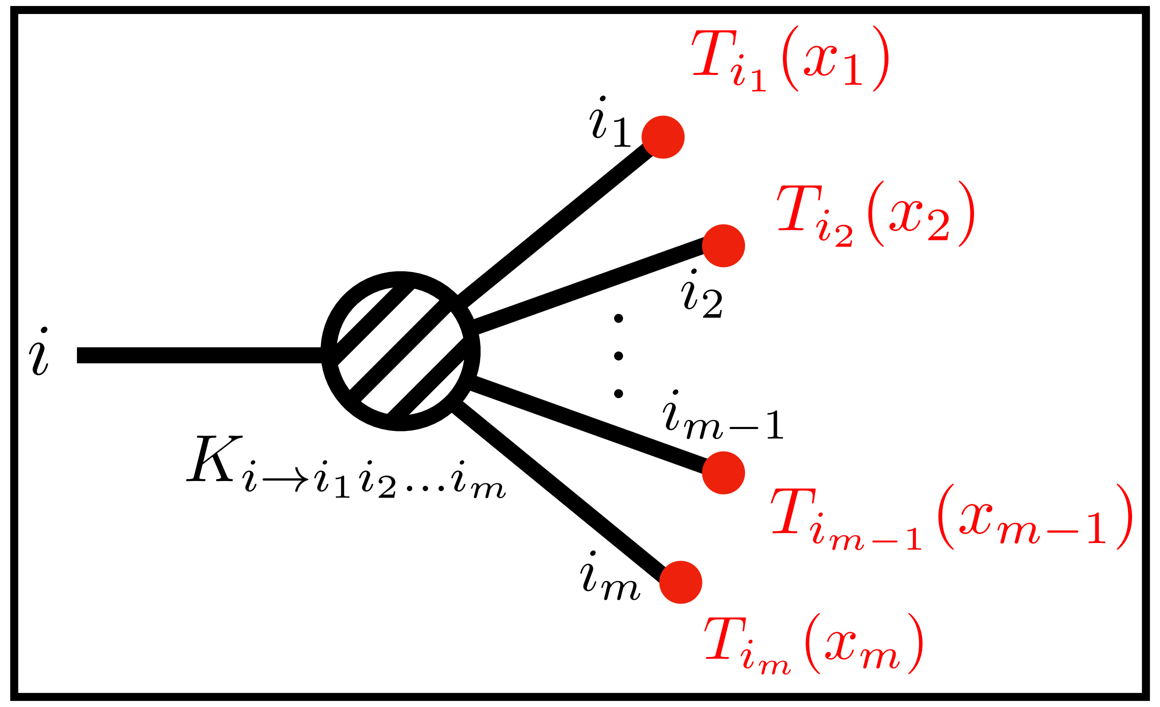
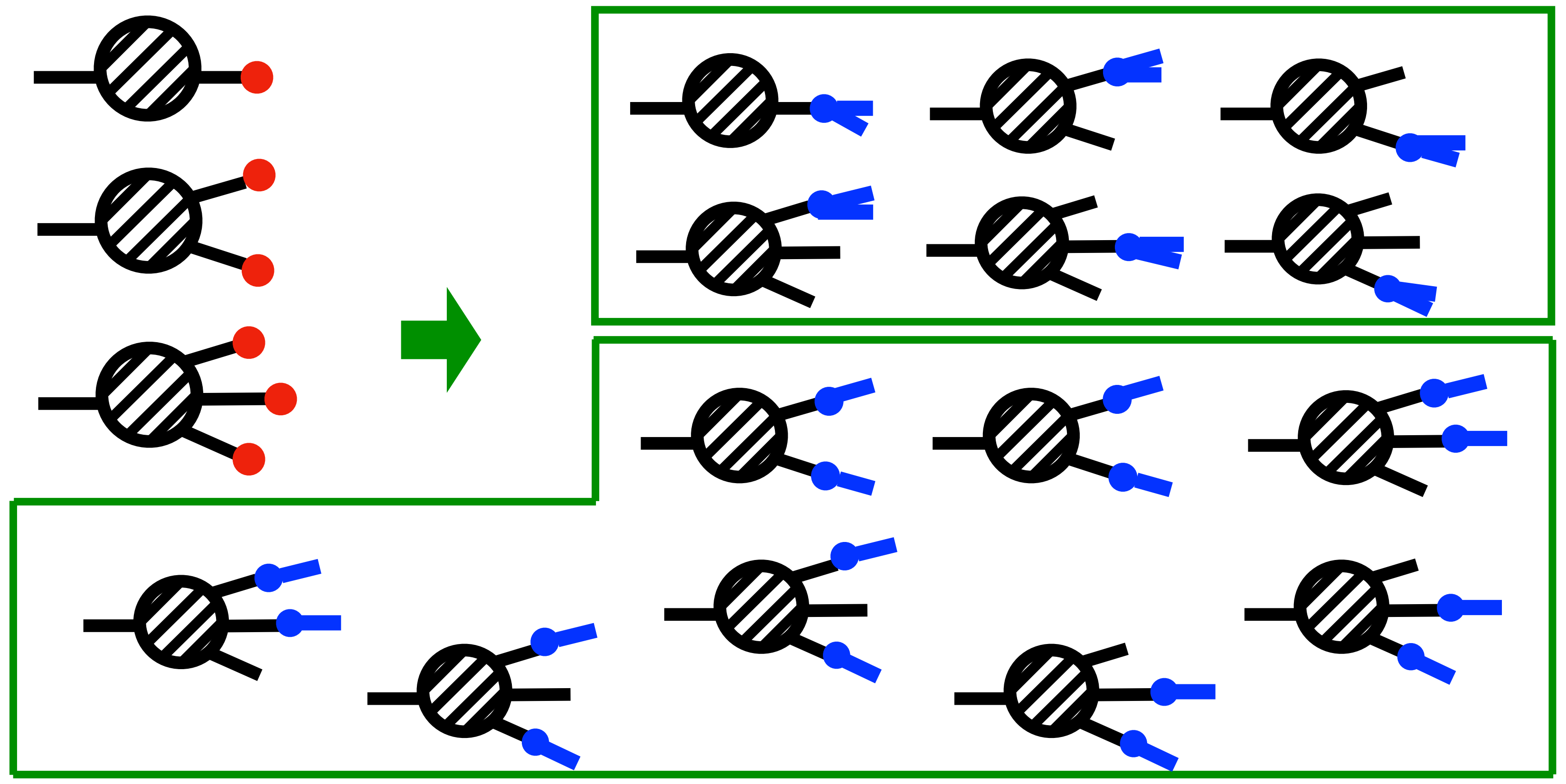
Extending Precision Perturbative QCD with Track Functions

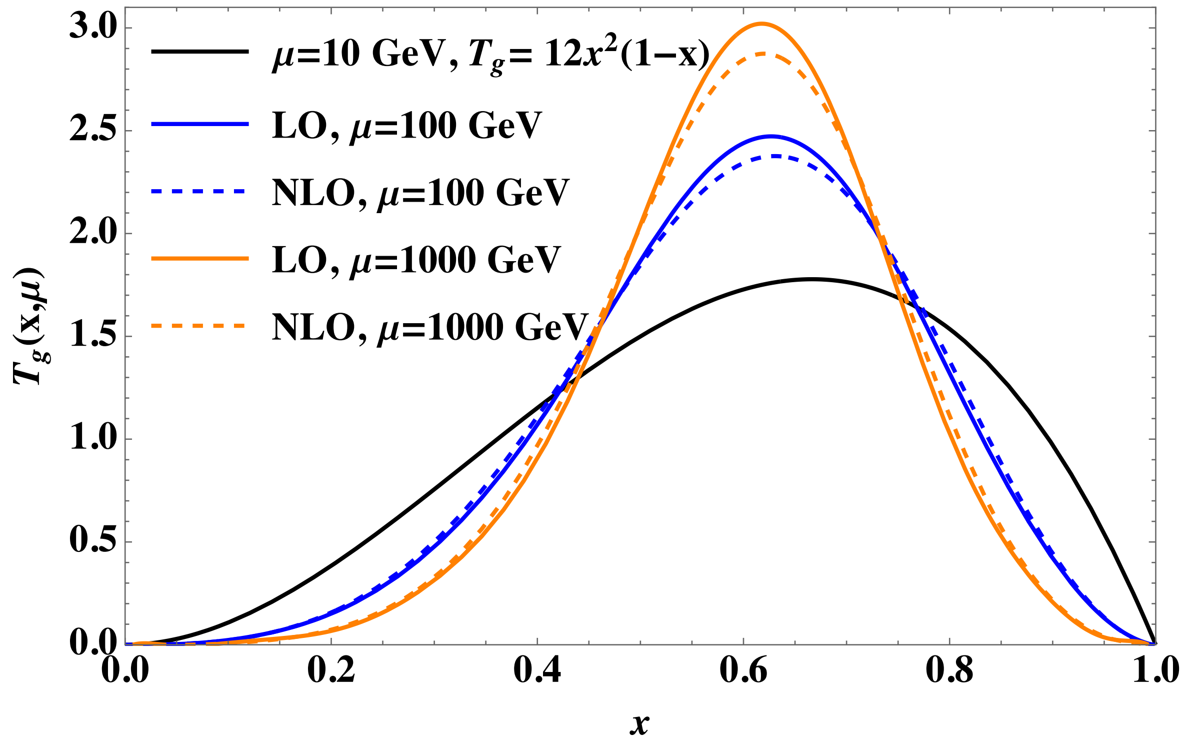
Yibei Li
Zhejiang University

Outline

- Introduction
- Incorporating track functions
- Track function evolution
- Calculational techniques & NLO results
- Reduction to DGLAP and multi-hadron fragmentation
- Numerical results







Advances & Challenges

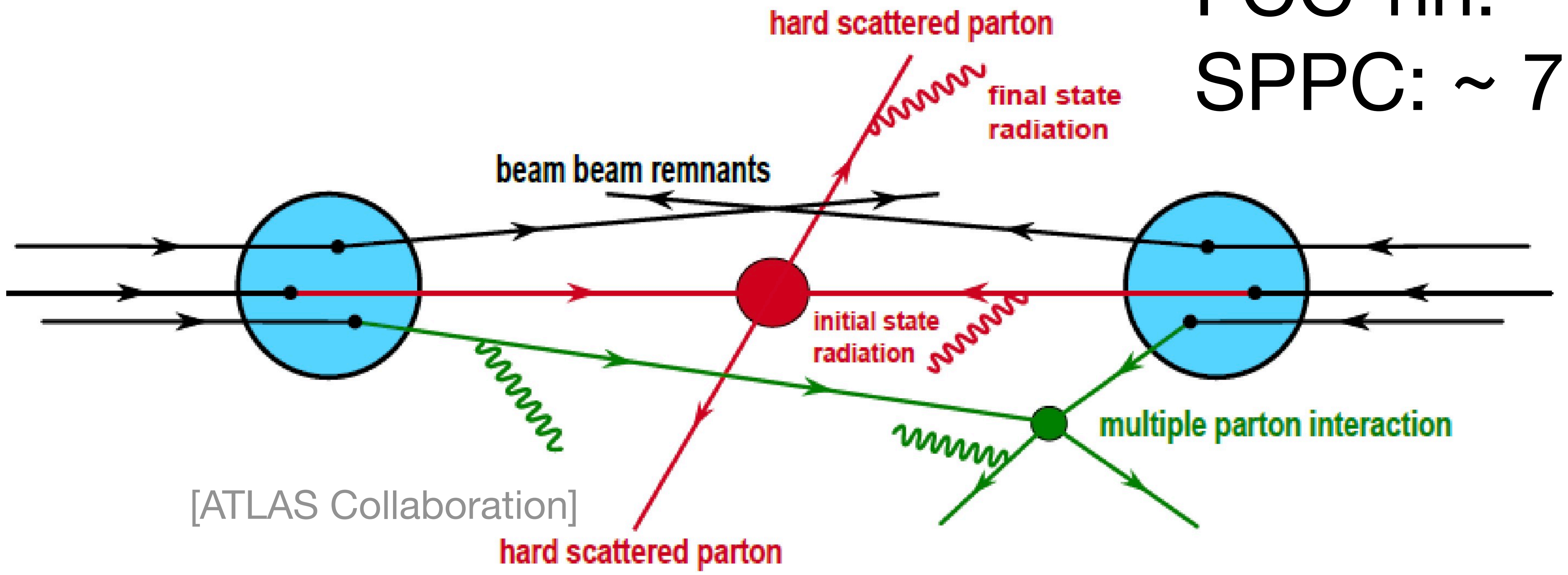
[S. Höche, arXiv:1411.4085]

