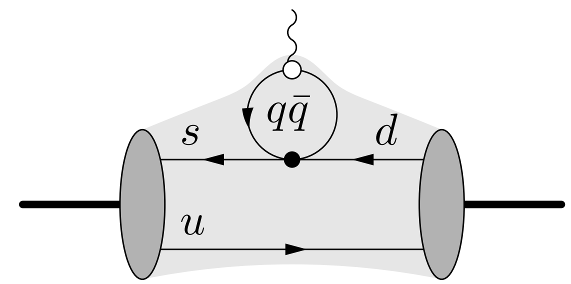
$$\frac{Q_u}{Q_d} = -2$$

$$J^Z_\mu = c^u_\nu(\bar{u}\gamma_\mu u + \bar{c}\gamma_\mu c) + c^d_\nu(\bar{d}\gamma_\mu d + \bar{s}\gamma_\mu \bar{s})$$

$$\frac{c^u_\nu}{c^d_\nu} = \frac{1/2 - 4/3\sin^2\theta_W}{-1/2 + 2/3\sin^2\theta_W} = -0.58$$

More information needed to isolate the isospin contribution unique to the weak current…

Nonlocal form factors



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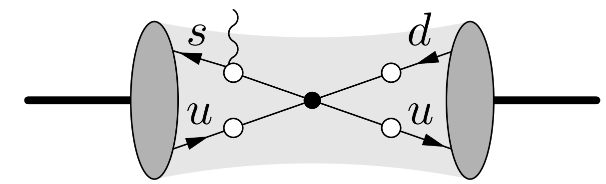


Weak neutral-current form factor in $K^+ \to \pi^+ \nu \bar{\nu}$

$$\int d^4x\, e^{-iqx} \langle \pi^+ | TQ(0) J^\mu_Z(x) | K^+ \rangle$$

$$= (q^\mu p \cdot q - p^\mu q^2) F_{Z\|}^{K^+\pi^+}(q^2) + q^\mu F_{Z\perp}^{K^+\pi^+}(q^2)$$

no contribution to rate ($m_\nu = 0$)

Weak and electromagnetic charges are not aligned

More information needed to isolate the isospin contribution unique to the weak current…

$$J^\gamma_\mu = \frac{2}{3}(\bar{u}\gamma_\mu u + \bar{c}\gamma_\mu c) - \frac{1}{3}(\bar{d}\gamma_\mu d + \bar{s}\gamma_\mu \bar{s}) \qquad \frac{Q_u}{Q_d} = -2$$

$$J^Z_\mu = c_v^u(\bar{u}\gamma_\mu u + \bar{c}\gamma_\mu c) + c_v^d(\bar{d}\gamma_\mu d + \bar{s}\gamma_\mu \bar{s}) \qquad \frac{c_v^u}{c_v^d} = \frac{1/2 - 4/3 \sin^2\theta_W}{-1/2 + 2/3 \sin^2\theta_W} = -0.58$$

$$F_{Z\|}(q^2) = \frac{3c_V^u}{2} F_\gamma(q^2) + (c_v^d + \frac{c_v^u}{2}) \int d^4x\, e^{-iqx} \langle \pi^+ | TQ(0) J^\mu_{d+s}(x) | K^+ \rangle$$

$$J_{d+s} = \bar{d}\gamma^\mu d + \bar{s}\gamma^\mu s$$

Absorbs u,c (UV)

Residual d,s (IR)

Hadronic amplitudes

Nonleptonic operators decompose into isospin $\Delta I = 1/2, 3/2$

$$\langle \pi_b \pi_c | Q_{\Delta I} | K_i \pi_a \rangle = C^{ia;bc}_{\Delta I} \langle I_{\pi\pi} \| Q_{\Delta I} \| I_{K\pi} \rangle$$

Recoupling + Wigner-Eckart + Crossing relations

$$i = \pm 1/2 : \quad (K^+, K^0)$$
$$a, b, c = 0, \pm 1 : \quad (\pi^0, \pi^\pm)$$

Reduced amplitudes are functions of s, t
and can be expanded in partial waves

$$p_K = p_a + p_b + p_c \qquad s = (p_K - p_a)^2 = (p_b + p_c)^2$$
$$t = (p_K - p_b)^2 = (p_a + p_c)^2$$

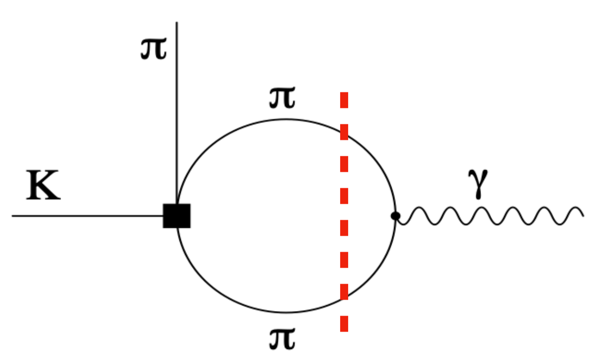
$$T^{I_{K\pi} I_{\pi\pi}}_{\ell, \Delta I}(s) = \int_{-1}^{+1} dz P_\ell(z) T^{I_{K\pi} I_{\pi\pi}}_{\Delta I}(s, t(z))$$

Discontinuity of nonlocal form factors from the
hadronic amplitude and pion vector form factor

$$\text{Disc}[F_{\gamma,Z}(s)] \sim \rho_\pi(s) T_1(s) F_\pi(s)$$

One subtraction:

$$F_{\gamma,Z}(s) = F_{\gamma,Z}(-Q_0^2) + \frac{s + Q_0^2}{\pi} \int_{4m\pi^2}^{\infty} dt \frac{\text{Disc}[F_{\gamma,Z}(t)]}{(t + Q_0^2)(t - s)} + \text{LH cuts}$$



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Only need $I_{\pi\pi} = 1$ amplitudes for
$K^+\pi^- \to \pi^+\pi^-$ in P-wave ($\to Z^*, \gamma^*$)

$\Delta I = 1/2$	$\Delta I = 3/2$
$\langle 0 \| Q_{1/2} \| 1/2 \rangle$	$\langle 0 \| Q_{3/2} \| 3/2 \rangle$
$\langle 1 \| Q_{1/2} \| 1/2 \rangle$	$\langle 1 \| Q_{3/2} \| 1/2 \rangle$
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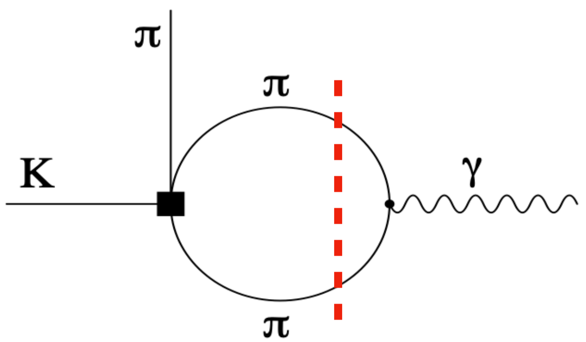

