FACTORIZATION OF NON-GLOBAL LHC OBSERVABLES

PART 2: THE GLAUBER SERIES

MATTHIAS NEUBERT

PRISMA+ CLUSTER OF EXCELLENCE & MITP

JOHANNES GUTENBERG UNIVERSITY MAINZ

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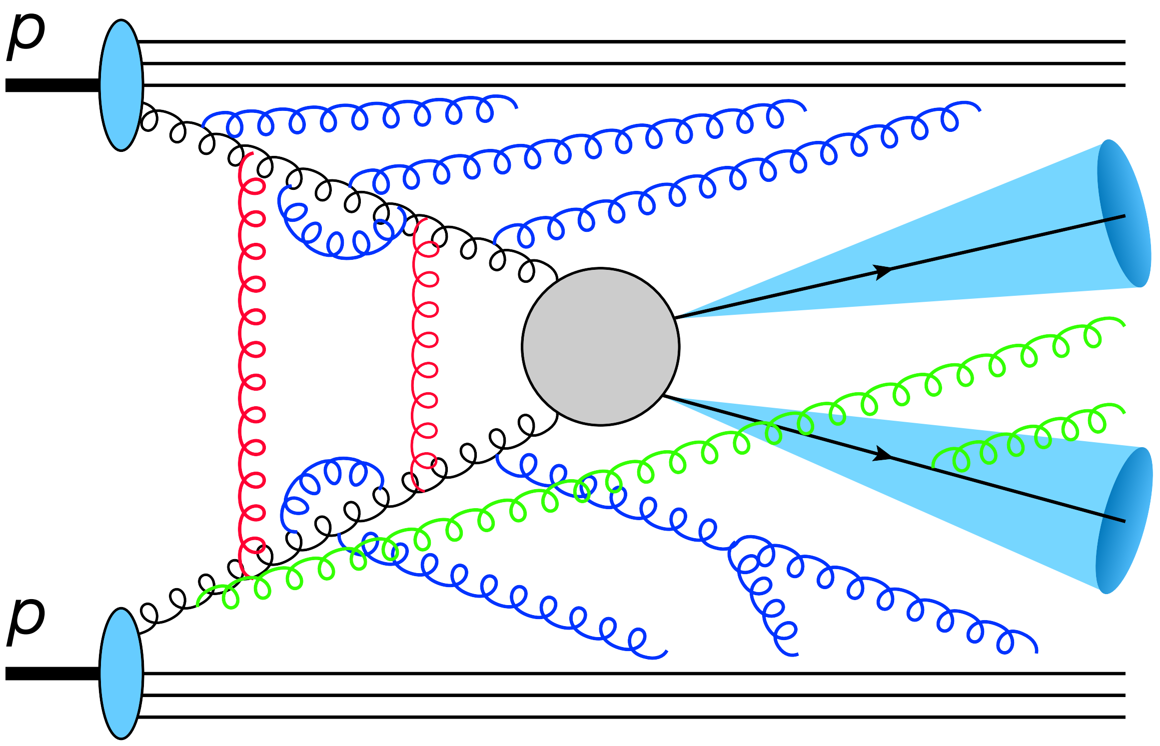
based on:

Thomas Becher, MN, Dingyu Shao, Michel Stillger [2307.06359]

Philipp Böer, Patrick Hager, MN, Michel Stillger, Xiaofeng Xu [2307.11089, 2311.18811 & 2405.05305]

THEORY OF JET PROCESSES AT LHC

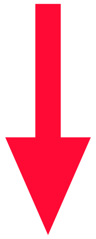


red: Coulomb gluons

blue: gluons emitted along beams

green: soft gluons between jets

Loss of color coherence from initial-state Coulomb interactions



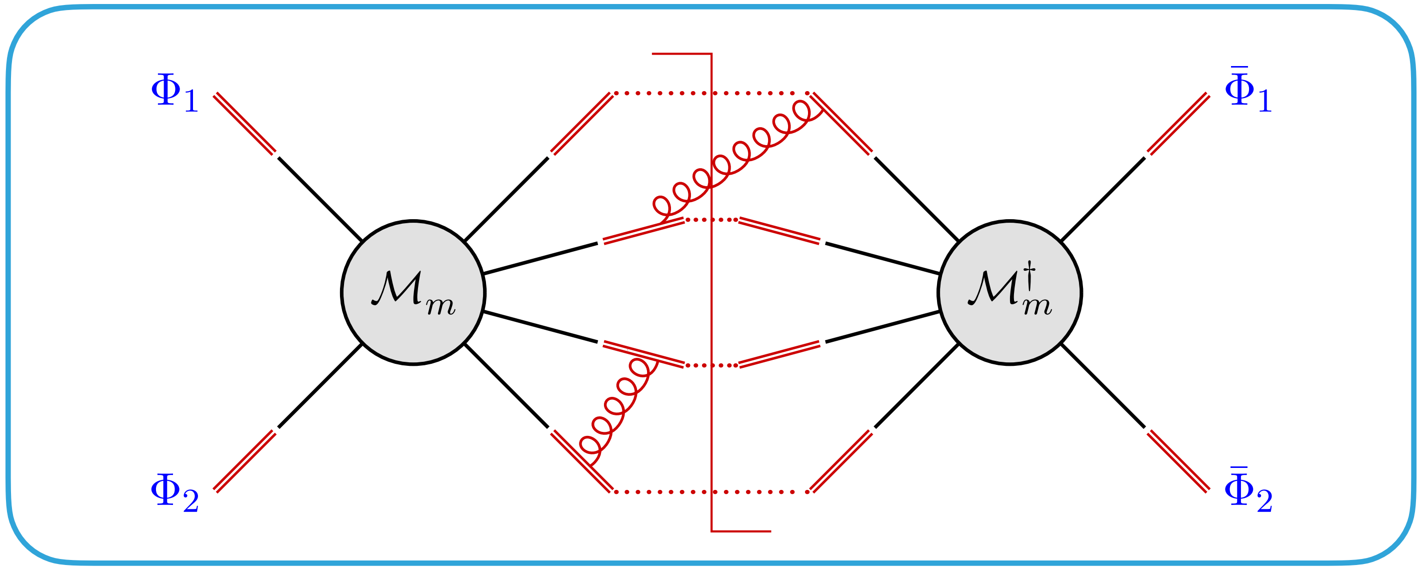
- Weird “super-leading logarithms”
- Breakdown of naive factorization
- Phenomenological consequences?

THEORY OF NON-GLOBAL LHC OBSERVABLES

SCET factorization theorem

$$\sigma_{2\to M}(Q,Q_0)=\sum_{a,b=q,\bar{q},g}\int dx_1dx_2\sum_{m=2+M}^{\infty}\big\langle \underbrace{\boldsymbol{\mathcal{H}}_m^{ab}(\{\underline{n}\},Q,\mu)}_{\text{high scale}}\otimes\underbrace{\boldsymbol{\mathcal{W}}_m^{ab}(\{\underline{n}\},Q_0,x_1,x_2,\mu)}_{\text{low scale}}\big\rangle$$

T. Becher, M. Neubert, D. Shao (2021)
[see also: T. Becher, M. Neubert, L. Rothen, D. Shao (2015, 2016)
R.A. Martínez, M. De Angelis, J.R. Forshaw, S. Plätzer, M.H. Seymour (2018)
J.R. Forshaw, J. Holguin, S. Plätzer (2019–2022)]



⇒ new perspective to think about non-global observables!

THEORY OF NON-GLOBAL LHC OBSERVABLES

SCET factorization theorem

$$\sigma_{2\to M}(Q,Q_0) = \sum_{a,b=q,\bar{q},g} \int dx_1 dx_2 \sum_{m=2+M}^{\infty} \big\langle \boldsymbol{\mathcal{H}}_m^{ab}(\{\underline{n}\},Q,\mu) \otimes \boldsymbol{\mathcal{W}}_m^{ab}(\{\underline{n}\},Q_0,x_1,x_2,\mu) \big\rangle$$

T. Becher, M. Neubert, D. Shao (2021)
[see also: T. Becher, M. Neubert, L. Rothen, D. Shao (2015, 2016)]

high scale

low scale

Renormalization-group equation:

$$\mu \frac{d}{d\mu} \boldsymbol{\mathcal{H}}_l^{ab}(\{\underline{n}\},Q,\mu) = -\sum_{m\le l} \boldsymbol{\mathcal{H}}_m^{ab}(\{\underline{n}\},Q,\mu)\, \boldsymbol{\Gamma}_{ml}^{H}(\{\underline{n}\},Q,\mu)$$

operator in color space and in the infinite space of parton multiplicities

All-order summation of large logarithmic corrections, including the super-leading logarithms!

RESUMMATION OF SUPER-LEADING LOGARITHMS

Evaluate factorization theorem at low scale $\mu_s \sim Q_0$

- Low-energy matrix element:

$$\boldsymbol{\mathcal{W}}_m^{ab}(\{\underline{n}\}, Q_0, x_1, x_2, \mu_s) = f_{a/p}(x_1)\, f_{b/p}(x_2)\, \mathbf{1} + \mathcal{O}(\alpha_\mathbf{s})$$

- Hard-scattering functions:

$$\boldsymbol{\mathcal{H}}_m^{ab}(\{\underline{n}\}, Q, \mu_s) = \sum_{l\le m} \boldsymbol{\mathcal{H}}_l^{ab}(\{\underline{n}\}, Q, Q)\, \mathbf{P} \exp\left[\int_{\mu_s}^{Q} \frac{d\mu}{\mu}\, \boldsymbol{\Gamma}^H(\{\underline{n}\}, Q, \mu)\right]_{lm}$$

- Expanding the solution in a power series generates arbitrarily high parton multiplicities starting from the $2 \to M$ Born process

RESUMMATION OF SUPER-LEADING LOGARITHMS

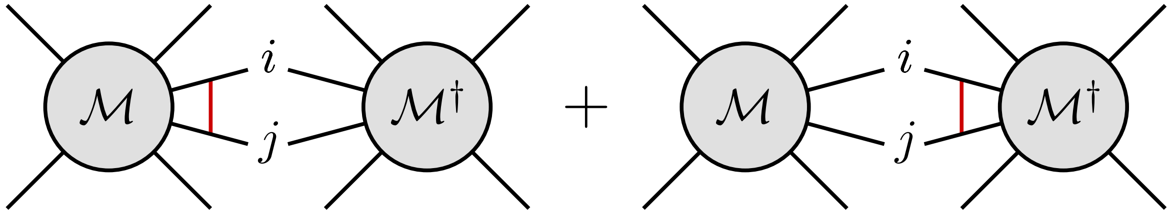
Evaluate factorization theorem at low scale $\mu_s \sim Q_0$

- Anomalous-dimension matrix:

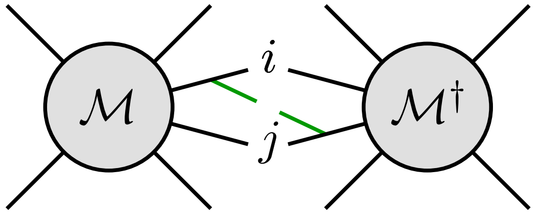
$$\mathbf{\Gamma}^H = \frac{\alpha_s}{4\pi} \begin{pmatrix} \boldsymbol{V}_{2+M} & \boldsymbol{R}_{2+M} & 0 & 0 & \dots \\ 0 & \boldsymbol{V}_{2+M+1} & \boldsymbol{R}_{2+M+1} & 0 & \dots \\ 0 & 0 & \boldsymbol{V}_{2+M+2} & \boldsymbol{R}_{2+M+2} & \dots \\ 0 & 0 & 0 & \boldsymbol{V}_{2+M+3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \mathcal{O}(\alpha_s^2)$$