LHC: ~ 14 TeV

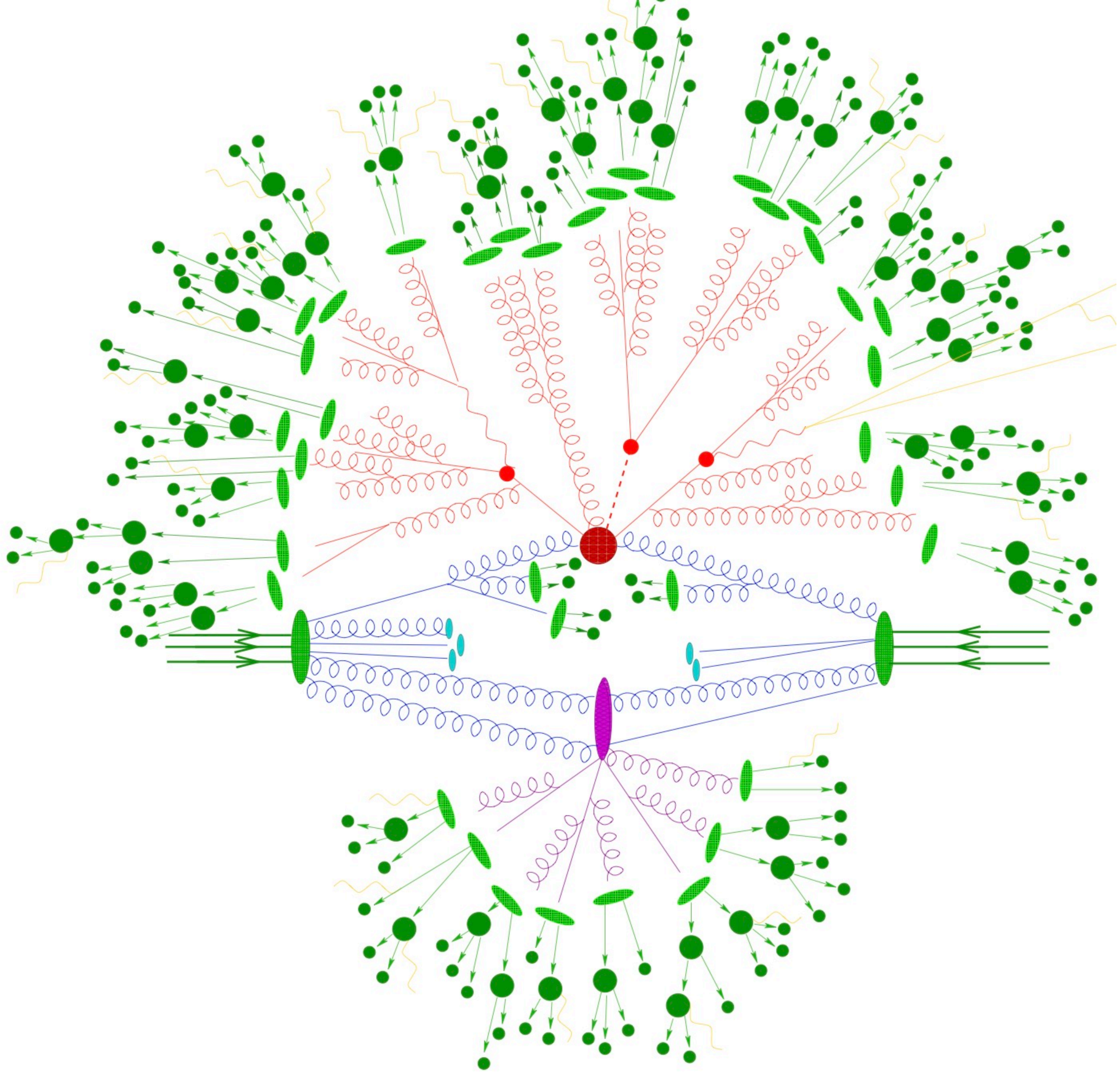
Future colliders

FCC-hh: ~ 100 TeV

SPPC: ~ 75 TeV



[ATLAS Collaboration]



Jet substructure observables

- Jet mass
- Jet angularity, thrust, broadening
- Energy correlation function observable
- N-subjettiness
- Electric charge of a jet

…

Use **track-based observables**!

(experimentally cleaner to measure)

Techniques:
Jet grooming, jet tagging, …

Track Functions

Motivation

- Track-based measurements offer:
  - superior angular resolution
  - pileup mitigation.
- One problem: Track-based calculations are not IR safe in perturbation theory.

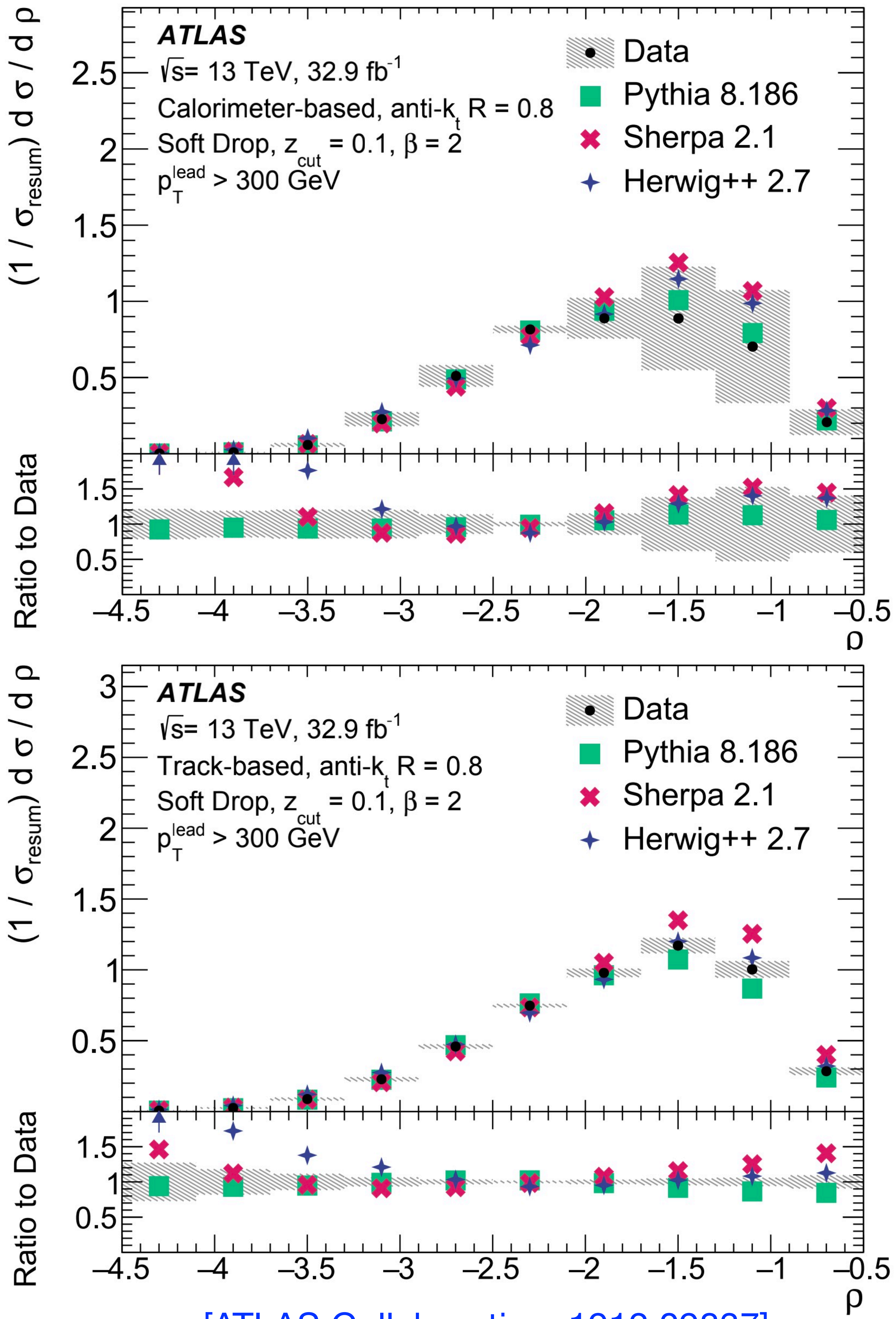
Track Functions

- IR divergences are absorbed into these **universal** non-perturbative functions.

(like the case of parton distribution functions and fragmentation functions)

calorimeter-based (all-particle)

track-based (charged-particle)



[ATLAS Collaboration, 1912.09837]

✓ Track functions introduced and studied at $\mathcal{O}(\alpha_s)$.

[H. Chang, M. Procura, J. Thaler, W. Waalewijn, arXiv:1303.6637, arXiv:1306.6630]

- Implementing track functions complicates perturbative calculations, which hinders people to apply that to higher order, while experimentalists urge predictions on tracks.

Eg. NNNLL+NNLO for all-particle thrusts but NLL+NLO for track thrusts.

observables. For all of these observables, the uncertainties for the track-based observables are significantly smaller than those for the calorimeter-based observables, particularly for higher values of β, where more soft radiation is included within the jet. However, since no track-based calculations exist at the present time, calorimeter-based measurements are still useful for precision QCD studies.

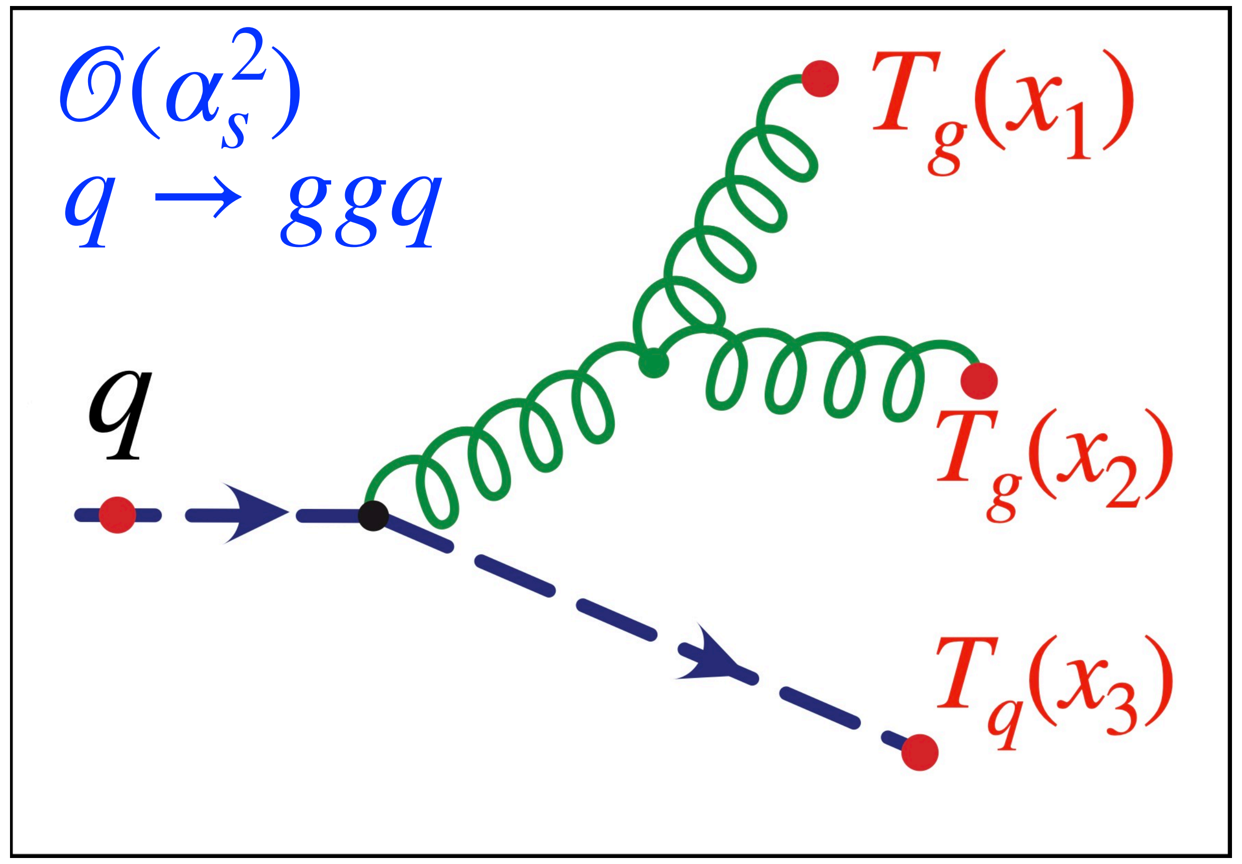
[ATLAS Collaboration, 1912.09837]

the selection of charged particle jets. Note that track-based observables are IRC-unsafe. In general, nonperturbative track functions can be used to directly compare track-based measurements to analytical calculations [67–69]; however, such an approach has not yet been developed for jet angularities. Two

[ALICE Collaboration, 2107.11303]

- The complication due to the RGE: The track function evolution encodes correlations in the hadronization process.