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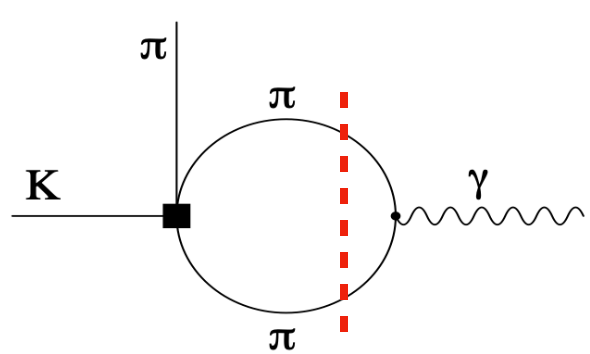

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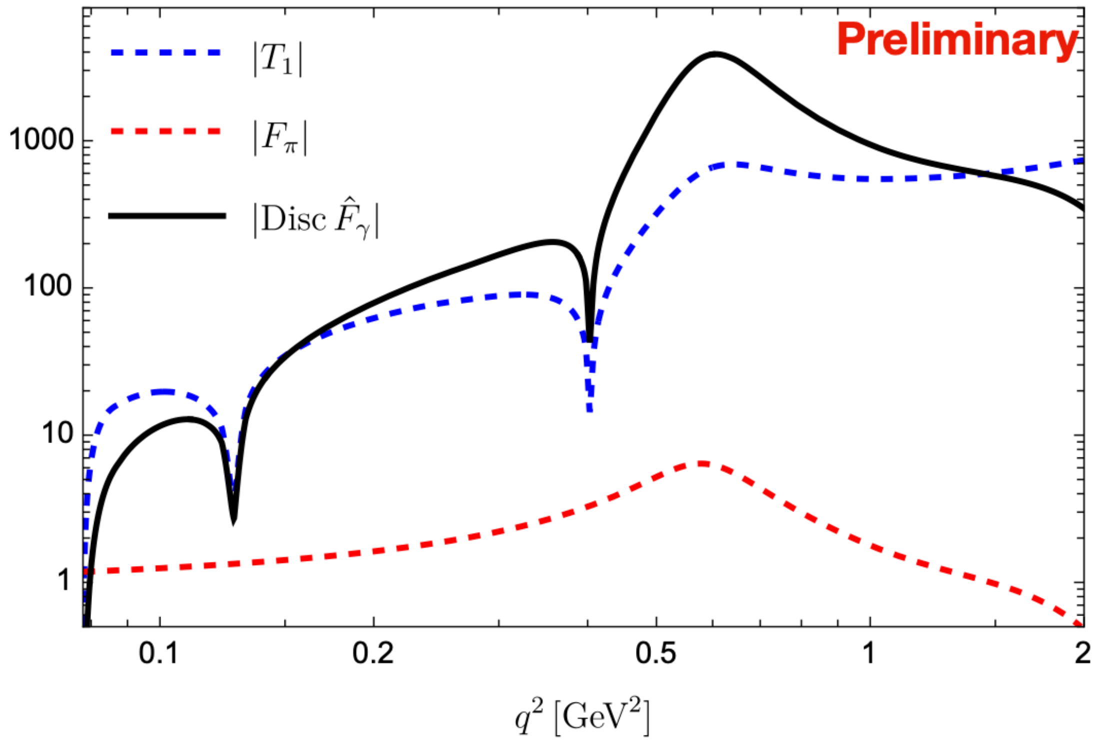
All 9 reduced amplitudes coupled in linear set of Khuri-Triemann equations (pion rescattering in s, t, u channels)

Overdetermined from fit to 14 observables (4 rates, 3 linear slopes, 7 quadratic slopes) to all CP-allowed kaon decays

$$K^+ \to \pi^+\pi^+\pi^-, \; K^+ \to \pi^0\pi^0\pi^+$$
$$K_L \to \pi^+\pi^-\pi^0, \; K_L \to \pi^0\pi^0\pi^0$$

Bernard, Descotes-Genon, Kneckt, Moussallam [2403.17570]

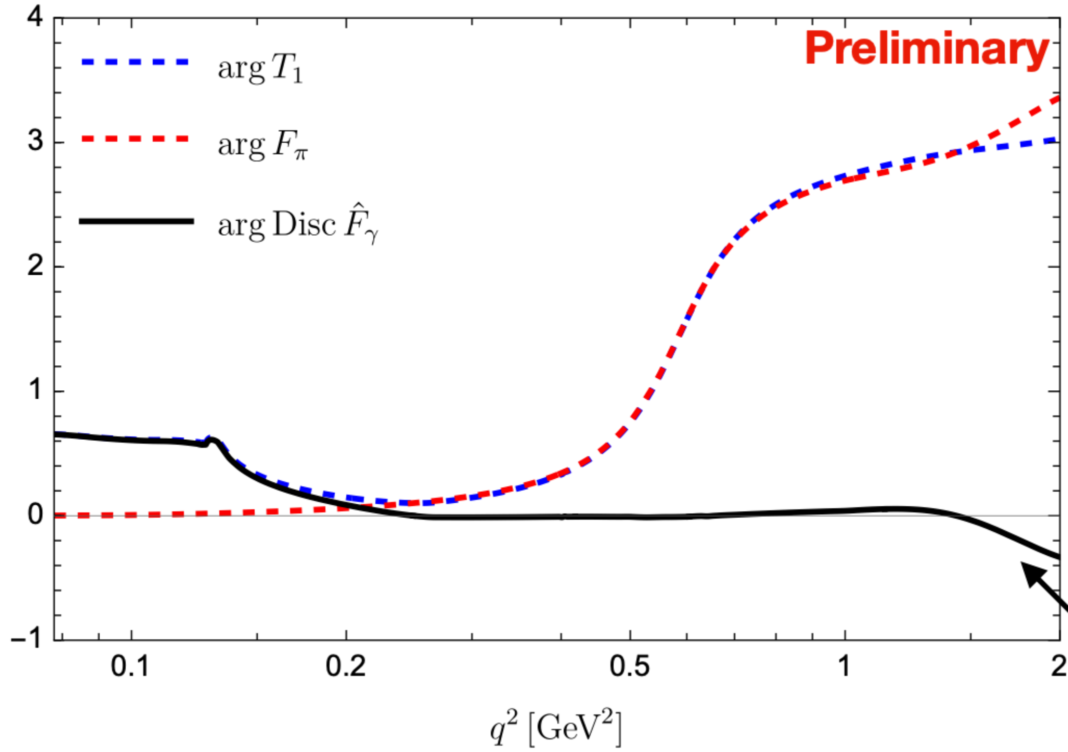
Discontinuity of FFs (Resonance Region)



