University of Vienna
Particle Physics Seminar

QCD scattering in the Regge limit

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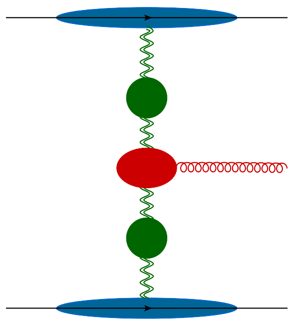


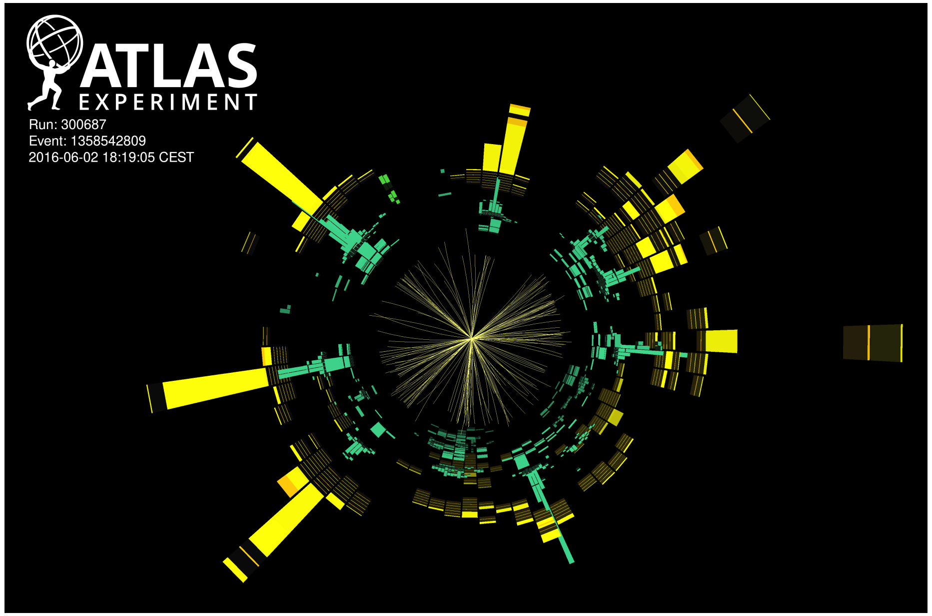
QCD Scattering in the Regge Limit

Abstract

Fixed-order computations of QCD amplitudes in general kinematics are limited to either one, two or three loops, depending on the number of particles produced. This strongly motivates our theoretical research programme aimed at understanding the behaviour of quark and gluon scattering amplitudes in special kinematic limits, in which new factorization and exponentiation properties arise. A particularly interesting limit is the Regge limit, where major simplifications take place. A remarkable property of this limit is the exponentiation of energy logarithms, a phenomenon known as *gluon Reggeization*, leading to power-like dependence on the energy. This phenomenon can be investigated by establishing rapidity evolution equations. The dynamics is markedly more complex in full QCD, where colour off-diagonal evolution, generated by multi-Reggeon interactions, gives rise to Regge cuts, as compared to the planar limit, where this structure collapses onto a single Regge pole. Understanding this evolution facilitated in recent years the formulation of an effective two-dimensional theory of Reggeized gluons. Combining this with recent progress in $2 \to 2$ and $2 \to 3$ amplitude computations, we are now able to determine key ingredients beyond the next-to-leading tower of logarithms, such as the three-loop gluon Regge trajectory and the two-loop central-emission vertex. These, in turn, can be used to predict the structure of other multi-leg amplitudes and to determine the kernel of rapidity evolution to the next unknown order.

Wealth of new experimental data on QCD

- The Large Hadron Collider provides a wealth of data at $\sqrt{s} \simeq 14\ \mathrm{TeV}$
- New energy regime is being explored
- Multi-jet events are abundant
- Heavy Ion programme explores high gluon density regime
- Much more data is expected in the high-luminosity phase (from 2028)

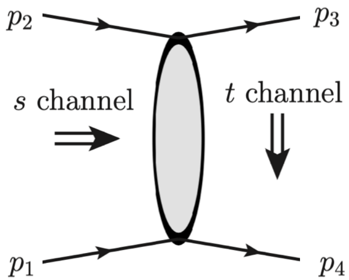


Excellent prospects to study QCD, e.g. jets, hadronization, kinematic limits,…

Strong motivation to develop thorough theoretical understanding and computation methods

Inspiration: cross section in proton-proton scattering

- The total cross section (measured in cosmic rays and colliders) rises with energy s
- $\sigma_{\text{tot}} \simeq \dfrac{\text{Im}\mathscr{M}}{s}$: the (forward) amplitude is growing as a power of the energy for $s \gg -t$.
- Despite the large energy s, σ_{tot} receives contributions of small momentum exchange t
- Elastic and inelastic (jet) cross sections can be computed perturbatively if $-t \gg \Lambda^2_{\text{QCD}}$.
- We consider $s \gg -t \gg \Lambda^2_{\text{QCD}}$



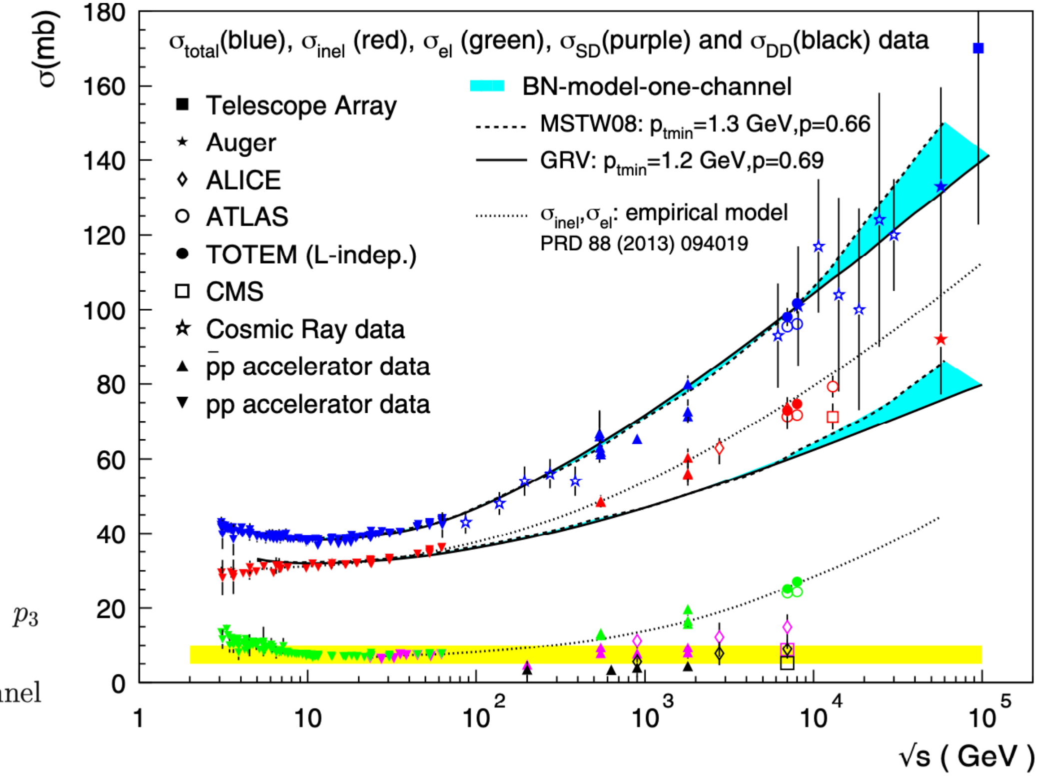


Figure from review by Pancheri and Srivastava.
Data and model compilation
by D. Fagundes, A. Grau and O. Shekhovtsova

We shall study $\mathscr{M}$ perturbatively, but will not be able to address the growth of σ_{tot}

parton density evolution in Q^2 and in rapidity

- DGLAP (Dokshitzer-Gribov-Lipatov-Altarelli-Parisi) evolution resums logarithms of Q^2/μ_F^2

- BFKL (Balitsky-Fadin-Kuraev-Lipatov) evolution resums energy logarithms (= rapidity Y)

The high-gluon-density saturation regime requires a generalisation: Balitsky-JIMWLK (Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov and Kovner) non-linear evolution in rapidity

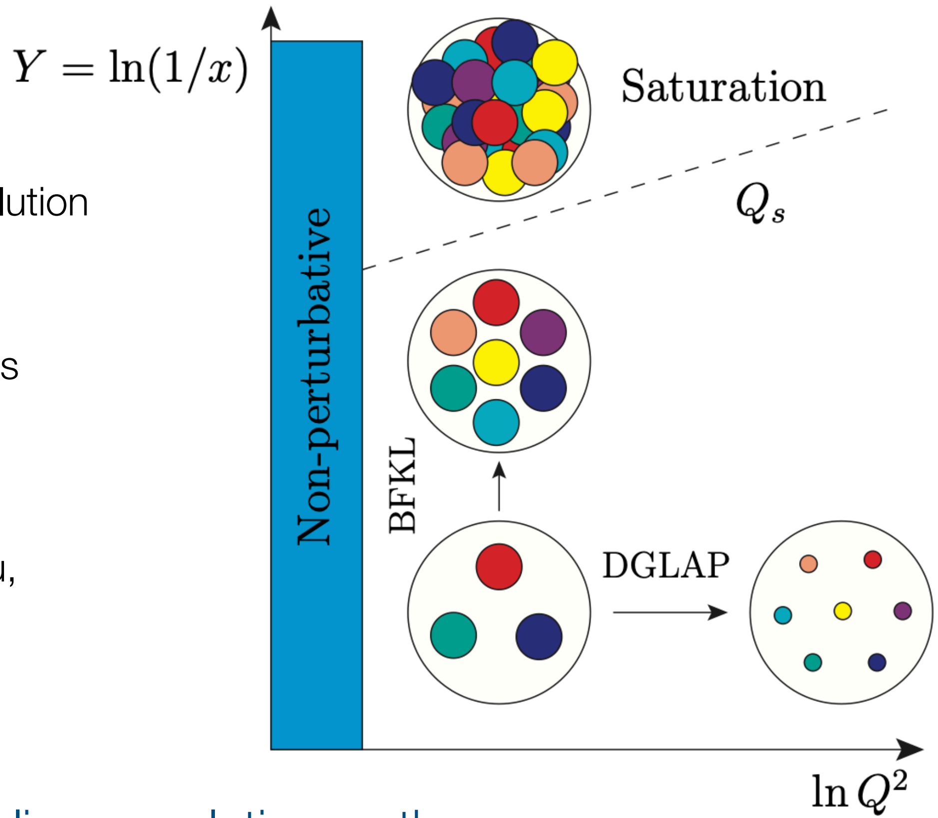


Figure by Michael Fucilla

We shall study the consequences of this non-linear evolution on the amplitude perturbatively, but not address the saturation regime

QCD scattering in the Regge limit

Motivation

- Understand the high-energy behaviour of **quark and gluon** scattering amplitudes in **full colour**
- Study the **exponentiation** of high-energy logarithms
- Connect **rapidity evolution equations** to properties of scattering amplitudes
- Establish connection with **Regge poles and cuts** in the complex angular momentum plane
- Understand the interplay between the Regge limit and **soft gluon exponentiation**

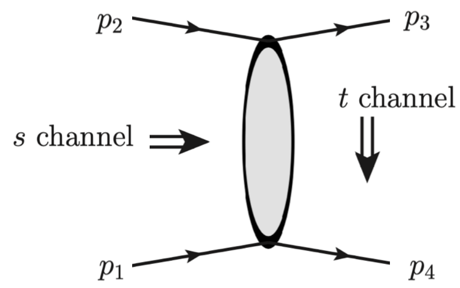
QCD Scattering in the Regge Limit

Outline of the talk

- Inspiration: QCD at LHC
- Amplitudes in the Regge limit:
  - ✓ Reggeization; factorization and its violation
  - ✓ Towards an effective two-dimensional theory from rapidity evolution (will not discuss alternative approaches, such as Glauber-SCET).
- Radiative corrections at the Loops and Legs frontier:
  - ✓ Disentangling pole from cut at NNLL in signature-odd 2 → 2 amplitudes
  - ✓ Determining the Lipatov Vertex at 2 loops

The Regge limit of 2 → 2 gauge-theory amplitudes

- **Regge theory:** the amplitude should be dominated by the t-channel exchange of the particle with the highest spin, $\mathcal{M} \sim s^{\ell}$.