- Action on hard functions:

$$\boldsymbol{\mathcal{H}}_m \, \boldsymbol{V}_m = \sum_{(ij)}$$



$$\boldsymbol{\mathcal{H}}_m \, \boldsymbol{R}_m = \sum_{(ij)}$$



RESUMMATION OF SUPER-LEADING LOGARITHMS

Detailed structure of the soft anomalous-dimension coefficients

$$\left.\begin{aligned} \boldsymbol{V}_m &= \overline{\boldsymbol{V}}_m + \boldsymbol{V}^G + \sum_{i=1,2} \boldsymbol{V}_i^c \ln\frac{\mu^2}{\hat{s}} \\ \boldsymbol{R}_m &= \overline{\boldsymbol{R}}_m + \sum_{i=1,2} \boldsymbol{R}_i^c \ln\frac{\mu^2}{\hat{s}} \end{aligned}\right\} \quad \boldsymbol{\Gamma} = \underset{\substack{\text{soft emission} \\ \text{(collinear div. subtracted)}}}{\overline{\boldsymbol{\Gamma}}} + \overset{\text{Glauber phase}}{\boldsymbol{V}^G} + \underset{\text{collinear emission}}{\boldsymbol{\Gamma}^c} \ln\frac{\mu^2}{\hat{s}}$$

with:

- color coherence without Glauber phases: $\boldsymbol{\mathcal{H}}_m\, \boldsymbol{\Gamma}^c\, \overline{\boldsymbol{\Gamma}} = \boldsymbol{\mathcal{H}}_m\, \overline{\boldsymbol{\Gamma}}\, \boldsymbol{\Gamma}^c$
- collinear safety: $\langle \boldsymbol{\mathcal{H}}_m\, \boldsymbol{\Gamma}^c \otimes \boldsymbol{1} \rangle = 0$
- cyclicity of the trace: $\big\langle \boldsymbol{\mathcal{H}}_m\, \boldsymbol{V}^G \otimes \boldsymbol{1} \big\rangle = 0$

RESUMMATION OF SUPER-LEADING LOGARITHMS

SLLs arise from the terms in $\mathbf{P}\exp\left[\int_{\mu_s}^{Q}\frac{d\mu}{\mu}\,\mathbf{\Gamma}^H(\{\underline{n}\},Q,\mu)\right]_{lm}$ with the highest number of insertions of Γᶜ

- Under the color trace, insertions of Γ_c are non-zero only if they come in conjunction with (at least) two Glauber phases and one $\overline{\Gamma}$
- Relevant color traces at $\mathcal{O}(\alpha_s^{n+3}L^{2n+3})$:

$$C_{rn} = \left\langle \boldsymbol{\mathcal{H}}_{2\to M}\,(\mathbf{\Gamma}^c)^r\,\boldsymbol{V}^G\,(\mathbf{\Gamma}^c)^{n-r}\,\boldsymbol{V}^G\,\overline{\mathbf{\Gamma}}\otimes\mathbf{1}\right\rangle$$

- Kinematic information contained in $(M+1)$ angular integrals from $\overline{\Gamma}$:

$$J_j = \int\frac{d\Omega(n_k)}{4\pi}\left(W_{1j}^k - W_{2j}^k\right)\Theta_{\rm veto}(n_k)\,;\quad \text{with}\quad W_{ij}^k = \frac{n_i\cdot n_j}{n_i\cdot n_k\,n_j\cdot n_k}$$

RESUMMATION OF SUPER-LEADING LOGARITHMS

General result for $2 \to M$ hard processes

$$C_{rn} = -256\pi^2 (4N_c)^{n-r} \left[\sum_{j=3}^{M+2} J_j \sum_{i=1}^{4} c_i^{(r)} \langle \boldsymbol{\mathcal{H}}_{2\to M}\, \boldsymbol{O}_i^{(j)} \rangle - J_2 \sum_{i=1}^{6} d_i^{(r)} \langle \boldsymbol{\mathcal{H}}_{2\to M}\, \boldsymbol{S}_i \rangle \right]$$

T. Becher, M. Neubert, D. Shao, M. Stillger (2023)

- Series of SLLs, starting at 3-loop order:

$$\sigma_{\text{SLL}} = \sigma_{\text{Born}} \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^{n+3} L^{2n+3} \frac{(-4)^n\, n!}{(2n+3)!} \sum_{r=0}^{n} \frac{(2r)!}{4^r\,(r!)^2}\, C_{rn}$$

from scale integrals (at fixed coupling)

- Reproduces all that is known about SLLs (and much more…)

GLAUBER SERIES

Structure of the cross section

- We found:

$$\sigma \sim \sum_{n=0}^{\infty} \left[c_{0,n} \left(\frac{\alpha_s}{\pi} L \right)^n + c_{1,n} \left(\frac{\alpha_s}{\pi} L \right) \left(\frac{\alpha_s}{\pi} i\pi L \right)^2 \left(\frac{\alpha_s}{\pi} L^2 \right)^n + \dots \right]$$

- Introduce two $O(1)$ parameters:

$$w = \frac{N_c \alpha_s(\bar{\mu})}{\pi} L^2 \,, \qquad w_\pi = \frac{N_c \alpha_s(\bar{\mu})}{\pi} \pi^2$$

- Including multiple Glauber insertions:

$$\sigma^{\text{SLL+G}} \sim \frac{\alpha_s L}{\pi N_c} \sum_{\ell=1}^{\infty} \sum_{n=0}^{\infty} c_{\ell,n} \, w_\pi^\ell \, w^{n+\ell}$$

- Relevant color traces:

$$C_{\{\underline{r}\}}^{\ell} \equiv \langle \boldsymbol{\mathcal{H}}_{2\to M} \, (\boldsymbol{\Gamma}^c)^{r_1} \, \boldsymbol{V}^G \, (\boldsymbol{\Gamma}^c)^{r_2} \, \boldsymbol{V}^G \dots (\boldsymbol{\Gamma}^c)^{r_{2\ell-1}} \, \boldsymbol{V}^G \, (\boldsymbol{\Gamma}^c)^{r_{2\ell}} \, \boldsymbol{V}^G \, \overline{\boldsymbol{\Gamma}} \rangle$$

P. Böer, M. Neubert, M. Stillger (2023)

GLAUBER SERIES

Structure of the cross section

- Including multiple Glauber insertions:

$$\sigma^{\text{SLL+G}} \sim \frac{\alpha_s\, L}{\pi\, N_c} \sum_{\ell=1}^{\infty} \sum_{n=0}^{\infty} c_{\ell,n}\, w_\pi^\ell\, w^{n+\ell}$$

- Relevant color traces:

$$C_{\{\underline{r}\}}^{\ell} \equiv \langle \boldsymbol{\mathcal{H}}_{2\to M}\, (\boldsymbol{\Gamma}^c)^{r_1}\, \boldsymbol{V}^G\, (\boldsymbol{\Gamma}^c)^{r_2}\, \boldsymbol{V}^G \dots (\boldsymbol{\Gamma}^c)^{r_{2\ell-1}}\, \boldsymbol{V}^G\, (\boldsymbol{\Gamma}^c)^{r_{2\ell}}\, \boldsymbol{V}^G\, \overline{\boldsymbol{\Gamma}} \rangle$$

  require an enlarged operator basis: **5** ($qq, \bar{q}\bar{q}, q\bar{q}$ scattering), **20** (gg scattering), and **14** ($qg, \bar{q}g$ scattering) P. Böer, P. Hager, M. Neubert, M. Stillger, X. Xu (2023)

- Leads to 2ℓ sums and complicated scale integrals
  $\Rightarrow$ no resummation possible beyond $\ell = 1$!

GLAUBER SERIES

Comments on the construction of the color basis

- Recall that $\boldsymbol{\Gamma}^c$ and $\boldsymbol{V}^G$ only depend on generators of partons 1 and 2, whereas $\overline{\boldsymbol{\Gamma}}$ brings in the generator of one additional parton *j*
- Hence, there are two types of structures:

$$\zeta\,\boldsymbol{\mathcal{C}}_1\,\widetilde{\boldsymbol{\mathcal{C}}}_2\,\boldsymbol{T}_j \qquad \text{or} \qquad \zeta\,\boldsymbol{\mathcal{C}}_1\,\widetilde{\boldsymbol{\mathcal{C}}}_2$$

- Color structures $\boldsymbol{\mathcal{C}}_i$ contain products of color generators of parton *i*; they carry two *matrix indices* (fundamental or adjoint) as well as an *open adjoint index* for each generator
- Such structures can be built from symmetric products:

$$\boldsymbol{\mathcal{C}}_i^{(k)a_1\dots a_k} = \frac{1}{k!}\sum_{\sigma\in S_k}\boldsymbol{T}_i^{a_{\sigma(1)}}\dots\boldsymbol{T}_i^{a_{\sigma(k)}}$$

GLAUBER SERIES

Comments on the construction of the color basis

- Open adjoint indices are contracted with ζ, which can be built from Kronecker δ, f- and d-symbols (higher d-symbols defined recursively):

$$\zeta^{(0)} = 1\,, \qquad \zeta^{(2)a_1a_2} = \delta^{a_1a_2}\,, \qquad \zeta^{(3)a_1a_2a_3} \in \{if^{a_1a_2a_3}\,,\, d^{a_1a_2a_3}\}$$

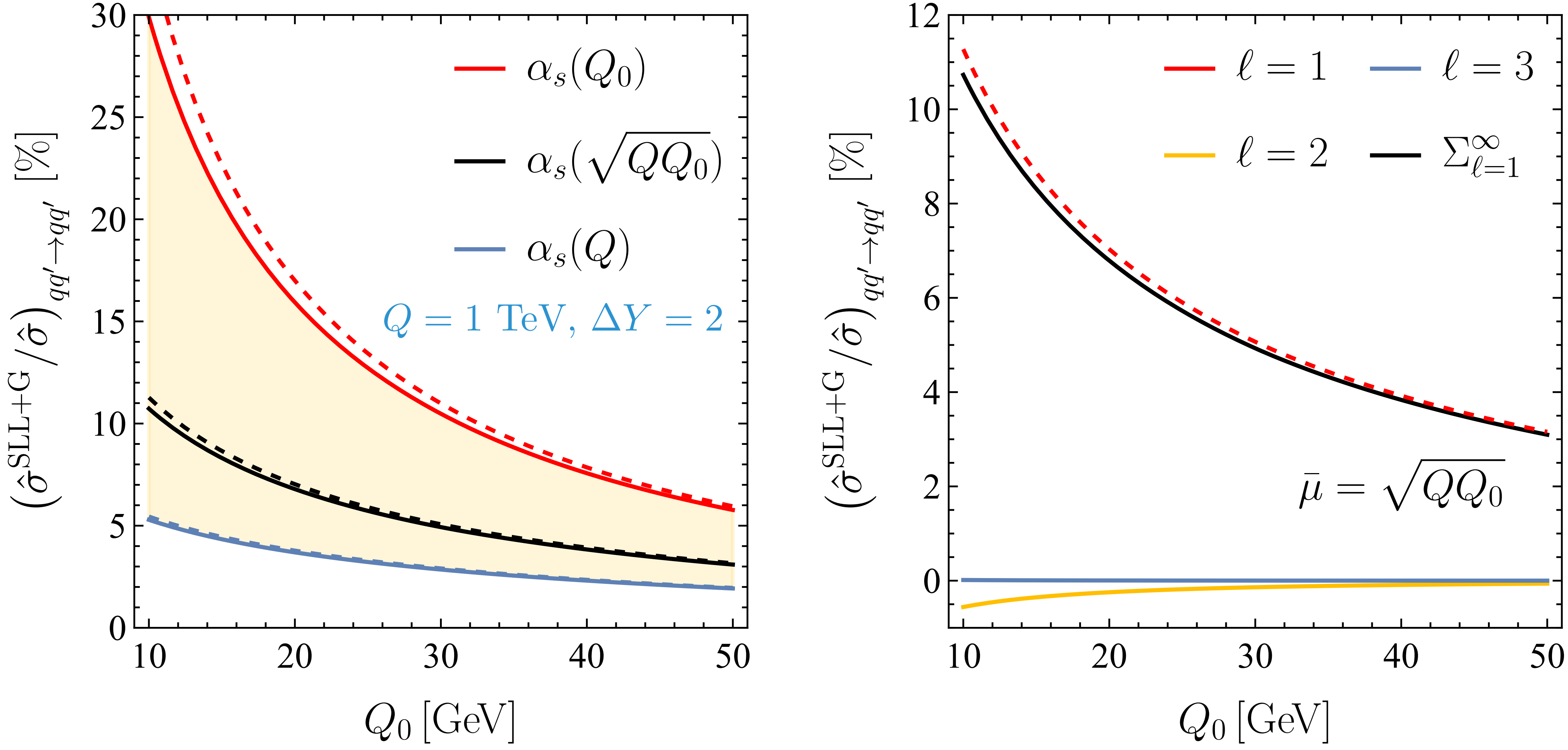
- For identical initial-state particles, the structures (including angular integrals J_j) need to be symmetric under $1 \leftrightarrow 2$
- For initial-state quarks or anti-quarks, symmetric products of generators can be reduced to linear form
- For initial-state gluons, all indices are adjoint ones

for more details, see **Section 2** in: P. Böer, P. Hager, M. Neubert, M. Stillger, X. Xu (arXiv:2311.18811)