➡ collinear evolution beyond DGLAP



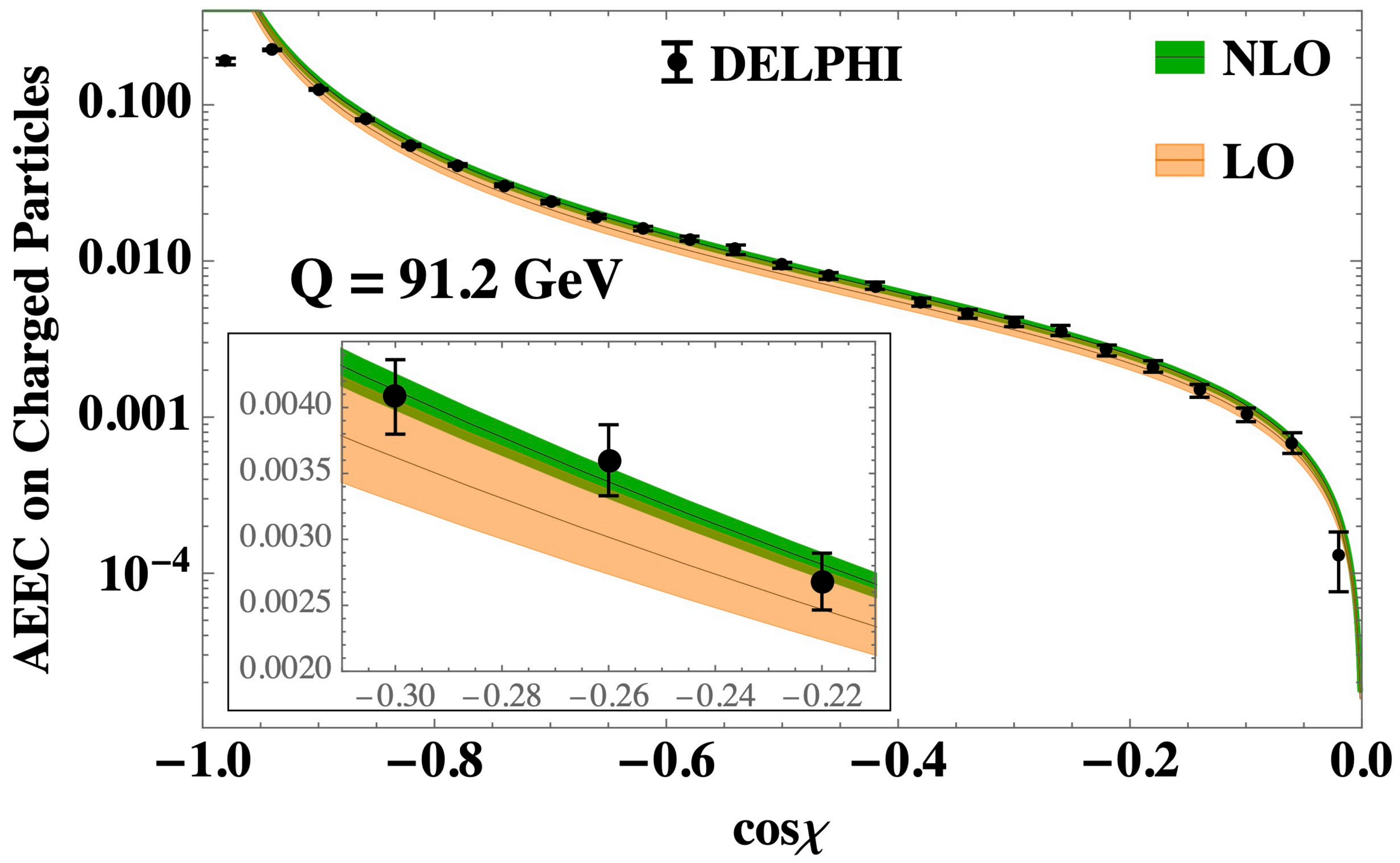
✓ Our work: Track function formalism beyond leading order.

[**Y. Li**, I. Moult, S. S. van Velzen, W. Waalewijn, H. X. Zhu, arXiv:2108.01674; M. Jaarsma, **Y. Li**, I. Moult, W. Waalewijn, H. X. Zhu, arXiv:2201.05166]

- Energy correlators are much simpler to interface with track functions.
- Moments of track functions have simple evolution.

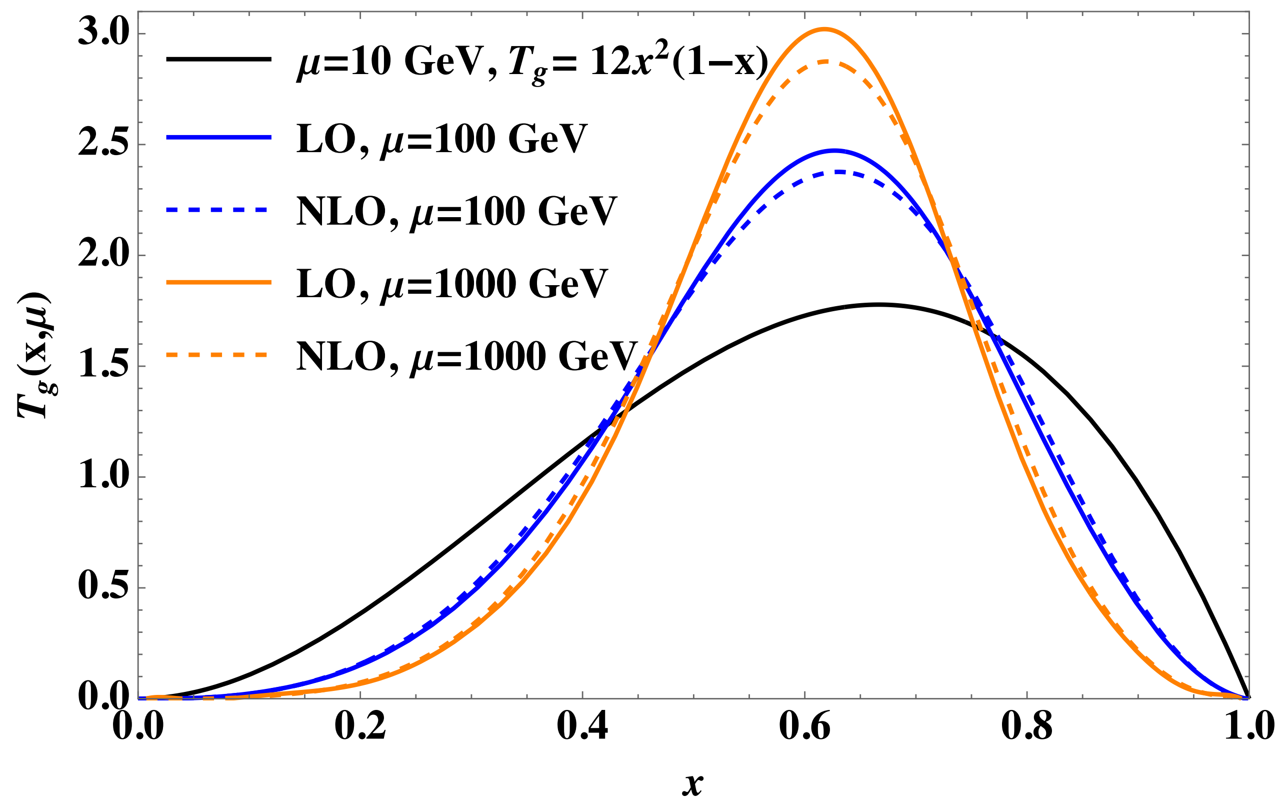
higher-order calculation

✦ Evolution of track functions in moment space and track EEC at $\mathcal{O}(\alpha_s^2)$.

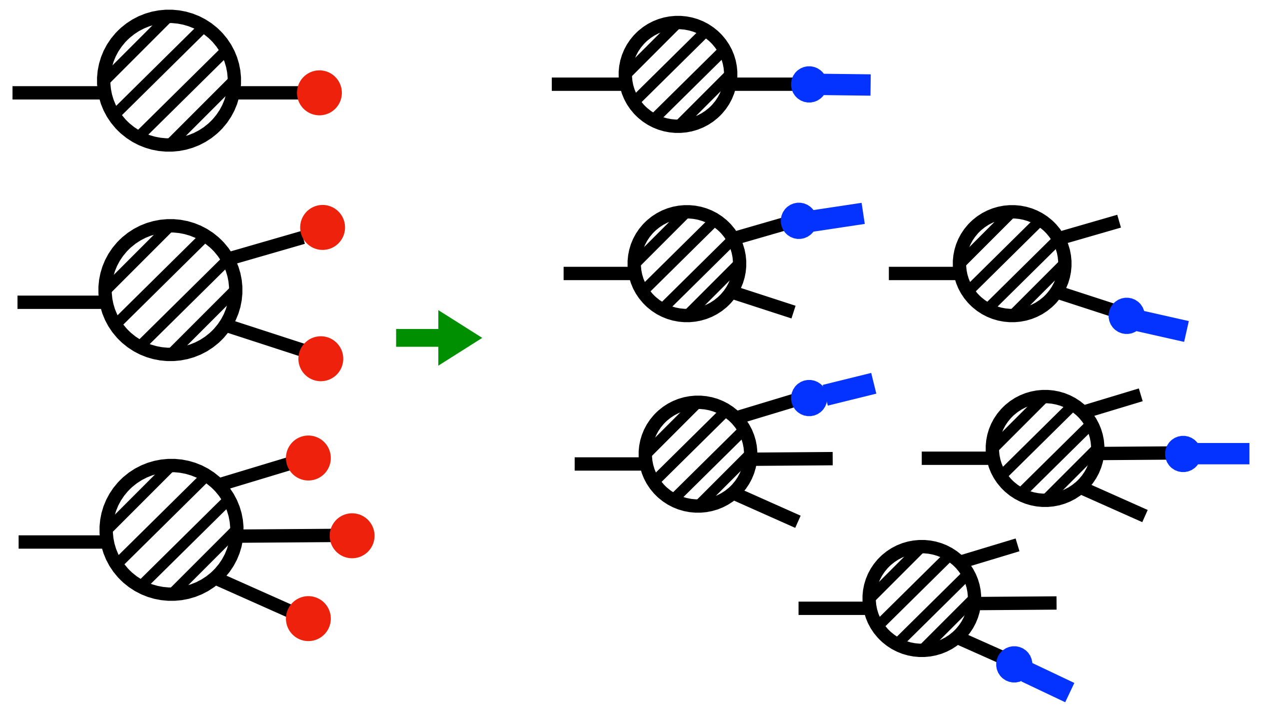


[H. Chen, **Y. Li**, M. Jaarsma, I. Moult, W. Waalewijn, H. X. Zhu, arXiv:2210.10058, arXiv:2210.10061]

