Precision in bound muon physics with EFT techniques

Institute for Theoretical Physics

Duarte Fontes

Karlsruher Institut für Technologie

based on 2412.05702 with Robert Szafron

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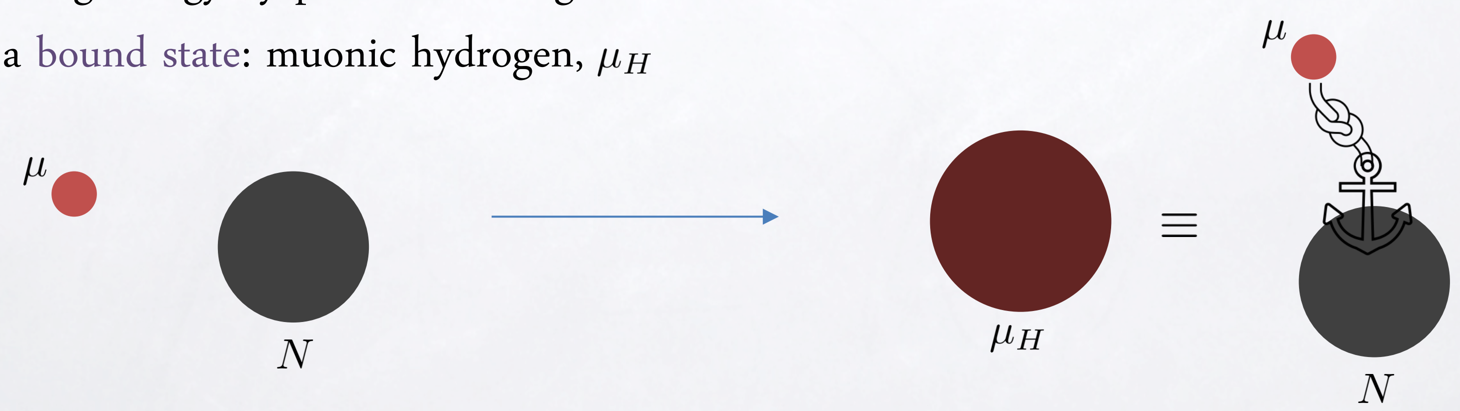
- We are looking for physics beyond the Standard Model (BSM)
- A great way to look for it is via **Charged Lepton Flavor Violation (CLFV)**

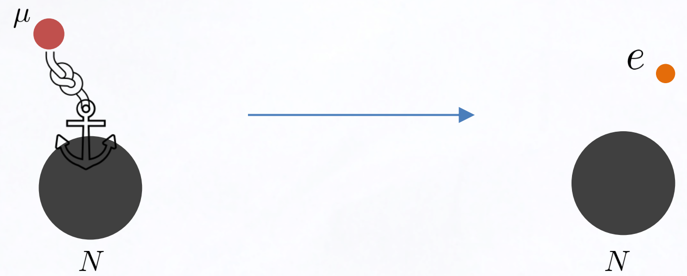
  [Calibbi, Signorelli, 1709.00294]
  - It is highly suppressed in the SM
  - If it were observed, it would clearly point to BSM physics
  - There are good chances that it might be observed very soon!
- In more detail: the most sensitive channels for CLFV are those involving a muon

  [Davidson, Echenard, 2204.00564]
- In particular, the processes in which the muon interacts with matter are very relevant:
  - After losing energy by photon exchange, the muon becomes bound to the nucleus, and forms a bound state: muonic hydrogen, μ_H

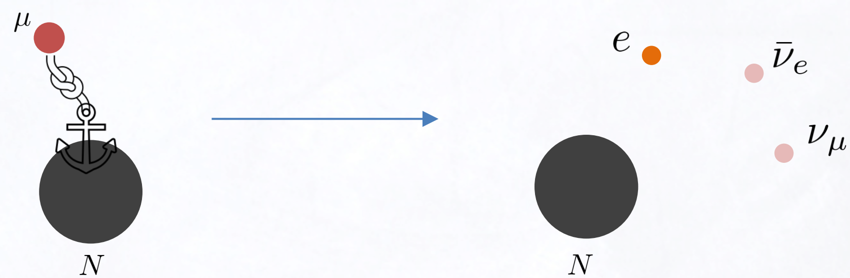


- After the muon reaches the 1S state, μ_H has three possible fates:

1) Muon conversion:

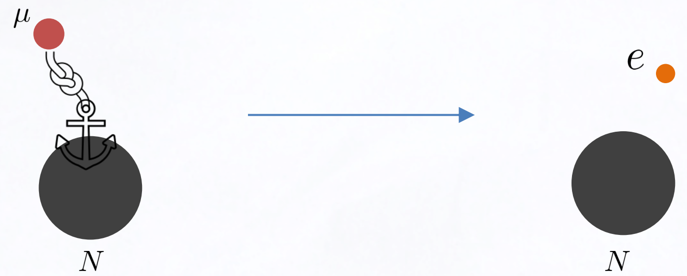


2) Muon decay-in-orbit (DIO):



3) Nuclear capture:

1) Muon conversion:



- Provides one of the most stringent limits on CLFV
  - The current best limit is $R_{\mu e} := \dfrac{\Gamma(\mu_H \to eN)}{\Gamma(\mu_H \to \nu_\mu N')} < 7 \times 10^{-13}$ at 90% CL

[SINDRUM II, Eur. Phys. J. C 47 (2006) 337]

- This limit is expected to be soon improved by *four orders of magnitude*

[Mu2e collaboration, 1501.05241] [COMET collaboration, 2308.14275]

- We should thus investigate muon conversion— as well as its main background, muon DIO

2) Muon decay-in-orbit (DIO):

- These spectacular experimental advances should be accompanied by theoretical progress.

  Many directions have been considered; e.g.:

  - improvement of wave functions
  - dependence on atomic number
  - analysis of spin-dependent structures

  [Rule et al, 2109.13503]
  [Heeck, Szafron, Uesaka, 2203.00702]
  [Cirigliano et al, 2203.09547]
  [Hoferichter, Menéndez, Noël, 2204.06005]
  [Haxton et al, 2208.07945]
  [Borrel, Hitlin, Middleton, 2401.15025]
  [Noël, Hoferichter, 2406.06677]
  [Haxton et al, 2406.13818]
  [Heinz et al, 2412.04545]
  [Kaygorodov et al, 2506.02416]

- What about precision?

  - Some work has been done in DIO, but not in a systematic fashion

    [Szafron, Czarnecki, 1505.05237]
    [Szafron, Czarnecki, 1506.00975]
    [Czarnecki et al, 1406.3575]

  - This is precisely my focus. The question is:

how to calculate precise predictions for muon conversion and muon DIO?

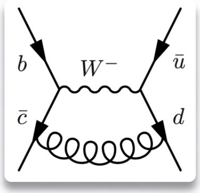
- We need a **framework**. We could start with a BSM model. But this has two defects:

  - It is a particular model (thus preventing a model-independent analysis)
  - It leads to large logs, of the form $\log(\Lambda/M_N)$, with: $\Lambda \gg M_N \sim 20$ GeV,

- An **Effective Field Theory** (EFT) is a stone that kills these two birds (and quite a few others)

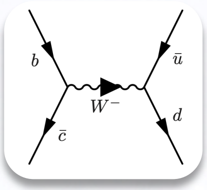
- How exactly does an EFT kill the bird of large logs? *By loop matching and RG running*

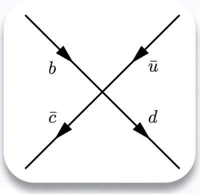
  Textbook example: Schwartz's section 31.3, $b \to c\bar{u}d$ in the SM

  - At one-loop in the SM, diagrams like 

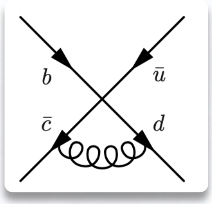
    generate large logs $\propto \alpha_s \log(m_W/m_b)$

  - Then, we consider an EFT without W: $\mathcal{L} \ni C_1 \mathcal{O}_1(x) = C_1 \left[\bar{c}^i_L(x)\gamma^\mu b^i_L(x)\right]\left[\bar{d}^j_L(x)\gamma^\mu u^j_L(x)\right]$

  - We calculate the matching at tree-level... 