PHENOMENOLOGICAL IMPACT (PARTON LEVEL)

Surprising suppression of higher Glauber contributions

P. Böer, M. Neubert, M. Stillger (2023)



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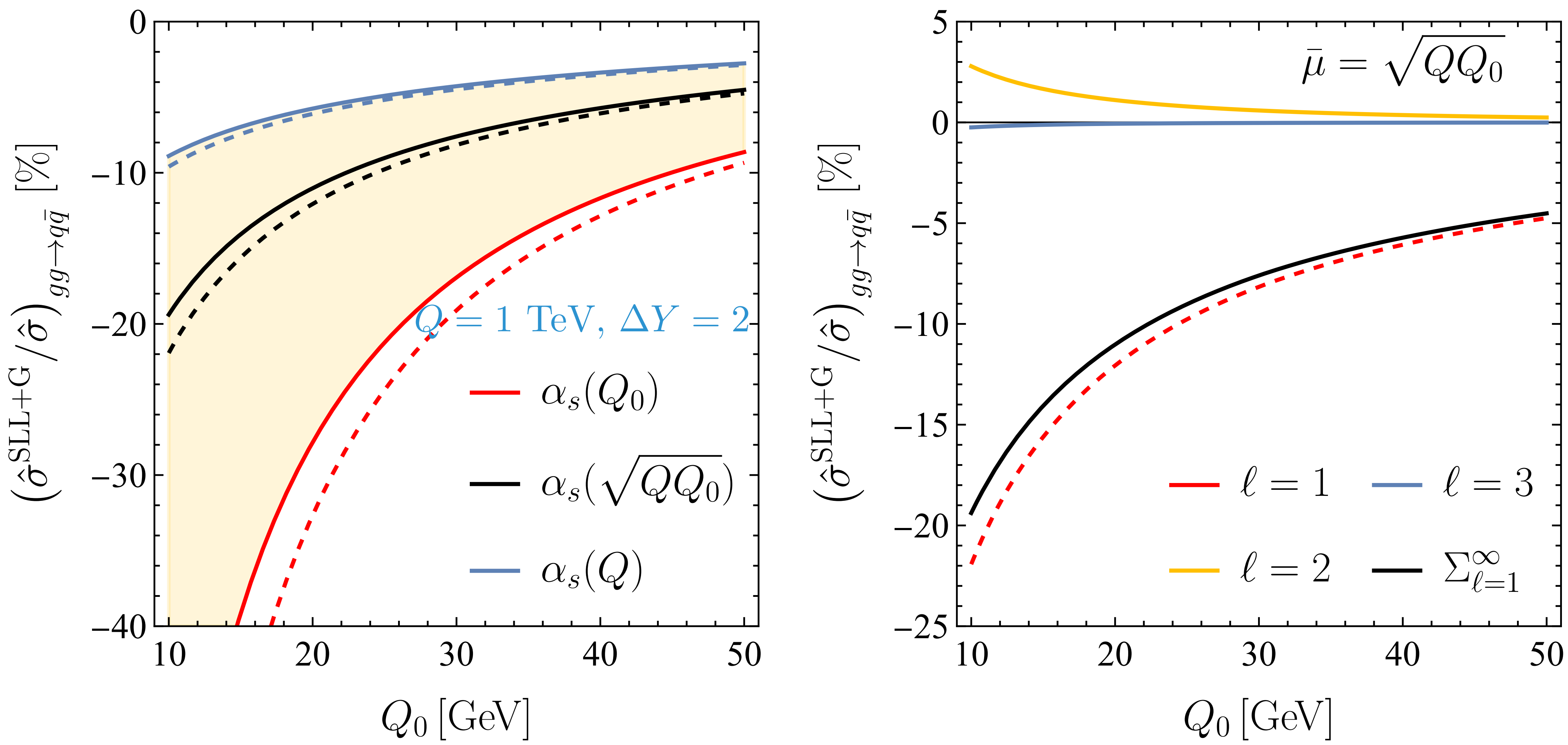
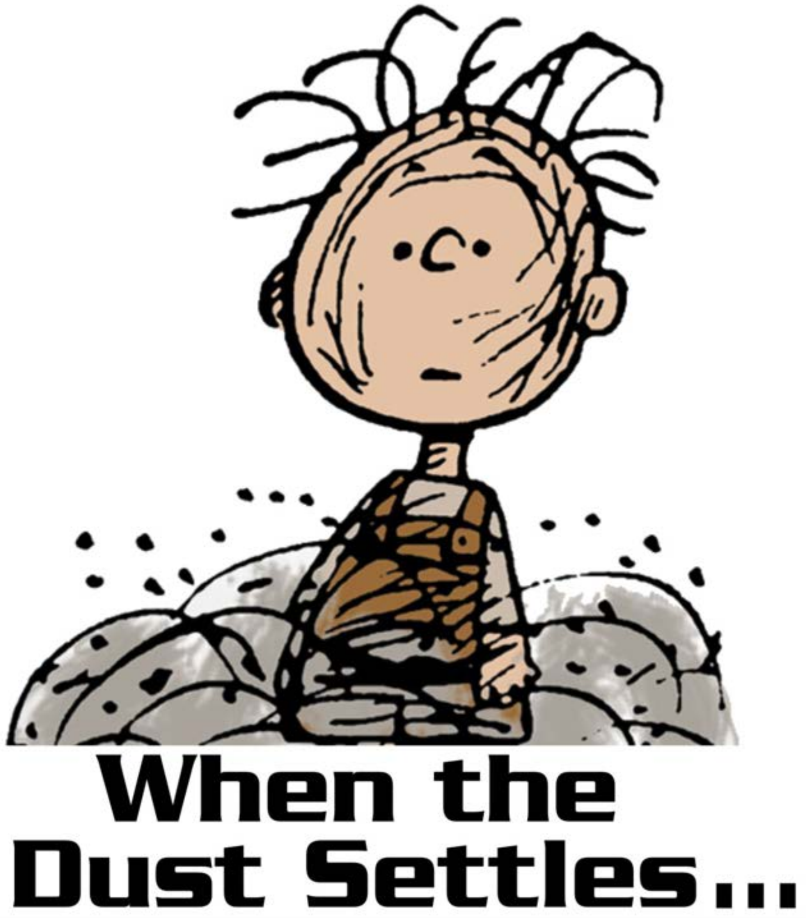


A MORE POWERFUL FORMALISM



P. Böer, P. Hager, M. Neubert, M. Stillger, X. Xu (2024)

A MORE POWERFUL FORMALISM

Master formula for the cross section

$$\sigma_{2\to M}^{\mathrm{SLL}}(Q_0) = \sum_{i\in\{q,\bar{q},g\}} \int dx_1 \int dx_2\, f_1(x_1,\mu_s)\, f_2(x_2,\mu_s) \sum_{n=0}^{\infty} \sum_{r=0}^{n} I_{rn}(\mu_h,\mu_s)\, C_{rn}$$

$\mu_h \simeq Q$

$\mu_s \simeq Q_0$

where:

cusp anomalous dimension

$$\begin{aligned} I_{rn}(\mu_h,\mu_s) = {} & \underbrace{\int_{\mu_s}^{\mu_h} \frac{d\mu_1}{\mu_1}\, \gamma_{\mathrm{cusp}}(\mu_1)\, \ln\frac{\mu_1^2}{\mu_h^2} \dots \int_{\mu_s}^{\mu_{r-1}} \frac{d\mu_r}{\mu_r}\, \overset{\downarrow}{\gamma_{\mathrm{cusp}}}(\mu_r)\, \ln\frac{\mu_r^2}{\mu_h^2}}_{r \text{ times}} \int_{\mu_s}^{\mu_r} \frac{d\mu_{r+1}}{\mu_{r+1}}\, \gamma_{\mathrm{cusp}}(\mu_r) \\ & \times \underbrace{\int_{\mu_s}^{\mu_{r+1}} \frac{d\mu_{r+2}}{\mu_{r+2}}\, \gamma_{\mathrm{cusp}}(\mu_{r+2})\, \ln\frac{\mu_{r+2}^2}{\mu_h^2} \dots \int_{\mu_s}^{\mu_n} \frac{d\mu_{n+1}}{\mu_{n+1}}\, \gamma_{\mathrm{cusp}}(\mu_{n+1})\, \ln\frac{\mu_{n+1}^2}{\mu_h^2}}_{(n-r) \text{ times}} \\ & \times \int_{\mu_s}^{\mu_{n+1}} \frac{d\mu_{n+2}}{\mu_{n+2}}\, \gamma_{\mathrm{cusp}}(\mu_{n+2}) \int_{\mu_s}^{\mu_{n+2}} \frac{d\mu_{n+3}}{\mu_{n+3}}\, \frac{\alpha_s(\mu_{n+3})}{4\pi} \end{aligned}$$

$$C_{rn} = \left\langle \boldsymbol{\mathcal{H}}_{2\to M}\, (\boldsymbol{\Gamma}^c)^r\, \boldsymbol{V}^G\, (\boldsymbol{\Gamma}^c)^{n-r}\, \boldsymbol{V}^G\, \overline{\boldsymbol{\Gamma}} \otimes \boldsymbol{1} \right\rangle$$

A MORE POWERFUL FORMALISM

$$\sum_{r=0}^{\infty}\sum_{n=r}^{\infty}=\sum_{r=0}^{\infty}\sum_{(n-r)=0}^{\infty}$$

Master formula for the cross section

$$\sigma_{2\to M}^{\rm SLL}(Q_0)=\sum_{i\in\{q,\bar q,g\}}\int dx_1\int dx_2\, f_1(x_1,\mu_s)\, f_2(x_2,\mu_s)\sum_{n=0}^{\infty}\sum_{r=0}^{n} I_{rn}(\mu_h,\mu_s)\, C_{rn}$$

$\mu_h \simeq Q$

$\mu_s \simeq Q_0$

where:

cusp anomalous dimension

$$\begin{aligned}I_{rn}(\mu_h,\mu_s)&=\underbrace{\int_{\mu_s}^{\mu_h}\frac{d\mu_1}{\mu_1}\gamma_{\rm cusp}(\mu_1)\ln\frac{\mu_1^2}{\mu_h^2}\dots\int_{\mu_s}^{\mu_{r-1}}\frac{d\mu_r}{\mu_r}\gamma_{\rm cusp}(\mu_r)\ln\frac{\mu_r^2}{\mu_h^2}}_{r\ \text{times}}\int_{\mu_s}^{\mu_r}\frac{d\mu_{r+1}}{\mu_{r+1}}\gamma_{\rm cusp}(\mu_r)\\&\times\underbrace{\int_{\mu_s}^{\mu_{r+1}}\frac{d\mu_{r+2}}{\mu_{r+2}}\gamma_{\rm cusp}(\mu_{r+2})\ln\frac{\mu_{r+2}^2}{\mu_h^2}\dots\int_{\mu_s}^{\mu_n}\frac{d\mu_{n+1}}{\mu_{n+1}}\gamma_{\rm cusp}(\mu_{n+1})\ln\frac{\mu_{n+1}^2}{\mu_h^2}}_{(n-r)\ \text{times}}\\&\times\int_{\mu_s}^{\mu_{n+1}}\frac{d\mu_{n+2}}{\mu_{n+2}}\gamma_{\rm cusp}(\mu_{n+2})\int_{\mu_s}^{\mu_{n+2}}\frac{d\mu_{n+3}}{\mu_{n+3}}\frac{\alpha_s(\mu_{n+3})}{4\pi}\end{aligned}$$

$$C_{rn}=\big\langle\boldsymbol{\mathcal{H}}_{2\to M}\,(\boldsymbol{\Gamma}^c)^r\,\boldsymbol{V}^G\,(\boldsymbol{\Gamma}^c)^{n-r}\,\boldsymbol{V}^G\,\overline{\boldsymbol{\Gamma}}\otimes\boldsymbol{1}\big\rangle$$