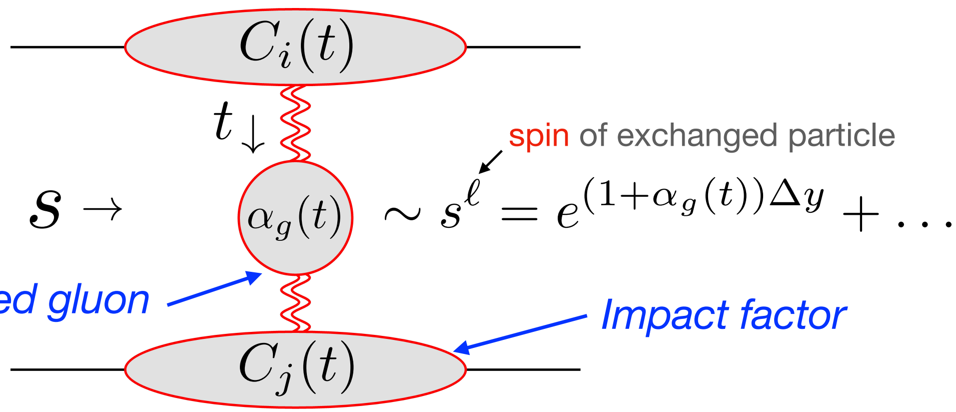


- **QCD**: Simplification at leading power in t/s: helicity is conserved, and indeed, t-channel **gluon** exchange is dominant.

- **Reggeization** (Regge-pole):
$$\frac{s}{t} \longrightarrow \frac{s}{t}\left(\frac{s}{-t}\right)^{\alpha_g(t)}$$

resumming all terms: $\left[\alpha_s(-t)\ \ln\left(\frac{s}{-t}\right)\right]^n$

- **Factorization:**



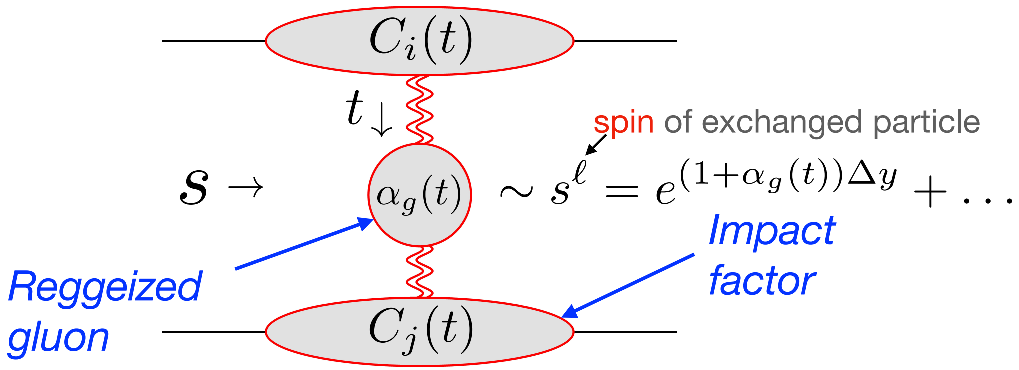
The high-energy limit of 2 → 2 gauge-theory amplitudes

- The gluon Regge trajectory can be computed in perturbation theory. At one loop:

$$\alpha_g(t) = -\alpha_s \mathbf{T}_t^2 (\mu^2)^\epsilon \int \frac{d^{2-2\epsilon}k_\perp}{(2\pi)^{2-2\epsilon}} \frac{q_\perp^2}{k_\perp^2 (q_\perp - k_\perp)^2} + \mathcal{O}(\alpha_s^2)$$
$$= \frac{\alpha_s}{\pi} \mathbf{T}_t^2 \left(\frac{-t}{\mu^2}\right)^{-\epsilon} \frac{B_0(\epsilon)}{2\epsilon} + \mathcal{O}(\alpha_s^2)$$

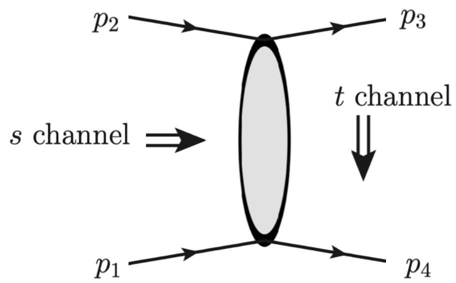
$$B_0(\epsilon) = e^{\epsilon\gamma_E} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} = 1 - \frac{\zeta_2}{2}\epsilon^2 - \frac{7\zeta_3}{3}\epsilon^3 + \dots$$

- Regge-pole factorization amounts to a **relation** between gg→gg, qg→qg, qq→qq



- This holds for the **real part** of the amplitude through NLL.
  Beyond that it is violated by **non-planar** corrections associated with **multi-Reggeon** exchange forming **Regge cuts.** These effects are now better understood.

$2 \to 2$ amplitudes: signature and reality properties



- Regge theory is based on expressing the t-channel amplitude as a sum over states with a given angular momentum ℓ, and analytically continuing to the s channel. The latter requires separating between even and odd values of ℓ, leading to **even/odd signature**.

- Defining **signature** *even* and *odd* amplitudes under $s \leftrightarrow u$

$$\mathcal{M}^{(\pm)}(s,t) = \tfrac{1}{2}\Big(\mathcal{M}(s,t) \pm \mathcal{M}(-s-t,t)\Big)$$

- The spectral representation of the amplitude implies:

$$\mathcal{M}^{(+)}(s,t) = i\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{2\sin(\pi j/2)}\, a_j^{(+)}(t)\, e^{jL}\,,$$

$$\mathcal{M}^{(-)}(s,t) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{2\cos(\pi j/2)}\, a_j^{(-)}(t)\, e^{jL}\,,$$

with $\left(a^{\pm}_{j^*}(t)\right)^* = a^{\pm}_j(t)$

$$L \equiv \log\left|\frac{s}{t}\right| - i\frac{\pi}{2} = \frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right)$$

- Expanding the amplitude in the signature-symmetric log, L, the coefficients in $\mathcal{M}^{(+)}$ are imaginary, while in $\mathcal{M}^{(-)}$ real.

[See 1701.05241 Caron-Huot, EG, Vernazza]

The singularity structure of 2 → 2 amplitudes in the complex angular momentum plane: pole vs. cut

- The signature-odd amplitude admits

$$\mathcal{M}^{(-)}(s,t) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{2\cos(\pi j/2)}\, a_j^{(-)}(t)\, e^{jL}\,,$$

	singularity	amplitude asymptotics
pole	$a_j^{(-)}(t) \simeq \frac{1}{j-1-\alpha(t)}$	$\mathcal{M}^{(-)}(s,t)\vert_{\text{Regge pole}} \simeq \frac{\pi}{\sin\frac{\pi\,\alpha(t)}{2}}\frac{s}{t}\, e^{L\,\alpha(t)} + \ldots,$
cut	$a_j^{(-)}(t) \simeq \frac{1}{(j-1-\alpha(t))^{1+\beta(t)}}$	$\mathcal{M}^{(-)}(s,t)\vert_{\text{Regge cut}} \simeq \frac{\pi}{\sin\frac{\pi\,\alpha(t)}{2}}\frac{s}{t}\frac{1}{\Gamma\left(1+\beta(t)\right)} L^{\beta(t)}\, e^{L\,\alpha(t)} + \text{subleading logs}$

- Reggeization of the **signature-odd** amplitude (NLL): a manifestation of a pure **Regge pole**.

Signature, number of Reggeons and t-channel colour flow

- The signature odd and even sectors decouple

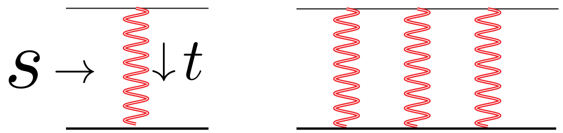
$$\mathcal{M}_{ij\to ij} \xrightarrow{\text{Regge}} \mathcal{M}^{(-)}_{ij\to ij} + \mathcal{M}^{(+)}_{ij\to ij}$$

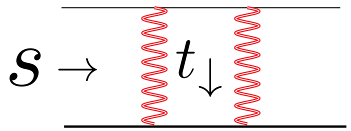
- odd/even signature amplitude is governed by the exchange of an odd/even number of Reggeons.

- Bose symmetry in $gg \to gg$ correlates odd/even signature with odd/even colour representations in the t channel.

$(\mathbf{8}\otimes\mathbf{8})_{gg}$

odd	$8_a \oplus (10 \oplus \overline{10})$	even	$0 \oplus 1 \oplus 8_s \oplus 27$





More generally we use channel colour operators: $\mathbf{T}_t^2$ is even, $\mathbf{T}_{s-u}^2 \equiv \frac{\mathbf{T}_s^2 - \mathbf{T}_u^2}{2}$ is odd

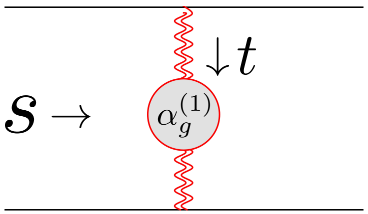
Signature-odd amplitudes: Regge-pole factorisation and its breaking

Regge factorization and violation:

$$\mathcal{M}^{(-)}_{ij\to ij} = C_i(t)\, e^{\alpha_g(t)\, C_A\, L}\, C_j(t) \mathcal{M}^{\text{tree}}_{ij\to ij} + \text{MR}$$

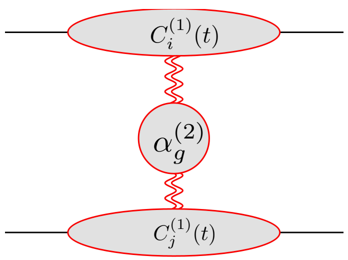
Colour **octet** exchange in the t channel: single Reggeon

✓ LL $\alpha_s^n \log^n\left(\frac{s}{-t}\right)$



one-loop Regge trajectory

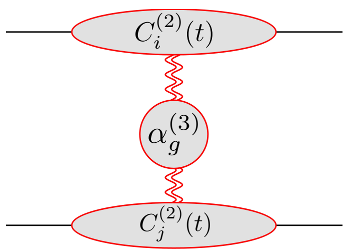
✓ NLL $\alpha_s^n \log^{n-1}\left(\frac{s}{-t}\right)$



two-loop Regge trajectory

one-loop impact factors

✗ NNLL $\alpha_s^n \log^{n-2}\left(\frac{s}{-t}\right)$



three-loop Regge trajectory

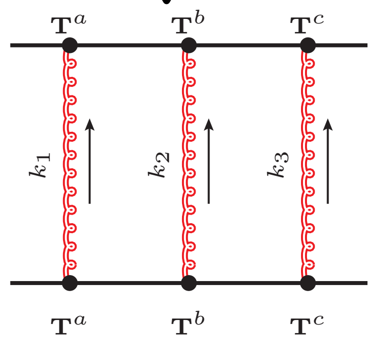
two-loop impact factors

Regge factorisation breaking (starting at 2 loops) can be inferred from comparing gg→gg, qg→qg, qq→qq amplitudes

[Del Duca, Glover '01]

[Del Duca, Falcioni, Magnea, Vernazza '14]

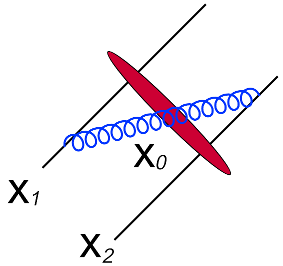
But until recently unknown how to account for it

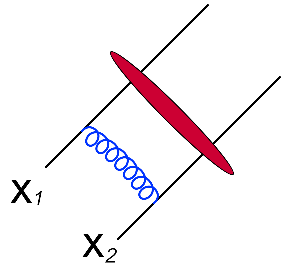


The shock-wave formalism and non-linear rapidity evolution

- The colliding particles are replaced by (sets of) infinite lightlike Wilson lines

$$U(\mathbf{x}) = \mathcal{P}\exp\left\{ig_s\int_{-\infty}^{\infty}dx^+ A^a_+(x^+, x^- = 0; \mathbf{x})T^a\right\}$$