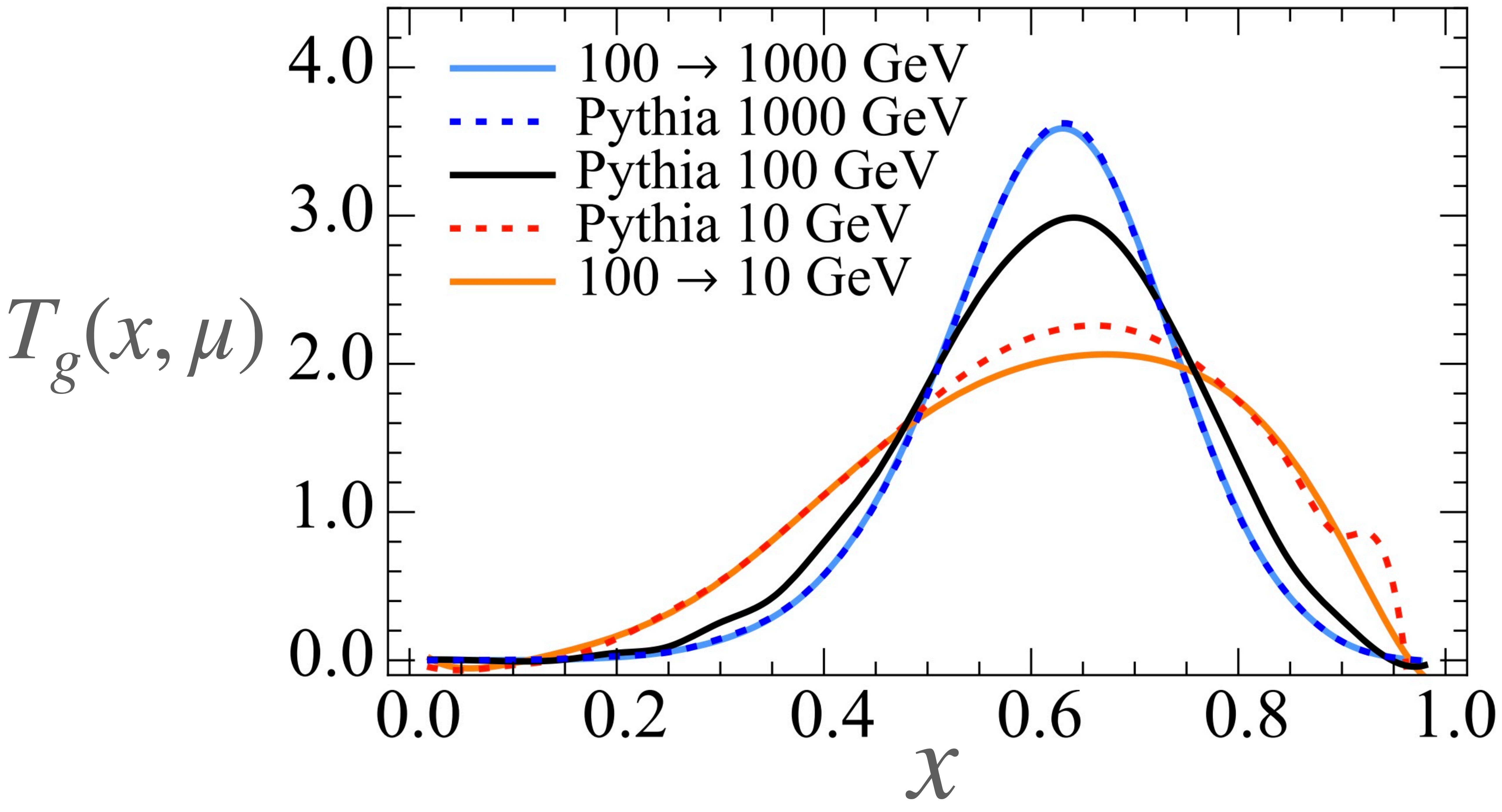
✦ Results for the NLO non-linear x-space evolution enabling the use of tracks for generic substructure observables!



✦ Correspondence between the evolution of track functions and that of single- or multi-hadron fragmentation functions.



Introduction to Track Functions

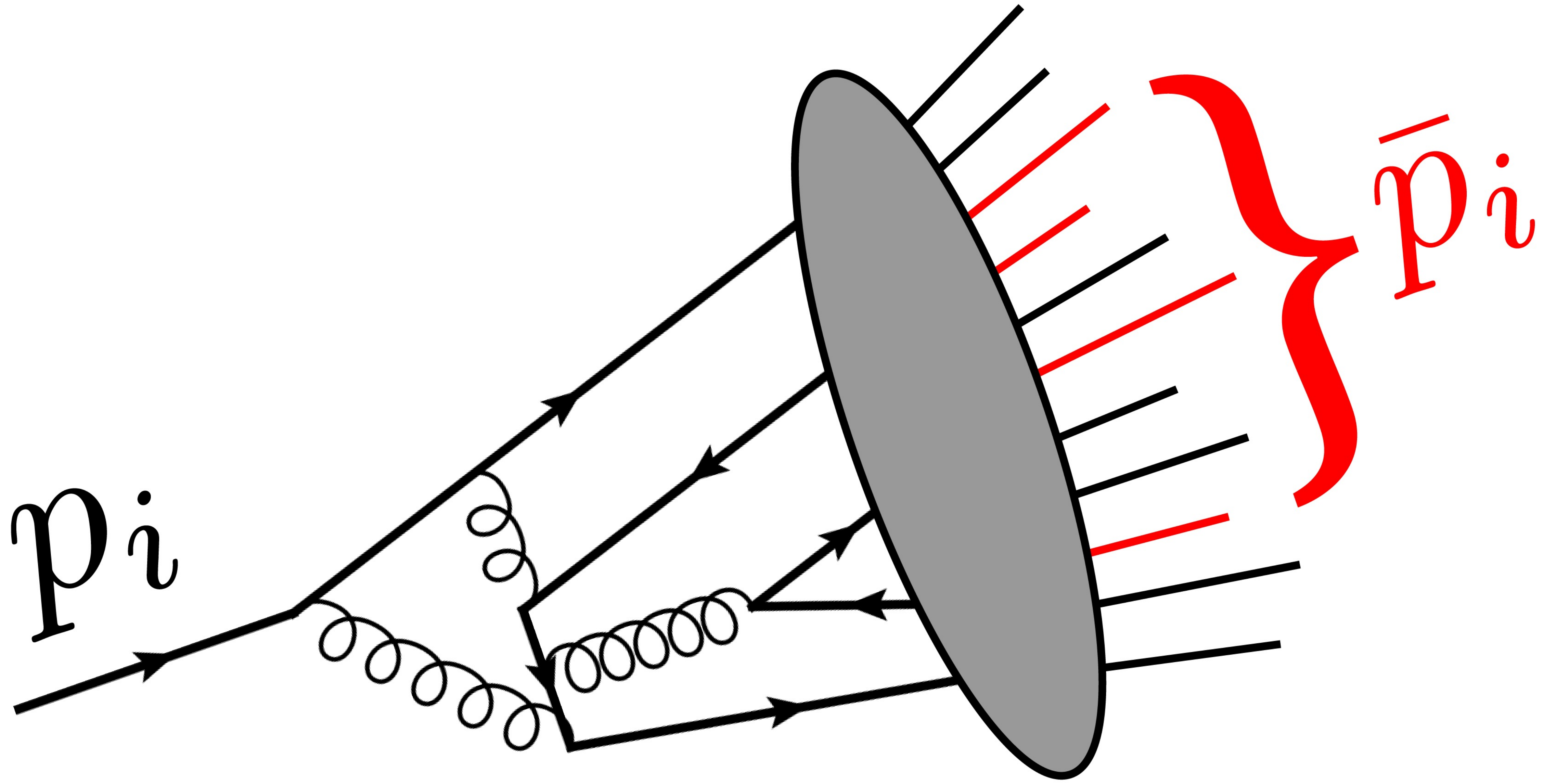


Track Functions $T_i(x, \mu)$ [H. Chang, M. Procura, J. Thaler, W. Waalewijn, 1303.6637, 1306.6630]

Definition

- The track function $T_i(x, \mu)$ describes the total momentum fraction x of *all* charged particles (tracks) in a jet initiated by a hard parton i.

$$\bar{p}_i^\mu = x p_i^\mu + O(\Lambda_{\text{QCD}}) \ , (0 \leq x \leq 1) \ .$$

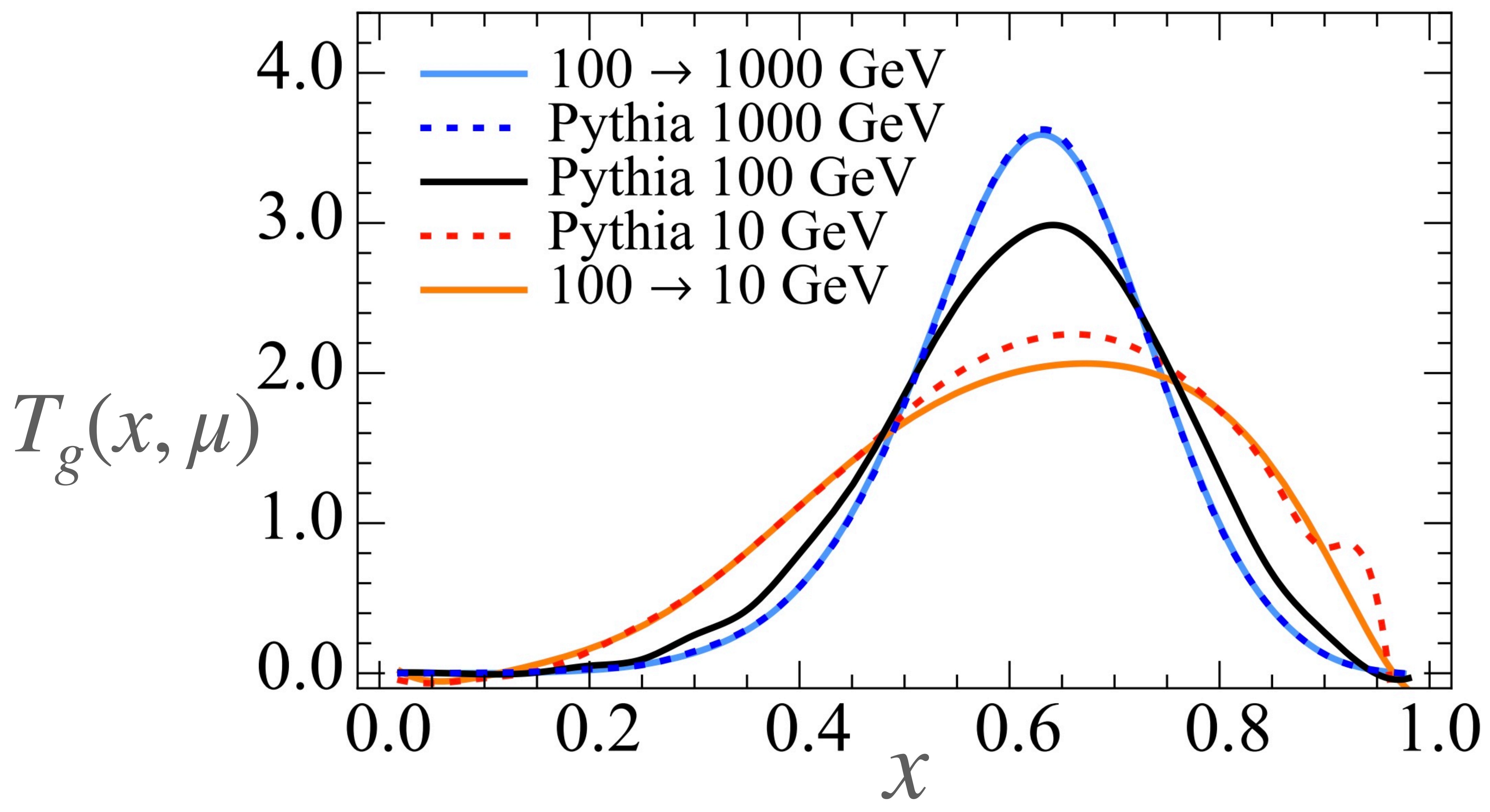


- This formalism applies to other subsets of hadrons (positively-charged, strange, etc).