    $\Longrightarrow C_1 = G_F$

  - ... as well at loop level 

    $\Longrightarrow C_1 = G_F \left[1 - \frac{\alpha_s}{2\pi}\left(\frac{1}{2}\log\frac{\mu^2}{m_W^2} + \frac{3}{4}\right)\right]$

  - We want to know $C_1(\mu = m_b)$. But that still leads to the undesired large logs

  - The solution has two simple steps:
    - We start by performing the matching with $\mu = m_W$
    - We calculate the RGEs, and run down the result: $C_1(m_b) = U(m_W, m_b)\, C_1(m_W)$

      where $U(m_W, m_b)$ resums the large logs

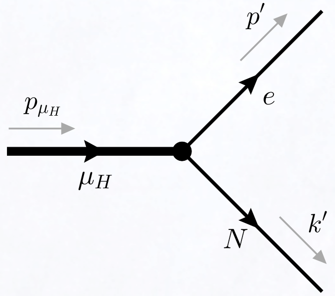
- The EFT allows to create **single-scale-objects** $\log \frac{\mu^2}{m_W^2}$ out of multiple-scale-objects, $\log \frac{m_b^2}{m_W^2}$
  $+\mu$

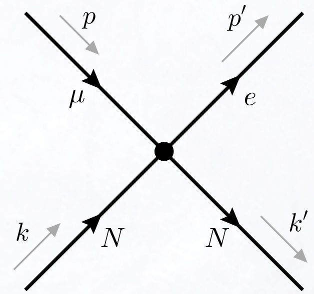
  By setting μ equal to the scale of the one-scale object, we *eliminate the large logs*

- The EFT usually considered for muon conversion is a form of LEFT: Low-Energy EFT

  [Rule et al, 2109.13503]
  [Haxton et al, 2208.07945]

  - One takes the SM d.o.f. and integrates out all those with mass $M \gtrsim m_W$
  - Though muon conversion has a bound muon, we start with the free scattering $\mu N \to eN$





  - The current can generically be written as

$$\mathcal{J} = -\frac{4G_F}{\sqrt{2}} \sum_{X=L,R} \Big\{ C_{SX}\, \bar{e} P_X \mu\, \bar{N} N + C_{PX} \bar{e} P_X \mu\, \bar{N}\gamma_5 N + C_{VX}\, \bar{e}\gamma^\alpha P_X \mu\, \bar{N}\gamma_\alpha N$$

$$+ C_{AX}\, \bar{e}\gamma^\alpha P_X \mu\, \bar{N}\gamma_\alpha\gamma_5 N + C_{\mathrm{Der}X}\, \bar{e}\gamma^\alpha P_X \mu\, (\bar{N} \overleftrightarrow{\partial}_\alpha i\gamma_5 N) + C_{TX}\, \bar{e}\sigma^{\alpha\beta} P_X \mu\, \bar{N}\sigma_{\alpha\beta} N \Big\} + \mathrm{h.c.}\,,$$

- I shall focus on the so-called coherent conversion, where the current reduces to:

$$\mathcal{J} = -\frac{4G_F}{\sqrt{2}} \sum_{X=L,R} \left\{ C_{SX}\, \bar{e} P_X \mu\, \bar{N} N + C_{VX}\, \bar{e}\gamma^\alpha P_X \mu\, \bar{N}\gamma_\alpha N \right\}$$

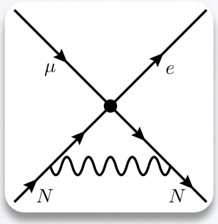
(a generalization to the incoherent case is straightforward)

- At first sight, LEFT is great: it solves our $\log(\Lambda/M_N)$ problems
- A closer look, however, reveals that it is still plagued by large logs!
  - In fact, in LEFT, there are still many scales in muon conversion:
    1. the nuclear mass $M_N \sim 20$ GeV
    2. the muon mass $m_\mu \sim 105$ MeV $\sim E_e$
    3. the muon momentum $|\vec{p}| \sim Z\alpha m_\mu \sim 10$ MeV
    4. the electron mass $m_e \sim 0.511$ MeV

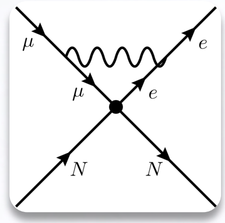
assuming aluminium, Z=13

  - We also need to consider a photon-energy cutoff $\Delta E = m_\mu - E_e$. We shall take $\Delta E \sim m_e$
  - Then, $M_N \gg m_\mu \sim E_e \gg Z\alpha m_\mu \gg (Z\alpha)^2 m_\mu \sim m_e \sim \Delta E$

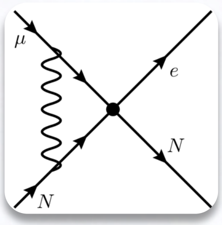
- Result: a mess of large logs!



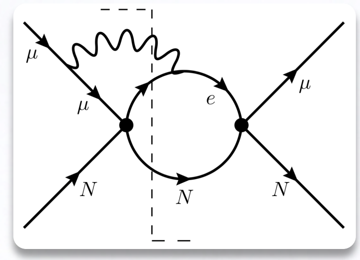
$\propto Z^2 \alpha \log \frac{\mu}{M_N}$,



$$\propto \frac{\alpha}{4\pi}\left[-4\log\left(\frac{m_e}{m_\mu}\right)+3\log\left(\frac{m_e}{\mu}\right)-8\log\left(\frac{\mu}{m_\mu}\right)\right]$$



$\propto Z\alpha \log \frac{\mu}{|\vec{p}|}$,



$$\propto \alpha \log \frac{E_e \mu}{\Delta E\, m_e}$$

- There is no μ that avoids large logs

- The different scales are intertwined:

- LEFT is not enough!

- The task is thus to build a *proper* EFT framework for precision calculations in muon conversion

- We will start by investigating the relevant scales: what physics describes each of them?

- This will allows us to build a sequence of EFTs, the last of which is finally free from large logs

  - Instead of integrating out just the scale m_W (like the textbook), we need to integrate out also M_N , and then m_μ , and then...

  - The goal is always: to find **single-scale-objects** — factorization — so that the final decay width looks like:

$$\Gamma_{\mu_H \to eN} \propto F_1(\mu/M_N)\, F_2(\mu/m_\mu)\, F_3(\mu/(Z\alpha m_\mu))\, F_4(\mu/m_e)\, F_5(\mu/(m_e \Delta E/m_\mu))$$
$$\times\, U_1(M_N,\mu_c)\, U_2(m_\mu,\mu_c)\, U_3(Z\alpha m_\mu,\mu_c)\, U_4(m_e,\mu_c)\, U_5(m_e\Delta E/m_\mu,\mu_c)$$

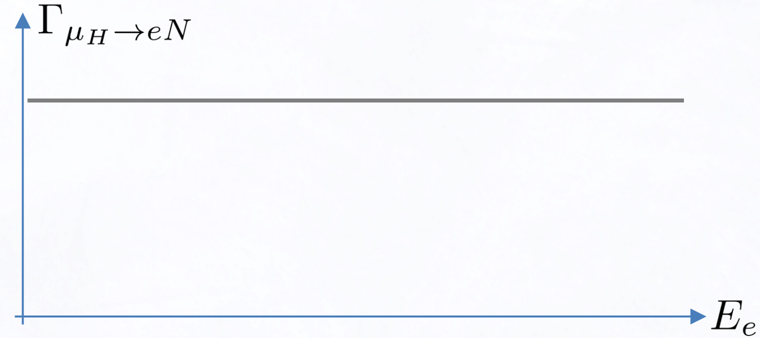
  Each one of the F functions is free from large logs, since it is a **single-scale-object**. By calculating all RGEs U, we can run all functions to a common scale μ_c

  - The result is *the* way to *calculate precise predictions for muon conversion.*

- My results will focus on the QED corrections (with no powers of Z)

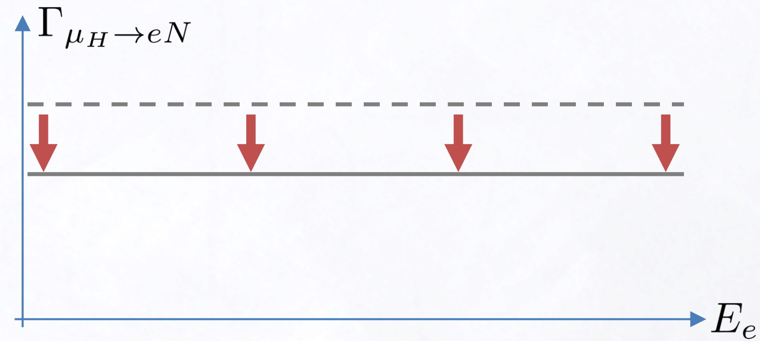
- Why QED corrections? Because of the **shape** of the spectrum. In more detail,

  - Suppose the LO rate vs. electron energy in muon conversion to be:



  - Then, higher order corrections have two effects:

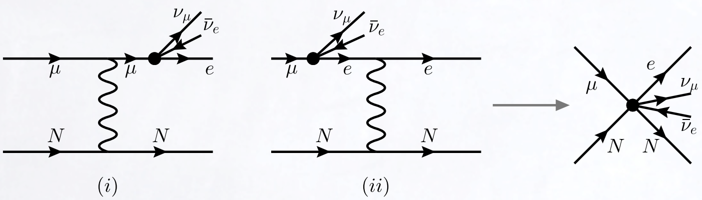
    1) Shifting the absolute **value** of the rate:
       - Crucial for BSM interpretations once muon conversion has been detected
       - Depends on nuclear effects



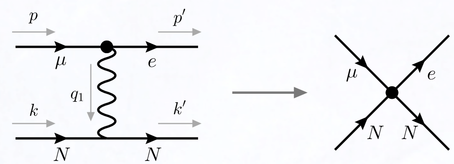
    2) Changing the **shape** of the rate:
       - Crucial for detection of muon conversion
       - Depends QED effects only

- An advantage of the EFT method is its **universality**
  - The EFT framework factorizes the decay rate into single-scale objects
  - Many of these objects are *the same* in different bound muon decays
  - So, the framework applies not only to direct muon conversion, but also to:

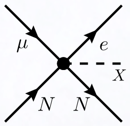
- DIO



- Photonic conversion



- $\mu_H \to eNX$

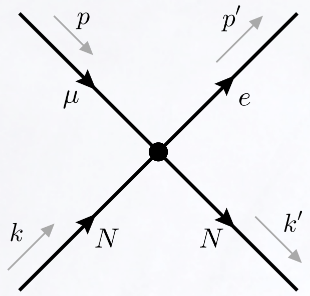


- ...