P-wave amplitude vanishes at various (pseudo)-thresholds

$$q^2 = 4m_\pi^2,\ (m_K - m_\pi)^2,\ (m_K + m_\pi)^2$$

Errors from $K \to 3\pi$ Dalitz parameters negligible, theory errors from KT should be scrutinized (PW truncation)

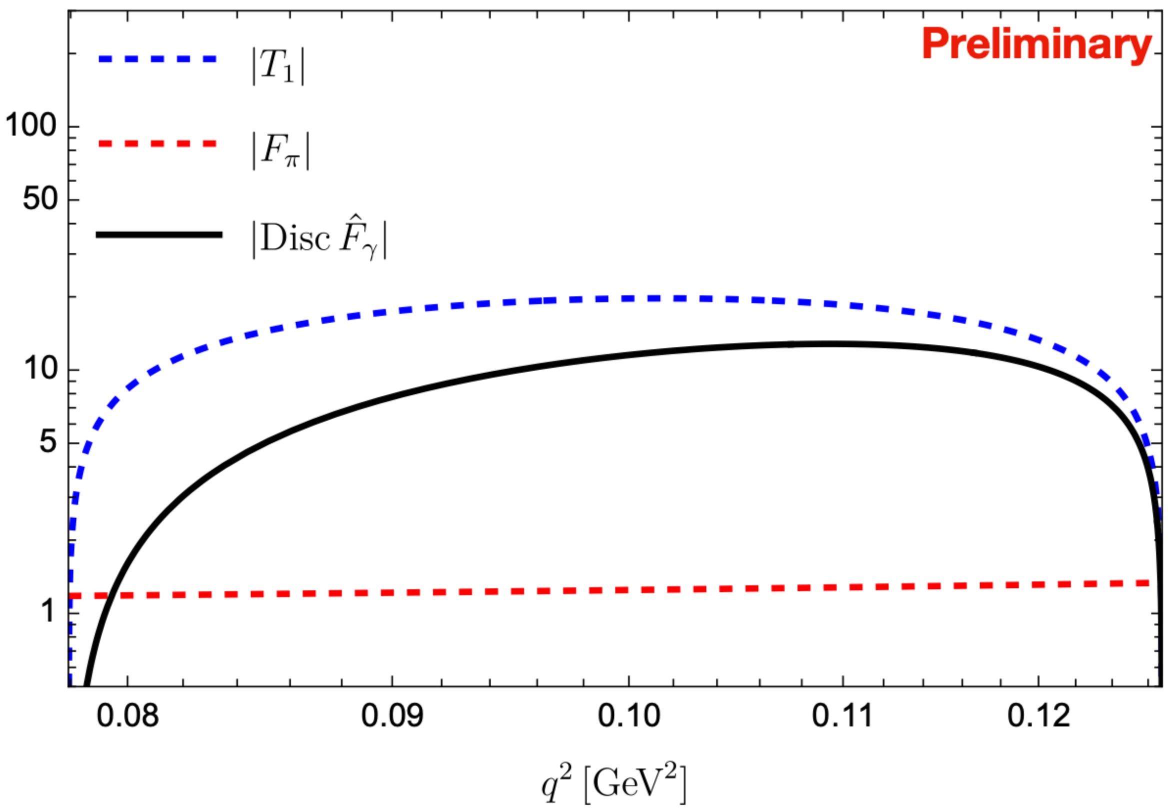


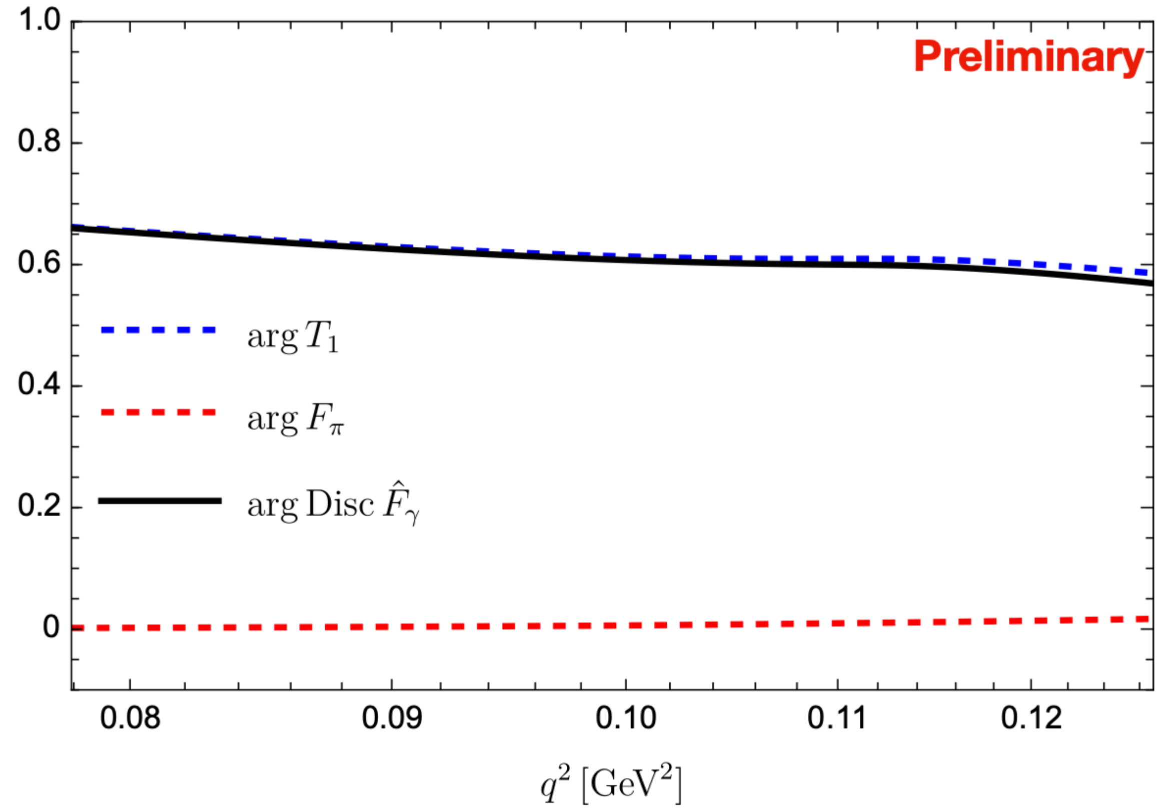
Phase of $K \to 3\pi$ amplitude and pion VFF are dominated by line shape of $\rho(770)$ above the semileptonic region