Track Functions

Features [H. Chang, M. Procura, J. Thaler, W. Waalewijn, 1303.6637, 1306.6630]

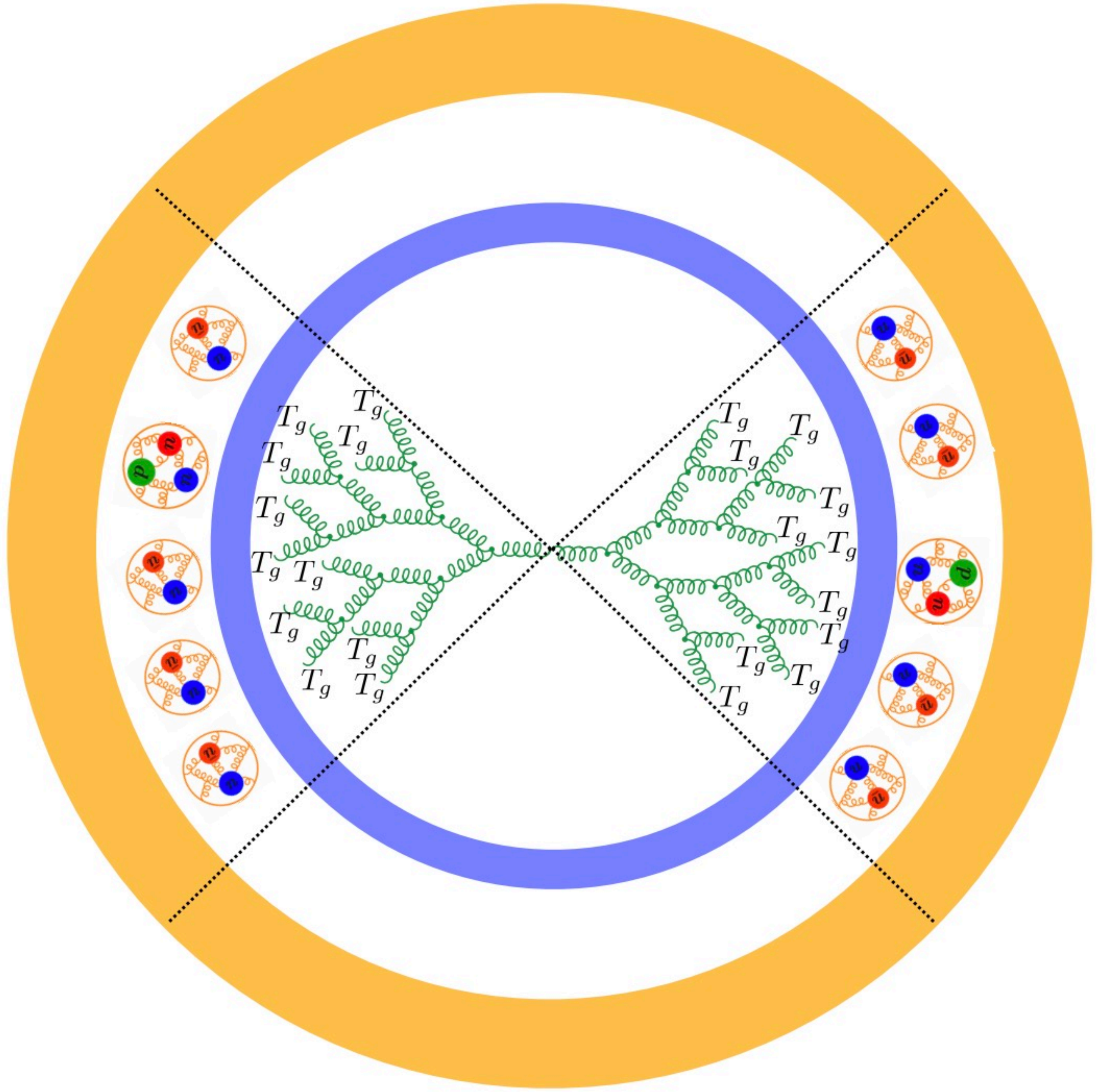
- A generalization of fragmentation functions (FFs).
  - Independent of hard process.
  - Fundamentally non-perturbative, with a calculable scale (μ) dependence.
  - Incorporating correlations between final-state hadrons, like multi-hadron FFs.
  - Sum rule: $\int_0^1 dx\ T_i(x,\mu) = 1\ .$

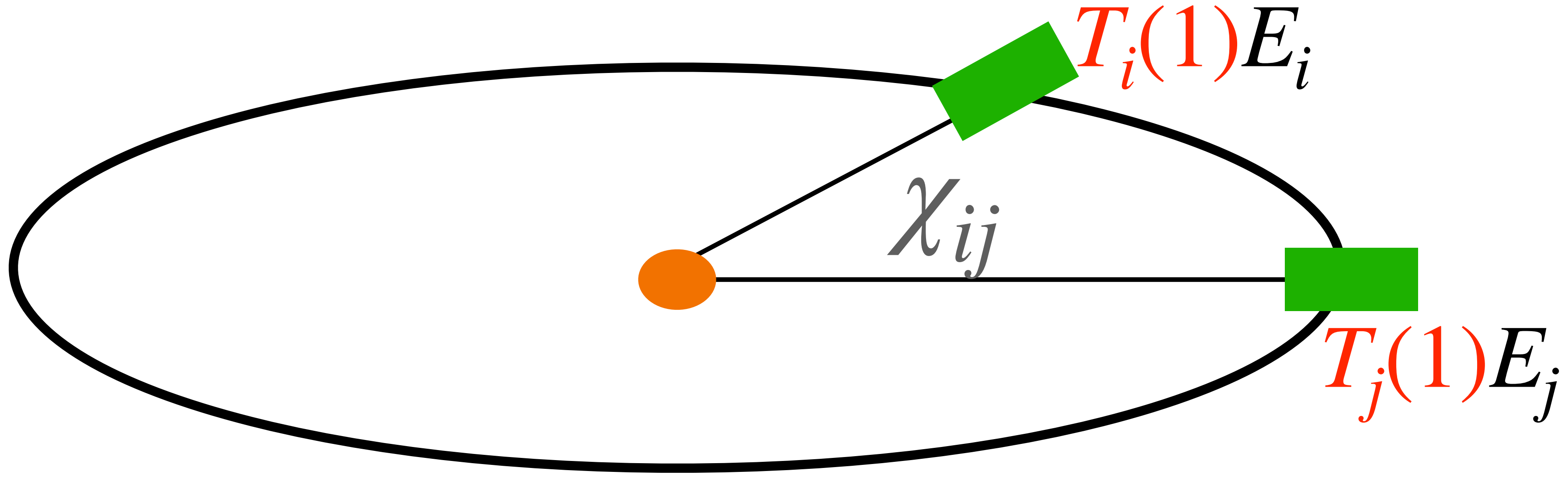


- The single-hadron fragmentation function:
  - The probability of a parton to produce a single-hadron state considered.
  - The momentum sum rule:

$$\sum_h \int_0^1 dz\ z\ D_{i\to h}(z,\mu) = 1\ .$$

Incorporating Tracks





Two Types of Observables

[1303.6637]

- For a δ-function type observable e measured using partons:

$$\frac{d\sigma}{de} = \sum_N \int d\Pi_N \frac{d\sigma_N}{d\Pi_N} \delta\left[e - \hat{e}(p_i^\mu)\right]$$

tracks

full functional form of T

$$\frac{d\sigma}{d\bar{e}} = \sum_N \int d\Pi_N \frac{d\bar{\sigma}_N}{d\Pi_N} \int \prod_{i=1}^{N} dx_i T_i(x_i) \delta\left[\bar{e} - \hat{e}(x_i p_i^\mu)\right]$$

- For correlations of energy flow: k-point correlation functions

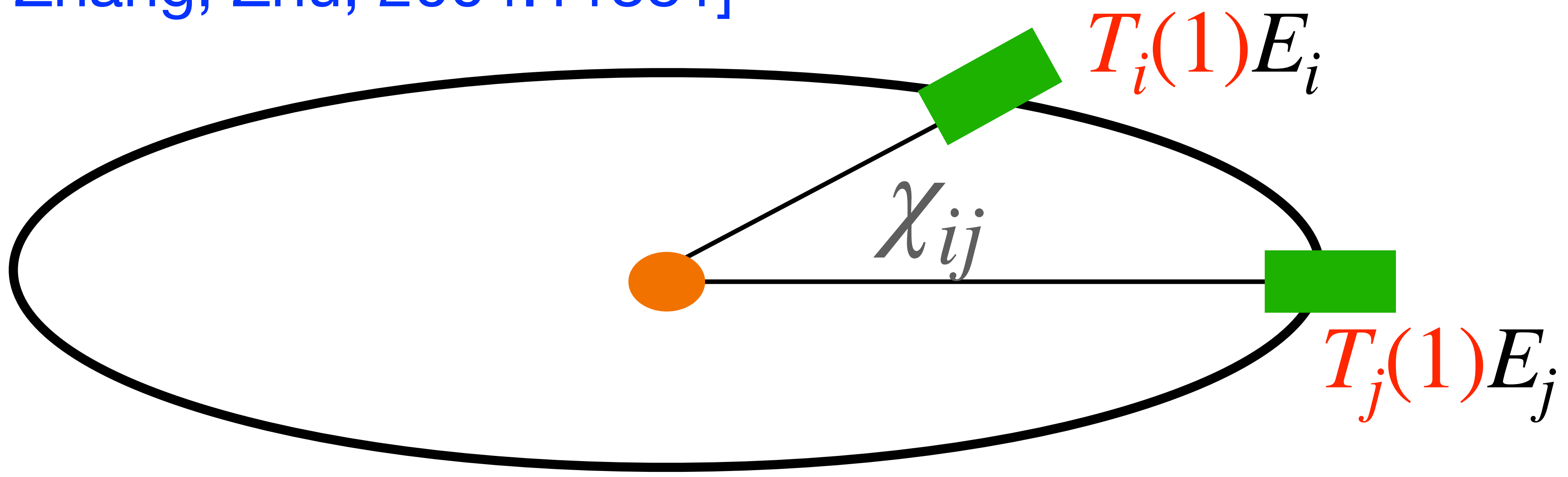
$$\langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\cdots\mathcal{E}(\vec{n}_k)\rangle$$

- The energy flow operator that measures energy flow on a restricted set R of final states: $\mathcal{E}_R$ e.g. charged hadrons
- Then, the k-point correlator on R is

$$\langle \mathcal{E}_R(\vec{n}_1)\mathcal{E}_R(\vec{n}_2)\cdots\mathcal{E}_R(\vec{n}_k)\rangle$$

Two Types of Observables

[Chen, Moult, Zhang, Zhu, 2004.11381]



[1303.6637]

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tracks

$$\frac{d\sigma}{d\bar{e}} = \sum_N \int d\Pi_N \frac{d\bar{\sigma}_N}{d\Pi_N} \int \prod_{i=1}^{N} dx_i T_i(x_i) \delta\left[\bar{e} - \hat{e}(x_i p_i^\mu)\right]$$

full functional form of T

- E.g., 2-point correlator (EEC)

$$\frac{d\Sigma}{d\cos\chi} = \sum_{i,j} \int \frac{E_i E_j}{Q^2} \delta\left(\cos\chi - \cos\chi_{ij}\right) d\sigma$$

$$E_i^n \to \int dx_i T_i(x_i) x_i^n E_i^n = T_i(n) E_i^n$$

Mellin moments

only moments of T

$$\left(\frac{d\Sigma}{d\cos\chi}\right)_{\mathrm{tr}} = \sum_{i\neq j} T_i(1) T_j(1) \int \frac{E_i E_j}{Q^2} \delta\left(\cos\chi - \cos\chi_{ij}\right) d\bar{\sigma} + \sum_k T_k(2) \int \frac{E_k^2}{Q^2} \delta\left(\cos\chi - 1\right) d\bar{\sigma}$$