A MORE POWERFUL FORMALISM

Recombine scale integrals and color structures

$$\sigma_{2\to M}^{\text{SLL}}(Q_0) = \sum_{i\in\{q,\bar{q},g\}} \int dx_1 \int dx_2\, f_1(x_1,\mu_s)\, f_2(x_2,\mu_s)\, \big\langle \boldsymbol{\mathcal{H}}_{2\to M}\, \boldsymbol{U}_{\text{SLL}}(\mu_h,\mu_s) \big\rangle$$

where:

$$\boldsymbol{U}_{\text{SLL}}(\mu_h,\mu_s) = \sum_{r=0}^{\infty} \underbrace{\boldsymbol{\Gamma}^c \int_{\mu_s}^{\mu_h} \frac{d\mu_1}{\mu_1}\, \gamma_{\text{cusp}}(\mu_1)\, \ln\frac{\mu_1^2}{\mu_h^2} \dots \boldsymbol{\Gamma}^c \int_{\mu_s}^{\mu_{r-1}} \frac{d\mu_r}{\mu_r}\, \gamma_{\text{cusp}}(\mu_r)\, \ln\frac{\mu_r^2}{\mu_h^2}}_{r\ \text{times}}\ \boldsymbol{V}^G \int_{\mu_s}^{\mu_r} \frac{d\mu_{r+1}}{\mu_{r+1}}\, \gamma_{\text{cusp}}(\mu_r)$$

$\mu_h > \mu_1 > \ldots > \mu_r > \mu_{r+1}$

$$\times \sum_{(n-r)=0}^{\infty} \underbrace{\boldsymbol{\Gamma}^c \int_{\mu_s}^{\mu_{r+1}} \frac{d\mu_{r+2}}{\mu_{r+2}}\, \gamma_{\text{cusp}}(\mu_{r+2})\, \ln\frac{\mu_{r+2}^2}{\mu_h^2} \dots \boldsymbol{\Gamma}^c \int_{\mu_s}^{\mu_n} \frac{d\mu_{n+1}}{\mu_{n+1}}\, \gamma_{\text{cusp}}(\mu_{n+1})\, \ln\frac{\mu_{n+1}^2}{\mu_h^2}}_{(n-r)\ \text{times}}$$

$$\times\, \boldsymbol{V}^G \int_{\mu_s}^{\mu_{n+1}} \frac{d\mu_{n+2}}{\mu_{n+2}}\, \gamma_{\text{cusp}}(\mu_{n+2})\, \overline{\boldsymbol{\Gamma}} \int_{\mu_s}^{\mu_{n+2}} \frac{d\mu_{n+3}}{\mu_{n+3}}\, \frac{\alpha_s(\mu_{n+3})}{4\pi}$$

A MORE POWERFUL FORMALISM

Recombine scale integrals and color structures

$$\sigma_{2\to M}^{\rm SLL}(Q_0) = \sum_{i\in\{q,\bar{q},g\}} \int dx_1 \int dx_2\, f_1(x_1,\mu_s)\, f_2(x_2,\mu_s)\, \big\langle \boldsymbol{\mathcal{H}}_{2\to M}\, \boldsymbol{U}_{\rm SLL}(\mu_h,\mu_s) \big\rangle$$

where:

ordered exponential $\boldsymbol{U}_c(\mu_h,\mu_{r+1})$

$$\boldsymbol{U}_{\rm SLL}(\mu_h,\mu_s) = \sum_{r=0}^{\infty} \underbrace{\boldsymbol{\Gamma}^c \int_{\mu_s}^{\mu_h} \frac{d\mu_1}{\mu_1}\, \gamma_{\rm cusp}(\mu_1)\, \ln\frac{\mu_1^2}{\mu_h^2} \dots \boldsymbol{\Gamma}^c \int_{\mu_s}^{\mu_{r-1}} \frac{d\mu_r}{\mu_r}\, \gamma_{\rm cusp}(\mu_r)\, \ln\frac{\mu_r^2}{\mu_h^2}}_{r\ \text{times}}\, \boldsymbol{V}^G \int_{\mu_s}^{\mu_r} \frac{d\mu_{r+1}}{\mu_{r+1}}\, \gamma_{\rm cusp}(\mu_r)$$

$\mu_h > \mu_1 > \dots > \mu_r > \mu_{r+1}$

$$\times \sum_{(n-r)=0}^{\infty} \underbrace{\boldsymbol{\Gamma}^c \int_{\mu_s}^{\mu_{r+1}} \frac{d\mu_{r+2}}{\mu_{r+2}}\, \gamma_{\rm cusp}(\mu_{r+2})\, \ln\frac{\mu_{r+2}^2}{\mu_h^2} \dots \boldsymbol{\Gamma}^c \int_{\mu_s}^{\mu_n} \frac{d\mu_{n+1}}{\mu_{n+1}}\, \gamma_{\rm cusp}(\mu_{n+1})\, \ln\frac{\mu_{n+1}^2}{\mu_h^2}}_{(n-r)\ \text{times}}$$

$$\times\, \boldsymbol{V}^G \int_{\mu_s}^{\mu_{n+1}} \frac{d\mu_{n+2}}{\mu_{n+2}}\, \gamma_{\rm cusp}(\mu_{n+2})\, \overline{\boldsymbol{\Gamma}} \int_{\mu_s}^{\mu_{n+2}} \frac{d\mu_{n+3}}{\mu_{n+3}}\, \frac{\alpha_s(\mu_{n+3})}{4\pi}$$

A MORE POWERFUL FORMALISM

Recombine scale integrals and color structures

$$\sigma^{\rm SLL}_{2\to M}(Q_0) = \sum_{i\in\{q,\bar{q},g\}} \int dx_1 \int dx_2\, f_1(x_1,\mu_s)\, f_2(x_2,\mu_s)\, \big\langle \boldsymbol{\mathcal{H}}_{2\to M}\, \boldsymbol{U}_{\rm SLL}(\mu_h,\mu_s) \big\rangle$$

where:

ordered exponential $\boldsymbol{U}_c(\mu_h,\mu_{r+1})$

$$\boldsymbol{U}_{\rm SLL}(\mu_h,\mu_s) = \sum_{r=0}^{\infty} \underbrace{\boldsymbol{\Gamma}^c \int_{\mu_s}^{\mu_h} \frac{d\mu_1}{\mu_1}\, \gamma_{\rm cusp}(\mu_1)\, \ln\frac{\mu_1^2}{\mu_h^2} \ldots \boldsymbol{\Gamma}^c \int_{\mu_s}^{\mu_{r-1}} \frac{d\mu_r}{\mu_r}\, \gamma_{\rm cusp}(\mu_r)\, \ln\frac{\mu_r^2}{\mu_h^2}}_{r\text{ times}}\, \boldsymbol{V}^G \int_{\mu_s}^{\mu_r} \frac{d\mu_{r+1}}{\mu_{r+1}}\, \gamma_{\rm cusp}(\mu_r)$$

$\mu_h > \mu_1 > \ldots > \mu_r > \mu_{r+1}$

$$\times \sum_{(n-r)=0}^{\infty} \underbrace{\boldsymbol{\Gamma}^c \int_{\mu_s}^{\mu_{r+1}} \frac{d\mu_{r+2}}{\mu_{r+2}}\, \gamma_{\rm cusp}(\mu_{r+2})\, \ln\frac{\mu_{r+2}^2}{\mu_h^2} \ldots \boldsymbol{\Gamma}^c \int_{\mu_s}^{\mu_n} \frac{d\mu_{n+1}}{\mu_{n+1}}\, \gamma_{\rm cusp}(\mu_{n+1})\, \ln\frac{\mu_{n+1}^2}{\mu_h^2}}_{(n-r)\text{ times}}$$

ordered exponential $\boldsymbol{U}_c(\mu_{r+1},\mu_{n+2})$

$$\times\, \boldsymbol{V}^G \int_{\mu_s}^{\mu_{n+1}} \frac{d\mu_{n+2}}{\mu_{n+2}}\, \gamma_{\rm cusp}(\mu_{n+2})\, \overline{\boldsymbol{\Gamma}} \int_{\mu_s}^{\mu_{n+2}} \frac{d\mu_{n+3}}{\mu_{n+3}}\, \frac{\alpha_s(\mu_{n+3})}{4\pi}$$

A MORE POWERFUL FORMALISM

Rewrite the evolution kernel (ordered exponential) for the SLLs

- Expand out all terms except the log-enhanced soft-collinear piece:

$$\boldsymbol{U}_{\rm SLL}(\{\underline{n}\},\mu_h,\mu_s) = \int_{\mu_s}^{\mu_h}\frac{d\mu_1}{\mu_1}\int_{\mu_s}^{\mu_1}\frac{d\mu_2}{\mu_2}\int_{\mu_s}^{\mu_2}\frac{d\mu_3}{\mu_3}$$

cusp anomalous dimension

$$\times\,\boldsymbol{U}_c(\mu_h,\mu_1)\,\gamma_{\rm cusp}\big(\alpha_s(\mu_1)\big)\,\boldsymbol{V}^G\,\boldsymbol{U}_c(\mu_1,\mu_2)\,\gamma_{\rm cusp}\big(\alpha_s(\mu_2)\big)\,\boldsymbol{V}^G\,\frac{\alpha_s(\mu_3)}{4\pi}\,\overline{\boldsymbol{\Gamma}}$$

where we define the Sudakov operator:

$$\boldsymbol{U}_c(\mu_i,\mu_j) = \exp\left[\boldsymbol{\Gamma}^c\int_{\mu_j}^{\mu_i}\frac{d\mu}{\mu}\,\gamma_{\rm cusp}\big(\alpha_s(\mu)\big)\ln\frac{\mu^2}{\mu_h^2}\right]$$

$\mu_h \simeq Q$
$\mu_s \simeq Q_0$

matrix on the space of basis operators

resums all double-logarithmic terms

A MORE POWERFUL FORMALISM

Rewrite the evolution kernel (ordered exponential) for the SLLs

- Expand out all terms except the log-enhanced soft-collinear piece:

$$\boldsymbol{U}_{\text{SLL}}(\{\underline{n}\},\mu_h,\mu_s) = \int_{\mu_s}^{\mu_h}\frac{d\mu_1}{\mu_1}\int_{\mu_s}^{\mu_1}\frac{d\mu_2}{\mu_2}\int_{\mu_s}^{\mu_2}\frac{d\mu_3}{\mu_3}$$

$$\times\,\boldsymbol{U}_c(\mu_h,\mu_1)\,\gamma_{\text{cusp}}\big(\alpha_s(\mu_1)\big)\,\boldsymbol{V}^G\,\boldsymbol{U}_c(\mu_1,\mu_2)\,\gamma_{\text{cusp}}\big(\alpha_s(\mu_2)\big)\,\boldsymbol{V}^G\,\frac{\alpha_s(\mu_3)}{4\pi}\,\overline{\boldsymbol{\Gamma}}$$

- All double-logarithmic terms are exponentiated!
- One scale integral for each insertion of $\boldsymbol{V}^G$ and $\overline{\boldsymbol{\Gamma}}$
- Easy to include running-coupling effects
- Asymptotic behavior of $\boldsymbol{U}_c(\mu_i,\mu_j)$ determines the asymptotic behavior of the resummed series