  - The higher-order corrections are essentially the same in all these processes

- So, our EFT framework applies to a *vast class* of bound muon decays
- In this talk, I will focus on direct muon conversion

- Kinematics of direct muon conversion:



$$p = \left(\sqrt{m_\mu^2 + |\vec{p}|^2}, \vec{p}\right), \qquad p' = \left(E_e, 0, 0, -\sqrt{E_e^2 - m_e^2}\right),$$

$$k = (M_N, \vec{0}), \qquad k' = \left(\sqrt{M_N^2 + |\vec{k'}|^2}, \vec{k'}\right),$$

with: $|\vec{p}| = \mathcal{O}(m_\mu Z\alpha), \qquad |\vec{k'}| = \mathcal{O}(m_\mu) = \mathcal{O}(E_e).$

- We define two expansions: the *recoil* expansion and the *power* expansion

$$\lambda_R \sim \frac{m_\mu}{M_N} \simeq 0.005 \qquad \lambda \sim Z\alpha \sim \sqrt{\frac{m_e}{m_\mu}} \simeq 0.1$$

- We also define $n_+ = (1,0,0,1), \quad n_- = (1,0,0,-1)$, such that:

$$n_- p' = \frac{m_e^2}{2E_e} + \mathcal{O}(\lambda), \qquad n_+ p' = 2E_e + \mathcal{O}(\lambda)$$

- In what follows, a 4-momentum l may be written in two ways:
  - Resorting to the light-cone basis: $l = (n_+ l, l_\perp, n_- l)$. Example: $p' = \left(2E_e, 0, \frac{m_e^2}{2E_e}\right)$
  - Separating time and space components: $l = (l_0, \vec{l})$

- The nucleus as dynamical field:

  [Kuno, Okada, 9909265], [Kosmas, Kovalenko, Schmidt, 0102101], [Cirigliano et al, 0904.0957], [Davidson, 1601.07166], [Bartolotta, Ramsey-Musolf, 1710.02129], [Rule et al, 2109.13503], [Haxton et al, 2208.07945]

  - The literature usually takes either nucleons or quarks as fields
  - That is reasonable for energies much higher than those of μ and e

    nucleus fields do not even exist at such energies

  - In my case, I consider the nucleus field because:
    - I focus on coherent conversion
    - I am especially interested in the shape of the spectrum, and not in its normalization
    - I focus on the scales of the μ and e, much smaller than the nucleus mass
  - The nucleus is thus a spectator, so that we treat it as **point-like particle** with mass M_N

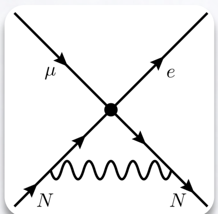
    is just a source of electron field

  - M_N is a placeholder for a proper nuclear description, whose essential elements are:
    - The nucleus must be first matched onto **nucleons**, using χPT. At LP, nucleus can be seen as the coherent sum of protons and neutrons, whose effects involve their densities in the nucleus
    - The nucleons must be matched onto **quarks**, involving non-perturbative physics. This can be done using nucleon form factors
  - Corrections to the point-like assumption can be accounted for with form factors

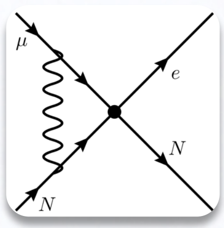
- To understand the physics, we resort to the method of regions [Beneke, Smirnov, 9711391]
  - This identifies the different scalings of the loop momentum that yield a non-vanishing contribution to the expanded result of the integral
- When we apply it to muon conversion, we find:

virtual corrections

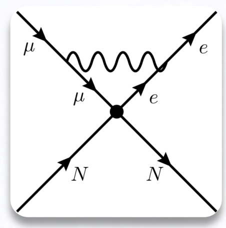
real corrections



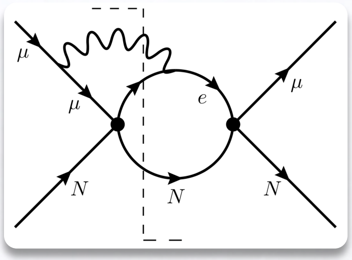
hard-nuclear



potential



hard, collinear



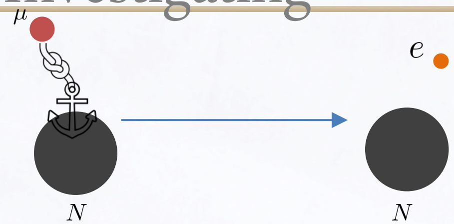
soft, soft-collinear

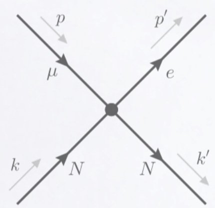
The photon emissions from the nucleus legs vanish in the Feynman gauge in the limit of infinite nucleus mass

Mode name	Abbreviation	Momentum scaling	Virtuality
Hard-nuclear	(hn)	$(1\ ,1\ ,1\ )\,M_N$	M_N^2
Hard	(h)	$(1\ ,1\ ,1\ )\,m_\mu$	m_μ^2
Semi-hard	(sh)	$(\lambda\ ,\lambda\ ,\lambda\ )\,m_\mu$	$\lambda^2\,m_\mu^2$
Hard-collinear	(hc)	$(1\ ,\lambda\ ,\lambda^2)\,m_\mu$	$\lambda^2\,m_\mu^2$
Potential	(p)	$(\lambda^2,\lambda)\,m_\mu$	$\lambda^2\,m_\mu^2$
Soft	(s)	$(\lambda^2,\lambda^2,\lambda^2)\,m_\mu$	$\lambda^4\,m_\mu^2$
Collinear	(c)	$(1\ ,\lambda^2,\lambda^4)\,m_\mu$	$\lambda^4\,m_\mu^2$
Soft-collinear	(sc)	$(\lambda^2,\lambda^4,\lambda^6)\,m_\mu$	$\lambda^8\,m_\mu^2$

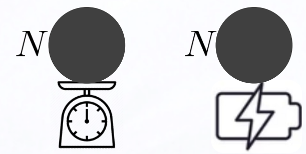
(“soft” in NRQCD terminology)

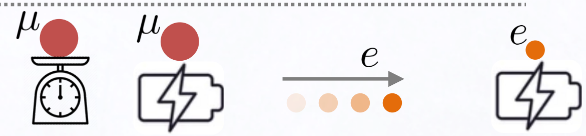
(“ultrasoft” in NRQCD term.)

- Muon conversion,  , thus involves 5 scales: $\mu_{hn} \gg \mu_h \gg \mu_{sh} \gg \mu_s \gg \mu_{sc}$

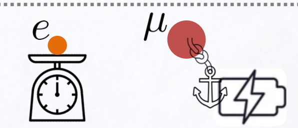


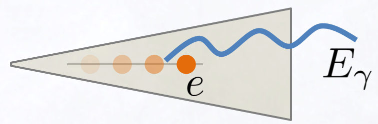
$$p = \left(\sqrt{m_\mu^2 + |\vec{p}|^2}, \vec{p}\right)$$
$$k = (M_N, \vec{0})$$
$$p' = \left(2E_e, 0, \frac{m_e^2}{2E_e}\right)$$
$$k' = (\sqrt{M_N^2 + |\vec{k'}|^2}, \vec{k'})$$

hard-nuclear scale: $\mu_{hn} \sim M_N$, 

hard scale: $\mu_h \sim m_\mu \simeq E_e = \mathcal{O}(\lambda_R M_N)$, 

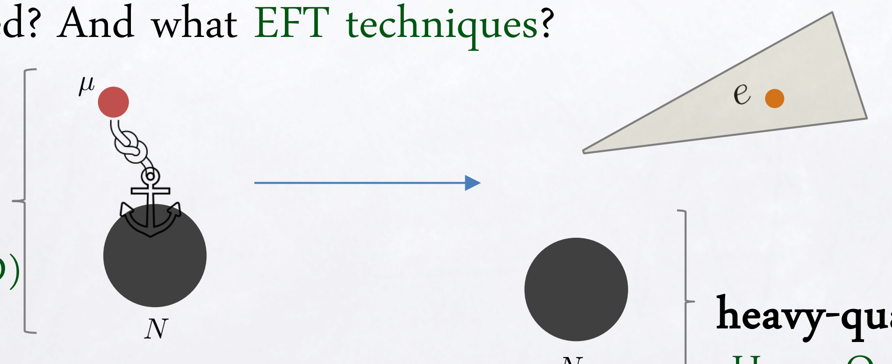
semi-hard scale: $\mu_{sh} \sim Z\alpha m_\mu = \mathcal{O}(\lambda\, \mu_h)$, 

soft scale: $\mu_s \sim (Z\alpha)^2 m_\mu \simeq m_e \simeq \Delta E = \mathcal{O}(\lambda^2\, \mu_h)$, 

soft-collinear scale: $\mu_{sc} \sim m_e \dfrac{\Delta E}{m_\mu} \sim \dfrac{m_e^2}{m_\mu} = \mathcal{O}(\lambda^4\, \mu_h)$, 