- Rapidity evolution equation [Balitsky-JIMWLK]

$$-\frac{d}{d\eta}\left[U(\mathbf{x}_1)\dots U(\mathbf{x}_n)\right] = H\left[U(\mathbf{x}_1)\dots U(\mathbf{x}_n)\right]$$

$$H = \frac{\alpha_s}{2\pi^2}\int d\mathbf{x}_i d\mathbf{x}_j d\mathbf{x}_0 \frac{\mathbf{x}_{0i}\cdot\mathbf{x}_{0j}}{\mathbf{x}_{0i}^2\mathbf{x}_{0j}^2}\left(T^a_{i,L}T^a_{j,L} + T^a_{i,R}T^a_{j,R} - U^{ab}_{\text{adj}}(\mathbf{x}_0)(T^a_{i,L}T^b_{j,R} + T^a_{j,L}T^b_{i,R})\right)$$

$$T^a_{i,L} \equiv T^a U(\mathbf{x}_i)\frac{\delta}{\delta U(\mathbf{x}_i)}, \qquad T^a_{i,R} \equiv U(\mathbf{x}_i)T^a\frac{\delta}{\delta U(\mathbf{x}_i)}$$

Provides complete separation between the light-cone directions and the transverse plane: **2-dimensional dynamics**

Towards an effective theory: Defining the Reggeon

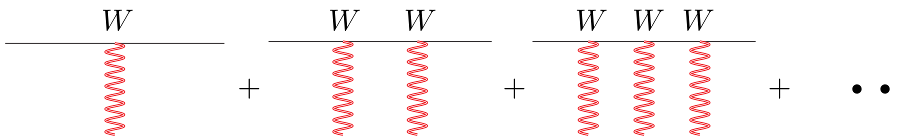
- In the perturbative regime $U(\mathbf{x}) \simeq 1$ it is natural to expand in terms of W Simon Caron-Huot (2013)

$$U(\mathbf{x}) = \mathcal{P} \exp \left\{ i g_s \int_{-\infty}^{\infty} dx^+ A_+^a(x^+, x^- = 0; \mathbf{x}) T^a \right\} = e^{i g_s T^a W^a(\mathbf{x})} .$$

W sources a Reggeon

- Scattered particles are expanded in states of a definite number of Reggeons

$$|\psi_i\rangle \equiv \frac{Z_i^{-1}}{2p_1^+} a_i(p_4) a_i^\dagger(p_1)|0\rangle \sim g_s\,|W\rangle + g_s^2\,|WW\rangle + g_s^3\,|WWW\rangle + \dots =$$

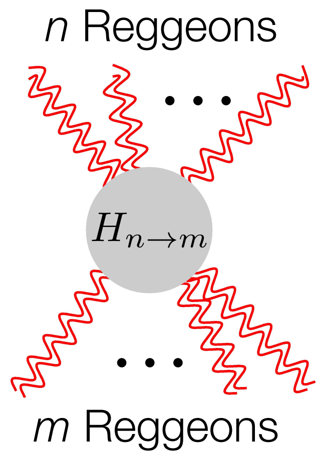


- Amplitudes are governed by rapidity evolution between the target and projectile:

$$\frac{i(Z_i Z_j)^{-1}}{2s} \mathcal{M}_{ij \to ij} = \langle \psi_j | e^{-HL} | \psi_i \rangle$$

$$-\frac{d}{d\eta}\,|\psi_i\rangle = H\,|\psi_i\rangle$$

$$H \begin{pmatrix} W \\ WW \\ WWW \\ \dots \end{pmatrix} \equiv \begin{pmatrix} H_{1\to1} & 0 & H_{3\to1} & \dots \\ 0 & H_{2\to2} & 0 & \dots \\ H_{1\to3} & 0 & H_{3\to3} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} W \\ WW \\ WWW \\ \dots \end{pmatrix}$$

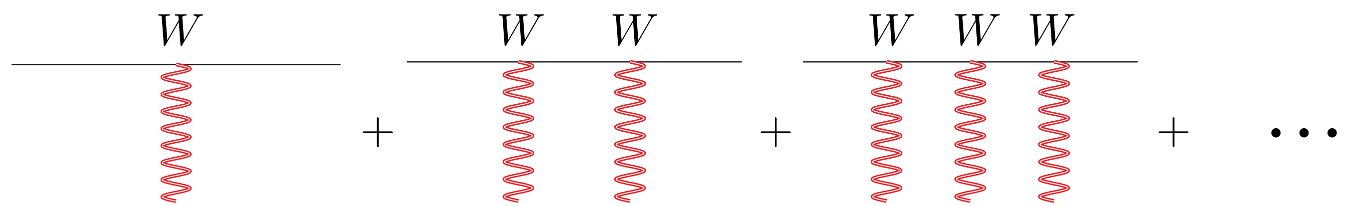


- Each action of the Hamiltonian generates an extra power of the high-energy log L

Computing multi-Regge exchanges using non-linear rapidity evolution

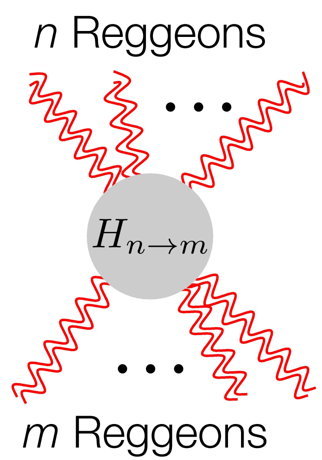
1701.05241 Caron-Huot, EG, Vernazza

Projectile $|\psi_i\rangle =$



n Reggeons to *m* Reggeons transition Hamiltonian [1701.05241]

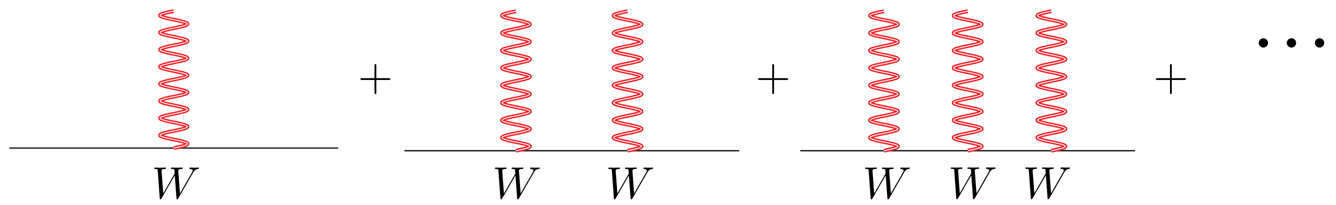
$\sum_{n,m}$



$$H\begin{pmatrix} W \\ WW \\ WWW \\ \cdots \end{pmatrix} \equiv \begin{pmatrix} H_{1\to1} & 0 & H_{3\to1} & \cdots \\ 0 & H_{2\to2} & 0 & \cdots \\ H_{1\to3} & 0 & H_{3\to3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}\begin{pmatrix} W \\ WW \\ WWW \\ \cdots \end{pmatrix}$$

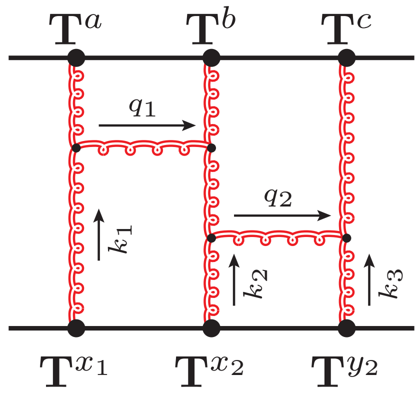
$$\sim \begin{pmatrix} g_s^2 & 0 & g_s^4 & \cdots \\ 0 & g_s^2 & 0 & \cdots \\ g_s^4 & 0 & g_s^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}\begin{pmatrix} W \\ WW \\ WWW \\ \cdots \end{pmatrix}$$

Target $\langle\psi_j| =$



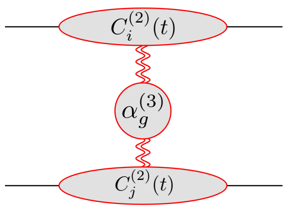
Signature-odd 2 → 2 amplitudes: understanding the NNLL tower

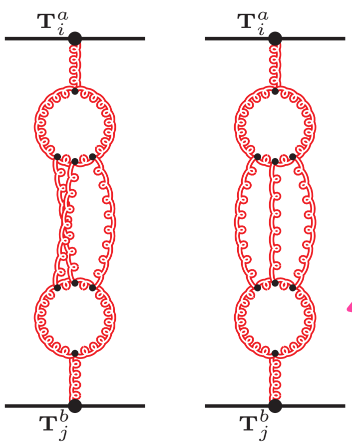
- Using **non-linear** rapidity evolution, the NNLL tower is determined to all orders in terms of **one** and **three** Reggeon exchanges

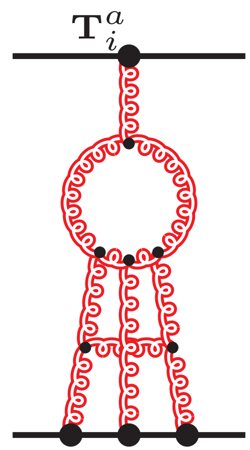


- Expanding in $X \equiv \dfrac{\alpha_s}{\pi} r_\Gamma L$

$$\frac{i}{2s}\hat{\mathcal{M}}^{(-),\mathrm{NNLL}}_{ij\to ij} = \left(\frac{\alpha_s}{\pi}\right)^2 \Bigg\{ \langle j_1|i_1\rangle^{\mathrm{NNLO}} + r_\Gamma^2\pi^2 \sum_{k=0}^{\infty} \frac{(-X)^k}{k!}\Bigg[\langle j_3|\hat{H}^k_{3\to3}|i_3\rangle + \Theta(k\ge1)\Big[\langle j_1|\hat{H}_{3\to1}\hat{H}^{k-1}_{3\to3}|i_3\rangle + \langle j_3|\hat{H}^{k-1}_{3\to3}\hat{H}_{1\to3}|i_1\rangle\Big] + \Theta(k\ge2)\langle j_1|\hat{H}_{3\to1}\hat{H}^{k-2}_{3\to3}\hat{H}_{1\to3}|i_1\rangle\Bigg]^{\mathrm{LO}}\Bigg\}$$







- All diagrams computed to four loops

Caron-Huot, EG, Vernazza
JHEP 06 (2017) 016
Falcioni, EG, Milloy, Vernazza
Phys. Rev. D 103 (2021) L111501

Signature odd 2 → 2 amplitude at NNLL: Regge pole and cut

Requiring that the Regge cut is strictly non-planar fixes the separation between Regge pole vs. Regge cut

$$\mathcal{M}^{(-)}_{ij\to ij} = \underbrace{\mathcal{M}^{(-)\,\mathrm{SR}}_{ij\to ij} + \mathcal{M}^{(-)\,\mathrm{MR}}_{ij\to ij}\Big|_{\mathrm{planar}}} + \mathcal{M}^{(-)\,\mathrm{MR}}_{ij\to ij}\Big|_{\mathrm{nonplanar}}$$

$$= \mathcal{M}^{(-)\,\mathrm{pole}}_{ij\to ij} + \mathcal{M}^{(-)\,\mathrm{cut}}_{ij\to ij}$$

$$\mathcal{M}^{(-)\,\mathrm{pole}}_{ij\to ij} = C_i(t)\, e^{\alpha_g(t)\, C_A\, L}\, C_j(t) \mathcal{M}^{\mathrm{tree}}_{ij\to ij}$$

Falcioni, EG, Maher, Milloy, Vernazza
Phys.Rev.Lett. 128 (2022) 13, 13;
JHEP 03 (2022) 053

$\mathcal{M}^{(-)\,\mathrm{MR}}_{ij\to ij}\Big|_{\mathrm{planar}}$ must be **universal** (gg, gq, qq) to be absorbed in the factorizing pole term.

$\mathcal{M}^{(-)\,\mathrm{MR}}_{ij\to ij}\Big|_{\mathrm{planar}}$ ***cannot*** contribute beyond 3 loops: the NNLL Regge pole term has **no** free parameters!

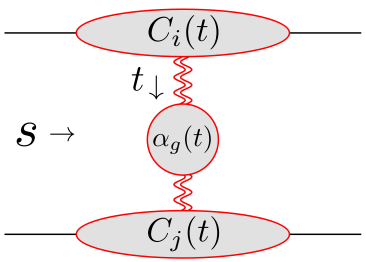
Indeed, at 4 loops **planar Multi Regge contributions conspire to cancel!**

Signature odd amplitude at NNLL: Regge pole and cut properties

All-order structure through NNLL for any gauge theory, any representation:

$$\mathcal{M}^{(-)}_{ij\to ij} = Z_i(t)\,\bar{D}_i(t)\,Z_j(t)\,\bar{D}_j(t)\left[\left(\frac{-s}{-t}\right)^{C_A\alpha_g(t)} + \left(\frac{-u}{-t}\right)^{C_A\alpha_g(t)}\right]\mathcal{M}^{\text{tree}}_{ij\to ij} + \sum_{n=2}^{\infty} a^n L^{n-2}\mathcal{M}^{(\pm,n,n-2)\,\text{cut}}$$

Regge pole-factorized



- ✓ single Reggeon; colour octet
- ✓ dominant in planar limit
- ✓ Trajectory and impact factors at NNLL are fully fixed by matching to (qq, gg, qg) scattering amplitudes*

Regge cut: breaks factorization

- ✓ multiple Reggeons; various colour reps.
- ✓ suppressed in planar limit
- ✓ proportional to $(i\pi)^2$
- ✓ no dependence on the matter content: the same for any gauge theory!
- ✓ Sensitive to soft singularities beyond the dipole formula.