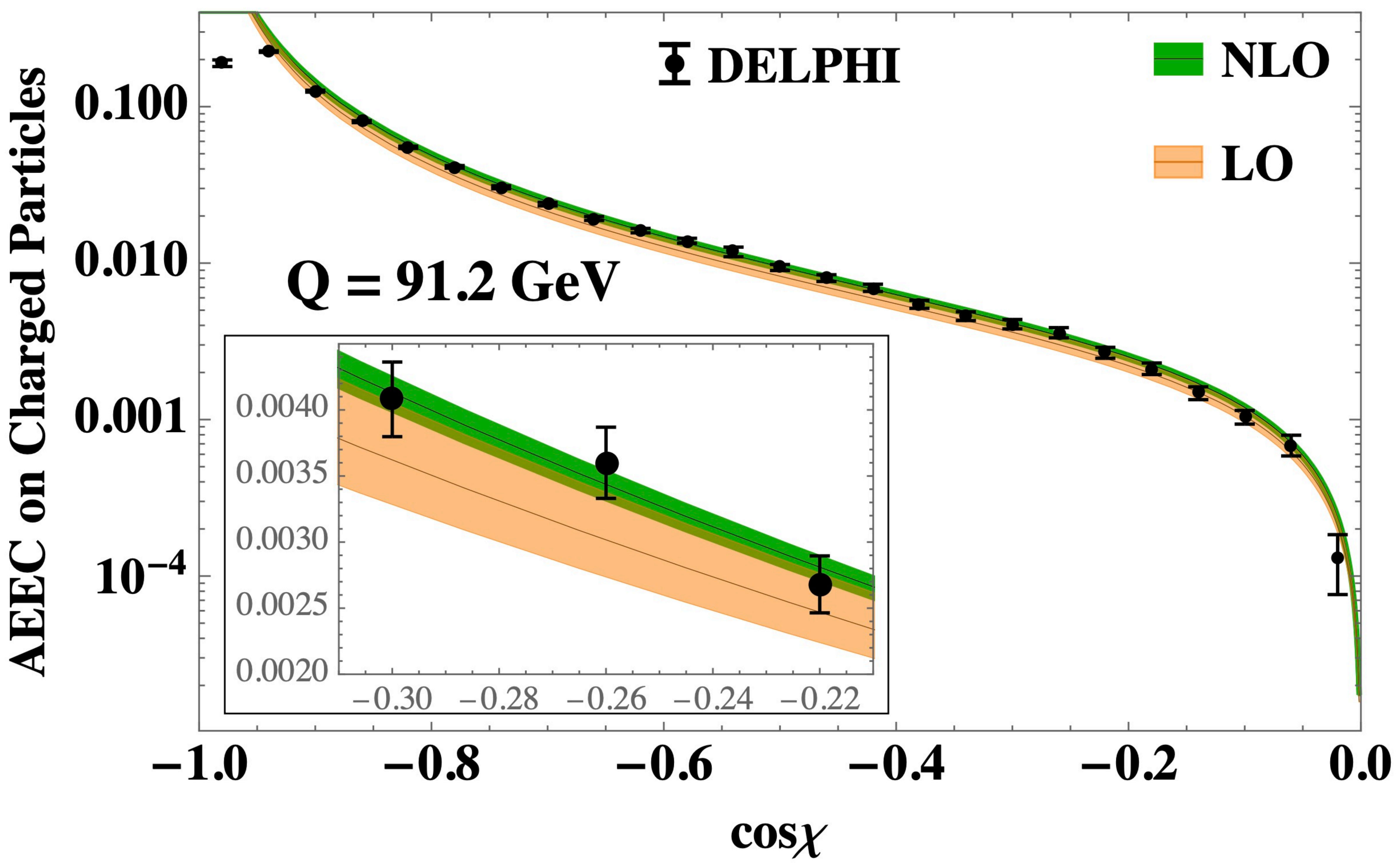
Track EEC

- **Energy correlators**: tracking easily included and can use modern fixed-order techniques.

→ Study of RG equations for moments of track functions, resummation of ECs.

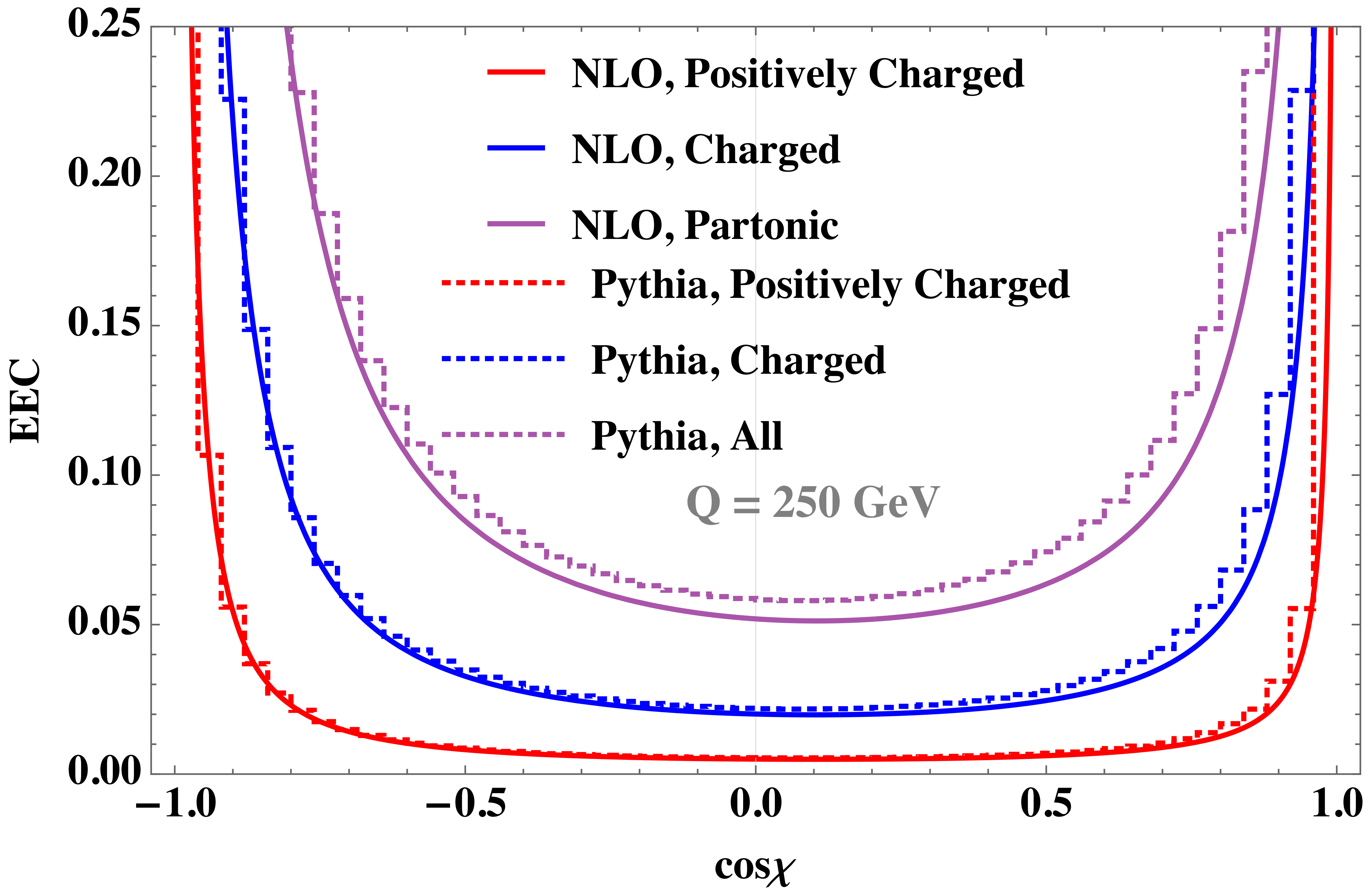
Track EEC for e^+e^- annihilation

- **First NLO ($\mathcal{O}(\alpha_s^2)$) calculations for track-based observables**
- **Results are available in completely analytical form.**



- The DELPHI data were published 27 years ago.

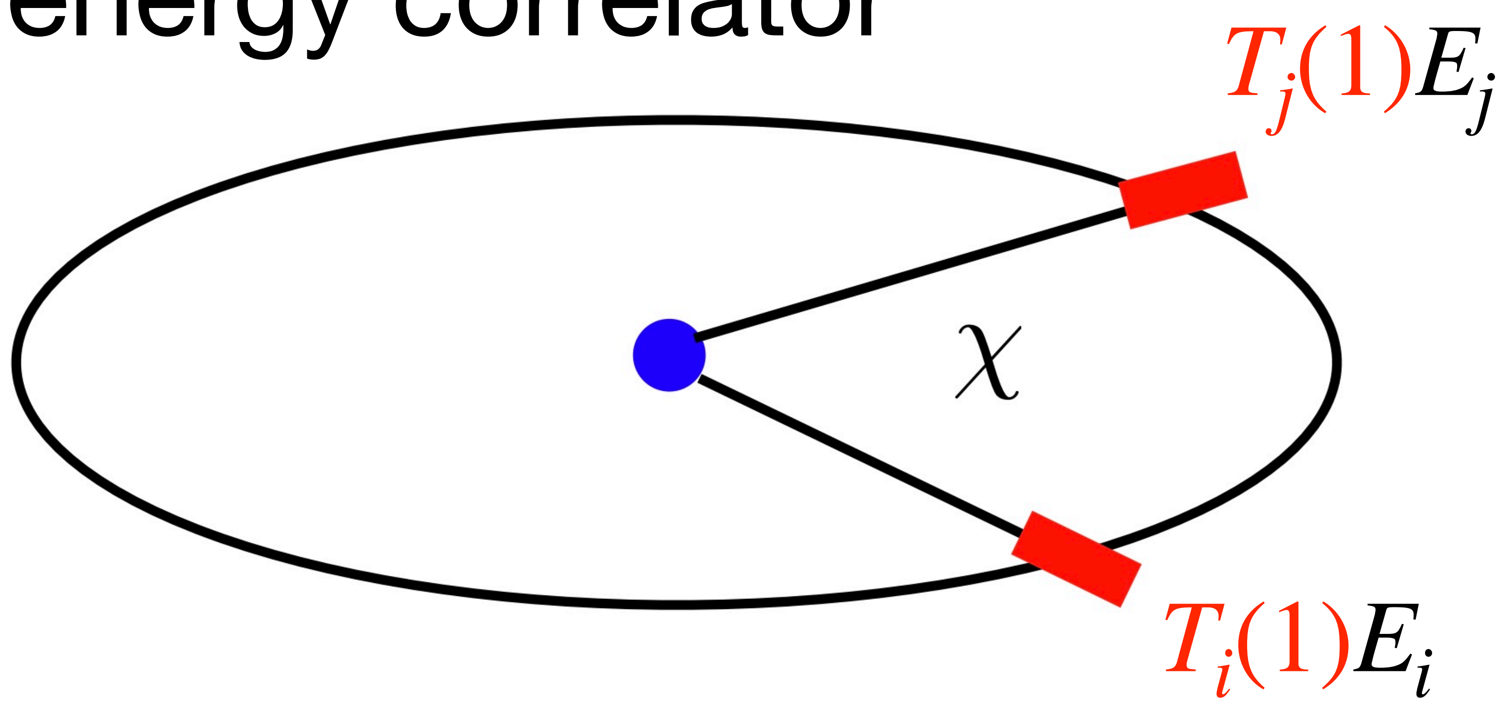
- Track function formalism can be applied to other subsets of hadrons specified by their quantum numbers.