In the semileptonic region, VFF phase is small

Only $\pi\pi$ rescattering (not $K\pi$) included in KT

Discontinuity of FFs (Semileptonic Region)





Chiral dynamics: Heavy to Light Decays

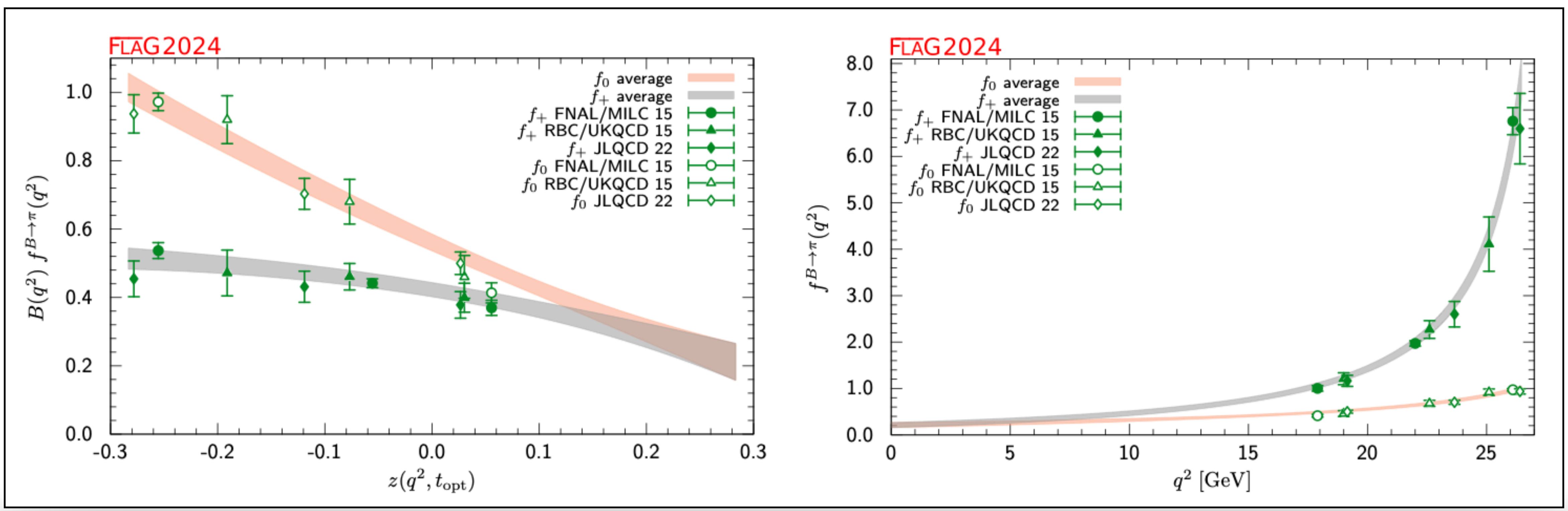
Thorsten Feldmann, JJ, Jaime del Palacio-Lirola
[in preparation]

Heavy to Light Form Factors

For V_{ub} from $B \to \pi \ell \nu$: form factors needed over full kinematic range

$$\langle \pi^+(k) \,|\, \bar{u}\gamma_\mu b \,|\, \bar{B}^0(p)\rangle$$
$$= f_+^{B\to\pi}(q^2) P_\mu + \left[f_0^{B\to\pi}(q^2) - f_+^{B\to\pi}(q^2) \right] \frac{P\cdot q}{q^2} q_\mu$$

State of the art: analytic parameterization
with constraints from lattice at low recoil

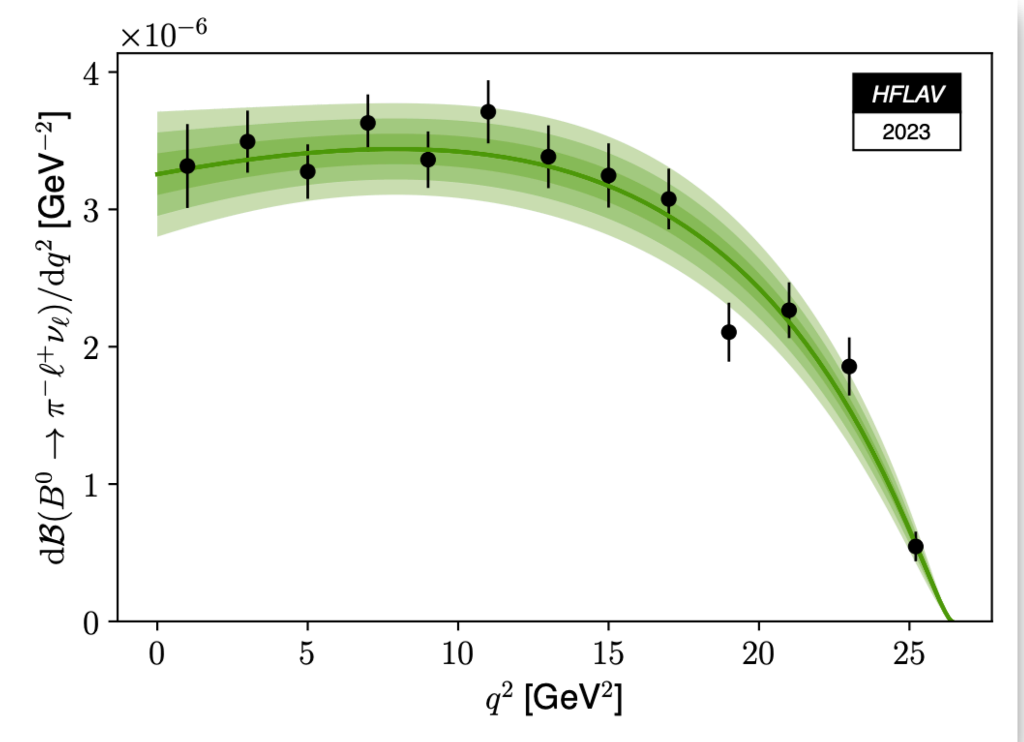


Heavy to Light Form Factors

Also constraints at high recoil
(light cone sum rules)