* 3-loop Amplitudes: Caola, Chakraborty, Gambuti, von Manteuffel, Tancredi, JHEP 10 (2021) 206]
Caola et al. Phys.Rev.Lett. 128 (2022) 21, 21

Falcioni, EG, Maher, Milloy, Vernazza, Phys.Rev.Lett. 128 (2022) 13, 13; *JHEP* 03 (2022) 053

Regge-pole factorisation for multi-leg amplitudes in MRK

Multi-Regge Kinematics (MRK)

4-momentum $p = (p^+, p^-; \mathbf{p})$

target $p_1 = (0, p_1^-; \mathbf{0})$

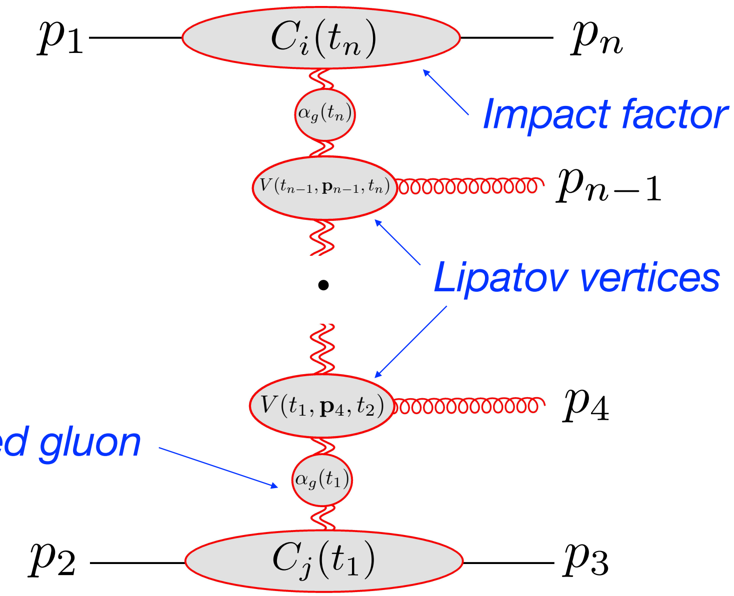
projectile $p_2 = (p_2^+, 0; \mathbf{0})$

strong hierarchy of light-cone components $p_3^+ \gg p_4^+ \gg ... \gg p_n^+$

$p_3^- \ll p_4^- \ll ... \ll p_n^-$

no ordering of transverse components $|\mathbf{p}_3| \sim |\mathbf{p}_4| \sim ... \sim |\mathbf{p}_n|$

Regge (pole) factorization in MRK



Regge (pole) factorization holds in MRK for the dispersive (real part) of the amplitudes through NLL; established using unitarity [Fadin et al. 2006]

Planar limit:

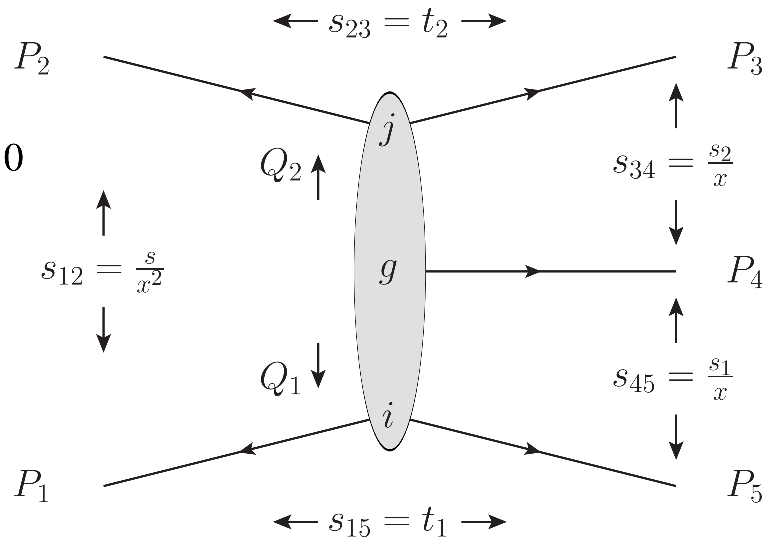
- Four- and five-point planar amplitudes have only Regge poles. Essential for the BDS ansatz in SYM.
- Six and higher-point planar amplitudes have also Regge cuts in some special kinematic regions [Bartels, Lipatov, Sabio Vera (2008)]. All multiplicity planar results are available [Del Duca *et al.* (2019)]

$2 \to 3$ amplitudes in multi-Regge kinematics

Del Duca and Schmidt (1998); Del Duca, Duhr, Glover (2009);
Caron-Huot, Chicherin, Henn, Zhang, Zoia, JHEP 10 (2020) 188;
Fadin, Fucilla, Papa (2023)

- Multi-Regge kinematics:

$$s_{12} \to \frac{s_{12}}{x^2} \qquad s_{45} \to \frac{s_1}{x} \qquad s_{34} \to \frac{s_2}{x} \qquad s_{15} \to t_1 \qquad s_{23} \to t_2 \qquad \text{for} \quad x \to 0$$



- Signature symmetry operations:

$$\begin{aligned} (1 \leftrightarrow 5) \quad &\to \quad \{s \to -s, \quad s_{45} \to -s_{45}\}, \\ (2 \leftrightarrow 3) \quad &\to \quad \{s \to -s, \quad s_{34} \to -s_{34}\}. \end{aligned}$$

- t-channel colour basis:
  diagonal operators:

$$\mathbf{T}_{t_1}^2 \equiv (\mathbf{T}_1 + \mathbf{T}_5)^2$$
$$\mathbf{T}_{t_2}^2 \equiv (\mathbf{T}_2 + \mathbf{T}_3)^2$$

Signature-preserving operator on line i, j: $\quad \mathbf{T}_{(++)} = (\mathbf{T}_1^a + \mathbf{T}_5^a) \cdot (\mathbf{T}_2^a + \mathbf{T}_3^a)$,

Signature-preserving on line i, inverting on j: $\quad \mathbf{T}_{(+-)} = (\mathbf{T}_1^a + \mathbf{T}_5^a) \cdot (\mathbf{T}_2^a - \mathbf{T}_3^a)$,

Signature-preserving on line j, inverting on i: $\quad \mathbf{T}_{(-+)} = (\mathbf{T}_1^a - \mathbf{T}_5^a) \cdot (\mathbf{T}_2^a + \mathbf{T}_3^a)$,

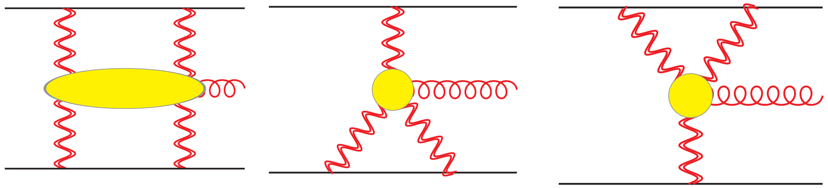
Signature-inverting operator on lines i, j: $\quad \mathbf{T}_{(--)} = (\mathbf{T}_1^a - \mathbf{T}_5^a) \cdot (\mathbf{T}_2^a - \mathbf{T}_3^a)$.

2 → 3 amplitudes at one loop: multi-Reggeon contributions

- A new feature compared to 2 → 2 scattering: even and odd signature mix

  These have now been eventuated also in the effective multi-Reggeon framework

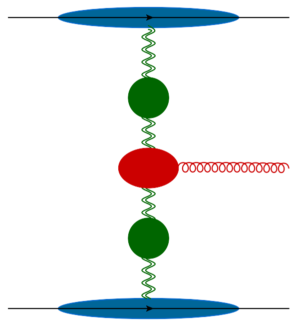
$$\mathcal{M}^{\mathrm{MR}\,(1)}_{ij\to i'gj'} = \mathcal{M}^{(1)}_{\mathcal{R}^2 g \mathcal{R}^2} + \mathcal{M}^{(1)}_{\mathcal{R} g \mathcal{R}^2} + \mathcal{M}^{(1)}_{\mathcal{R}^2 g \mathcal{R}}$$



$$= \frac{i\pi}{4}\left\{ \frac{1}{\epsilon}\left(\mathbf{T}_{(--)} + \mathbf{T}_{(+-)} + \mathbf{T}_{(-+)}\right) + \log\frac{p_4^2}{p_3^2 p_5^2}\mathbf{T}_{(--)} + \log\frac{p_3^2}{p_4^2 p_5^2}\mathbf{T}_{(-+)} + \log\frac{p_5^2}{p_3^2 p_4^2}\mathbf{T}_{(+-)} + \mathcal{O}(\epsilon)\right\}\mathcal{M}^{\mathrm{tree}}_{ij\to i'gj'}$$

- As in 2 → 2 scattering, at one-loop multi-Reggeon exchanges **do not affect** the odd-odd signature part of the amplitude, hence factorization (for the [8,8] component) holds just as at tree level:

$$\frac{\mathcal{M}^{(-,-)}_{ij\to i'gj'}\Big|^{\text{1-Reggeon}}}{\mathcal{M}^{\mathrm{tree}}_{ij\to i'gj'}} = c_i(t_1,\tau)\, e^{\omega_1\,\eta_1}\, v(t_1,t_2,\mathbf{p}_4^2,\tau)\, e^{\omega_2\,\eta_2}\, c_j(t_2,\tau)$$

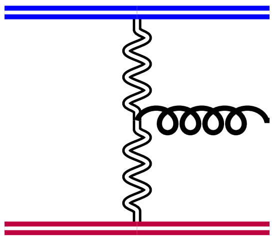


Extracting the Lipatov vertex from one-loop amplitudes

1-loop vertex extracted/computed in Del Duca and Schmidt (1998); Del Duca, Duhr, Glover (2009); Fadin, Fucilla, Papa (2023)

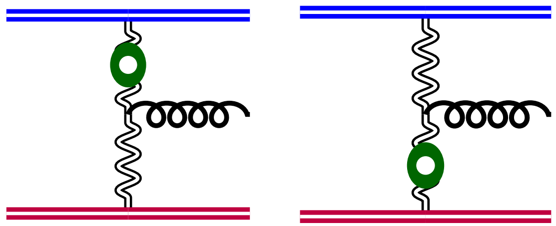
Figures from: Buccioni, Caola, Devoto, Gambuti — 2411.14050 [hep-ph]

Tree



One-loop: $(-,-)$ signature

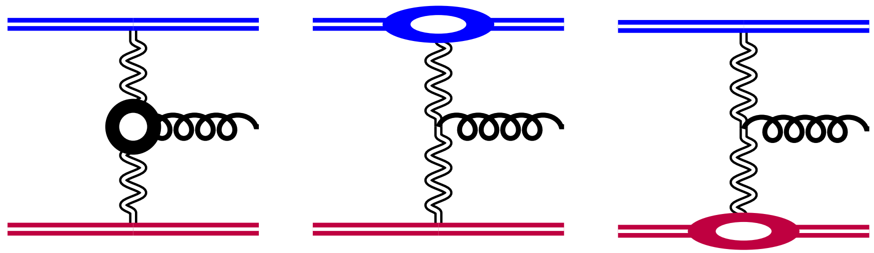
Leading Logarithms



One-loop

$(-,-)$ signature

No Logarithms ("Next to Leading")