A MORE POWERFUL FORMALISM

Rewrite the evolution kernel for the Glauber series

- Analogous relation holds for higher-order terms in the Glauber series (more $\boldsymbol{V}^G$ factors and additional integrals):

$$\boldsymbol{U}_{\rm SLL}^{(l)}(\{\underline{n}\},\mu_h,\mu_s) = \int_{\mu_s}^{\mu_h}\frac{d\mu_1}{\mu_1}\dots\int_{\mu_s}^{\mu_l}\frac{d\mu_{l+1}}{\mu_{l+1}}\left[\prod_{i=1}^{l}\boldsymbol{U}_c(\mu_{i-1},\mu_i)\,\gamma_{\rm cusp}\big(\alpha_s(\mu_i)\big)\,\boldsymbol{V}^G\right]\frac{\alpha_s(\mu_{l+1})}{4\pi}\,\overline{\boldsymbol{\Gamma}}$$

$$\boldsymbol{U}_{\rm SLL+G}(\{\underline{n}\},\mu_h,\mu_s) = \sum_{l=1}^{\infty}\boldsymbol{U}_{\rm SLL}^{(l)}(\{\underline{n}\},\mu_h,\mu_s)$$

- Structure share similarities with a parton shower, but the Sudakov operator and Glauber phases imply a non-trivial operator mixing in color space

A MORE POWERFUL FORMALISM

Introduce a color basis

- Simplest case of (anti-)quark-initiated scattering processes:

$$\boldsymbol{X}_1 = \sum_{j>2} J_j \, i f^{abc} \, \boldsymbol{T}_1^a \boldsymbol{T}_2^b \boldsymbol{T}_j^c \,, \qquad \boldsymbol{X}_4 = \frac{1}{N_c} J_{12} \, \boldsymbol{T}_1 \cdot \boldsymbol{T}_2 \,,$$

$$\boldsymbol{X}_2 = \sum_{j>2} J_j \, (\sigma_1 - \sigma_2) \, d^{abc} \, \boldsymbol{T}_1^a \boldsymbol{T}_2^b \boldsymbol{T}_j^c \,, \qquad \boldsymbol{X}_5 = J_{12} \, \boldsymbol{1} \,,$$

$$\boldsymbol{X}_3 = \frac{1}{N_c} \sum_{j>2} J_j \, (\boldsymbol{T}_1 - \boldsymbol{T}_2) \cdot \boldsymbol{T}_j \,,$$

where $\sigma_i = -1 \ (+1)$ for an initial-state quark (anti-quark), and all structures are normalized such that their trace with a hard function is at most of $O(N_c^0)$ in the large-N_c limit

A MORE POWERFUL FORMALISM

Introduce a color basis

- Represent $\boldsymbol{\Gamma}^c$, $\boldsymbol{V}^G$ and $\boldsymbol{V}^G\,\overline{\boldsymbol{\Gamma}}$ as objects acting in that basis:

$$\boldsymbol{\Gamma}^c \to N_c\,\mathbb{\Gamma}^c \qquad \text{with} \qquad \mathbb{\Gamma}^c = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -\frac{C_F}{N_c} & 0 & 0 \end{pmatrix}$$

Positive eigenvalues: {0, 1/2, 1}
(additional ones for initial-state gluons)

Recall:

$$\boldsymbol{U}_{\text{SLL}}(\{\underline{n}\},\mu_h,\mu_s) = \int_{\mu_s}^{\mu_h}\frac{d\mu_1}{\mu_1}\int_{\mu_s}^{\mu_1}\frac{d\mu_2}{\mu_2}\int_{\mu_s}^{\mu_2}\frac{d\mu_3}{\mu_3}$$

$$\times\,\boldsymbol{U}_c(\mu_h,\mu_1)\,\gamma_{\text{cusp}}\big(\alpha_s(\mu_1)\big)\,\boldsymbol{V}^G\,\boldsymbol{U}_c(\mu_1,\mu_2)\,\gamma_{\text{cusp}}\big(\alpha_s(\mu_2)\big)\,\boldsymbol{V}^G\,\frac{\alpha_s(\mu_3)}{4\pi}\,\overline{\boldsymbol{\Gamma}}$$

A MORE POWERFUL FORMALISM

Introduce a color basis

- Represent $\boldsymbol{\Gamma}^c$, $\boldsymbol{V}^G$ and $\boldsymbol{V}^G\,\overline{\boldsymbol{\Gamma}}$ as objects acting in that basis:

$$\boldsymbol{U}_c(\mu_i,\mu_j) \;\rightarrow\; \mathbb{U}_c(\mu_i,\mu_j) = \begin{pmatrix} U_c(1;\mu_i,\mu_j) & 0 & 0 & 0 & 0 \\ 0 & U_c(1;\mu_i,\mu_j) & 0 & 0 & 0 \\ 0 & 0 & U_c(\frac{1}{2};\mu_i,\mu_j) & 0 & 0 \\ 0 & 0 & 2\left[U_c(\frac{1}{2};\mu_i,\mu_j) - U_c(1;\mu_i,\mu_j)\right] & U_c(1;\mu_i,\mu_j) & 0 \\ 0 & 0 & \frac{2C_F}{N_c}\left[1 - U_c(\frac{1}{2};\mu_i,\mu_j)\right] & 0 & 1 \end{pmatrix}$$

Generalized Sudakov factors: $U_c(v;\mu_i,\mu_j) = \exp\left[v\,N_c \int_{\mu_j}^{\mu_i} \frac{d\mu}{\mu}\,\gamma_{\text{cusp}}\big(\alpha_s(\mu)\big)\ln\frac{\mu^2}{\mu_h^2}\right] \le 1$

Recall:

$$\boldsymbol{U}_{\text{SLL}}(\{\underline{n}\},\mu_h,\mu_s) = \int_{\mu_s}^{\mu_h}\frac{d\mu_1}{\mu_1}\int_{\mu_s}^{\mu_1}\frac{d\mu_2}{\mu_2}\int_{\mu_s}^{\mu_2}\frac{d\mu_3}{\mu_3}$$

$$\times\,\boldsymbol{U}_c(\mu_h,\mu_1)\,\gamma_{\text{cusp}}\big(\alpha_s(\mu_1)\big)\,\boldsymbol{V}^G\,\boldsymbol{U}_c(\mu_1,\mu_2)\,\gamma_{\text{cusp}}\big(\alpha_s(\mu_2)\big)\,\boldsymbol{V}^G\,\frac{\alpha_s(\mu_3)}{4\pi}\,\overline{\boldsymbol{\Gamma}}$$

double-log terms (SLLs)
always lead to suppression!

A MORE POWERFUL FORMALISM

Introduce a color basis

- Represent $\boldsymbol{\Gamma}^c$, $\boldsymbol{V}^G$ and $\boldsymbol{V}^G\,\overline{\boldsymbol{\Gamma}}$ as objects acting in that basis:

$$\boldsymbol{V}^G \to i\pi N_c \mathbb{V}^G \quad \text{with} \quad \mathbb{V}^G = \begin{pmatrix} 0 & -2\delta_{q\bar{q}}\,\frac{N_c^2-4}{N_c^2} & \frac{4}{N_c^2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Recall:

$$\boldsymbol{U}_{\rm SLL}(\{\underline{n}\},\mu_h,\mu_s) = \int_{\mu_s}^{\mu_h}\frac{d\mu_1}{\mu_1}\int_{\mu_s}^{\mu_1}\frac{d\mu_2}{\mu_2}\int_{\mu_s}^{\mu_2}\frac{d\mu_3}{\mu_3}$$

$$\times\,\boldsymbol{U}_c(\mu_h,\mu_1)\,\gamma_{\rm cusp}\big(\alpha_s(\mu_1)\big)\,\boldsymbol{V}^G\,\boldsymbol{U}_c(\mu_1,\mu_2)\,\gamma_{\rm cusp}\big(\alpha_s(\mu_2)\big)\,\boldsymbol{V}^G\,\frac{\alpha_s(\mu_3)}{4\pi}\,\overline{\boldsymbol{\Gamma}}$$

A MORE POWERFUL FORMALISM

Introduce a color basis

- Represent $\boldsymbol{\Gamma}^c, \boldsymbol{V}^G$ and $\boldsymbol{V}^G\,\overline{\boldsymbol{\Gamma}}$ as objects acting in that basis:

$$\boldsymbol{V}^G\,\overline{\boldsymbol{\Gamma}} \to 16i\pi X_1 \equiv 16i\pi X^T\varsigma \qquad \varsigma = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Recall:

$$\boldsymbol{U}_{\rm SLL}(\{\underline{n}\},\mu_h,\mu_s) = \int_{\mu_s}^{\mu_h}\frac{d\mu_1}{\mu_1}\int_{\mu_s}^{\mu_1}\frac{d\mu_2}{\mu_2}\int_{\mu_s}^{\mu_2}\frac{d\mu_3}{\mu_3}$$

$$\times\,\boldsymbol{U}_c(\mu_h,\mu_1)\,\gamma_{\rm cusp}\big(\alpha_s(\mu_1)\big)\,\boldsymbol{V}^G\,\boldsymbol{U}_c(\mu_1,\mu_2)\,\gamma_{\rm cusp}\big(\alpha_s(\mu_2)\big)\,\boldsymbol{V}^G\,\frac{\alpha_s(\mu_3)}{4\pi}\,\overline{\boldsymbol{\Gamma}}$$

A MORE POWERFUL FORMALISM

Introduce a color basis

- This yields:

$$\sigma_{2\to M}^{\text{SLL+G}}(Q_0) = \sum_{\text{partonic channels}} \int d\xi_1 \int d\xi_2\, f_1(\xi_1,\mu_s)\, f_2(\xi_2,\mu_s)$$

$$\times \sum_{l=1}^{\infty} \big\langle \underset{\uparrow}{\boldsymbol{\mathcal{H}}_{2\to M}(\mu_h)}\, \boldsymbol{X}^T \big\rangle\, \mathbb{U}_{\text{SLL}}^{(l)}(\mu_h,\mu_s)\, \varsigma\,,$$

5 process-dependent color traces

with:

$$\mathbb{U}_{\text{SLL}}^{(l)}(\mu_h,\mu_s) = 16\,(i\pi)^l\, N_c^{l-1} \int_{\mu_s}^{\mu_h} \frac{d\mu_1}{\mu_1} \dots \int_{\mu_s}^{\mu_l} \frac{d\mu_{l+1}}{\mu_{l+1}}\, \mathbb{U}_c(\mu_h,\mu_1)$$

$$\times \left[\prod_{i=1}^{l-1} \gamma_{\text{cusp}}\big(\alpha_s(\mu_i)\big)\, \mathbb{V}^G\, \mathbb{U}_c(\mu_i,\mu_{i+1}) \right] \gamma_{\text{cusp}}\big(\alpha_s(\mu_l)\big)\, \frac{\alpha_s(\mu_{l+1})}{4\pi}$$

A MORE POWERFUL FORMALISM

Perform the scale integrals in terms of the running coupling

- Generalized Sudakov factors in RG-improved perturbation theory:

$$U_c(v;\mu_i,\mu_j) = \exp\left[v\,N_c \int_{\mu_j}^{\mu_i} \frac{d\mu}{\mu}\,\gamma_{\rm cusp}\big(\alpha_s(\mu)\big)\ln\frac{\mu^2}{\mu_h^2}\right]$$

$$= \exp\left\{ \frac{\gamma_0\,v N_c}{2\beta_0^2}\left[\frac{4\pi}{\alpha_s(\mu_h)}\left(\frac{1}{x_i}-\frac{1}{x_j}-\ln\frac{x_j}{x_i}\right) + \left(\frac{\gamma_1}{\gamma_0}-\frac{\beta_1}{\beta_0}\right)\left(x_i - x_j + \ln\frac{x_j}{x_i}\right) + \frac{\beta_1}{2\beta_0}\left(\ln^2 x_j - \ln^2 x_i\right)\right]\right\}$$