Exp + Lattice

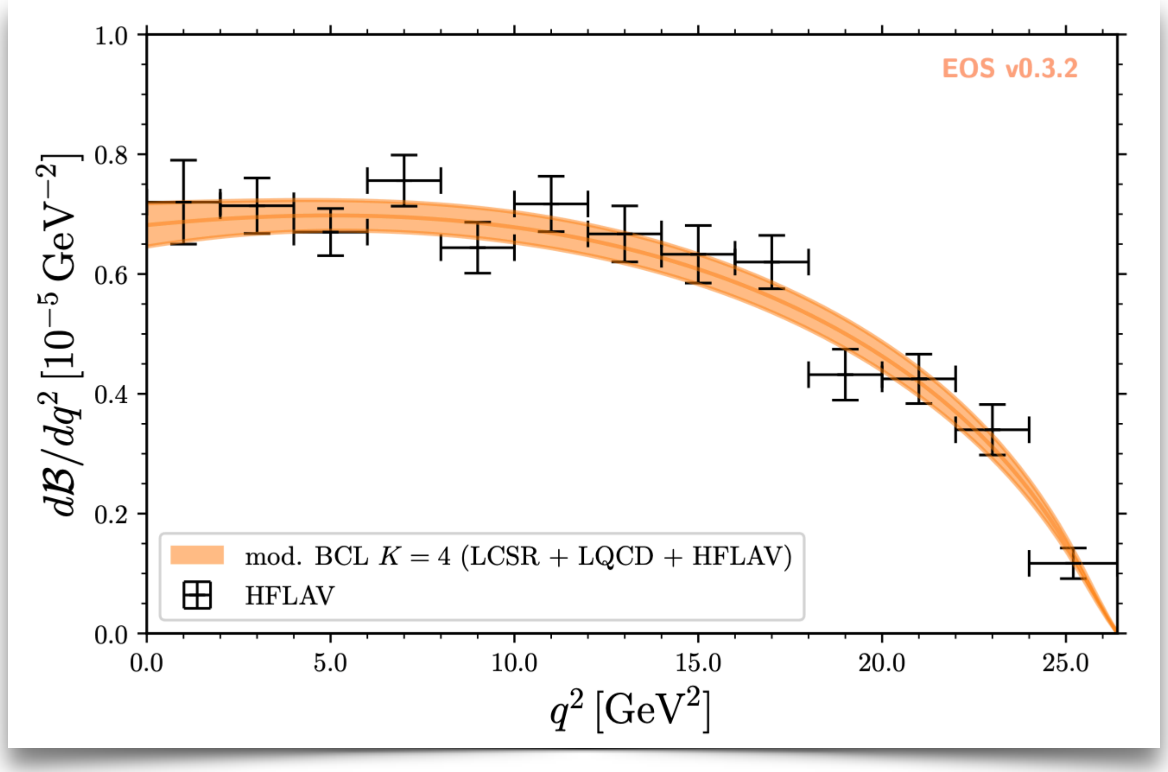
$$|V_{ub}|^{excl}_{B\to\pi} = (3.75 \pm 0.06_{exp} \pm 0.19_{th}) \times 10^{-3}$$



[2206.07501]

Exp + Lattice + LCSR

$$|V_{ub}|^{excl}_{B\to\pi} = (3.77 \pm 0.15) \times 10^{-3}$$



Leljak, Melic, van Dyk [2102.07233]

Develop an alternative / complementary approach to LCSR making use of PCAC

Covariant formulation of ChPT

Pions appear in spinoral irreps. of $SU(2)_L \times SU(2)_R$.
Corresponds to the 'square root' of standard CCSW nonlinear representation (adjoint, vector rep.)

$$\xi(x) = \exp\left[\frac{i\pi^a(x)t^a}{f_\pi}\right]$$

$$\Sigma(x) = \xi^2(x) \to L\xi^2(x)R^\dagger$$

$$\xi(x) \to L\xi(x)U^\dagger(x) = U(x)\xi(x)R^\dagger$$

Compensator field specifies coset of ChSB:
$SU(2)_L \times SU(2)_R / SU(2)_V$

For the standard choice $L = R\,(= V)$,
solution is simply $U(x) = V$

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Can define covariant objects with homogenous (U) transformation properties under background (ext) fields

$$V^\mu = \frac{1}{2}\left[\xi^\dagger i D_L^\mu \xi + \xi i D_R^\mu \xi^\dagger\right] \qquad V^\mu \to U V^\mu U^\dagger + U[iD^\mu, U^\dagger]$$

$$A^\mu = \frac{1}{2}\left[\xi^\dagger i D_L^\mu \xi - \xi i D_R^\mu \xi^\dagger\right] \qquad A^\mu \to U A^\mu U^\dagger$$

$$D_L^\mu = \partial^\mu - i(v-a)^\mu_{ext}$$

$$D_R^\mu = \partial^\mu - i(v+a)^\mu_{ext}$$

$$D^\mu = \partial^\mu - i v^\mu_{ext}$$

$$S = \frac{1}{2}\left[\xi(s+ip)\xi + \xi^\dagger(s-ip)\xi^\dagger\right] \qquad S \to USU^\dagger$$

$$P = \frac{1}{2}\left[\xi(s+ip)\xi - \xi^\dagger(s-ip)\xi^\dagger\right] \qquad P \to UPU^\dagger$$

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$$P = \frac{1}{2}\left[\xi(s+ip)\xi - \xi^\dagger(s-ip)\xi^\dagger\right] \qquad P \to UPU^\dagger$$

Standard kinetic term of chiral Lagrangian is recovered on expanding the exponential:

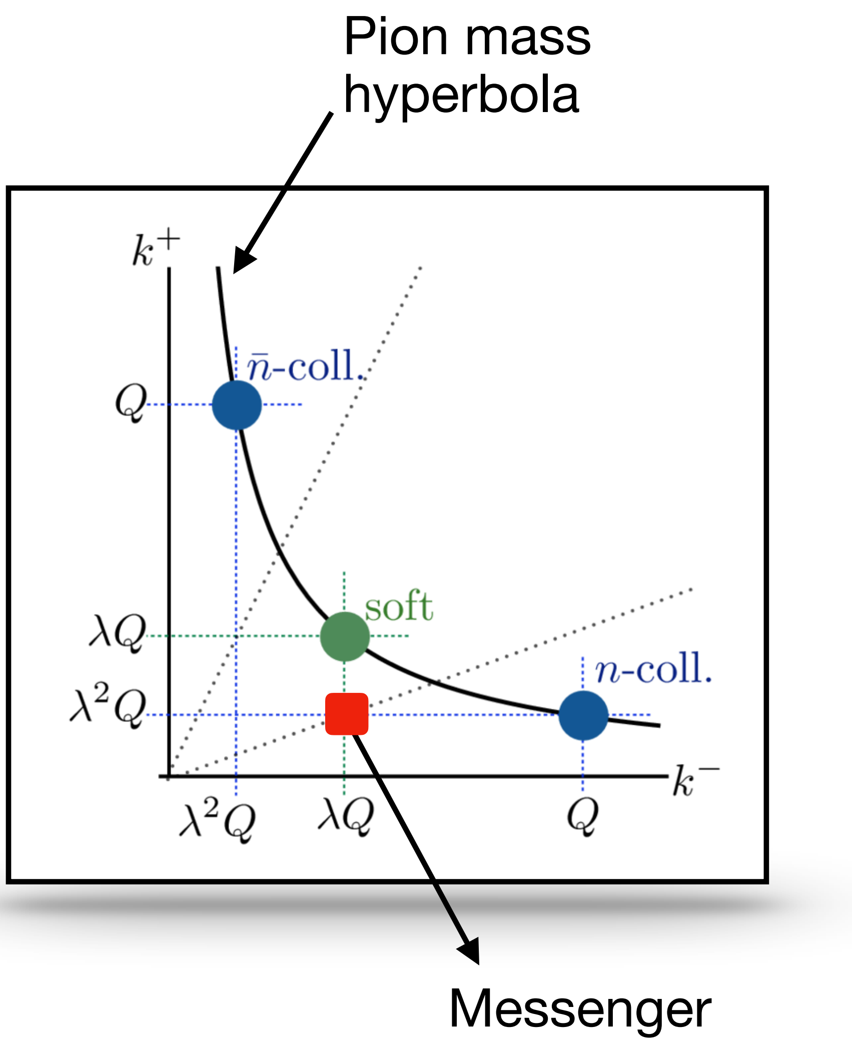
scalar spurion (mass)

$$\mathscr{L}_\chi = f_\pi^2\,\mathrm{Tr}[A_\mu A^\mu + S]$$
$$\sim \frac{1}{2}\partial_\mu\pi^a\partial^\mu\pi^a - \frac{1}{2}m_\pi^2\pi^a\pi^a$$

Soft-Collinear Factorization

$$\mathscr{L}_\chi = f_\pi^2 \,\mathrm{Tr}[A_\mu A^\mu + S] \qquad A_\mu = -\frac{1}{f_\pi}\partial_\mu \pi^a t^a + \ldots$$

Pions are derivatively coupled
in the chiral limit ($S \to 0$)



A novel aspect of our approach is a multipole expansion of the chiral Lagrangian into a soft sector (s) and one or more collinear sectors (c)

light cone decomp.

$$\xi \to \{\xi_s, \xi_c\} \qquad A^\mu \simeq A_c^\mu + A_s^\mu \to \{\bar{n}\cdot A_c, A_c^\perp, n\cdot A_c, A_s^\mu\}$$

Ansatz: chiral vector field should transform in analogy to QCD, with the soft and collinear fields transforming with respect to a background soft-collinear field (messenger mode)

Soft:

$$V_s^\mu \to U_s V_s^\mu U_s^\dagger + U_s[iD_{sc}^\mu, U_s^\dagger]$$
$$V_c^\mu \to V_c^\mu$$
$$V_{sc}^\mu \to V_{sc}^\mu$$

Collinear:

$$V_s^\mu \to V_s^\mu$$
$$V_c^\mu \to U_c V_c^\mu U_c^\dagger + U_c[iD_{sc}^\mu, U_c^\dagger]$$
$$V_{sc}^\mu \to V_{sc}^\mu$$

Background:

$$V_s^\mu \to U_{sc} V_s^\mu U_{sc}^\dagger$$
$$V_c^\mu \to U_{sc} V_c^\mu U_{sc}^\dagger$$
$$V_{sc}^\mu \to U_{sc} V_{sc}^\mu U_{sc}^\dagger + U_{sc}[i\partial^\mu, U_{sc}^\dagger]$$

$$D_{sc}^\mu = \partial_\mu - iV_{sc}^\mu$$

Soft-Collinear Factorization

To complete the analogy with QCD, introduce
chiral Wilson lines

$$S_s(x) = P\exp\left[\int_{-\infty}^{0} dt\, n\cdot V_s(x+tn)\right] \qquad S_s \to U_s S_s$$

$$W_c(x) = P\exp\left[\int_{-\infty}^{0} dt\, n\cdot V_n(x+tn)\right] \qquad W_c \to U_c W_c$$

The only difference is that chiral symmetry is explicitly broken by the quark masses, and QCD color gauge symmetry is exact..

.. but the spurion just sits in the scalar field S, which is not multipole expanded (benign)

Leading order current

HQET → ChPT (only soft pions)

$$\bar{q}_L \Gamma b_v = C(\mu) \text{Tr}[H_v \Gamma] \xi^\dagger$$

Wilson coefficient is non-perturbative, related to **decay constant** in chiral and HQ limits

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SCET → ChPT (includes collinear pions)

$$\mathcal{J}(tn) = \bar{q}_{cL}(tn)W_n^\dagger(tn)\not{n}S_n(0)b_v(0)$$

$$\mathcal{K}(tn) = \mathrm{Tr}[S_n(0)H_v(0)\gamma_5]W_n(tn)\xi_c^\dagger(tn)$$

$$\mathcal{J}_n(\omega) = C(\omega;\mu)\mathcal{K}_n(\omega)$$

Wilson coefficient related to **form factor**

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Wilson coefficient related to **form factor**

But… $\langle 0 | \mathcal{K}_n(\omega) | H_v \rangle = 0$

(No collinear particles in initial state)

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$O(\lambda^0), O(1/f_\pi)$

The effective theory has both ChPT power counting (operator mass dimensions and loops) and SCET pc. (multipole expansion of collinear fields)

Hard Pion Chiral Perturbation Theory for $B \to \pi$ and $D \to \pi$ Formfactors

Johan Bijnens and Ilaria Jemos

Department of Astronomy and Theoretical Physics, Lund University,
Sölvegatan 14A, SE 223-62 Lund, Sweden

We should thus be able to describe the hard part of any diagram by an effective Lagrangian. This effective Lagrangian should include the most general terms allowed consistent with all the symmetries and have coefficients that depend on the hard kinematical quantities and can even be complex. A two-loop example will be given in [15]. We expect that a proof along the lines of SCET [20] should be possible. Once it is accepted that one can do this, a second step is to prove that the effective Lagrangian one uses is sufficient to describe the neighbourhood of the hard process and calculate chiral logarithms.