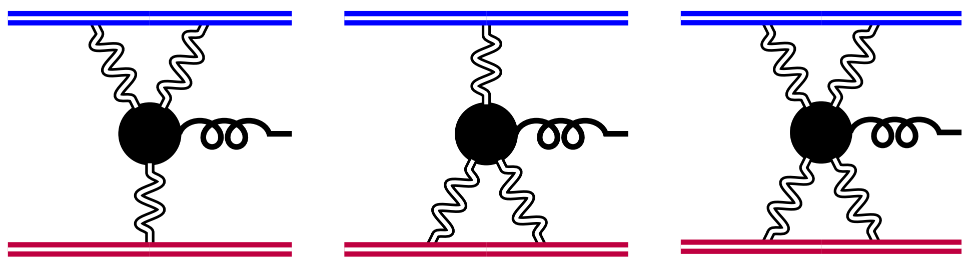


One-loop

$(-,+)$, $(+-)$, $(+,+)$ signature

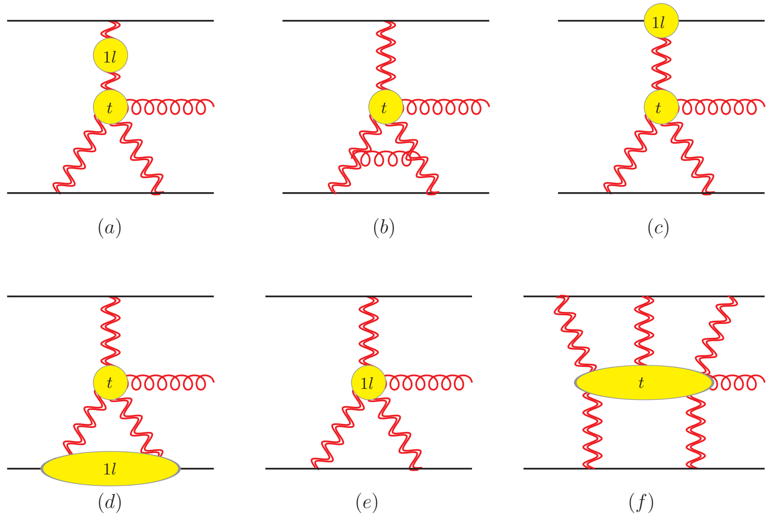
No Logarithms

No [8,8] colour component

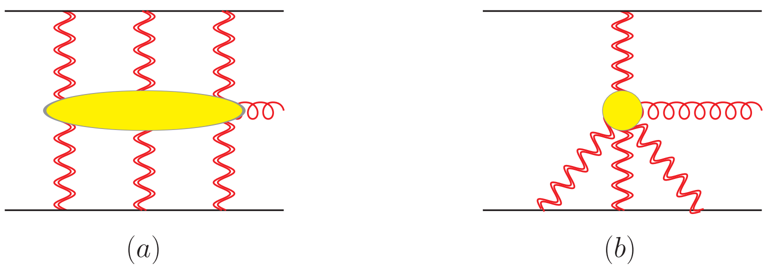


Multiple-Reggeon effect in 2 → 3 scattering: two loops

- At two loops there are many contributions of mixed odd-even signature



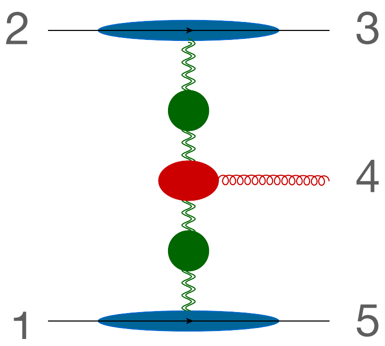
- But importantly, **there are also odd-odd contributions from multiple Reggeons**

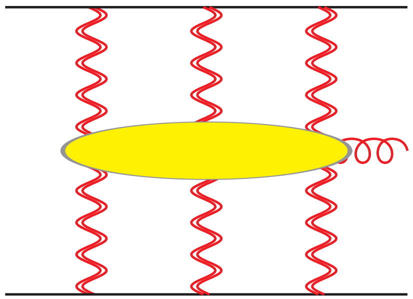


These break factorization!

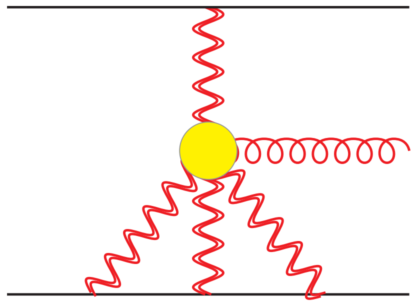
Signature odd-odd 2 → 3 amplitude at two loops

- Two-loop contributions of odd-odd signature: $\mathcal{M}^{(-,-)\,(2)}_{ij\to i'gj'} = \mathcal{M}^{(2)}_{\mathcal{R}g\mathcal{R}} + \mathcal{M}^{(2)}_{\mathcal{R}^3g\mathcal{R}^3} + \mathcal{M}^{(2)}_{\mathcal{R}g\mathcal{R}^3} + \mathcal{M}^{(2)}_{\mathcal{R}^3g\mathcal{R}}$





(a)



(b)

$$\begin{aligned} C_{\mathcal{R}^3g\mathcal{R}^3} &= \mathbf{T}_i^{\{a,b,c\}}\, i f^{ca_4d}\, \mathbf{T}_j^{\{a,b,d\}} \\ &= \frac{1}{144}\left\{9\mathbf{T}^2_{(--)} + \mathbf{T}^2_{(++)} + 4N_c\mathbf{T}_{(++)} + 3\left(\mathbf{T}^2_{(-+)} + \mathbf{T}^2_{(+-)}\right)\right\}\mathcal{C}^{(0)}_{ij} \\ &= \begin{cases} \frac{1}{72}\left(N_c^2 - 6 + \frac{18}{N_c^2}\right)\, c^{[8,8]_a} & \text{for } qq \\ \frac{1}{72}\left(N_c^2 + 6\right)\, c^{[8,8_a]_a} & \text{for } qg \\ \frac{1}{72}\left(N_c^2 + 36\right)\, c^{[8_a,8_a]} - \frac{1}{4}\sqrt{N_c^2-4}\, c^{[10,\overline{10}]_1} & \text{for } gg \end{cases} \end{aligned}$$

$$\begin{aligned} C_{\mathcal{R}g\mathcal{R}^3} &= \mathbf{T}_i^b f^{bck} f^{kge} f^{eda_4}\mathbf{T}_j^{\{c,d,g\}} \\ &= \frac{1}{24}\left(2N_c\,\mathbf{T}_{(++)} + 2(\mathbf{T}_{(++)})^2 + 6(\mathbf{T}_{(-+)})^2\right)\mathcal{C}^{(0)}_{ij} \\ &= \begin{cases} \left(\frac{N_c^2}{24} + \frac{3}{2}\right) c^{[8_a,8_a]} - \frac{3\sqrt{N_c^2-4}}{4\sqrt{2}} c^{[10+\overline{10},8_a]} & \text{for } gg \\ \left(\frac{N_c^2}{24} + \frac{1}{4}\right) c^{[8,8]_a} & \text{for } qq \\ \left(\frac{N_c^2}{24} + \frac{1}{4}\right) c^{[8,8_a]_a} & \text{for } qg \\ \left(\frac{N_c^2}{24} + \frac{3}{2}\right) c^{[8,8_a]_a} - \frac{3\sqrt{N_c^2-4}}{4\sqrt{2}} c^{[8,10+\overline{10}]} & \text{for } gq \end{cases} \end{aligned}$$

- These affects the [8,8] colour channel
- Their large-Nc limit is universal

Factorizable and non-factorizable contributions in 2 → 3 amplitudes

- The [8,8] odd-odd component of the Multi-Reggeon (MR) amplitude, splits into Regge-factorizable (planar) terms and non-factorizable terms

$$\mathcal{M}_{\mathrm{MR}}^{(2),\,[8,8]} = \frac{(i\pi)^2}{72}\left(\frac{\mu^2}{|\mathbf{p}_4|^2}\right)^{2\epsilon}\mathcal{M}^{(0),\,[8,8]} \times \begin{cases} (N_c^2+36)F_{\mathrm{fact}}(z,\bar{z}) & \text{for } gg \\ N_c^2 F_{\mathrm{fact}}(z,\bar{z}) \,+\, F^{qq}_{\text{non-fact}}(z,\bar{z}) & \text{for } qq \\ N_c^2 F_{\mathrm{fact}}(z,\bar{z}) \,+\, F^{qg}_{\text{non-fact}}(z,\bar{z}) & \text{for } qg \end{cases}$$

$$F_{\mathrm{fact}}(z,\bar{z}) = \frac{1}{\epsilon^2} - \frac{1}{2\epsilon}\log|z|^2|1-z|^2 + 3\,D_2(z,\bar{z}) - \zeta_2 + \frac{5}{4}\log^2|z|^2 + \frac{5}{4}\log^2|1-z|^2 - \frac{1}{2}\log|z|^2\log|1-z|^2$$

$D_2(z,\bar{z})$: Block-Wigner Dilogarithm

$$F^{qq}_{\text{non-fact}} = \frac{9}{\epsilon}\log|z|^2|1-z|^2 + \frac{9}{2}\Big(12D_2(z,\bar{z}) - \log^2|z|^2 - 2\log|z|^2\log|1-z|^2 - \log^2|1-z|^2\Big)$$
$$+\frac{3}{N_c^2}\left(\frac{3}{\epsilon^2} - \frac{6}{\epsilon}\log|z|^2|1-z|^2 - 18D_2(z,\bar{z}) + 6\log^2|z|^2 + 3\log|z|^2\log|1-z|^2 + 6\log^2|1-z|^2 - \frac{\pi^2}{2}\right)$$

$$F^{qg}_{\text{non-fact}} = \frac{27}{2\epsilon^2} - \frac{9}{\epsilon}\Big(2\log|z|^2 - 3\log|1-z|^2\Big) + \frac{9}{4}\Big(48D_2(z,\bar{z}) + 10\log^2|z|^2 - 8\log|z|^2\log|1-z|^2 - \pi^2\Big)$$

Abreu, Falcioni, EG, Milloy and Vernazza — Regge poles and cuts and the Lipatov vertex *PoS* LL2024 (2024) 085

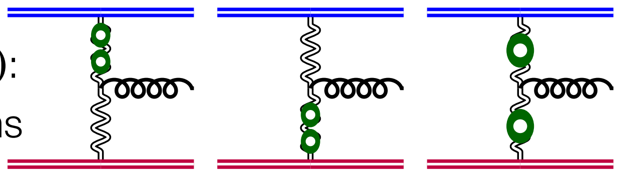
Buccioni, Caola, Devoto, Gambuti — 2411.14050 [hep-ph]

Abreu, De Laurentis, Falcioni, EG, Milloy and Vernazza — to appear

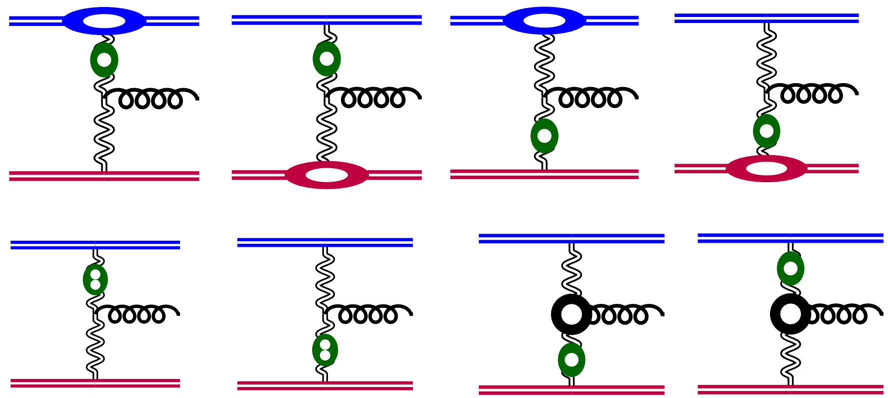
Extracting the Lipatov vertex from 2 → 3 $(-,-)$ signature amplitudes

Two-loop amplitudes are available since last year: G. De Laurentis, H. Ita , M. Klinkert, V. Sotnikov 2311.10086, 2311.18752,
B. Agarwal, F. Buccioni, F. Devoto, G. Gambuti, A. von Manteuffel, L. Tancredi, 2311.09870

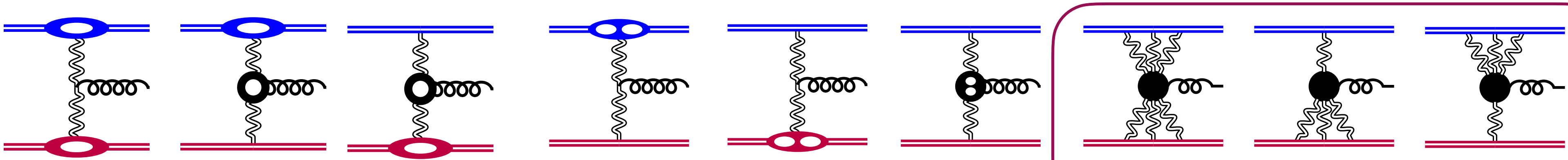
Two-loop $(-,-)$: Leading Logarithms



Two-loop $(-,-)$: NLL



Two-loop $(-,-)$: Next-to-next-to Leading Logarithms



Multi-Reggeon [8,8] contributions

Robust check: **we obtained the (same) expression for the 2-loop QCD Lipatov Vertex from all three partonic channels!**

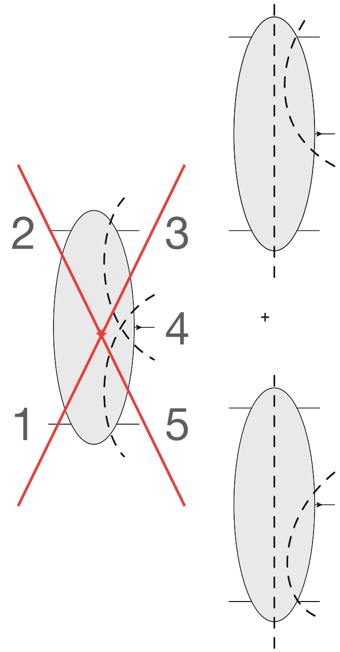
Abreu, Falcioni, EG, Milloy and Vernazza — Regge poles and cuts and the Lipatov vertex *PoS* LL2024 (2024) 085

Buccioni, Caola, Devoto, Gambuti — 2411.14050 [hep-ph]; Abreu, De Laurentis, Falcioni, EG, Milloy and Vernazza: to appear

Odd-Odd 2 → 3 amplitude: discontinuity structure

- Steinmann relations forbid unitarity cuts in partially overlapping channels.