- What physics is involved? And what EFT techniques?

bound state physics
- Non-Relativistic QED (NRQED)
- potential NRQED (pNRQED)



soft-collinear physics
- Soft Collinear Effective Theory (SCET)
- boosted HQET (bHQET)

heavy-quark physics
- Heavy Quark Effective Theory (HQET)

- We can now build a proper EFT framework:

μ		EFT	Radiation	Nucleus	Muon	Electron
$M_N \sim$	μ_{hn}	I	$\mathcal{L}_A(A^{(hn)})$	$\mathcal{L}^{(m\neq 0)}_{\text{Dirac}}(N^{(hn)}, A^{(hn)})$	$\mathcal{L}^{(m=0)}_{\text{Dirac}}(\mu^{(hn)}, A^{(hn)})$	$\mathcal{L}^{(m=0)}_{\text{Dirac}}(e^{(hn)}, A^{(hn)})$
$m_\mu \sim$	μ_h	II	$\mathcal{L}_A(A^{(h)})$	$\mathcal{L}_{\text{HQET}}(h_N^{(h)}, A^{(h)})$	$\mathcal{L}^{(m\neq 0)}_{\text{Dirac}}(\mu^{(h)}, A^{(h)})$	$\mathcal{L}^{(m=0)}_{\text{Dirac}}(e^{(h)}, A^{(h)})$
$Z\alpha m_\mu \sim$	μ_{sh}	III	$\mathcal{L}_A(A^{(sh,hc)})$	$\mathcal{L}_{\text{HQET}}(h_N^{(sh)}, A^{(sh)})$	$\mathcal{L}_{\text{NRQED}}(\psi^{(p)}, A^{(sh)})$	$\mathcal{L}_{\text{SCET}_{\text{I}}}(\xi^{(hc)}, A^{(hc,s)})$
$m_e \sim$	μ_s	IV	$\mathcal{L}_A(A^{(s,sc)})$	$\mathcal{L}_{\text{HQET}}(h_N^{(s)}, A^{(s,sc)})$	$\mathcal{L}_{\text{pNRQED}}(\Psi^{(p)}, A^{(s,sc)})$	$\mathcal{L}_{\text{SCET}_{\text{II}}}(\xi^{(c)}, A^{(c)})$
$m_e \dfrac{\Delta E}{m_\mu} \sim$	μ_{sc}	V	$\mathcal{L}_A(A^{(sc)})$	$\mathcal{L}_{\text{HQET}}(h_N^{(s)}, A^{(sc)})$	$\mathcal{L}_{\text{pNRQED}}(\Psi^{(p)}, A^{(sc)})$	$\mathcal{L}_{\text{bHQET}}(h_e^{(sc)}, A^{(sc)})$

- For each EFT, we write the Lagrangian, the current, the matching and the RGEs.

μ_{hn} | I | $\mathcal{L}_A(A^{(hn)})$ | $\mathcal{L}^{(m\neq 0)}_{\text{Dirac}}(N^{(hn)}, A^{(hn)})$ | $\mathcal{L}^{(m=0)}_{\text{Dirac}}(\mu^{(hn)}, A^{(hn)})$ | $\mathcal{L}^{(m=0)}_{\text{Dirac}}(e^{(hn)}, A^{(hn)})$

The Lagrangian is:

$$\mathcal{L}^{(\mathrm{I})} = \mathcal{L}^{(\mathrm{I}),\mathrm{LR}} + \mathcal{O}(\lambda_R), \qquad \mathcal{L}^{(\mathrm{I}),\mathrm{LR}} = \mathcal{L}_A^{(\mathrm{I}),\mathrm{LR}} + \mathcal{L}_N^{(\mathrm{I}),\mathrm{LR}} + \mathcal{L}_\mu^{(\mathrm{I}),\mathrm{LR}} + \mathcal{L}_e^{(\mathrm{I}),\mathrm{LR}},$$

with

$$\mathcal{L}_A^{(\mathrm{I}),\mathrm{LR}} = -\frac{1}{4} F_{\mu\nu}^{(hn)} F^{\mu\nu (hn)}, \qquad \mathcal{L}_N^{(\mathrm{I}),\mathrm{LR}} = \bar{N}^{(hn)} (i \not{D}^{(hn)} - M_N) N^{(hn)},$$

$$\mathcal{L}_\mu^{(\mathrm{I}),\mathrm{LR}} = \bar{\mu}^{(hn)} i \not{D}^{(hn)} \mu^{(hn)}, \qquad \mathcal{L}_e^{(\mathrm{I}),\mathrm{LR}} = \bar{e}^{(hn)} i \not{D}^{(hn)} e^{(hn)}.$$

We have a very clear counting. Example with the muon mass:

$$\langle 0 | T\{\mu^{(hn)}(0) \bar{\mu}^{(hn)}(x)\} | 0 \rangle = \int \frac{d^4 L}{(2\pi)^4} \frac{i}{\not{L} - m_\mu + i\varepsilon} e^{iL.x} \sim M_N^3$$

So, $\mu^{(hn)} \sim M_N^{3/2}$ and $\bar{\mu}^{(hn)} \not{\partial} \mu^{(hn)} \sim M_N^4$. Therefore, $\bar{\mu}^{(hn)} \mu^{(hn)} m_\mu \sim M_N^3 m_\mu \sim M_N^4 \lambda_R$

μ_{hn}	I	$\mathcal{L}_A(A^{(hn)})$	$\mathcal{L}_{\text{Dirac}}^{(m\neq 0)}(N^{(hn)}, A^{(hn)})$	$\mathcal{L}_{\text{Dirac}}^{(m=0)}(\mu^{(hn)}, A^{(hn)})$	$\mathcal{L}_{\text{Dirac}}^{(m=0)}(e^{(hn)}, A^{(hn)})$

The current is:

$$\mathcal{J}^{(\mathrm{I})} = -\frac{4G_F}{\sqrt{2}}\left\{C_{SX}^{(\mathrm{I})}\mathcal{O}_{SX}^{(\mathrm{I})} + C_{VX}^{(\mathrm{I})}\mathcal{O}_{SX}^{(\mathrm{I})}\right\} + \text{h.c.},$$

with

$$\mathcal{O}_{SX}^{(\mathrm{I})} \equiv \bar{N}^{(hn)}N^{(hn)}\,\bar{e}^{(hn)}P_X\mu^{(hn)}, \qquad \mathcal{O}_{VX}^{(\mathrm{I})} \equiv \bar{N}^{(hn)}\gamma_\alpha N^{(hn)}\,\bar{e}^{(hn)}\gamma_\alpha P_X\mu^{(hn)}.$$

This is the starting point of my analysis

μ_h II $\mathcal{L}_A(A^{(h)})$ $\mathcal{L}_{\mathrm{HQET}}(h_N^{(h)}, A^{(h)})$ $\mathcal{L}_{\mathrm{Dirac}}^{(m\neq 0)}(\mu^{(h)}, A^{(h)})$ $\mathcal{L}_{\mathrm{Dirac}}^{(m=0)}(e^{(h)}, A^{(h)})$

The Lagrangian is:

$$\mathcal{L}^{(\mathrm{II})} = \mathcal{L}^{(\mathrm{II}),\mathrm{LP}} + \mathcal{O}(\lambda), \qquad \mathcal{L}^{(\mathrm{II}),\mathrm{LP}} = \mathcal{L}_A^{(\mathrm{II}),\mathrm{LP}} + \mathcal{L}_{h_N}^{(\mathrm{II}),\mathrm{LP}} + \mathcal{L}_\mu^{(\mathrm{II}),\mathrm{LP}} + \mathcal{L}_e^{(\mathrm{II}),\mathrm{LP}},$$

with

$$\mathcal{L}_A^{(\mathrm{II}),\mathrm{LP}} = -\frac{1}{4} F_{\mu\nu}^{(h)} F^{\mu\nu(h)}, \qquad \mathcal{L}_{h_N}^{(\mathrm{II}),\mathrm{LP}} = \bar{h}_N^{(h)} iv\cdot D^{(h)} h_N^{(h)}, \qquad v \equiv (1,0,0,0)$$

$$\mathcal{L}_\mu^{(\mathrm{II}),\mathrm{LP}} = \bar{\mu}^{(h)}(i\not{D}^{(h)} - m_\mu)\mu^{(h)}, \qquad \mathcal{L}_e^{(\mathrm{II}),\mathrm{LP}} = \bar{e}^{(h)} i\not{D}^{(h)} e^{(h)}.$$

The current is such that:

$$C_{SX}^{(\mathrm{I})}\mathcal{O}_{SX}^{(\mathrm{I})} + C_{VX}^{(\mathrm{I})}\mathcal{O}_{VX}^{(\mathrm{I})} = C_X^{(\mathrm{II})}\,\mathcal{O}_X^{(\mathrm{II})}, \qquad \text{with} \qquad \mathcal{O}_X^{(\mathrm{II})} \equiv \bar{h}_N^{(h)} h_N^{(h)}\,\bar{e}^{(h)} P_X \mu^{(h)}$$

and the matching is:

$$C_X^{(\mathrm{II})}(\mu_{hn}) = C_{SX}^{(\mathrm{I})}(\mu_{hn}) + C_{V\bar{X}}^{(\mathrm{I})}(\mu_{hn}) + \frac{Z\alpha}{2\pi}\left[2\,C_{SX}^{(\mathrm{I})}(\mu_{hn})\ln\frac{M_N^2}{\mu_{hn}^2} - 5\,C_{SX}^{(\mathrm{I})}(\mu_{hn}) + 7\,C_{V\bar{X}}^{(\mathrm{I})}(\mu_{hn})\right]$$

These loops proportional to Z cannot be calculated perturbatively

$$-\frac{Z^2\alpha}{4\pi} C_{SX}^{(\mathrm{I})}(\mu_{hn})\left(3\ln\frac{M_N^2}{\mu_{hn}^2} + 2\right)$$

(they should be replaced by a generalization of EFT I, with non-perturbative matching)

μ_h II $\mathcal{L}_A(A^{(h)})$ $\mathcal{L}_{\text{HQET}}(h_N^{(h)}, A^{(h)})$ $\mathcal{L}_{\text{Dirac}}^{(m\neq 0)}(\mu^{(h)}, A^{(h)})$ $\mathcal{L}_{\text{Dirac}}^{(m=0)}(e^{(h)}, A^{(h)})$

Such loops belong to the **nuclear effects**. For our purposes, it is enough to use nuclear form factors

In practice, we take the hard scale as the starting point of a perturbative description.