Summary

- Towards V_{ub}/V_{cb} from inclusive B decays..
  - Closer look at the fully inclusive (u+c) kinematical distributions
  - Would be nice to implement this in a mass-independent scheme for HQET parameters
- Rare decays:
  - Different choices for (theory) normalization of $B \to X_s \ell \ell$ are sensitive to V_{cb} (or) V_{ub}. Closely related to V_{ub}/V_{cb} issue
  - “Irreducible” hadronic effects in rare kaon decays within reach (off the lattice / complementary to lattice)
- Exclusive V_{ub}: chiral extrapolation for $B \to \pi$ form factors
  - Demonstrated soft-collinear factorization of ChPT in the covariant representation
  - The lore that (hard pion ChPT = standard ChPT) seems to work at one loop
  - Generalization to baryon decays, nonleptonic decays ($B \to \pi\pi$), QED corrections

Backup

Hard Pion ChPT

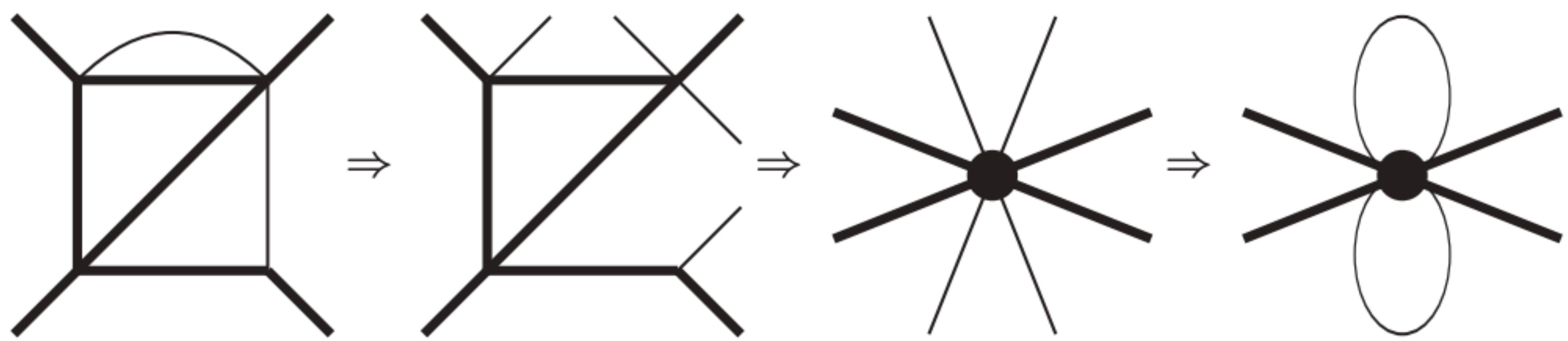

Figure 1: An example of the argument used. The thick lines contain a large momentum, the thin lines a soft momentum. Left: a general Feynman diagram with hard and soft lines. Middle-left: we cut the soft lines to remove the soft singularity. Middle-right: The contracted version where the hard part is assumed to be correctly described by a "vertex" of an effective Lagrangian. Right: the contracted version as a loop diagram. This is expected to reproduce the chiral logarithm of the left diagram. Figure from [14].

QED: Charged currents

Bigi et. al. [2309.02849]

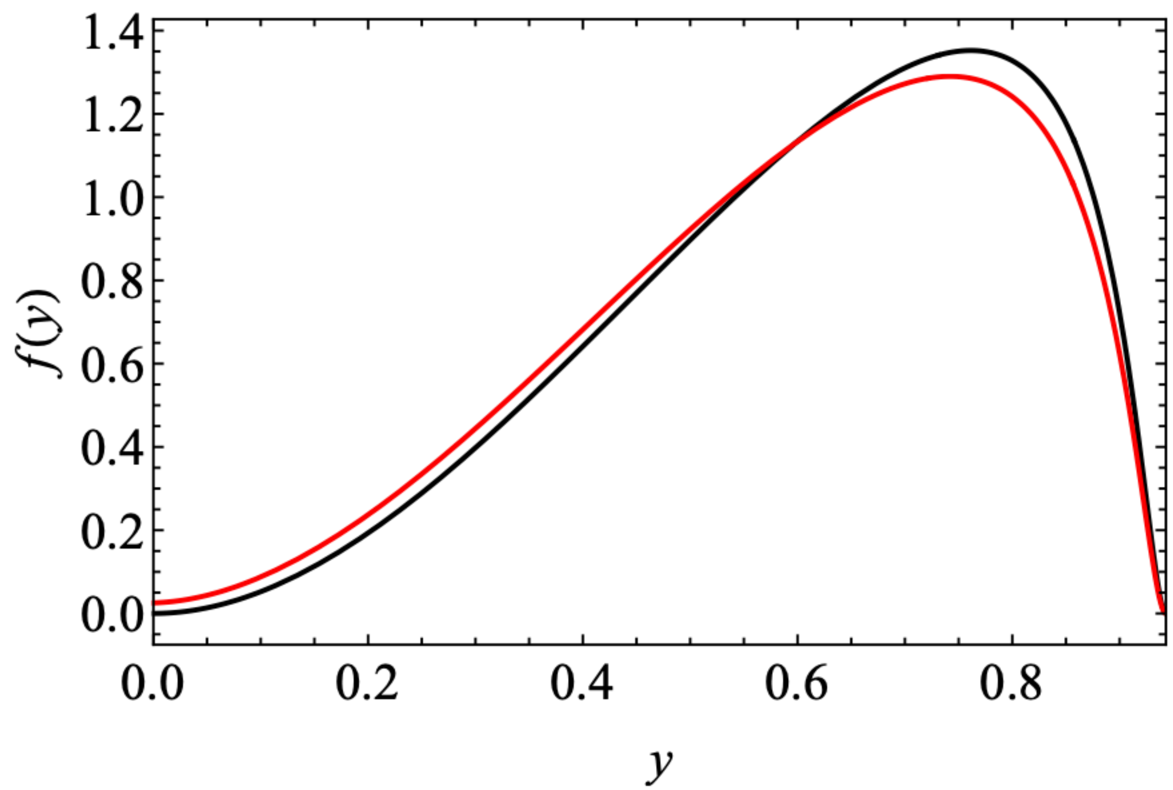

Figure 1. The red curve corresponds to $f(y)$ defined in (2.7) while the black curve represents the LO contribution $f^{(0)}(y)$ as given in (2.6). A kinetic bottom- and $\overline{\text{MS}}$ charm-quark mass is employed and final states containing electrons are considered.