2-loop cusp anomalous dimension
and β-function

with $x_i \equiv \alpha_s(\mu_i)/\alpha_s(\mu_h)$ and:

$$U_c(v;\mu_i,\mu_j)\,U_c(v;\mu_j,\mu_k) = U_c(v;\mu_i,\mu_k)\,, \qquad U_c(0;\mu_i,\mu_j) = 1$$

- Encounter products of Sudakov factors:

$$U_c(v_1,\dots,v_l;\mu_h,\mu_1,\dots,\mu_l) \equiv U_c(v_1;\mu_h,\mu_1)\,U_c(v_2;\mu_1,\mu_2)\dots U_c(v_l;\mu_{l-1},\mu_l)$$

A MORE POWERFUL FORMALISM

Evolution functions with two and four Glauber insertions

- *l*=2:

$$\mathbb{U}^{(2)}_{\rm SLL}(\mu_h,\mu_s)\,\varsigma = -\frac{32\pi^2}{\beta_0^3}\,N_c \int_1^{x_s}\frac{dx_2}{x_2}\,\ln\frac{x_s}{x_2}\int_1^{x_2}\frac{dx_1}{x_1}\begin{pmatrix} 0 \\ -\frac{1}{2}\,U_c(1;\mu_h,\mu_2) \\ U_c(\frac{1}{2},1;\mu_h,\mu_1,\mu_2) \\ 2\left[U_c(\frac{1}{2},1;\mu_h,\mu_1,\mu_2) - U_c(1;\mu_h,\mu_2)\right] \\ \frac{2C_F}{N_c}\left[U_c(1;\mu_1,\mu_2) - U_c(\frac{1}{2},1;\mu_h,\mu_1,\mu_2)\right] \end{pmatrix}$$

- *l*=4

$$\mathbb{U}^{(4)}_{\rm SLL}(\mu_h,\mu_s)\,\varsigma = \frac{128\pi^4}{\beta_0^5}\,N_c^3 \int_1^{x_s}\frac{dx_4}{x_4}\,\ln\frac{x_s}{x_4}\int_1^{x_4}\frac{dx_3}{x_3}\int_1^{x_3}\frac{dx_2}{x_2}\int_1^{x_2}\frac{dx_1}{x_1}\begin{pmatrix} 0 \\ -\frac{1}{2}\left[K_{12}\,U_c(1;\mu_h,\mu_4) + \frac{4}{N_c^2}\,U_c(1,\frac{1}{2},1;\mu_h,\mu_2,\mu_3,\mu_4)\right] \\ K_{12}\,U_c(\frac{1}{2},1;\mu_h,\mu_1,\mu_4) + \frac{4}{N_c^2}\,U_c(\frac{1}{2},1,\frac{1}{2},1;\mu_h,\mu_1,\mu_2,\mu_3,\mu_4) \\ 2\left[K_{12}\,U_c(\frac{1}{2},1;\mu_h,\mu_1,\mu_4) + \frac{4}{N_c^2}\,U_c(\frac{1}{2},1,\frac{1}{2},1;\mu_h,\mu_1,\mu_2,\mu_3,\mu_4)\right] \\ -2\left[K_{12}\,U_c(1;\mu_h,\mu_4) + \frac{4}{N_c^2}\,U_c(1,\frac{1}{2},1;\mu_h,\mu_2,\mu_3,\mu_4)\right] \\ \frac{2C_F}{N_c}\left[K_{12}\,U_c(1;\mu_1,\mu_4) + \frac{4}{N_c^2}\,U_c(1,\frac{1}{2},1;\mu_1,\mu_2,\mu_3,\mu_4)\right] \\ -\frac{2C_F}{N_c}\left[K_{12}\,U_c(\frac{1}{2},1;\mu_h,\mu_1,\mu_4) + \frac{4}{N_c^2}\,U_c(\frac{1}{2},1,\frac{1}{2},1;\mu_h,\mu_1,\mu_2,\mu_3,\mu_4)\right] \end{pmatrix}$$

$$K_{12} \equiv (\sigma_1-\sigma_2)^2\,\frac{N_c^2-4}{4N_c^2} = \frac{N_c^2-4}{N_c^2}\,\delta_{q\bar q}$$

A MORE POWERFUL FORMALISM

Resummation of the Glauber series in the large-N_c limit

- Closed analytic expression in terms of a double integral:

$$\sum_{l=2,4,6,\dots} \mathbb{U}_{\mathrm{SLL}}^{(l)}(\mu_h,\mu_s)\,\varsigma = -\frac{32\pi^2 N_c}{\beta_0^3}\int_1^{x_s}\frac{dx}{x}\,\ln\frac{x_s}{x}\int_1^{x}\frac{dx_1}{x_1}\left[1-2\delta_{q\bar{q}}\,\sin^2\left(\frac{\pi N_c}{\beta_0}\,\ln\frac{x}{x_1}\right)\right]\begin{pmatrix} 0 \\ -\frac{1}{2}\,U_c(1;\mu_h,\mu_x) \\ U_c(\frac{1}{2},1;\mu_h,\mu_1,\mu_x) \\ 2\left[U_c(\frac{1}{2},1;\mu_h,\mu_1,\mu_x)-U_c(1;\mu_h,\mu_x)\right] \\ \frac{2C_F}{N_c}\left[U_c(1;\mu_1,\mu_x)-U_c(\frac{1}{2},1;\mu_h,\mu_1,\mu_x)\right] \end{pmatrix}$$

- Super-leading logarithms (l=2 term) are exact

- First RG-improved resummation of SLLs and the Glauber series!

- Analogous results can be derived for processes with initial-state gluons

ASYMPTOTIC BEHAVIOR

$$L_s = \ln\frac{\mu_h}{\mu_s} \approx \ln\frac{Q}{Q_0}$$

Asymptotics for $\alpha_s L_s \sim 1\,,\ \alpha_s L_s^2 \gg 1$ derived using a fixed coupling

- Analytic expression in terms of Σ-functions:

$$\mathbb{U}_{\text{SLL}}^{(2)}(\mu_h,\mu_s)\,\varsigma = -\frac{2\pi^2}{3}\,N_c\left(\frac{\alpha_s}{\pi}\,L_s\right)^3 \begin{pmatrix} 0 \\ -\frac{1}{2}\,\Sigma(1,1;w) \\ \Sigma(\frac{1}{2},1;w) \\ 2\left[\Sigma(\frac{1}{2},1;w)-\Sigma(1,1;w)\right] \\ \frac{2C_F}{N_c}\left[\Sigma(0,1;w)-\Sigma(\frac{1}{2},1;w)\right] \end{pmatrix}$$

Kampé de Fériet functions

$$w = \frac{N_c\,\alpha_s(\bar{\mu})}{\pi}\,L_s^2$$

$$\mathbb{U}_{\text{SLL}}^{(4)}(\mu_h,\mu_s)\,\varsigma = \frac{\pi^4}{30}\,N_c^3\left(\frac{\alpha_s}{\pi}\,L_s\right)^5 \begin{pmatrix} 0 \\ -\frac{1}{2}\left[K_{12}\,\Sigma(1,1,1,1;w)+\frac{4}{N_c^2}\,\Sigma(1,1,\frac{1}{2},1;w)\right] \\ K_{12}\,\Sigma(\frac{1}{2},1,1,1;w)+\frac{4}{N_c^2}\,\Sigma(\frac{1}{2},1,\frac{1}{2},1;w) \\ 2\left[K_{12}\,\Sigma(\frac{1}{2},1,1,1;w)+\frac{4}{N_c^2}\,\Sigma(\frac{1}{2},1,\frac{1}{2},1;w)\right] \\ -2\left[K_{12}\,\Sigma(1,1,1,1;w)+\frac{4}{N_c^2}\,\Sigma(1,1,\frac{1}{2},1;w)\right] \\ \frac{2C_F}{N_c}\left[K_{12}\,\Sigma(0,1,1,1;w)+\frac{4}{N_c^2}\,\Sigma(0,1,\frac{1}{2},1;w)\right] \\ -\frac{2C_F}{N_c}\left[K_{12}\,\Sigma(\frac{1}{2},1,1,1;w)+\frac{4}{N_c^2}\,\Sigma(\frac{1}{2},1,\frac{1}{2},1;w)\right] \end{pmatrix}$$

integrals of
Kampé de Fériet functions

ASYMPTOTIC BEHAVIOR

$L_s = \ln\frac{\mu_h}{\mu_s} \approx \ln\frac{Q}{Q_0}$

Asymptotics for $\alpha_s L_s \sim 1\,,\ \alpha_s L_s^2 \gg 1$ derived using a fixed coupling

- Analytic expression in terms of Σ-functions:

$$\mathbb{U}^{(2)}_{\rm SLL}(\mu_h,\mu_s)\,\varsigma = -\frac{2\pi^2}{3}\,N_c\left(\frac{\alpha_s}{\pi}\,L_s\right)^3 \left(\begin{array}{r} - \\ \Sigma \\ 2\,\big[\Sigma(\tfrac{1}{2},1 \\ \tfrac{2C_F}{N_c}\,\big[\Sigma(0, \end{array}\right.$$

$$\Sigma(1,1;w) = \frac{3}{w} - \frac{3\sqrt{\pi}}{2w^{3/2}} + \mathcal{O}(e^{-w})\,,$$

$$\Sigma(\tfrac{1}{2},1;w) = \frac{3\sqrt{2}\,\ln(1+\sqrt{2})}{w} - \frac{3\sqrt{\pi}}{\sqrt{2}\,w^{3/2}} + \mathcal{O}(w^{-2})\,,$$

$$\Sigma(0,1;w) = \frac{3}{2}\,\frac{\ln(4w)+\gamma_E-2}{w} + \frac{3}{4w^2} + \mathcal{O}(w^{-3})\,,$$

$$\mathbb{U}^{(4)}_{\rm SLL}(\mu_h,\mu_s)\,\varsigma = \frac{\pi^4}{30}\,N_c^3\left(\frac{\alpha_s}{\pi}\,L_s\right)^5 \left(\begin{array}{r} -\tfrac{1}{2}\,\Big[K_{12}\,\Sigma(\\ K_{12}\,\Sigma(\tfrac{1}{2}, \\ 2\,\Big[K_{12}\,\Sigma(\tfrac{1}{2} \\ -2\,\Big[K_{12}\,\Sigma(\\ \tfrac{2C_F}{N_c}\,\Big[K_{12}\,\Sigma(\\ -\tfrac{2C_F}{N_c}\,\Big[K_{12}\,\Sigma \end{array}\right.$$

$$\Sigma(1,1,1,1;w) = \frac{10}{w^2} - \frac{15\sqrt{\pi}}{2w^{5/2}} + \mathcal{O}(e^{-w})\,,$$

$$\Sigma(\tfrac{1}{2},1,1,1;w) = \frac{15\,\big[2-\sqrt{2}\,\ln(1+\sqrt{2})\big]}{w^2} - \frac{15\sqrt{\pi}\,(2-\sqrt{2})}{w^{5/2}} + \mathcal{O}(e^{-\frac{w}{2}})\,,$$

$$\Sigma(0,1,1,1;w) = \frac{15}{w^2} - \frac{15\sqrt{\pi}}{w^{5/2}} + \mathcal{O}(w^{-3})\,,$$

$$\Sigma(1,1,\tfrac{1}{2},1;w) = \frac{60}{w^2}\,\Big[\sqrt{2}\ln(1+\sqrt{2})-1\Big] - \frac{30\sqrt{\pi}\,(\sqrt{2}-1)}{w^{5/2}} + \mathcal{O}(e^{-\frac{w}{2}})\,,$$

$$\Sigma(\tfrac{1}{2},1,\tfrac{1}{2},1;w) = \frac{15\sqrt{2}}{w^2}\left[\frac{5\pi^2}{4} - \frac{3}{2}\ln^2 2 - 12\,{\rm Li}_2\!\left(\frac{1}{\sqrt{2}}\right)\right] - \frac{15\sqrt{2\pi}\,\ln 2}{w^{5/2}} + \mathcal{O}(e^{-\frac{w}{2}})\,,$$

$$\Sigma(0,1,\tfrac{1}{2},1;w) = \frac{30\,\ln^2(1+\sqrt{2})}{w^2} - \frac{30\sqrt{\pi}\,\ln(1+\sqrt{2})}{w^{5/2}} + \mathcal{O}(w^{-3})\,.$$