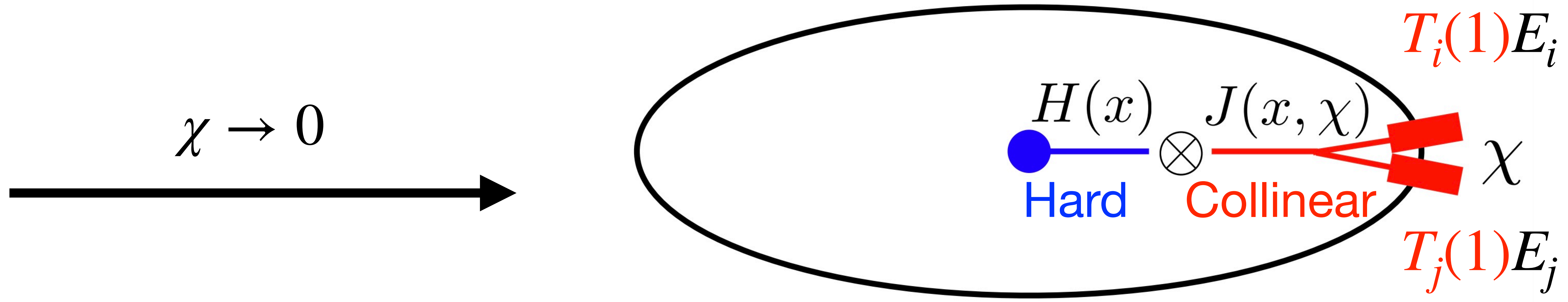


Energy Correlators Within Jets

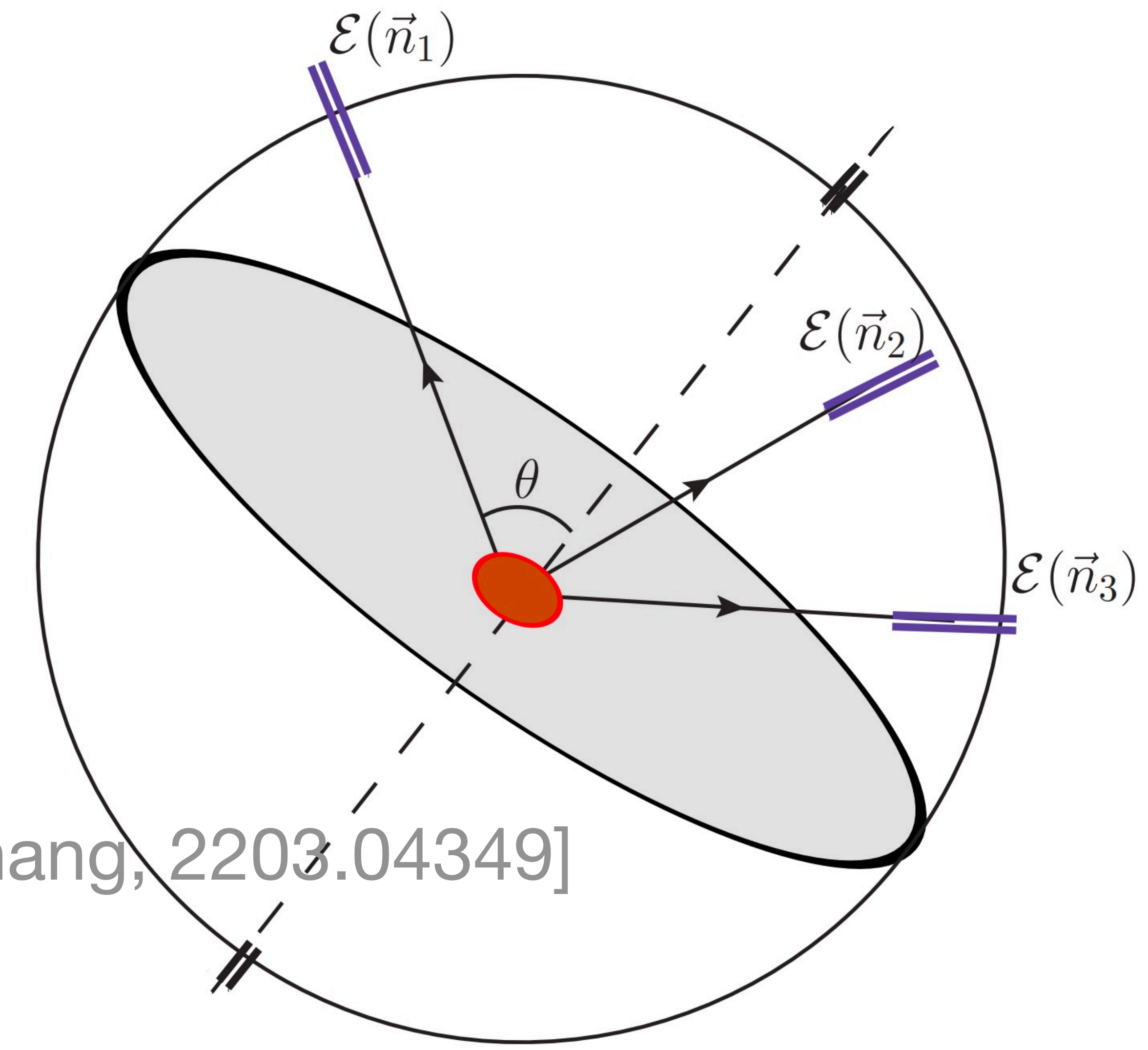
2-point energy correlator

In the collinear limit:

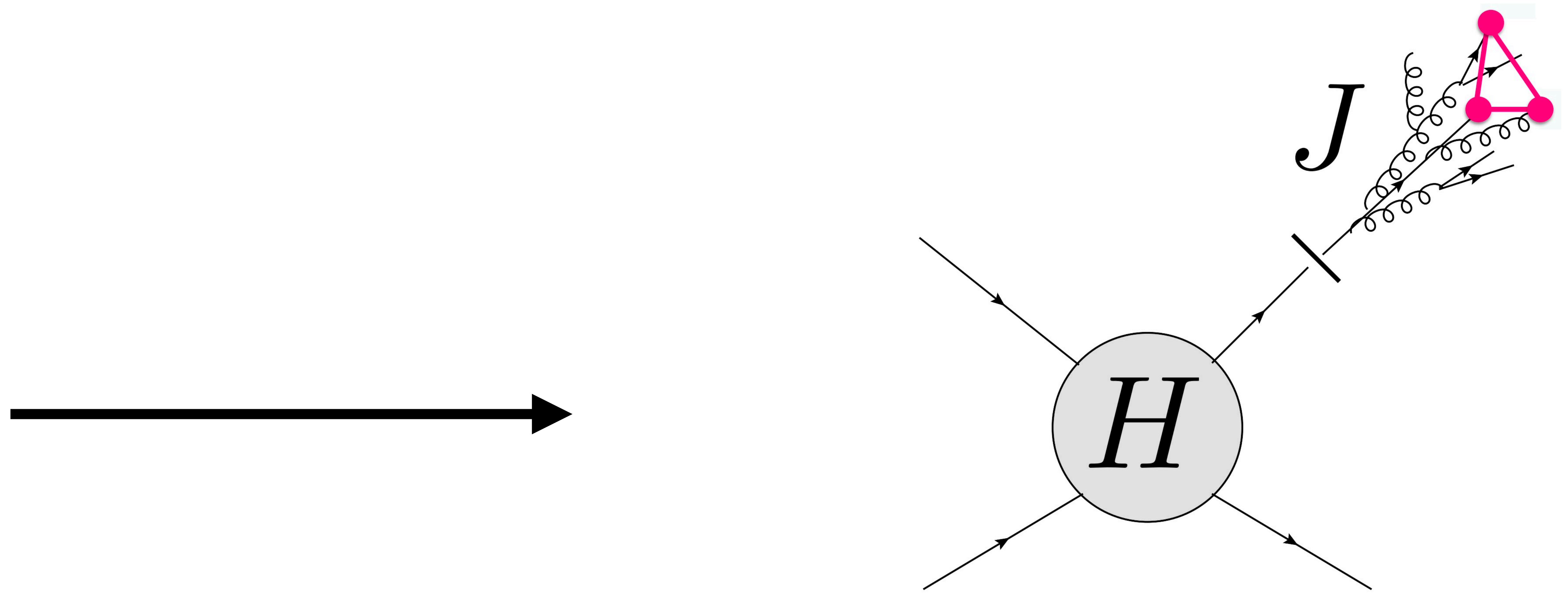




3-point energy correlator



[K. Yan, X. Zhang, 2203.04349]



[H. Chen, M.X. Luo, I. Moult, T.Z. Yang, X. Zhang, H.X. Zhu, 1912.11050]

- The energy correlator is a jet substructure observable: $\Sigma(x_L) = \vec{J} \otimes \vec{H}$.

Energy Correlators Within Jets

- Projected energy correlators are single logarithmic collinear (soft insensitive) observables, like the groomed jet mass.

The longest side definition

$$\frac{d\sigma^{[k]}}{dx_L} = \int d\vec{\Omega}\, \delta\left(x_L - \frac{1 - \vec{n}_1 \cdot \vec{n}_2}{2}\right) \prod_{\substack{1 \le i < j \le k \\ i + j > 3}} \theta(|\vec{n}_1 - \vec{n}_2| - |\vec{n}_i - \vec{n}_j|) \langle \mathcal{E}(\vec{n}_1) \cdots \mathcal{E}(\vec{n}_k) \rangle$$

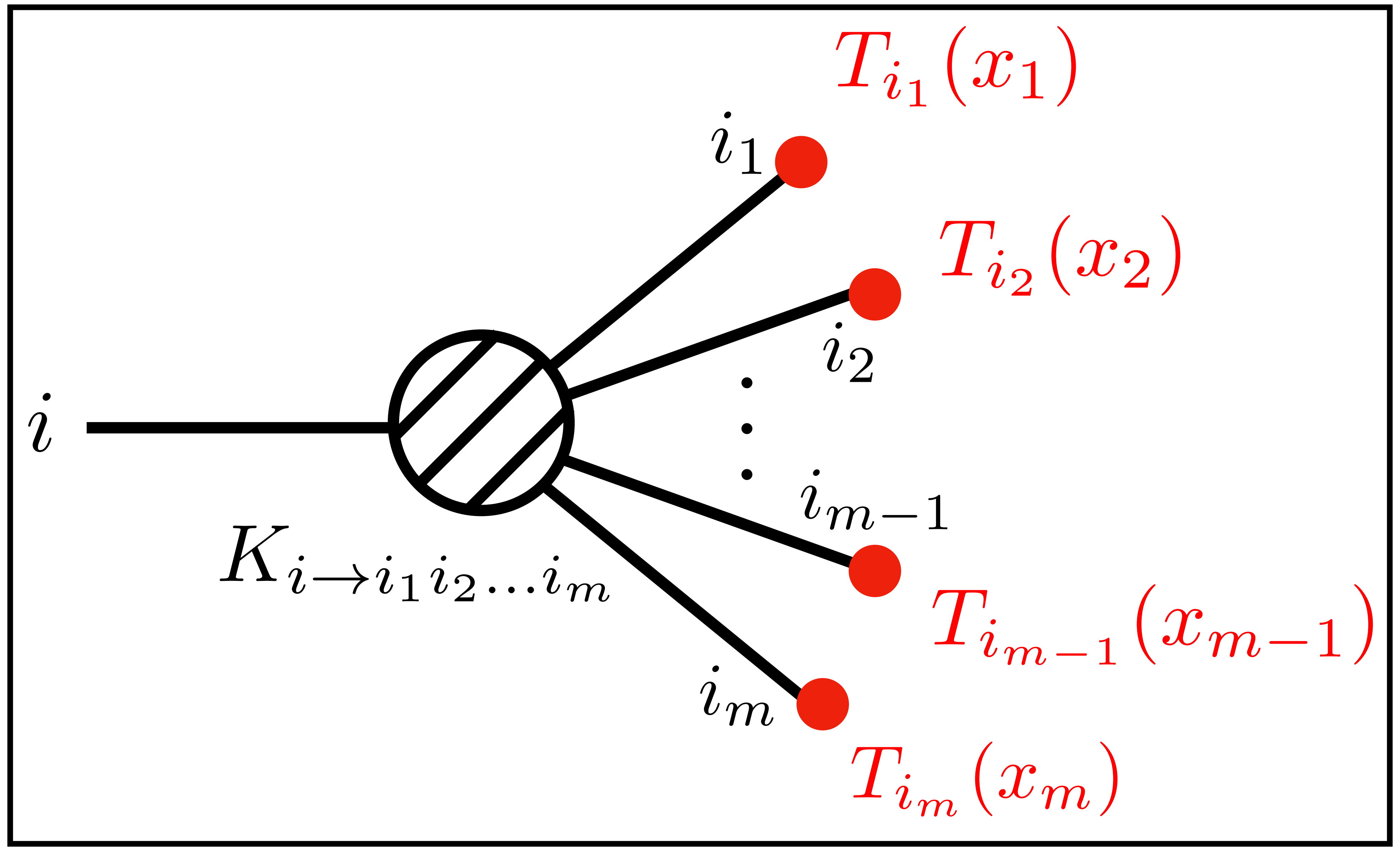
- Jet functions for projected energy correlators on tracks, $\vec{J}_{\text{tr}}\left(\ln \frac{x_L Q^2}{\mu^2}, T_i(n,\mu), a_s(\mu)\right)$: Integer moments $T_i(n, \mu)$ appear as the coefficients. Resummation convenient!
  - The jet function constants (the jet functions with the logarithmic dependence excluded): e.g. for track EECs, up to $\mathcal{O}(\alpha_s^2)$

$$j^g = \frac{1}{4} T_g(2) + a_s \left\{ T_g(1) T_g(1) C_A \left(-\frac{449}{150}\right) + \sum_q T_q(1) T_{\bar{q}}(1) T_F \left(-\frac{7}{25}\right) \right\}$$

$$+ a_s^2 \left\{ T_g(1) T_g(1) \left\{ C_A^2 \left(-\frac{527\zeta_3}{10} + \frac{133639871}{3240000} - \frac{2159\pi^2}{1800} + \frac{19\pi^4}{90} \right) + C_A n_f T_F \frac{139}{270} \right\} + \sum_q T_q(1) T_{\bar{q}}(1) \cdots \right\}$$

  - Matches the state-of-the-art calculation for jet substructure, but now on tracks.