- Allowed iterated discontinuities: s_{12} and s_{45} or s_{12} and s_{34}
  compatible with the signature



- All-order factorization formula for 2 → 3 amplitudes in Multi-Regge kinematics in terms of two real-valued vertex functions

$$\frac{\mathcal{M}^{(-,-)}_{ij\to i'gj'}\Big|^{\text{1-Reggeon}}}{\mathcal{M}^{\text{tree}}_{ij\to i'gj'}} = c_i(t_1,\tau)\frac{1}{4}\Bigg\{\left[\left(\frac{s_{34}}{\tau}\right)^{\omega_2-\omega_1}+\left(\frac{-s_{34}}{\tau}\right)^{\omega_2-\omega_1}\right]\left[\left(\frac{s}{\tau}\right)^{\omega_1}+\left(\frac{-s}{\tau}\right)^{\omega_1}\right]v_R(t_1,t_2,|\mathbf{p}_4|^2,\tau)+ \left[\left(\frac{s_{45}}{\tau}\right)^{\omega_1-\omega_2}+\left(\frac{-s_{45}}{\tau}\right)^{\omega_1-\omega_2}\right]\left[\left(\frac{s}{\tau}\right)^{\omega_2}+\left(\frac{-s}{\tau}\right)^{\omega_2}\right]v_L(t_1,t_2,|\mathbf{p}_4|^2,\tau)\Bigg\}c_j(t_2,\tau)$$

[Bartels (1980), Fadin and Lipatov (1993),…, Fadin, Fucilla, Papa (2023)]

Odd-Odd 2 → 3 amplitude

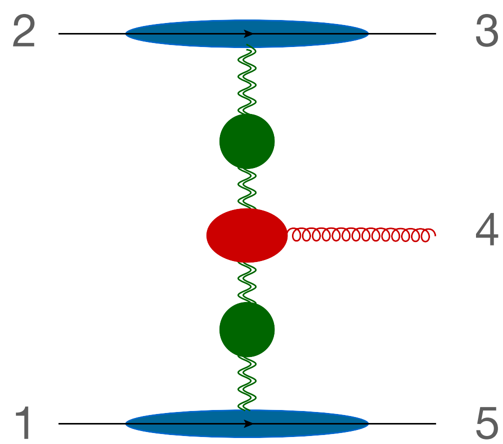
- All-order factorization formula for 2 → 3 amplitudes in Multi-Regge kinematics in terms of two real-valued vertex functions v_R, v_L

$$\omega_1 = C_A \alpha_g(t_1), \qquad \omega_2 = C_A \alpha_g(t_2)$$

$$\frac{\mathcal{M}^{(-,-)}_{ij\to i'gj'}\Big|^{\text{1-Reggeon}}}{\mathcal{M}^{\text{tree}}_{ij\to i'gj'}} = c_i(t_1,\tau)\frac{1}{4}\Bigg\{\left[\left(\frac{s_{34}}{\tau}\right)^{\omega_2-\omega_1}+\left(\frac{-s_{34}}{\tau}\right)^{\omega_2-\omega_1}\right]\left[\left(\frac{s}{\tau}\right)^{\omega_1}+\left(\frac{-s}{\tau}\right)^{\omega_1}\right]v_R(t_1,t_2,|\mathbf{p}_4|^2,\tau)+$$
$$\left[\left(\frac{s_{45}}{\tau}\right)^{\omega_1-\omega_2}+\left(\frac{-s_{45}}{\tau}\right)^{\omega_1-\omega_2}\right]\left[\left(\frac{s}{\tau}\right)^{\omega_2}+\left(\frac{-s}{\tau}\right)^{\omega_2}\right]v_L(t_1,t_2,|\mathbf{p}_4|^2,\tau)\Bigg\}c_j(t_2,\tau)$$

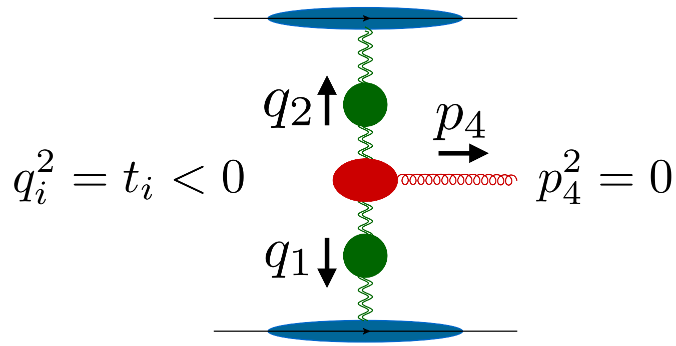
- **Equivalently**: a single complex-valued vertex rapidity variables absorb a phase: $\eta_1 = \log\frac{s_{45}}{\tau} - \frac{i\pi}{4}, \qquad \eta_2 = \log\frac{s_{34}}{\tau} - \frac{i\pi}{4}$

$$\frac{\mathcal{M}^{(-,-)}_{ij\to i'gj'}\Big|^{\text{1-Reggeon}}}{\mathcal{M}^{\text{tree}}_{ij\to i'gj'}} = c_i(t_1,\tau)\, e^{\omega_1\,\eta_1}\, v(t_1,t_2,\mathbf{p}_4^2,\tau)\, e^{\omega_2\,\eta_2}\, c_j(t_2,\tau)$$



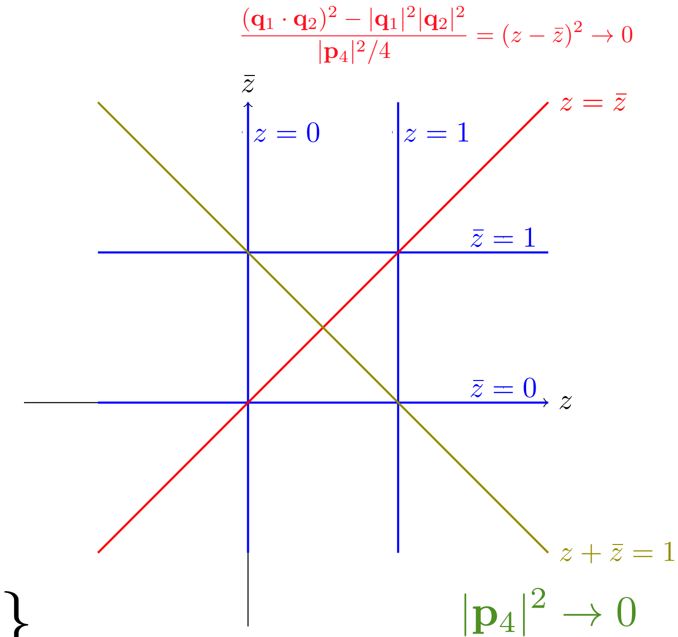
Complex-valued vertex: properties

$$v(t_1, t_2, |\mathbf{p}_4|^2, \tau) = \frac{\mathcal{M}^{(-,-)}_{ij\to i'gj'}\Big|^{\text{1-Reggeon}}}{c_i(t_1,\tau)\, e^{\omega_1\,\eta_1}\, e^{\omega_2\,\eta_2}\, c_j(t_2,\tau)\mathcal{M}^{\text{tree}}_{ij\to i'gj'}}$$



- 2-dim momenta: $\frac{-t_1}{|\mathbf{p}_4|^2} = (1-z)(1-\bar{z}), \qquad \frac{-t_2}{|\mathbf{p}_4|^2} = z\bar{z}$

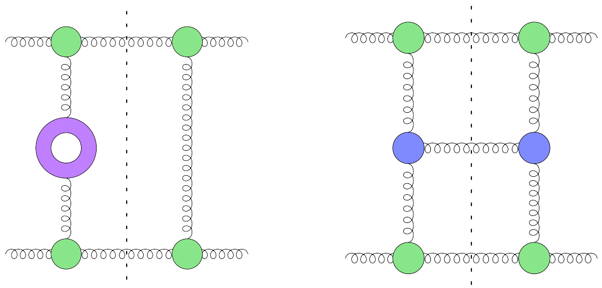
$$v(t_1, t_2, |\mathbf{p}_4|^2, \tau) = v(z, \bar{z}) \left(\frac{\tau}{|\mathbf{p}_4|^2}\right)^{\frac{1}{2}(\omega_1+\omega_2)}$$



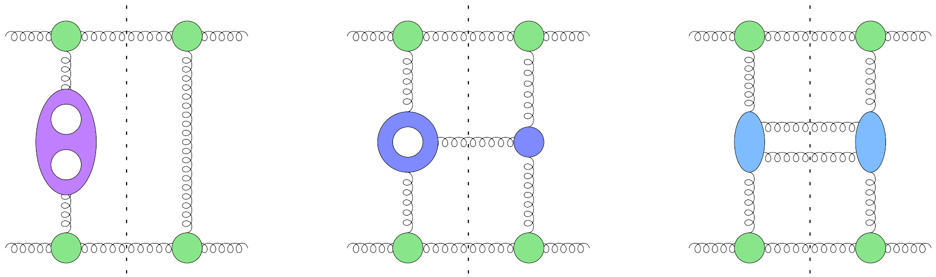
- Absence of discontinuities in physical kinematics ($z = \bar{z}^*$) implies that the transcendental functions $f(z, \bar{z})$ in the complex vertex should be **Single-Valued GPLs**
- The **reality** of v_L and v_R and **target-projectile symmetry** imply definite $z \leftrightarrow \bar{z}$ and $z \leftrightarrow 1-z$ symmetry properties
- Symbol alphabet: $\{z,\ \bar{z},\ 1-z,\ 1-\bar{z},\ z-\bar{z},\ 1-z-\bar{z}\}$
- Rational factors may have <u>spurious</u> singularities on the lines $z = \bar{z},\ z + \bar{z} = 1$

Contributions to BFKL kernel at increasing logarithmic accuracy

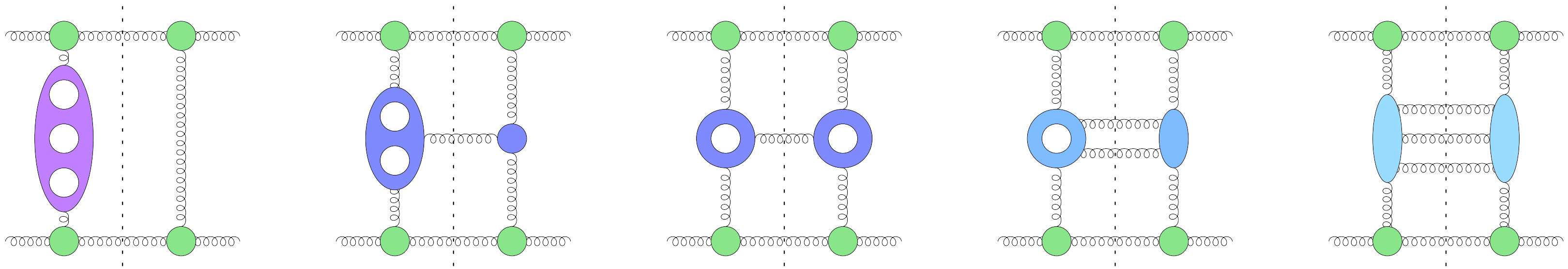
Leading
Order



Next to
Leading Order



Next to next to
Leading Order



Multiple parton Central Emission Vertices at : 2-gluon CeV in sYM: Byrne, Del Duca, Dixon, EG, Smillie 2204.12459
and on-going work with De Laurentis, Byrne, Del Duca, Mo, Smillie