That is, we take $C_X^{(\text{II})}(\mu_{hn})$ as the input parameters of our analysis. Their RGEs are:

$$\mu \frac{d}{d\mu} C_X^{(\text{II})}\Big(\mu, \alpha(\mu), C^{(\text{I})}(\mu)\Big) = -\frac{3\alpha(\mu)}{2\pi} C_X^{(\text{II})}\Big(\mu, \alpha(\mu), C^{(\text{I})}(\mu)\Big) + \mathcal{O}(\alpha^2).$$

The nuclear effects (i.e. terms with Z) ends up cancelling in the running

As suggested, nuclear effects will play a crucial role in a possible interpretation of a muon conversion signal in terms of a BSM model. Yet, my focus here is not on such interpretation, but on the shape of the signal for muon conversion rate

μ_{sh} **III** $\mathcal{L}_A(A^{(sh,hc)})$ $\mathcal{L}_{\mathrm{HQET}}(h_N^{(sh)}, A^{(sh)})$ $\mathcal{L}_{\mathrm{NRQED}}(\psi^{(p)}, A^{(sh)})$ $\mathcal{L}_{\mathrm{SCET_I}}(\xi^{(hc)}, A^{(hc,s)})$

The Lagrangian is:

$$\mathcal{L}^{(\mathrm{III})} = \mathcal{L}^{(\mathrm{III}),\mathrm{LP}} + \mathcal{O}(\lambda), \quad \mathcal{L}^{(\mathrm{III}),\mathrm{LP}} = \mathcal{L}_A^{(\mathrm{III}),\mathrm{LP}} + \mathcal{L}_{h_N}^{(\mathrm{III}),\mathrm{LP}} + \mathcal{L}_\psi^{(\mathrm{III}),\mathrm{LP}} + \mathcal{L}_e^{(\mathrm{III}),\mathrm{LP}} + \mathcal{L}_\xi^{(\mathrm{III}),\mathrm{LP}}$$

with

$$\mathcal{L}_A^{(\mathrm{III}),\mathrm{LP}} = -\frac{1}{4}F_{\mu\nu}^{(sh)}F^{\mu\nu(sh)} - \frac{1}{4}F_{\mu\nu}^{(hc)}F^{\mu\nu(hc)}, \qquad \mathcal{L}_{h_N}^{(\mathrm{III}),\mathrm{LP}} = \bar{h}_N^{(sh)} iv\cdot D^{(sh)} h_N^{(sh)},$$

$$\mathcal{L}_\psi^{(\mathrm{III}),\mathrm{LP}} = \bar{\psi}^{(p)}\left(iv\cdot D^{(sh)} + \frac{(\vec{D}^{(sh)})^2}{2m_\mu}\right)\psi^{(p)}, \qquad \mathcal{L}_e^{(\mathrm{III}),\mathrm{LP}} = \bar{e}^{(sh)} i\not{D}^{(sh)} e^{(sh)},$$

$$\mathcal{L}_\xi^{(\mathrm{III}),\mathrm{LP}} = \bar{\xi}^{(hc)}\frac{\not{n}_+}{2}\left[\boxed{in_-D^{(hc+s)}} + i\not{D}_\perp^{(hc)}\frac{1}{in_+D^{(hc)}}i\not{D}_\perp^{(hc)}\right]\xi^{(hc)},$$

$$= in_-\partial - e\left[n_-A^{(hc)}(x) + n_-A^{(s)}(x_-)\right]$$

The matching condition is: $C_X^{(\mathrm{II})}\mathcal{O}_X^{(\mathrm{II})}(0) = \int dt\, C_X^{(\mathrm{III})}(t)\,\mathcal{O}_X^{(\mathrm{III})}(t),$

with: $\mathcal{O}_X^{(\mathrm{III})}(t) \equiv \bar{h}_N^{(sh)}(0)h_N^{(sh)}(0)\left[\bar{\xi}^{(hc)}W^{(hc)}\right](tn_+)P_X Y_{n_-}^{(sh)\dagger}(0)\psi^{(p)}(0)$

$\left(W^{(hc)}(x) \equiv \exp\left[ieQ\int_{-\infty}^0 dsn_+A^{(hc)}(x+sn_+)\right]\right)$ $\left(Y_{n_-}^{(sh)\dagger}(x) \equiv \exp\left[iQe\int_0^\infty ds\, n_-\, A^{(sh)}(x+s\,n_-)\, e^{-\varepsilon s}\right]\right)$

μ_{sh} III $\mathcal{L}_A(A^{(sh,hc)})$ $\mathcal{L}_{\text{HQET}}(h_N^{(sh)}, A^{(sh)})$ $\mathcal{L}_{\text{NRQED}}(\psi^{(p)}, A^{(sh)})$ $\mathcal{L}_{\text{SCET}_\text{I}}(\xi^{(hc)}, A^{(hc,s)})$

Defining the Fourier transform as $\int dt e^{itn_+p'} C_X^{(\text{III})}(t) = C_X^{(\text{III})}(n_+p', m_\mu)$, the matching is:

$$C_X^{(\text{III})}(2E_e, m_\mu; \mu_h) = C_X^{(\text{II})}(\mu_h)\, \mathcal{H}(2E_e, m_\mu; \mu_h),$$

where I define $\mathcal{H}$ to be the *hard function*, given by:

$$\mathcal{H}(2E_e, m_\mu; \mu_h) = 1 - \frac{\alpha}{4\pi}\left\{ \frac{m_\mu \ln\left(\frac{16E_e^4\mu_h^2}{m_\mu^6}\right)}{4E_e - 2m_\mu} - \ln\left(\frac{\mu_h}{m_\mu}\right)\ln\left(\frac{\mu_h^7}{4E_e^2 m_\mu^5}\right) - 2\,\text{Li}_2\left(1 - \frac{m_\mu}{2E_e}\right) \right.$$
$$\left. + \ln^2\left(\frac{2E_e m_\mu^2}{\mu_h^3}\right) + \frac{2E_e \ln\left(\frac{m_\mu}{\mu_h}\right)}{2E_e - m_\mu} + \frac{\pi^2}{12} \right\} + \mathcal{O}(\alpha^2).$$

The RGEs are:

$$\mu\frac{d}{d\mu} C_X^{(\text{III})}(2E_e, m_\mu; \mu) = \left[\Gamma_{\text{cusp}}^{(\text{III})} \log\left(\frac{2E_e}{\mu}\right) + \gamma^{(\text{III})}\right] C_X^{(\text{III})}(2E_e, m_\mu; \mu),$$

with:

$$\Gamma_{\text{cusp}}^{(\text{III})} = \frac{\alpha}{\pi} - \left(\frac{\alpha}{\pi}\right)^2 \frac{5}{9} + \mathcal{O}(\alpha^3), \qquad \gamma^{(\text{III})} = -\frac{5\alpha}{4\pi} + \mathcal{O}(\alpha^2).$$

μ_{sh} III $\mathcal{L}_A(A^{(sh,hc)})$ $\mathcal{L}_{\text{HQET}}(h_N^{(sh)}, A^{(sh)})$ $\mathcal{L}_{\text{NRQED}}(\psi^{(p)}, A^{(sh)})$ $\mathcal{L}_{\text{SCET}_\text{I}}(\xi^{(hc)}, A^{(hc,s)})$

Finally, the soft modes in $\mathcal{L}_\xi^{(\text{III}),\text{LP}}$ can be decoupled via:

$$\bar{\xi}^{(hc)}(x) = \bar{\xi}^{(hc)}_{(0)}(x) Y_{n_+}^{(s)\dagger}(x_-), \qquad \xi^{(hc)}(x) = \overline{Y}_{n_+}^{(s)}(x_-)\xi^{(hc)}_{(0)}(x),$$

with:

$$Y_u^{(s)\dagger}(x) \equiv \exp\left(iQe\int_0^\infty ds\, u\cdot A^{(s)}(x+s\,u)\, e^{-\varepsilon s}\right),$$

$$\overline{Y}_u^{(s)}(x) \equiv \exp\left(iQ\,e\int_{-\infty}^0 ds\, u\cdot A^{(s)}(x+s\,u)\, e^{\varepsilon s}\right).$$