Asymmetries in b->sll

$\bar{B} \to X_s \ell^+ \ell^-$ ($\ell = e, \mu$ average)

q^2 range [GeV^2]	[1, 6]	[1, 3.5]	[3.5, 6]
$\mathcal{B}$ [10^{-7}]	17.41 ± 1.31	9.58 ± 0.65	7.83 ± 0.67
$\mathcal{H}_T$ [10^{-7}]	4.77 ± 0.40	2.50 ± 0.18	2.27 ± 0.22
$\mathcal{H}_L$ [10^{-7}]	12.65 ± 0.92	7.085 ± 0.48	5.56 ± 0.45
$\mathcal{H}_A$ [10^{-7}]	-0.10 ± 0.21	-0.989 ± 0.080	0.89 ± 0.16
q^2 range [GeV^2]	> 14.4		
$\mathcal{B}$ [10^{-7}]	2.66 ± 0.70		
$\mathcal{R}(q_0^2)$ [10^{-4}]	22.27 ± 1.83		

Table 2: Phenomenological results including log-enhanced QED corrections to the $\bar{B} \to X_s \ell^+ \ell^-$ process. All quantities are obtained by averaging $\ell = e, \mu$. The denominator of the ratio $\mathcal{R}(q_0^2)$ (i.e. the $\bar{B} \to X_u \ell \bar{\nu}$ rate for $q^2 > q_0^2$), on the other hand, does not include effects which correspond to log-enhanced QED corrections on the theory side. See text for further details.

q^2 range [GeV^2]	[1, 6]	[1, 3.5]	[3.5, 6]
$\mathcal{B}$ [10^{-7}]	16.87 ± 1.25	9.17 ± 0.61	7.70 ± 0.65
$\mathcal{H}_T$ [10^{-7}]	3.14 ± 0.25	1.49 ± 0.09	1.65 ± 0.17
$\mathcal{H}_L$ [10^{-7}]	13.65 ± 1.00	7.63 ± 0.54	6.02 ± 0.49
$\mathcal{H}_A$ [10^{-7}]	-0.27 ± 0.21	-1.08 ± 0.08	0.81 ± 0.16
q^2 range [GeV^2]	> 14.4	> 15	
$\mathcal{B}$ [10^{-7}]	3.04 ± 0.69	2.59 ± 0.68	
$\mathcal{R}(q_0^2)$ [10^{-4}]	26.02 ± 1.76	27.00 ± 1.94	

Table 1: Phenomenological results without logarithmically enhanced electromagnetic effects. The slight changes compared to [9] are due to the change in the input parameters.

Kinetic mass

$$\frac{m^{\text{kin}}}{m^{\text{OS}}} = 1 - \frac{\alpha_s^{(n_l)}}{\pi} C_F \left(\frac{4}{3} \frac{\mu}{m^{\text{OS}}} + \frac{1}{2} \frac{\mu^2}{(m^{\text{OS}})^2} \right) + \left(\frac{\alpha_s^{(n_l)}}{\pi} \right)^2 C_F \left\{ \frac{\mu}{m^{\text{OS}}} \left[C_A \left(-\frac{215}{27} + \frac{2\pi^2}{9} + \frac{22}{9} l_\mu \right) + n_l T_F \left(\frac{64}{27} - \frac{8}{9} l_\mu \right) \right] \right.$$
$$\left. + \frac{\mu^2}{(m^{\text{OS}})^2} \left[C_A \left(-\frac{91}{36} + \frac{\pi^2}{12} + \frac{11}{12} l_\mu \right) + n_l T_F \left(\frac{13}{18} - \frac{1}{3} l_\mu \right) \right] \right\} + \left(\frac{\alpha_s^{(n_l)}}{\pi} \right)^3 C_F \left\{ \frac{\mu}{m^{\text{OS}}} \left[C_A^2 \left(-\frac{130867}{1944} \right. \right. \right.$$
$$+ \frac{511\pi^2}{162} + \frac{19\zeta_3}{2} - \frac{\pi^4}{18} + \left(\frac{2518}{81} - \frac{22\pi^2}{27} \right) l_\mu - \frac{121}{27} l_\mu^2 \Big) + C_A n_l T_F \left(\frac{19453}{486} - \frac{104\pi^2}{81} - 2\zeta_3 \right.$$
$$\left. + \left(-\frac{1654}{81} + \frac{8\pi^2}{27} \right) l_\mu + \frac{88}{27} l_\mu^2 \right) + C_F n_l T_F \left(\frac{11}{4} - \frac{4\zeta_3}{3} - \frac{2}{3} l_\mu \right) + n_l^2 T_F^2 \left(-\frac{1292}{243} + \frac{8\pi^2}{81} + \frac{256}{81} l_\mu - \frac{16}{27} l_\mu^2 \right) \Big]$$
$$+ \frac{\mu^2}{(m^{\text{OS}})^2} \left[C_A^2 \left(-\frac{96295}{5184} + \frac{445\pi^2}{432} + \frac{57\zeta_3}{16} - \frac{\pi^4}{48} + \left(\frac{2155}{216} - \frac{11\pi^2}{36} \right) l_\mu - \frac{121}{72} l_\mu^2 \right) + C_A n_l T_F \left(\frac{13699}{1296} - \frac{23\pi^2}{54} \right. \right.$$
$$\left. \left. \left. - \frac{3\zeta_3}{4} + \left(-\frac{695}{108} + \frac{\pi^2}{9} \right) l_\mu + \frac{11}{9} l_\mu^2 \right) + C_F n_l T_F \left(\frac{29}{32} - \frac{\zeta_3}{2} - \frac{1}{4} l_\mu \right) + n_l^2 T_F^2 \left(-\frac{209}{162} + \frac{\pi^2}{27} + \frac{26}{27} l_\mu - \frac{2}{9} l_\mu^2 \right) \right] \right\},$$