Conclusions

(1) Rapidity evolution equations (2 dim!) facilitate efficient computation in the (multi) Regge limit

NLL for signature even 2 → 2 amplitudes (all orders)

NNLL for signature odd 2 → 2 amplitudes (so far to four loops)

NNLL for signature odd-odd 2 → 3 amplitudes (so far to two loops)

(2) **Regge-pole factorization violations in 2 → 2 and 2 → 3 amplitudes -** Regge cut contributions **- are non-planar**

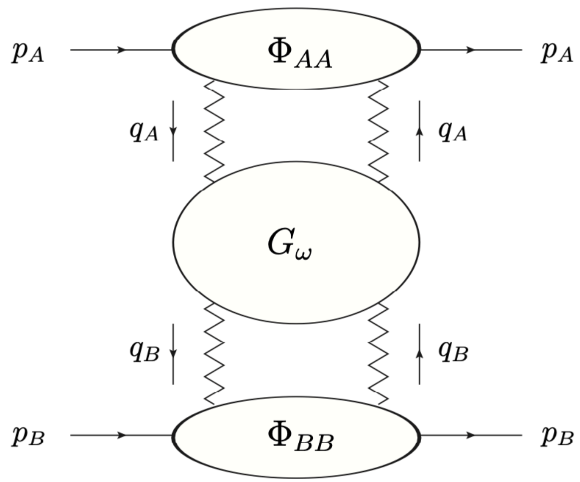
(3) Based on (1), (2) and recent 3-loop 4-point calculations **we now know all Regge-pole parameters to NNLO.**

(4) Based on (1), (2) and (3) and recent 2-loop 5-point calculations **we determined the 2-loop** Lipatov vertex in QCD**.**

This vertex is one of the building blocks of **NNLO BFKL Kernel**. The remaining ones will be available soon.

Great prospects to further exploiting the interplay between the Regge limit, fixed-order computations and the study of IR singularities

k_T factorization in forward elastic scattering

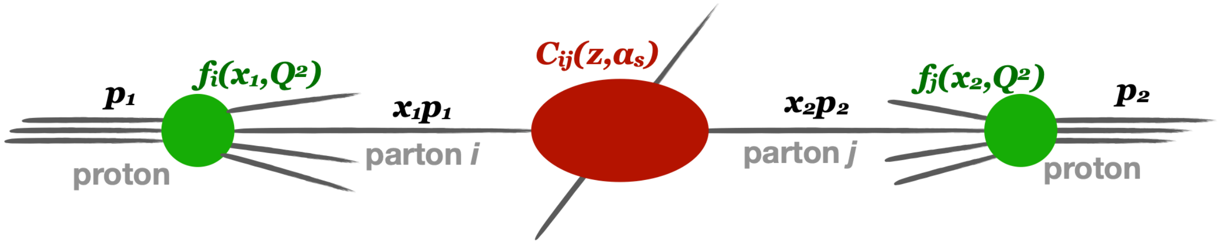


$$\sigma(s) = \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} \left(\frac{s}{\tau}\right)^{\omega} \int d^{D-2}\mathbf{q_A} \int d^{D-2}\mathbf{q_B} \frac{\Phi_A(\mathbf{q_A})}{|\mathbf{q_A}|^4} G_\omega(\mathbf{q_A}, \mathbf{q_B}) \frac{\Phi_B(\mathbf{q_B})}{|\mathbf{q_B}|^4}$$

Figure by Michael Fucilla

Collinear factorization

Separation of process-dependent short-distance physics (perturbative partonic cross sections) and universal long-distance physics (non-perturbative Parton Density Functions) in hard scattering:



Factorization schematic by Federico Silvetti

$$\sigma(s,Q^2) = \sum_{i,j} \int dx_1 dx_2\, \hat{\sigma}_{ij}\left(\frac{Q^2}{s x_1 x_2}, \frac{Q^2}{\mu_F^2}\right) f_i(x_1,\mu_F^2)\, f_j(x_2,\mu_F^2) + \frac{\Lambda_{\rm QCD}^2}{Q^2}$$

Proven to all order orders for inclusive measurements (e.g. Drell-Yan)

[Collins-Soper-Sterman, hep-ph/0409313]

DGLAP evolution resums logarithms of Q^2/μ_F^2, but effects due to large rapidity separations appear in both partonic cross sections and PDFs

TMD factorization

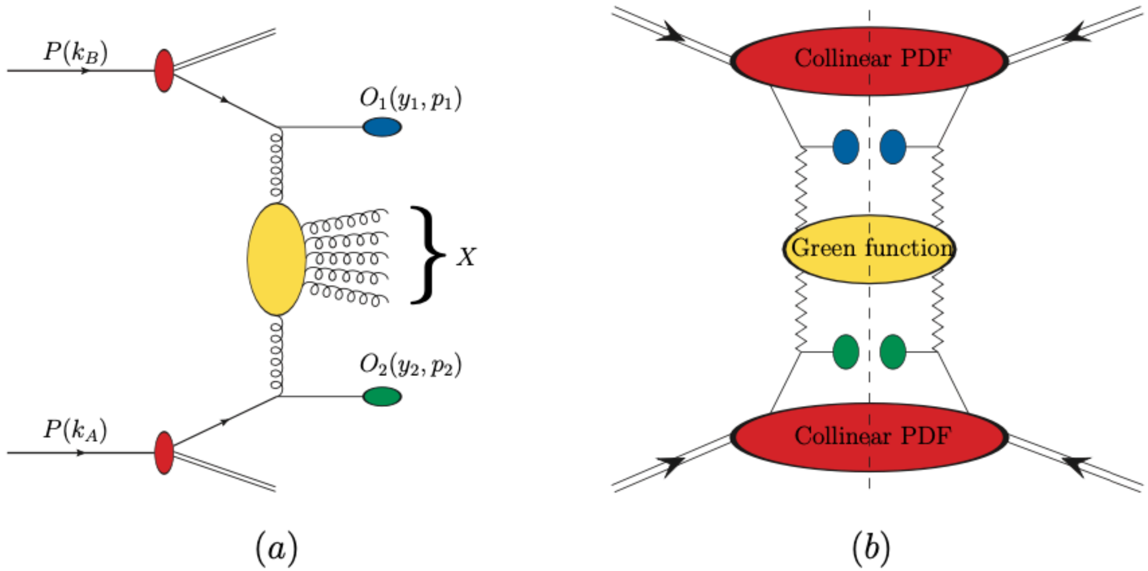


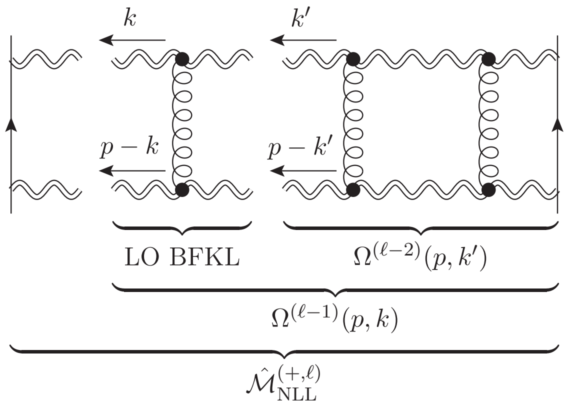
Figure by Michael Fucilla

Signature-even 2 → 2 amplitude in full colour: NLL to any order

Caron-Huot, EG, Reichel, Vernazza, JHEP 1803 (2018) 098; JHEP 08 (2020) 116

Defining the *Reduced Amplitude*

$$\frac{i}{2s}\hat{\mathcal{M}}_{ij\to ij} \equiv \langle\psi_j|e^{-(H-H_{1\to 1})L}|\psi_i\rangle = \langle\psi_j|e^{-\hat{H}X}|\psi_i\rangle$$



The evolution of the signature-even amplitude (NLL) is governed by the BFKL equation:

$$\Omega^{(\ell-1)}(p,k) = \hat{H}\,\Omega^{(\ell-2)}(p,k), \qquad \hat{H} = \underbrace{(2C_A - \mathbf{T}_t^2)}_{C_1}\,\hat{H}_{\rm i} + \underbrace{(C_A - \mathbf{T}_t^2)}_{C_2}\,\hat{H}_{\rm m}$$

Integrate ($\hat{H}_{\rm i}$) Multiply ($\hat{H}_{\rm m}$)

$$\hat{H}_{\rm i}\,\Psi(p,k) = \int [{\rm D}k']\,f(p,k,k')\Big[\Psi(p,k') - \Psi(p,k)\Big],$$

$$\hat{H}_{\rm m}\,\Psi(p,k) = J(p,k)\,\Psi(p,k)$$

$$f(p,k,k') \equiv \frac{k^2}{k'^2(k-k')^2} + \frac{(p-k)^2}{(p-k')^2(k-k')^2} - \frac{p^2}{k'^2(p-k')^2}$$

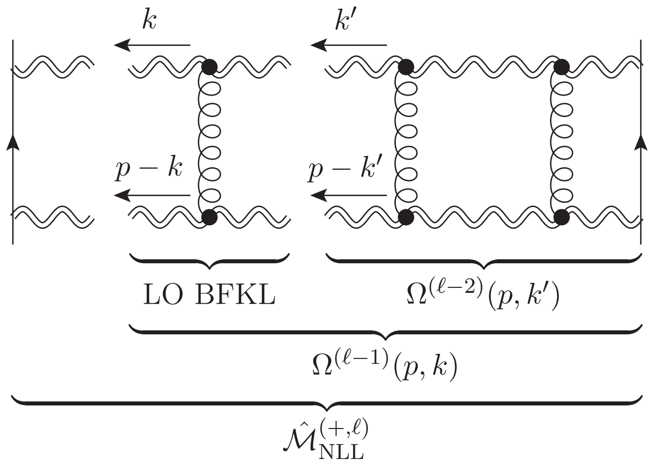
The equation can be solved iteratively.

$$J(p,k) = \frac{1}{2\epsilon} + \int [{\rm D}k']\,f(p,k,k')$$
$$= \frac{1}{2\epsilon}\left[2 - \left(\frac{p^2}{k^2}\right)^{\epsilon} - \left(\frac{p^2}{(p-k)^2}\right)^{\epsilon}\right].$$

While the amplitude has IR poles,
the wavefunction is IR finite to any order!