ASYMPTOTIC BEHAVIOR

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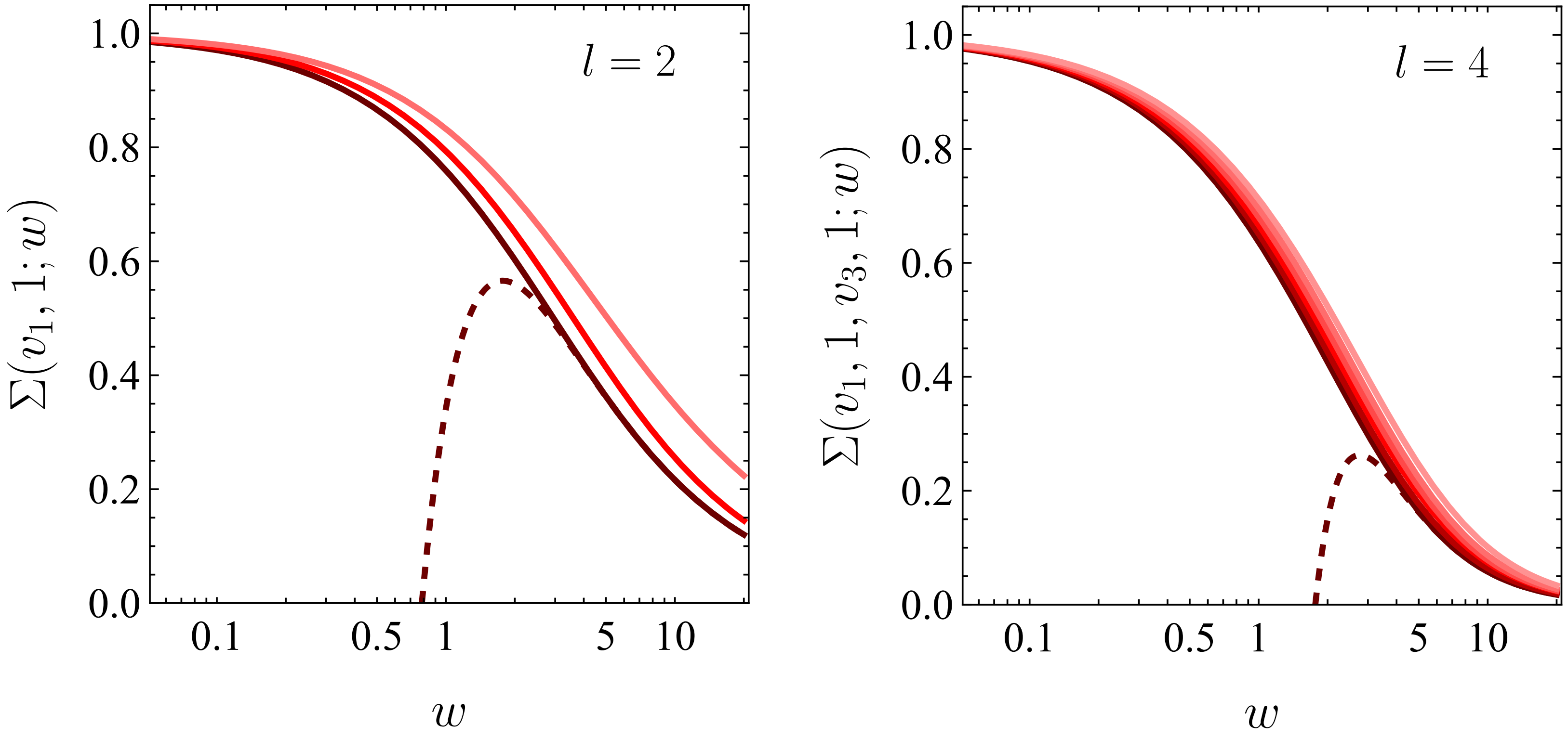
Asymptotics for $\alpha_s L_s \sim 1\,,\ \alpha_s L_s^2 \gg 1$ derived using a fixed coupling

- Analytic expression in terms of Σ-functions:

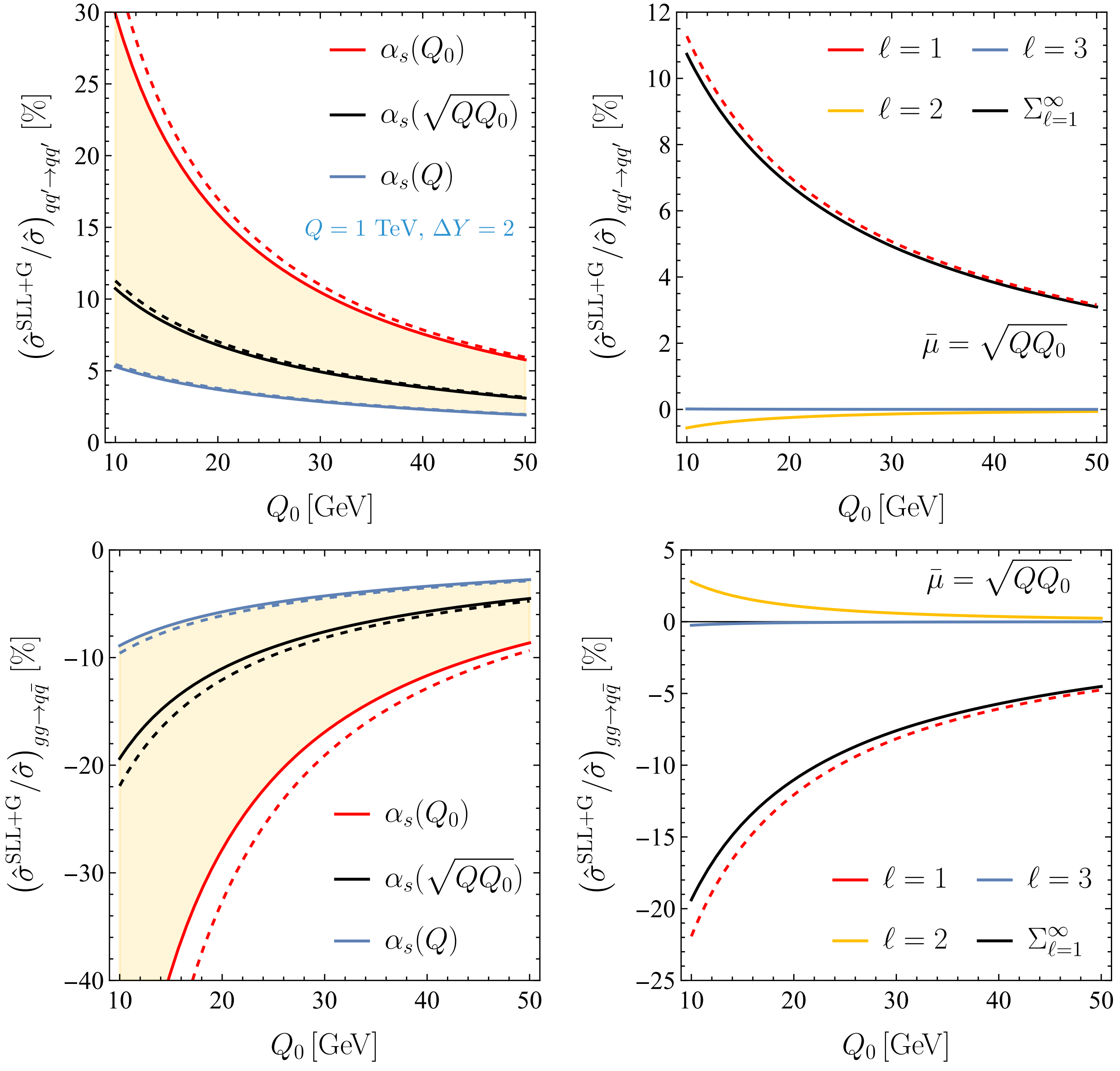


- Parametric suppression:

$$\mathbb{U}_{\mathrm{SLL}}^{(l)}(\mu_h, \mu_s) \sim \frac{(i\pi)^l}{(l+1)!}\, N_c^{l-1} \left(\frac{\alpha_s L_s}{\pi}\right)^{l+1} \frac{1}{w^{l/2}}$$

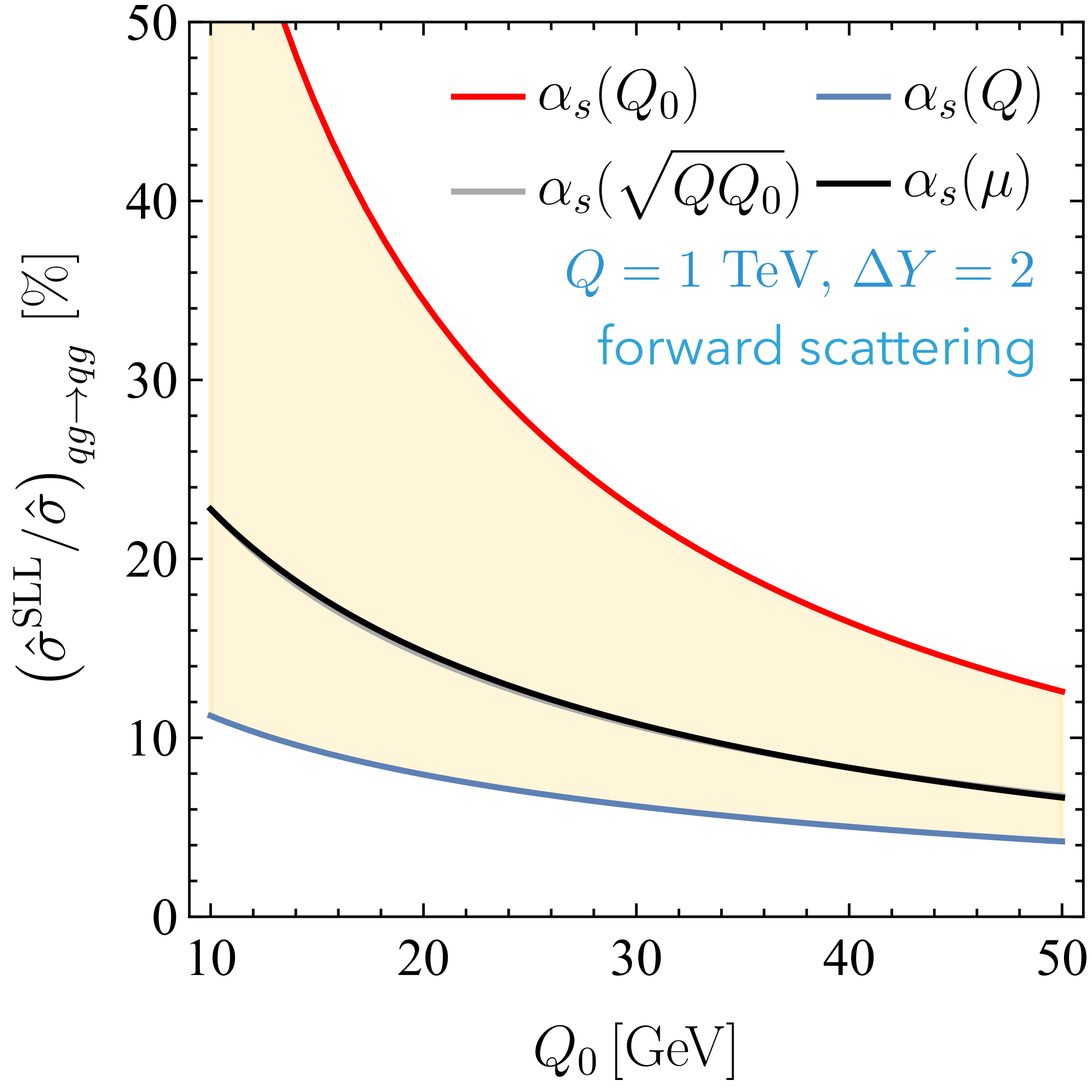
PHENOMENOLOGICAL IMPACT (PARTON LEVEL)