So,

$$\mathcal{L}_\xi^{(\text{III}),\text{LP}} = \bar{\xi}^{(hc)}_{(0)}\frac{\not{n}_+}{2}\left[in_-D^{(hc)} + i\not{D}_\perp^{(hc)}\frac{1}{in_+D^{(hc)}}i\not{D}_\perp^{(hc)}\right]\xi^{(hc)}_{(0)},$$

and

$$\mathcal{O}_X^{(\text{III})}(t) = \bar{h}_N^{(sh)}(0)\, h_N^{(sh)}(0)\left[\bar{\xi}^{(hc)}_{(0)} W^{(hc)}\right](tn_+) P_X Y_{n_+}^{(s)\dagger}(0) Y_{n_-}^{(sh)\dagger}(0)\psi^{(p)}(0).$$

μ_s **IV** $\mathcal{L}_A(A^{(s,sc)})$ $\mathcal{L}_{\text{HQET}}(h_N^{(s)}, A^{(s,sc)})$ $\mathcal{L}_{\text{pNRQED}}(\Psi^{(p)}, A^{(s,sc)})$ $\mathcal{L}_{\text{SCET}_{\text{II}}}(\xi^{(c)}, A^{(c)})$

The Lagrangian is:

$$\mathcal{L}^{(\text{IV})} = \mathcal{L}^{(\text{IV}),\text{LP}} + \mathcal{O}(\lambda), \qquad \mathcal{L}^{(\text{IV}),\text{LP}} = \mathcal{L}_A^{(\text{IV}),\text{LP}} + \mathcal{L}_{h_N}^{(\text{IV}),\text{LP}} + \mathcal{L}_e^{(\text{IV}),\text{LP}} + \mathcal{L}_\Psi^{(\text{IV}),\text{LP}} + \mathcal{L}_\xi^{(\text{IV}),\text{LP}}$$

with:

$$\mathcal{L}_A^{(\text{IV}),\text{LP}} = -\frac{1}{4} F_{\mu\nu}^{(s)} F^{\mu\nu(s)}, \qquad \mathcal{L}_{h_N}^{(\text{IV}),\text{LP}} = \bar{h}_N^{(s)} iv \cdot D^{(s+sc)} h_N^{(s)}, \qquad \mathcal{L}_e^{(\text{IV}),\text{LP}} = \bar{e}^{(s)} (i\not{D}^{(s+sc)} - m_e) e^{(s)},$$

$$\mathcal{L}_{\Psi,h_N}^{(\text{IV}),\text{LP}} = \bar{\Psi}^{(p)}(x) \left(iv \cdot D^{(s+sc)}(x_0) + \frac{\vec{\nabla}^2}{2m_\mu} \right) \Psi^{(p)}(x) + \int d^3r \ \bar{h}_N^{(s)}(x) h_N^{(s)}(x) V(\vec{r}) \bar{\Psi}^{(p)}(x+\vec{r}) \Psi^{(p)}(x+\vec{r}),$$

$$\mathcal{L}_\xi^{(\text{IV}),\text{LP}} = \bar{\xi}^{(c)} \frac{\not{n}_+}{2} \left[in_- D^{(c+us)} + \left(i\not{D}_\perp^{(c)} - m_e \right) \frac{1}{in_+ D^{(c)}} \left(i\not{D}_\perp^{(c)} + m_e \right) \right] \xi^{(c)},$$

where:

$$D_\mu^{(s+sc)}(x) = \partial_\mu + iQe \left[A_\mu^{(s)}(x) + n_+ \cdot A^{(sc)}(x_+) \frac{n_{-\mu}}{2} \right],$$

$$V(\vec{r}) = -\frac{Z\alpha}{r} \left(1 + \frac{2\alpha}{3\pi} \int_1^\infty dx e^{-2m_e r x} \frac{2x^2+1}{2x^4} \sqrt{x^2-1} \right)$$

μ_s IV $\mathcal{L}_A(A^{(s,sc)})$ $\mathcal{L}_{\text{HQET}}(h_N^{(s)}, A^{(s,sc)})$ $\mathcal{L}_{\text{pNRQED}}(\Psi^{(p)}, A^{(s,sc)})$ $\mathcal{L}_{\text{SCET}_{\text{II}}}(\xi^{(c)}, A^{(c)})$

The matching condition is: $\int dt\, C_X^{(\text{III})}(t)\, \mathcal{O}_X^{(\text{III})}(t) = \int dt\, C_X^{(\text{IV})}(t)\, \mathcal{O}_X^{(\text{IV})}(t),$

with

$$\mathcal{O}_X^{(\text{IV})}(t) \equiv \bar{h}_N^{(s)}(0) h_N^{(s)}(0) \left[\bar{\xi}^{(c)} W^{(c)}\right](tn_+) P_X Y_{n_+}^{(s)\dagger} \Psi^{(p)}(0)$$

$$\left(W^{(c)}(x) \equiv \exp\left[ieQ \int_{-\infty}^{0} dsn_+ A^{(c)}(x + sn_+)\right]\right)$$

The matching is trivial: $C_X^{(\text{IV})}(2E_e, m_\mu; \mu_{sh}) = C_X^{(\text{III})}(2E_e, m_\mu; \mu_{sh})$

and so is the running: $C_X^{(\text{IV})}(2E_e, m_\mu; \mu_s) = U_h(\mu_h, \mu_s)\, C_X^{(\text{III})}(2E_e, m_\mu; \mu_h)$

The remaining soft modes in $\mathcal{L}^{(\text{IV}),\text{LP}}$ can be decoupled via:

$$\bar{h}_N^{(s)} = \bar{h}_{N(0)}^{(s)} Y_{h_N}^{(s)\dagger}, \qquad h_N^{(s)} = \overline{Y}_{h_N}^{(s)} h_{N(0)}^{(s)}, \qquad \Psi^{(p)} = \overline{Y}_{\Psi}^{(s)} \Psi_{(0)}^{(p)}, \quad \text{with:}$$

Then,

$$\mathcal{O}_X^{(\text{IV})}(t) = \mathcal{O}_s(0)\, \mathcal{O}_{X(0)}^{(\text{IV})}(t),$$

with:

$$\mathcal{O}_s(x) = \left[Y_v^{(s)\dagger} \overline{Y}_v^{(s)}\, Y_{n_+}^{(s)\dagger} \overline{Y}_v^{(s)}\right](x), \qquad \mathcal{O}_{X(0)}^{(\text{IV})}(t) = \bar{h}_{N(0)}^{(s)}(0)\, h_{N(0)}^{(s)}(0) \left[\bar{\xi}^{(c)} W^{(c)}\right](tn_+) P_X \Psi_{(0)}^{(p)}(0).$$

μ_{sc}	V	$\mathcal{L}_A(A^{(sc)})$	$\mathcal{L}_{\mathrm{HQET}}(h_N^{(s)}, A^{(sc)})$	$\mathcal{L}_{\mathrm{pNRQED}}(\Psi^{(p)}, A^{(sc)})$	$\mathcal{L}_{\mathrm{bHQET}}(h_e^{(sc)}, A^{(sc)})$

The Lagrangian is:

$$\mathcal{L}^{(\mathrm{V})} = \mathcal{L}^{(\mathrm{V}),\mathrm{LP}} + \mathcal{O}(\lambda), \qquad \mathcal{L}^{(\mathrm{V}),\mathrm{LP}} = \mathcal{L}_A^{(\mathrm{V}),\mathrm{LP}} + \mathcal{L}_{h_N}^{(\mathrm{V}),\mathrm{LP}} + \mathcal{L}_{\Psi}^{(\mathrm{V}),\mathrm{LP}} + \mathcal{L}_{h_e}^{(\mathrm{V}),\mathrm{LP}},$$

with:

$$\mathcal{L}_A^{(\mathrm{V}),\mathrm{LP}} = -\frac{1}{4} F_{\mu\nu}^{(sc)} F^{\mu\nu(sc)}, \qquad \mathcal{L}_{h_N(0)}^{(\mathrm{V}),\mathrm{LP}} = \bar{h}_{N(0)}^{(s)} iv \cdot D^{(sc)}(x_+) h_{N(0)}^{(s)},$$

$$\mathcal{L}_{\Psi}^{(\mathrm{V}),\mathrm{LP}} = \bar{\Psi}_{(0)}^{(p)}(x) \left(iv \cdot D^{(sc)}(x_0) + \frac{\vec{\nabla}^2}{2m_\mu} \right) \Psi_{(0)}^{(p)}(x) + \int d^3 r \; \bar{h}_N^{(s)}(x) h_N^{(s)}(x) V(\vec{r}) \bar{\Psi}_{(0)}^{(p)}(x+\vec{r}) \Psi_{(0)}^{(p)}(x+\vec{r}),$$

$$\mathcal{L}_{h_e}^{(\mathrm{V}),\mathrm{LP}} = \bar{h}_e^{(sc)} \, iv_e \cdot \boxed{D_e^{(sc)}} \, h_e^{(sc)}.$$

$$= i\partial_\mu + Qe \frac{(n_-)_\mu}{2} n_+ \cdot A_{(sc)}(x_+) \qquad v_e \equiv \left(\frac{2m_\mu}{m_e}, 0, \frac{m_e}{2m_\mu} \right)$$