All orders solution in the soft approximation

Solving for the wavefunction in the soft approximation:



$$J_s(p,k) = \frac{1}{2\epsilon}\left[1-\left(\frac{p^2}{k^2}\right)^{\epsilon}\right] \qquad \xi \equiv \left(p^2/k^2\right)^{\epsilon}$$

$$\int [{\rm D}k'] \frac{2(k\cdot k')}{k'^2(k-k')^2}\left(\frac{p^2}{k'^2}\right)^{n\epsilon} = -\frac{1}{2\epsilon}\frac{B_n(\epsilon)}{B_0(\epsilon)}\left(\frac{p^2}{k^2}\right)^{(n+1)\epsilon}$$

$$\text{with} \qquad B_n(\epsilon) = e^{\epsilon\gamma_{\rm E}}\frac{\Gamma(1-\epsilon)}{\Gamma(1+n\epsilon)}\frac{\Gamma(1+\epsilon+n\epsilon)\Gamma(1-\epsilon-n\epsilon)}{\Gamma(1-2\epsilon-n\epsilon)}\,.$$

$$\hat{B}_n(\epsilon) = 1-\frac{B_n(\epsilon)}{B_0(\epsilon)} = 2n(2+n)\zeta_3\epsilon^3 + 3n(2+n)\zeta_4\epsilon^4 + \ldots.$$

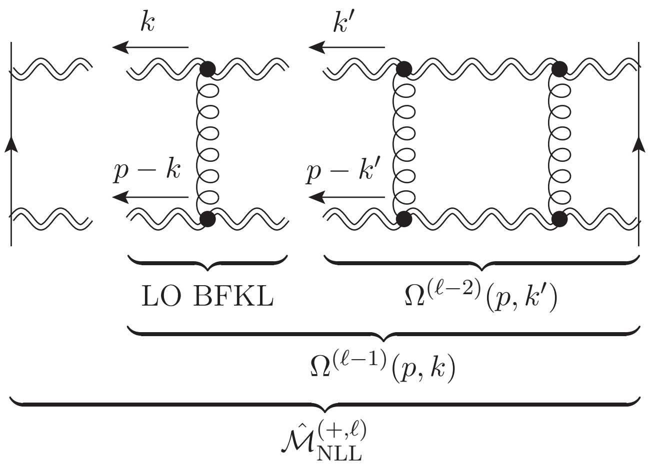
$$\begin{aligned}
\Omega^{(0)}(\xi) =& 1,\\
\Omega^{(1)}(\xi) =& \frac{(C_A-{\bf T}_t)}{2\epsilon}\Big(1-\xi\Big),\\
\Omega^{(2)}(\xi) =& \frac{(C_A-{\bf T}_t)^2}{(2\epsilon)^2}\left\{1-2\xi+\xi^2\left[1-\hat{B}_1(\epsilon)\frac{2C_A-{\bf T}_t}{C_A-{\bf T}_t}\right]\right\},\\
\Omega^{(3)}(\xi) =& \frac{(C_A-{\bf T}_t)^3}{(2\epsilon)^3}\left\{1-3\xi+3\xi^2\left[1-\hat{B}_1(\epsilon)\frac{2C_A-{\bf T}_t}{C_A-{\bf T}_t}\right]\right.\\
&\left. -\,\xi^3\left[1-\hat{B}_1(\epsilon)\frac{2C_A-{\bf T}_t}{C_A-{\bf T}_t}\right]\left[1-\hat{B}_2(\epsilon)\frac{2C_A-{\bf T}_t}{C_A-{\bf T}_t}\right]\right\}
\end{aligned}$$

All-order result:

$$\Omega^{(\ell-1)}(p,k) = \frac{(C_A-{\bf T}_t)^{\ell-1}}{(2\epsilon)^{\ell-1}}\sum_{n=0}^{\ell-1}(-1)^n\left(\begin{array}{c}\ell-1\\ n\end{array}\right)\left(\frac{p^2}{k^2}\right)^{n\epsilon}\prod_{m=0}^{n-1}\left\{1-\hat{B}_m(\epsilon)\frac{2C_A-{\bf T}_t}{C_A-{\bf T}_t}\right\}$$

The amplitude in the soft approximation

Having solved for the wavefunction, we can compute the amplitude!



Summing over the two soft limits, we get (at any given order):

$$\hat{\mathcal{M}}^{(+,\ell)}_{\rm NLL} = -i\pi \frac{(B_0)^\ell}{(\ell-1)!}\int [{\rm D}k]\,\frac{p^2}{k^2(p-k)^2}\,\Omega^{(\ell-1)}(p,k)\,\mathbf{T}^2_{s-u}\,\mathcal{M}^{(\rm tree)}$$

All IR divergences can be resumed into a closed form expression:

$$\hat{\mathcal{M}}^{(+)}_{\rm NLL}\Big|_{\rm IR} = \frac{i\pi}{L(C_A-\mathbf{T}_t^2)}\left(1-R(\epsilon)\frac{C_A}{C_A-\mathbf{T}_t^2}\right)^{-1}\left[\exp\left\{\frac{B_0(\epsilon)}{2\epsilon}\frac{\alpha_s}{\pi}L(C_A-\mathbf{T}_t)\right\}-1\right]\mathbf{T}^2_{s-u}\,\mathcal{M}^{(\rm tree)}+\mathcal{O}(\epsilon^0).$$

$$R(\epsilon)\equiv\frac{B_0(\epsilon)}{B_{-1}(\epsilon)}-1=\frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}-1=-2\zeta_3\,\epsilon^3-3\zeta_4\,\epsilon^4-6\zeta_5\epsilon^5-\left(10\zeta_6-2\zeta_3^2\right)\epsilon^6+\mathcal{O}(\epsilon^7).$$

The soft anomalous dimension at NLL is thus solved to all order:

$$\mathbf{\Gamma}^{(-)}_{\rm NLL}=i\pi\frac{\alpha_s}{\pi}\,G\left(\frac{\alpha_s}{\pi}L\right)\mathbf{T}^2_{s-u}$$

$$G^{(l)}=\frac{1}{(l-1)!}\left[\frac{(C_A-\mathbf{T}_t^2)}{2}\right]^{l-1}\left(1-\frac{C_A}{C_A-\mathbf{T}_t^2}R(\epsilon)\right)^{-1}\Bigg|_{\epsilon^{l-1}}$$

Signature-even 2 → 2 amplitude: Iterating the BFKL Hamiltonian in two dimensions

Caron-Huot, EG, Reichel, Vernazza, JHEP 08 (2020) 116

The 2d wavefunction computed in terms of pure Single-Valued Harmonic Polylogarithms (SVHPLs)

$$\Omega_{2d}^{(\ell-1)}(z,\bar{z}) = \hat{H}_{2d}\Omega_{2d}^{(\ell-2)}(z,\bar{z})$$

The action of the Hamiltonian on SVHPL amounts to the following DEs:

$$\begin{aligned}
\frac{d}{dz}\hat{H}_{2\mathrm{d,i}}\mathcal{L}_{0,\sigma}(z,\bar{z}) &= \frac{\hat{H}_{2\mathrm{d,i}}\mathcal{L}_{\sigma}(z,\bar{z})}{z} \\
\frac{d}{dz}\hat{H}_{2\mathrm{d,i}}\mathcal{L}_{1,\sigma}(z,\bar{z}) &= \frac{\hat{H}_{2\mathrm{d,i}}\mathcal{L}_{\sigma}(z,\bar{z})}{1-z} - \frac{1}{4}\frac{\mathcal{L}_{1,\sigma}(z,\bar{z})}{z} \\
&\quad - \frac{1}{4}\frac{\mathcal{L}_{0,\sigma}(z,\bar{z}) + 2\mathcal{L}_{1,\sigma}(z,\bar{z}) - [\mathcal{L}_{0,\sigma}(w,\bar{w}) + \mathcal{L}_{1,\sigma}(w,\bar{w})]_{w,\bar{w}\to\infty}}{1-z}
\end{aligned}$$

An algorithm is set up to iteratively determine the wavefunction to any loop order. Computed explicitly to 12 loops.

$$\begin{aligned}
\Omega_{2\mathrm{d}}^{(1)} &= \frac{1}{2}C_2\left(\mathcal{L}_0 + 2\mathcal{L}_1\right) \\
\Omega_{2\mathrm{d}}^{(2)} &= \frac{1}{2}C_2^2\left(\mathcal{L}_{0,0} + 2\mathcal{L}_{0,1} + 2\mathcal{L}_{1,0} + 4\mathcal{L}_{1,1}\right) + \frac{1}{4}C_1C_2\left(-\mathcal{L}_{0,1} - \mathcal{L}_{1,0} - 2\mathcal{L}_{1,1}\right) \\
\Omega_{2\mathrm{d}}^{(3)} &= \frac{3}{4}C_2^3\left(\mathcal{L}_{0,0,0} + 2\mathcal{L}_{0,0,1} + 2\mathcal{L}_{0,1,0} + 4\mathcal{L}_{0,1,1} + 2\mathcal{L}_{1,0,0} + 4\mathcal{L}_{1,0,1} + 4\mathcal{L}_{1,1,0} + 8\mathcal{L}_{1,1,1}\right) \\
&\quad + \frac{1}{4}C_1C_2^2\left(2\zeta_3 - 2\mathcal{L}_{0,0,1} - 3\mathcal{L}_{0,1,0} - 7\mathcal{L}_{0,1,1} - 2\mathcal{L}_{1,0,0} - 7\mathcal{L}_{1,0,1} - 7\mathcal{L}_{1,1,0} - 14\mathcal{L}_{1,1,1}\right) \\
&\quad + \frac{1}{16}C_1^2C_2\left(\mathcal{L}_{0,0,1} + 2\mathcal{L}_{0,1,0} + 4\mathcal{L}_{0,1,1} + \mathcal{L}_{1,0,0} + 4\mathcal{L}_{1,0,1} + 4\mathcal{L}_{1,1,0} + 8\mathcal{L}_{1,1,1}\right)
\end{aligned}$$

The full signature-even amplitude at NLL

Caron-Huot, EG, Reichel, Vernazza
JHEP 1803 (2018) 098
JHEP 08 (2020) 116

The **soft** wavefunction alone generates all IR singularities in the amplitude.
We can therefore split the wavefunction into **soft** and **hard**: $\Omega(p,k) = \Omega_{\text{hard}}(p,k) + \Omega_{\text{soft}}(p,k)$
and use dim. reg. only for the **soft**: $\Omega^{(2\text{d})}_{\text{hard}}(z,\bar{z}) \equiv \lim_{\epsilon\to 0}\Omega_{\text{hard}} = \Omega^{(2\text{d})}(z,\bar{z}) - \Omega^{(2\text{d})}_{\text{soft}}(z,\bar{z})$

The full amplitude is recovered by summing two integrals:

$$\hat{\mathcal{M}}^{(+,\,\text{NLL})}_{ij\to ij}\left(\frac{s}{-t}\right) = -i\pi\left[\int [\text{D}k]\frac{p^2}{k^2(p-k)^2}\Omega_{\text{soft}}(p,k) + \frac{1}{4\pi}\int\frac{d^2z}{z\bar{z}}\Omega^{(2\text{d})}_{\text{hard}}(z,\bar{z})\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(\text{tree})}_{ij\to ij}$$

We explicitly computed it to **13 loops**.
The first few orders are:

$$\hat{\mathcal{M}}^{(+,1,0)} = i\pi\, r_\Gamma\left\{\frac{1}{2\epsilon}\right\}\mathbf{T}^2_{s-u}\mathcal{M}^{\text{tree}},$$

$$\hat{\mathcal{M}}^{(+,2,1)} = i\pi\,\frac{r_\Gamma^2}{2}\left\{-\frac{1}{4\epsilon^2}\right\}[\mathbf{T}^2_t,\mathbf{T}^2_{s-u}]\mathcal{M}^{\text{tree}},$$

$$\hat{\mathcal{M}}^{(+,3,2)} = i\pi\,\frac{r_\Gamma^3}{3!}\left\{\frac{1}{8\epsilon^3} - \frac{11\zeta_3}{4}\right\}[\mathbf{T}^2_t,[\mathbf{T}^2_t,\mathbf{T}^2_{s-u}]]\mathcal{M}^{\text{tree}},$$

$$\hat{\mathcal{M}}^{(+,4,3)} = i\pi\,\frac{r_\Gamma^4}{4!}\left\{-\left(\frac{\zeta_3}{8\epsilon} + \frac{3\zeta_4}{16}\right)[\mathbf{T}^2_t,[\mathbf{T}^2_t,\mathbf{T}^2_{s-u}]]\mathbf{T}^2_t - \frac{1}{16\epsilon^4}[\mathbf{T}^2_t,[\mathbf{T}^2_t,[\mathbf{T}^2_t,\mathbf{T}^2_{s-u}]]]\right\}\mathcal{M}^{\text{tree}}$$

Constraint on the
4-loop soft anom. dim.

The soft amplitude can be resummed to all orders in $x = \frac{\alpha_s}{\pi}L$:

$$\hat{\mathcal{M}}_{\text{NLL,s}} = \frac{i\pi}{L(C_A - \mathbf{T}^2_t)}\left\{\left(e^{\frac{B_0}{2\epsilon}(C_A - \mathbf{T}^2_t)x} - 1\right)\left(1 - \frac{C_A}{(C_A - \mathbf{T}^2_t)}R(\epsilon)\right)^{-1} + 1 - e^{-\gamma_E(2C_A - \mathbf{T}^2_t)\,x}\,\frac{\Gamma\left(1-(C_A-\mathbf{T}^2_t)x\right)}{\Gamma\left(1+(C_A-\mathbf{T}^2_t)x\right)}\left(\frac{\Gamma\left(1+(C_A-\mathbf{T}^2_t)\frac{x}{2}\right)}{\Gamma\left(1-(C_A-\mathbf{T}^2_t)\frac{x}{2}\right)}\right)^{-\frac{\mathbf{T}^2_t}{C_A-\mathbf{T}^2_t}}\right\}\mathbf{T}^2_{s-u}\,\mathcal{M}^{\text{tree}}$$

$$R(\epsilon) = \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} - 1$$

A resummed result for the finite, hard part
(with SV MZVs) is yet unknown.