Suppression of higher Glauber contributions now explained



PHENOMENOLOGICAL IMPACT OF RG IMPROVEMENT

Partonic channels contributing to $pp \to$ 2 jets (gap between jets)



PHENOMENOLOGICAL IMPACT OF RG IMPROVEMENT

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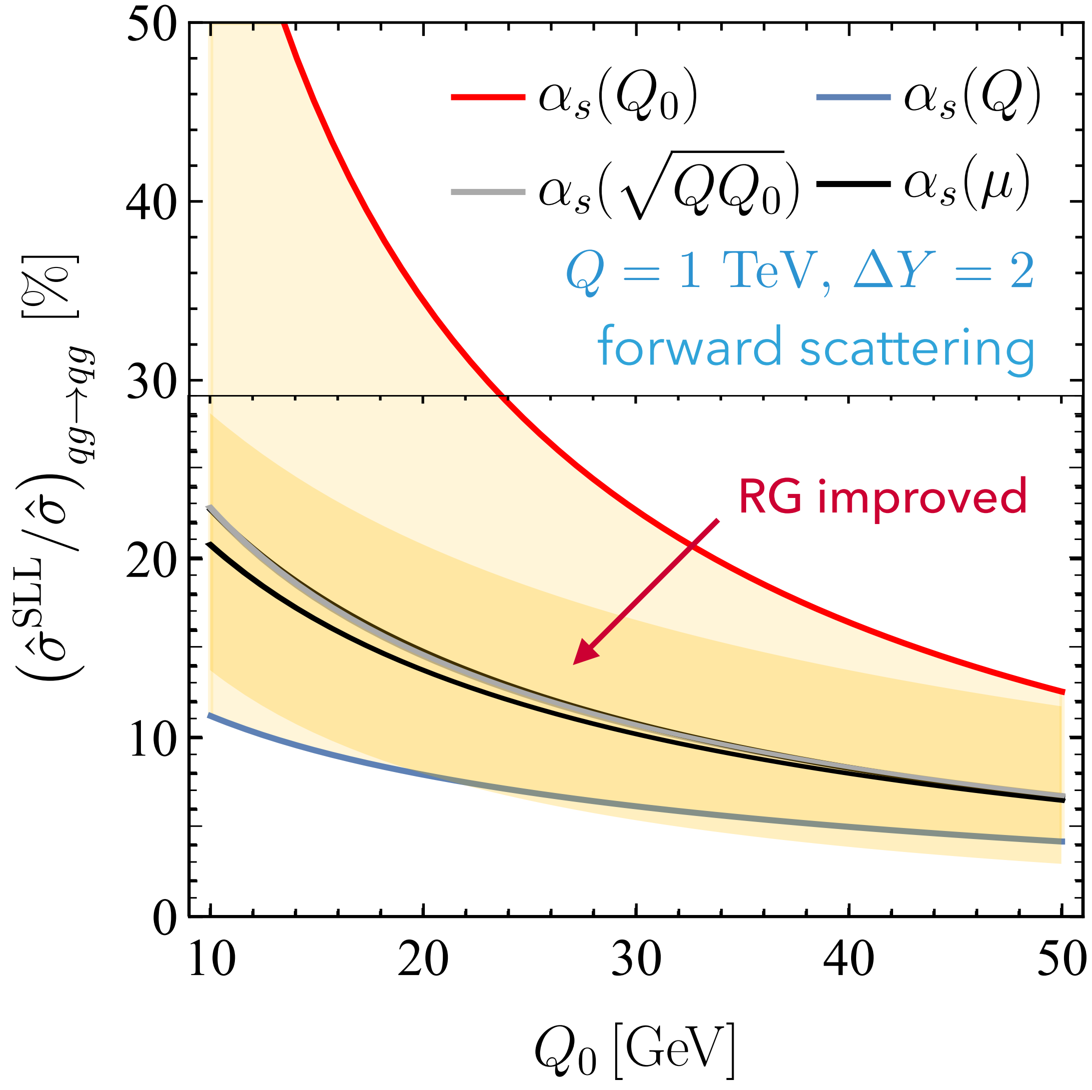

FUTURE CHALLENGES

Important open questions

- How to include multiple soft emissions (single-log effects), and how large is their effect? Does large-N_c help?

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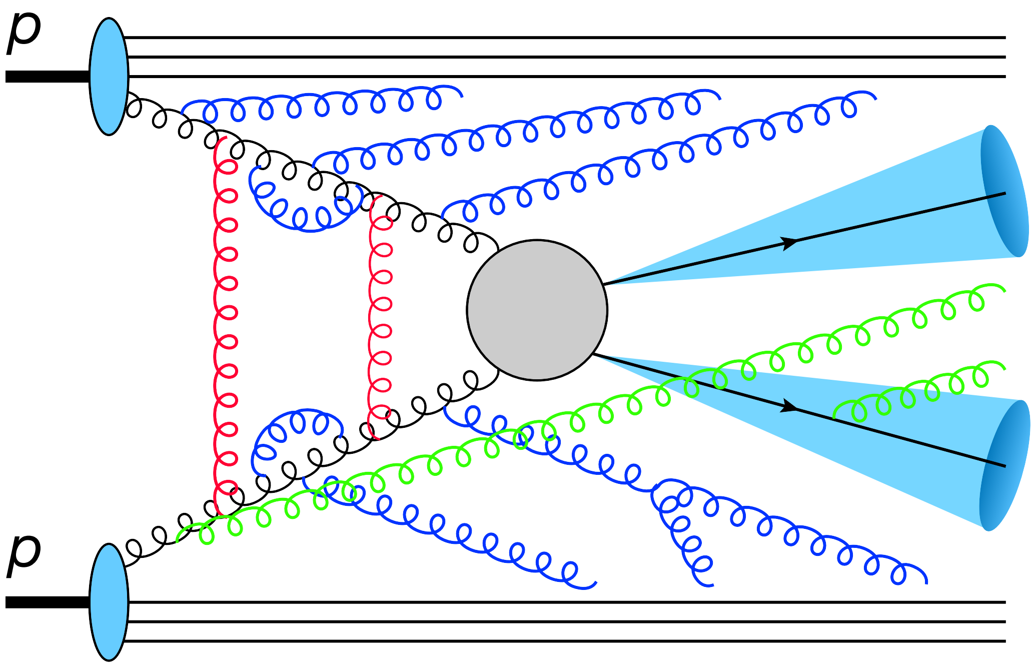

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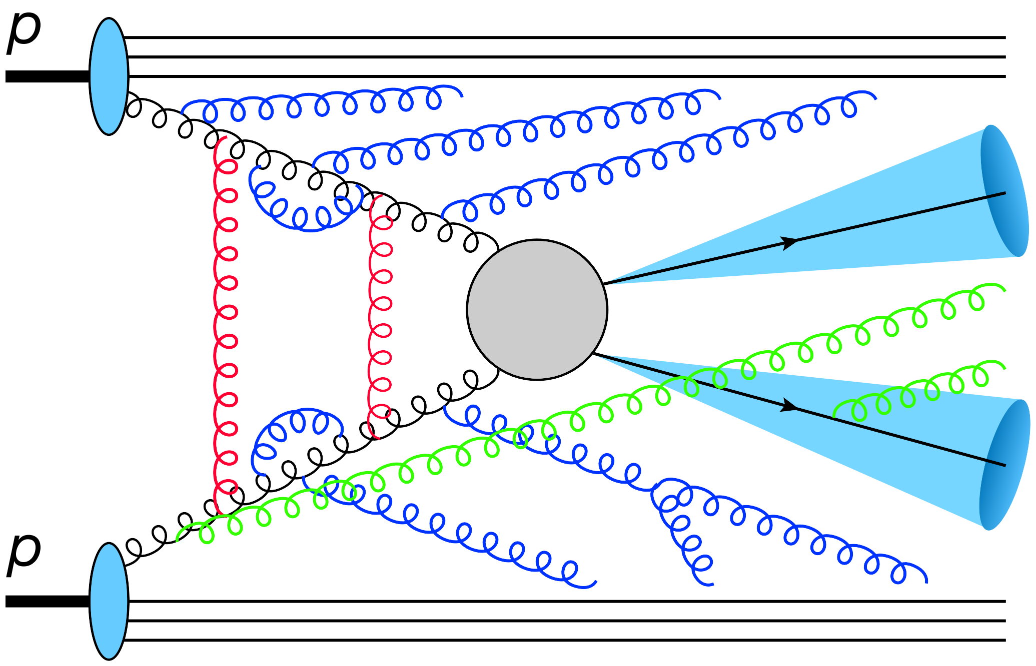
- How to include multiple soft emissions (single-log effects), and how large is their effect? Does large-N_c help?
- Can collinear factorization violations be understood in a quantitative way, and at which scale (Q_0 or Λ_{QCD}) do they occur?
- Implications for LHC phenomenology?



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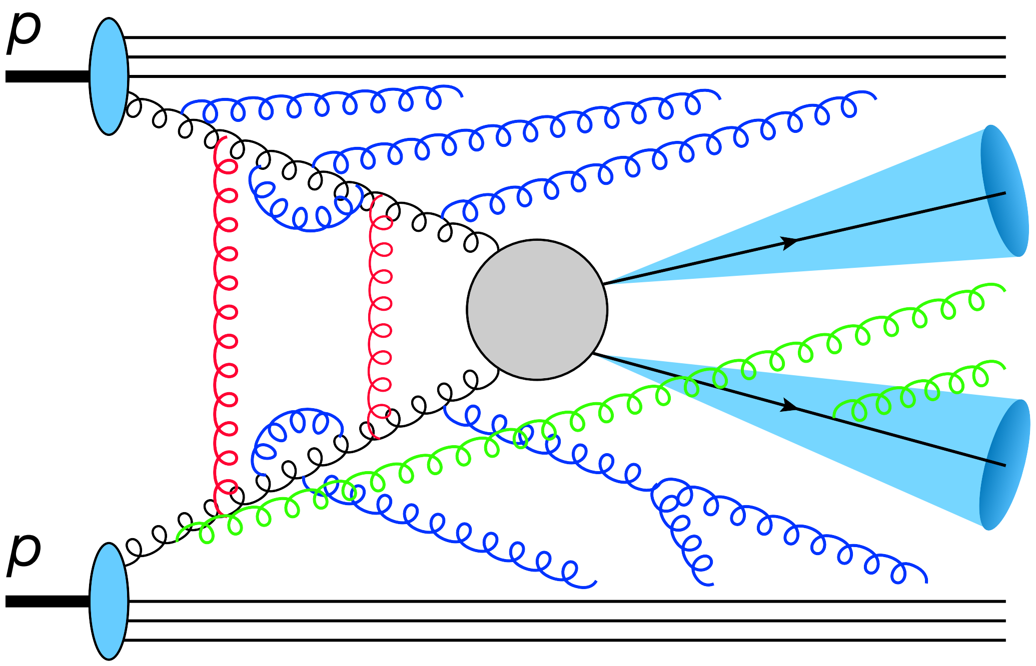
Important open questions

- How to include multiple soft emissions (single-log effects), and how large is their effect? Does large-N_c help?



- Can collinear factorization violations be understood in a quantitative way, and at which scale (Q_0 or Λ_{QCD}) do they occur?
- Implications for LHC phenomenology?
- Our analytical results may be relevant for validations of parton showers with quantum interference

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R.A. Martínez, M. De Angelis, J.R. Forshaw, S. Plätzer, M.H. Seymour (2018); J.R. Forshaw, J. Holguin, S. Plätzer (2019–2022)
M. Dasgupta, F.A. Dreyer, K. Hamilton, P.F. Monni, G.P. Salam (2020); M. van Beekveld, S.F. Ravasio, G.P. Salam, A. Soto-Ontoso, G. Soyez *et al.* (2022, 2023)