The matching condition reads:

$$\mathcal{O}_s(0) \int dt \, C_X^{(\mathrm{IV})}(t) \mathcal{O}_{X(0)}^{(\mathrm{IV})}(t) = \mathcal{O}_s(0) \int dt C_X^{(\mathrm{IV})}(t) e^{i m_e n_+ \cdot v_e t} C_m(m_e) \, \mathcal{O}_X^{(\mathrm{V})}(0)$$

with

$$\mathcal{O}_X^{(\mathrm{V})} \equiv \bar{h}_{N(0)}^{(s)} h_{N(0)}^{(s)} \, \bar{h}_e^{(sc)} P_X \Psi_{(0)}^{(p)}.$$

(This operator is now local)

μ_{sc} V $\mathcal{L}_A(A^{(sc)})$ $\mathcal{L}_{\text{HQET}}(h_N^{(s)}, A^{(sc)})$ $\mathcal{L}_{\text{pNRQED}}(\Psi^{(p)}, A^{(sc)})$ $\mathcal{L}_{\text{bHQET}}(h_e^{(sc)}, A^{(sc)})$

The matching is:

$$C_m(m_e;\mu_s) = 1 + \frac{\alpha}{4\pi}\left\{2\ln^2\left(\frac{m_e}{\mu_s}\right) - \ln\left(\frac{m_e}{\mu_s}\right) + \frac{\pi^2}{12} + 2\right\}$$

and we define:

$$C_X^{(\text{V})}(\mu_s) = C_X^{(\text{IV})}(2m_\mu, m_\mu;\mu_s)C_m(m_e;\mu_s)$$

We simplify the analysis by choosing μ_s to be our ultimate scale, to which everything is evolved. In this case, we do not need to calculate the RGEs here

The soft-collinear modes can be decoupled via:

$$\bar{h}^{(s)}_{N(0)} = \bar{h}^{(s)}_{N(00)} Y^{(sc)\dagger}_{n_+}, \qquad \bar{\Psi}^{(p)}_{(0)} = \bar{\Psi}^{(p)}_{(00)} Y^{(sc)\dagger}_{n_+}, \qquad \bar{h}^{(sc)}_{e} = \bar{h}^{(s)}_{e(0)} Y^{(sc)\dagger}_{v_e},$$

$$h^{(s)}_{N(0)} = \overline{Y}^{(sc)}_{n_+} h^{(s)}_{N(00)}, \qquad \Psi^{(p)}_{(0)} = \overline{Y}^{(sc)}_{n_+} \Psi^{(p)}_{(00)}, \qquad h^{(sc)}_{e} = \overline{Y}^{(sc)}_{v_e} h^{(sc)}_{e(0)},$$

Then,

$$\mathcal{O}_X^{(\text{V})} = \mathcal{O}_{sc}(0)\,\mathcal{O}_{X(0)}^{(\text{V})},$$

with

$$\mathcal{O}_{sc}(x) = \left[Y^{(sc)\dagger}_{n_+}\overline{Y}^{(sc)}_{n_+} Y^{(sc)\dagger}_{v_e}\overline{Y}^{(sc)}_{n_+}\right](x), \qquad \mathcal{O}_{X(0)}^{(\text{V})} = \bar{h}^{(s)}_{N(00)} h^{(s)}_{N(00)}\,\bar{h}^{(sc)}_{e(0)} P_X \Psi^{(p)}_{(00)}.$$

μ_{sc}	V	$\mathcal{L}_A(A^{(sc)})$	$\mathcal{L}_{\mathrm{HQET}}(h_N^{(s)}, A^{(sc)})$	$\mathcal{L}_{\mathrm{pNRQED}}(\Psi^{(p)}, A^{(sc)})$	$\mathcal{L}_{\mathrm{bHQET}}(h_e^{(sc)}, A^{(sc)})$

The Lagrangian after this reads:

$$\mathcal{L}_{h_N}^{(\mathrm{V}),\mathrm{LP}} = \bar{h}_{N(00)}^{(s)} iv\cdot\partial h_{N(00)}^{(s)},$$

$$\mathcal{L}_{\Psi}^{(\mathrm{V}),\mathrm{LP}} = \bar{\Psi}_{(00)}^{(p)}(x)\left(iv\cdot\partial+\frac{\vec{\nabla}^2}{2m_\mu}\right)\Psi_{(00)}^{(p)}(x) + \int d^3r\ \bar{h}_{N(00)}^{(s)}(x)h_{N(00)}^{(s)}(x)V(\vec{r})\bar{\Psi}_{(00)}^{(p)}(x+\vec{r})\Psi_{(00)}^{(p)}(x+\vec{r}),$$

$$\mathcal{L}_{h_e}^{(\mathrm{V}),\mathrm{LP}} = \bar{h}_{e(0)}^{(sc)}\, iv_e\cdot\partial\, h_{e(0)}^{(sc)}.$$

We achieved *factorization* at LP: the different sectors do not interact at LP, to all orders in α

- This allows us to describe bound muon decays in a consistent and improvable way
- We consider the amplitude for muon conversion, with final arbitrary radiation $\mathcal{X}$:

$$i\mathcal{M}_{\mu_H \to eN\mathcal{X}} = \left\langle e^{(sc)} N^{(s)} \mathcal{X} \middle| J(0) \middle| \mu_H \right\rangle$$

where $|\mu_H\rangle = \left|\mu^{(p)} N^{(s)}\right\rangle$ and the current is:

$$\mathcal{J}(0) = -\frac{4G_F}{\sqrt{2}} C_X^{(\mathrm{V})} \mathcal{O}_s(0) \mathcal{O}_{sc}(0) \mathcal{O}_{X(0)}^{(\mathrm{V})}(0),$$

$$\begin{cases} \mathcal{O}_s(x) = \left[Y_v^{(s)\dagger} \overline{Y}_v^{(s)} \, Y_{n_+}^{(s)\dagger} \overline{Y}_v^{(s)}\right](x), \\ \mathcal{O}_{sc}(x) = \left[Y_{n_+}^{(sc)\dagger} \overline{Y}_{n_+}^{(sc)} Y_{v_e}^{(sc)\dagger} \overline{Y}_{n_+}^{(sc)}\right](x), \\ \mathcal{O}_{X(0)}^{(\mathrm{V})} = \bar{h}_{N(00)}^{(s)} h_{N(00)}^{(s)} \, \bar{h}_{e(0)}^{(sc)} P_X \Psi_{(00)}^{(p)}. \end{cases}$$

- Because the real emission has two different modes (soft and soft-collinear), we have:

$$|\mathcal{X}\rangle = \left|\mathcal{X}^{(s)}\right\rangle \otimes \left|\mathcal{X}^{(sc)}\right\rangle$$

- Then, the matrix element is completely factorized:

$$\begin{aligned} i\mathcal{M}_{\mu_H \to eN\mathcal{X}} = {} & -i\frac{4G_F}{\sqrt{2}} C_X^{(\mathrm{V})} \left\langle \mathcal{X}^{(sc)} \middle| \mathcal{O}_{sc}(0) \middle| 0 \right\rangle \left\langle \mathcal{X}^{(s)} \middle| \mathcal{O}_s(0) \middle| 0 \right\rangle \\ & \times \left\langle e^{(sc)} \middle| \left[\bar{h}_{e(0)}^{(sc)}(0)\right]_\alpha \middle| 0 \right\rangle \left\langle N^{(s)} \middle| \bar{h}_{N(00)}^{(s)}(0) h_{N(00)}^{(s)}(0) \left[P_X \Psi_{(00)}^{(p)}(0)\right]_\alpha \middle| \mu_H \right\rangle. \end{aligned}$$

(at last!)

- The decay width is:

$$\Gamma_{\mu_H \to eN\mathcal{X}} = \frac{1}{2M_{\mu_H}} \int (2\pi)^d \delta^{(d)} \left(p_{\mu_H} - p' - k' - \sum_i p_{\mathcal{X}_i} \right) \frac{d^{d-1}k'}{(2\pi)^{d-1} 2M_N} \frac{d^{d-1}p'}{(2\pi)^3 2E_e} \prod_i \frac{d^{d-1}p_{\mathcal{X}_i}}{(2\pi)^{d-1} 2E_{\mathcal{X}_i}} \, |\overline{\mathcal{M}}_{\mu_H \to eN\mathcal{X}}|^2 .$$

- The factorized NLO differential rate is:

$$\boxed{\begin{aligned} \frac{1}{\Gamma_0} \frac{\Gamma_{\mu_H \to eN\mathcal{X}}}{dE_e} &= |\psi_{\text{corr}}|^2 |\mathcal{H}(2m_\mu, m_\mu; \mu_h)|^2 \, |U_h(\mu_h, \mu_s)|^2 \, |C_m(m_e, \mu_s)|^2 \\ &\times \int_0^\infty dE_{sc} \int_0^\infty dE_s \delta(\Delta E - E_{sc} - E_s) \mathcal{S}(E_s) \mathcal{SC}(E_{sc}) \end{aligned}}$$

$$\Gamma_0 = \frac{2G_F^2 m_\mu^2}{\pi} |\psi_{\text{Schr.}}(0)|^2_{\text{LO}} \left(\left| C^{(\text{II})}_{L,\text{LO}} \right|^2 + \left| C^{(\text{II})}_{R,\text{LO}} \right|^2 \right)$$

$$|\psi_{\text{corr}}|^2 \equiv \frac{|\psi_{\text{Schr.}}(0)|^2}{|\psi_{\text{Schr.}}(0)|^2_{\text{LO}}} = 1 + \frac{\alpha}{\pi} \delta_{\text{pot}}$$

$$\begin{cases} \mathcal{S}(E_s) \equiv \sum_{\mathcal{X}^{(s)}} \delta(E_s - E_{\mathcal{X}^{(s)}}) \langle 0 | \mathcal{O}_s^\dagger(0) | \mathcal{X}^{(s)} \rangle \langle \mathcal{X}^{(s)} | \mathcal{O}_s(0) \, |0\rangle \\ \mathcal{SC}(E_{sc}) \equiv \sum_{\mathcal{X}^{(sc)}} \delta(E_{sc} - E_{\mathcal{X}^{(sc)}}) \langle 0 | \mathcal{O}_{sc}^\dagger(0) | \mathcal{X}^{(sc)} \rangle \langle \mathcal{X}^{(sc)} | \mathcal{O}_{sc}(0) \, |0\rangle \end{cases}$$

- To discuss results, we consider the cumulant, $\Gamma^{\text{cumul}} = \int \frac{d\Gamma}{dE'_e} dE'_e$
  - At LO, $\frac{d\Gamma}{dE'_e} = \delta(E'_e)$, so that the LO cumulant is a horizontal line — trivial shape
  - At NLO, we consider two approaches:
    - The fixed-order (FO) one: no RG running, and all scales set to the same generic scale

$$\frac{\Gamma_{\text{cumul}}}{\Gamma_{\text{LO}}}\bigg|_{\text{FO}} = |\psi_{\text{corr}}|^2_{\text{NLO}} \times |\mathcal{H}(2m_\mu, m_\mu;\mu)|^2_{\text{NLO}} \times |C_m(m_e,\mu)|^2_{\text{NLO}} \times \left[\int_0^{\Delta E} d\mathcal{E}\, \mathcal{S}(\mathcal{E})\right]_{\text{NLO}} \times \widehat{\mathcal{SC}}(\Delta E)_{\text{NLO}},$$

with:

$$|\mathcal{H}(2m_\mu, m_\mu;\mu)|^2_{\text{NLO}} = 1 - \frac{\alpha}{2\pi}\left\{\ln^2\left(\frac{m_\mu}{\mu}\right) + \ln^2\left(\frac{4m_\mu}{\mu}\right) + \ln\left(\frac{4m_\mu}{\mu}\right) - \frac{\pi^2}{12} - 2\ln^2(2)\right\} + \mathcal{O}(\alpha^2),$$

$$|C_m(m_e,\mu)|^2_{\text{NLO}} = 1 + \frac{\alpha}{2\pi}\left\{2\ln^2\left(\frac{m_e}{\mu}\right) - \ln\left(\frac{m_e}{\mu}\right) + \frac{\pi^2}{12} + 2\right\} + \mathcal{O}(\alpha^2),$$

$$\left[\int_0^{\Delta E} d\mathcal{E}\, \mathcal{S}(\mathcal{E})\right]_{\text{NLO}} = 1 + \frac{\alpha}{\pi}\left\{\ln^2\left(\frac{2\Delta E}{\mu}\right) - \ln\left(\frac{2\Delta E}{\mu}\right) + 1 - \frac{\pi^2}{8}\right\} + \mathcal{O}(\alpha^2),$$

$$\widehat{\mathcal{SC}}(\Delta E)_{\text{NLO}} = 1 - \frac{\alpha}{\pi}\left\{\ln^2\left(\frac{\Delta E m_e}{m_\mu\, \mu}\right) + \ln\left(\frac{\Delta E\, m_e}{m_\mu\, \mu}\right) + \frac{\pi^2}{24}\right\} + \mathcal{O}(\alpha^2).$$

$$\left[\widehat{\mathcal{SC}}(E_{sc}) \equiv \sum_{\mathcal{X}^{(sc)}} \int \prod_j \frac{d^{d-1}p_{\mathcal{X}_j^{(sc)}}}{(2\pi)^{d-1} 2E_{\mathcal{X}_j^{(sc)}}} \theta(E_{sc} - E_{\mathcal{X}^{(sc)}}) \langle 0|\mathcal{O}^\dagger_{sc}(0)|\mathcal{X}^{(sc)}\rangle\langle\mathcal{X}^{(sc)}|\mathcal{O}_{sc}(0)\,|0\rangle\right]$$

- The resummed result: scales are set equal to the canonical scales

$$\frac{\Gamma_{\text{cumul}}}{\Gamma_{\text{LO}}}\bigg|_{\text{Resu.}} = \left\{1 + \frac{\alpha}{\pi}\left[\delta_{\text{pot}} + \ln^2\left(\frac{m_e}{2\Delta E}\right) + \ln\left(\frac{m_e}{2\Delta E}\right) - \frac{\ln(2)}{2} - \frac{\pi^2}{12} + 2\right]\right\}$$
$$\times \exp\left\{\frac{\alpha}{2\pi}\left[\left(\frac{10\alpha}{9\pi} - 2\right)\ln^2\left(\frac{2m_\mu}{m_e}\right) + 5\ln\left(\frac{2m_\mu}{m_e}\right) - 2\ln^2\left(\frac{m_\mu}{\Delta E}\right) + 2\ln\left(\frac{m_\mu}{\Delta E}\right) - \frac{\pi^2}{12}\right]\right\}.$$

- For the resummed result, it is convenient to consider different approximations.

  We assume the logarithm counting:

$$d\Gamma = \sum_n \underbrace{\alpha^n L^{2n}}_{\text{LL}} + \underbrace{\alpha^n L^{2n-1} +}_{\text{NLL}} \cdots$$

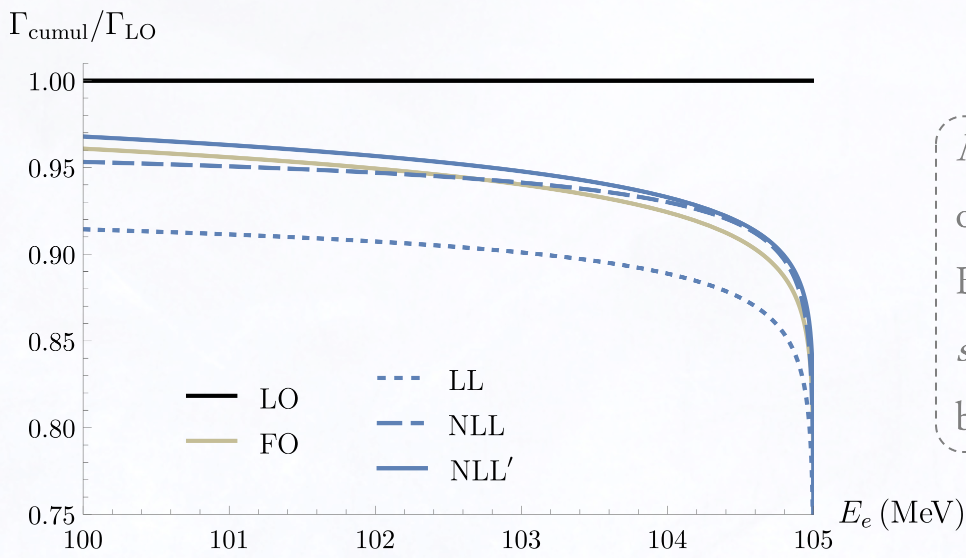
  We also consider NLL', which includes the finite remains of the FO approach

- The generic scale of the FO approach is set at the hard scale, and a proper phenomenological investigation is left for future work

- Without real emission, $E_e = E_e^{\max} := m_\mu$. A non-zero emission implies $E_e = E_e^{\max} - \Delta E$

  We want to study $\Delta E \sim m_e$

- We find:



Nota bene: the absolute value of the rate will matter for BSM interpretations. But the *shape* of the rate is crucial to be able to *observe* conversion

- The FO result leads to a correction of around -9% relative to the LO result
- The resummed results show a perturbative character, such that NLL' prime is close to FO

- Is the EFT framework worth it? **Yes**. Besides addessing the large logs,
  - Is systematically improvable
  - Has a transparent and homogenous counting
  - Provides Feynman rules
  - Is not restricted to QED
  - Avoids double counting
  - Provides proper QFT definitions
  - Allows the derivation of all-order theorems

- CLFV might be observed soon in muon conversion. But precise predictions are challenging
- I developed a consistent (EFT) framework for precise predictions for bound muon decays
- Due to the presence of both bound state and collinear physics, the framework is not trivial: it involves a sequence of 5 EFTs, comprising HQET, NRQED, pNRQED, SCET and bHQET, as well as many different modes

(free from large logs!)

- The final result is a factorization theorem, composed only of **single-scale objects**
- Besides the theorem, I derived the one-loop matchings and runnings for muon conversion
- This allowed precise theoretical predictions for the upcoming experiments Mu2e and COMET
- Several future directions:
  - Determine the shape for DIO. Do phenonomenology for DIO and conversion
  - Explore nuclear effects and finite-nucleus size corrections
  - Consider particular BSM models, perform the complete matching, find absolute values
  - Explore recoil/power corrections