Track Function Evolution

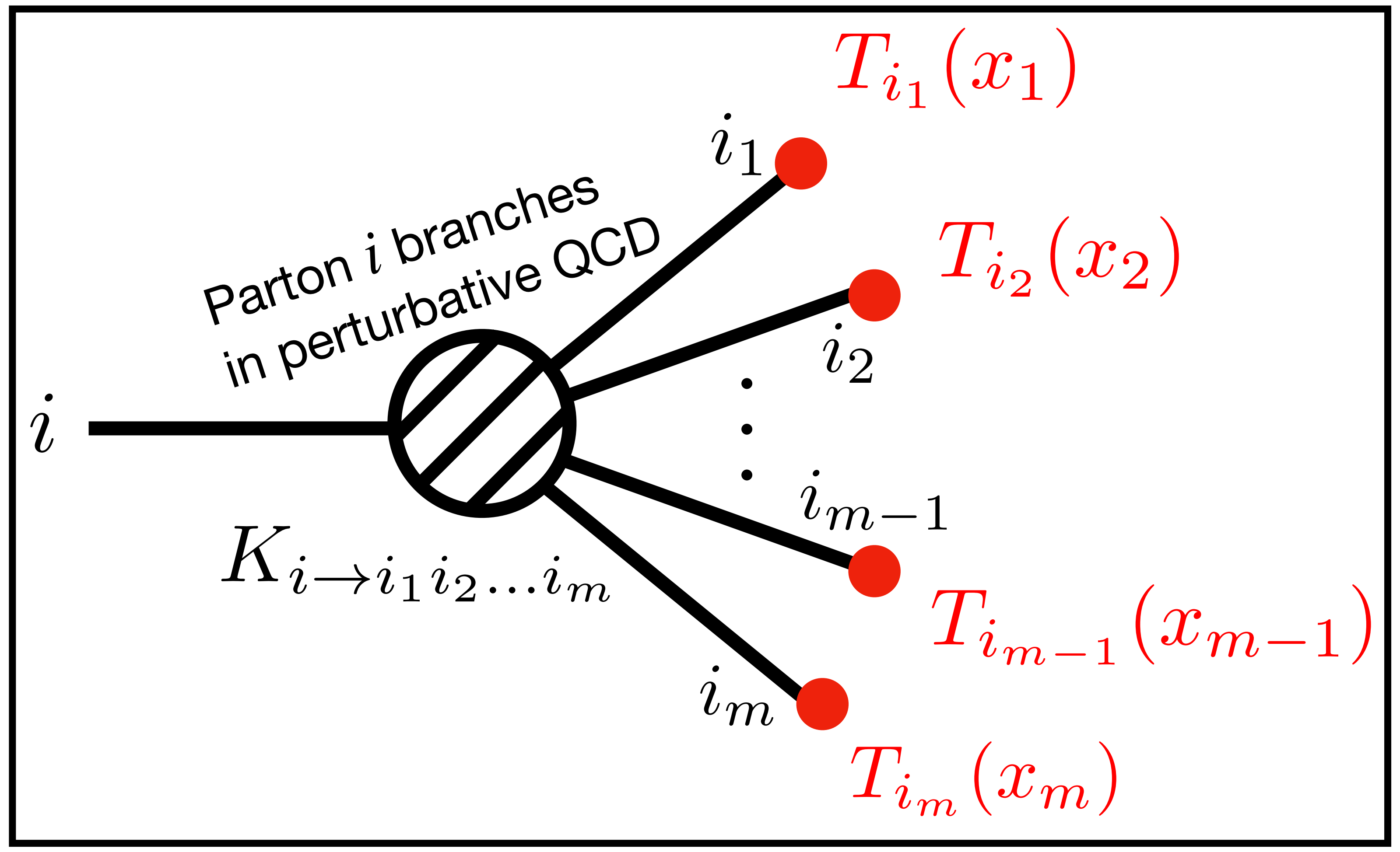


Track Function Evolution

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}T_i(x) = \sum_M \sum_{\{i_f\}} \left[\prod_{m=1}^{M}\int_0^1 \mathrm{d}z_m\right]\delta\left(1-\sum_{m=1}^{M} z_m\right) K_{i\to\{i_f\}}(\{z_f\})$$

$$\times \left[\prod_{m=1}^{M}\int_0^1 \mathrm{d}x_m\, T_{i_m}(x_m)\right]\delta\left(x-\sum_{m=1}^{M} z_m x_m\right)$$

$(i, i_f = g, u, \bar{u}, d, \cdots)$

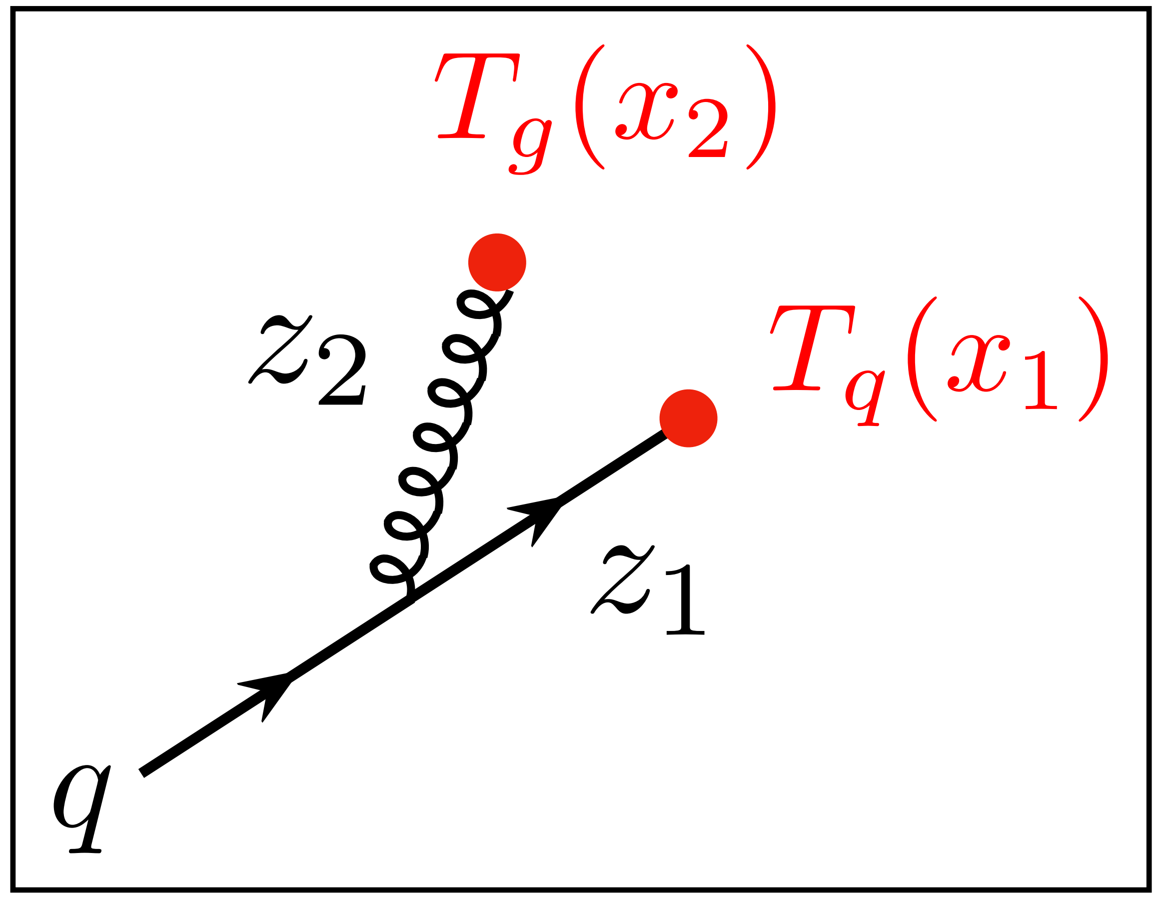


- **Nonlinear**, involving contributions from all branches of splittings.
- E.g., LO evolution:

- For single-hadron FFs: Only one branch observed → **Linearity**

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}T_i(x,\mu) = a_s(\mu)\sum_{\{jk\}}\int \mathrm{d}z_1\mathrm{d}z_2\; K^{(0)}_{i\to jk}(z_1,z_2)\delta(1-z_1-z_2)$$

$$\times \int \mathrm{d}x_1\mathrm{d}x_2 T_j(x_1,\mu)T_k(x_2,\mu)\delta[x-z_1x_1-z_2x_2]\ .$$



Involving contributions from both the branches of the splitting.

Track Function Evolution

In Mellin Space

$\int_0^1 dx\ x^n$ →

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}T_i(x) = \sum_N \sum_{\{i_f\}} \left[\prod_{m=1}^{N}\int_0^1 \mathrm{d}z_m\right]\delta\left(1-\sum_{m=1}^{N} z_m\right)P_{i\to\{i_f\}}(\{z_f\})$$
$$\times\left[\prod_{m=1}^{N}\int_0^1 \mathrm{d}x_m\, T_{i_m}(x_m)\right]\delta\left(x-\sum_{m=1}^{N} z_m x_m\right)$$

- RG equations for $\mathbf{T} =$

$$\{T_i(n), \cdots, T_{i_1}(k)T_{i_2}(n-k), \cdots, T_{i_1}(1)\cdots T_{i_n}(1)\}^{\mathrm{t}}$$

  - Matrix form: $\dfrac{d}{d\ln\mu^2}\mathbf{T} = \mathbb{R}\mathbf{T}$
  - $\mathbb{R}$: related to moments of timelike splitting functions.

- For fragmentation functions:

$$\frac{d}{d\ln\mu^2}D_{i\to h}(n) = -\sum_j D_{j\to h}(n)\gamma_{ji}^T(n+1)$$

- $\dfrac{d}{d\ln\mu^2}T_i(n) = -\sum_j T_j(n)\ \gamma_{ji}^T(n+1)$+terms of products of lower moments

A Surprising Symmetry:

- Energy conservation implies the evolution is **shift-symmetric:** $x \to x + a$

$$\frac{d}{d\ln\mu^2} T_i(x+a) = \sum_X \int \left(\prod_m dx_m dz_m T_{i_m}(x_m + a) \right) P_{i\to i_1\cdots i_m\cdots}(\{z_m\})\, \delta\left(1 - \sum_m z_m\right) \delta\left(x - \sum_m x_m z_m\right)$$

- This uniquely fixes the form of the evolution of the first three moments:

$$\frac{d}{d\ln\mu^2}\Delta = [-\gamma_{qq}(2) - \gamma_{gg}(2)]\,\Delta\,,$$

$$\frac{d}{d\ln\mu^2}\begin{bmatrix}\sigma_g(2)\\ \sigma_q(2)\end{bmatrix} = \begin{bmatrix}-\gamma_{gg}(3) & -\gamma_{qg}(3)\\ -\gamma_{gq}(3) & -\gamma_{qq}(3)\end{bmatrix}\begin{bmatrix}\sigma_g(2)\\ \sigma_q(2)\end{bmatrix} + \begin{bmatrix}\gamma_g^{\Delta^2}\\ \gamma_q^{\Delta^2}\end{bmatrix}\Delta^2\,,$$

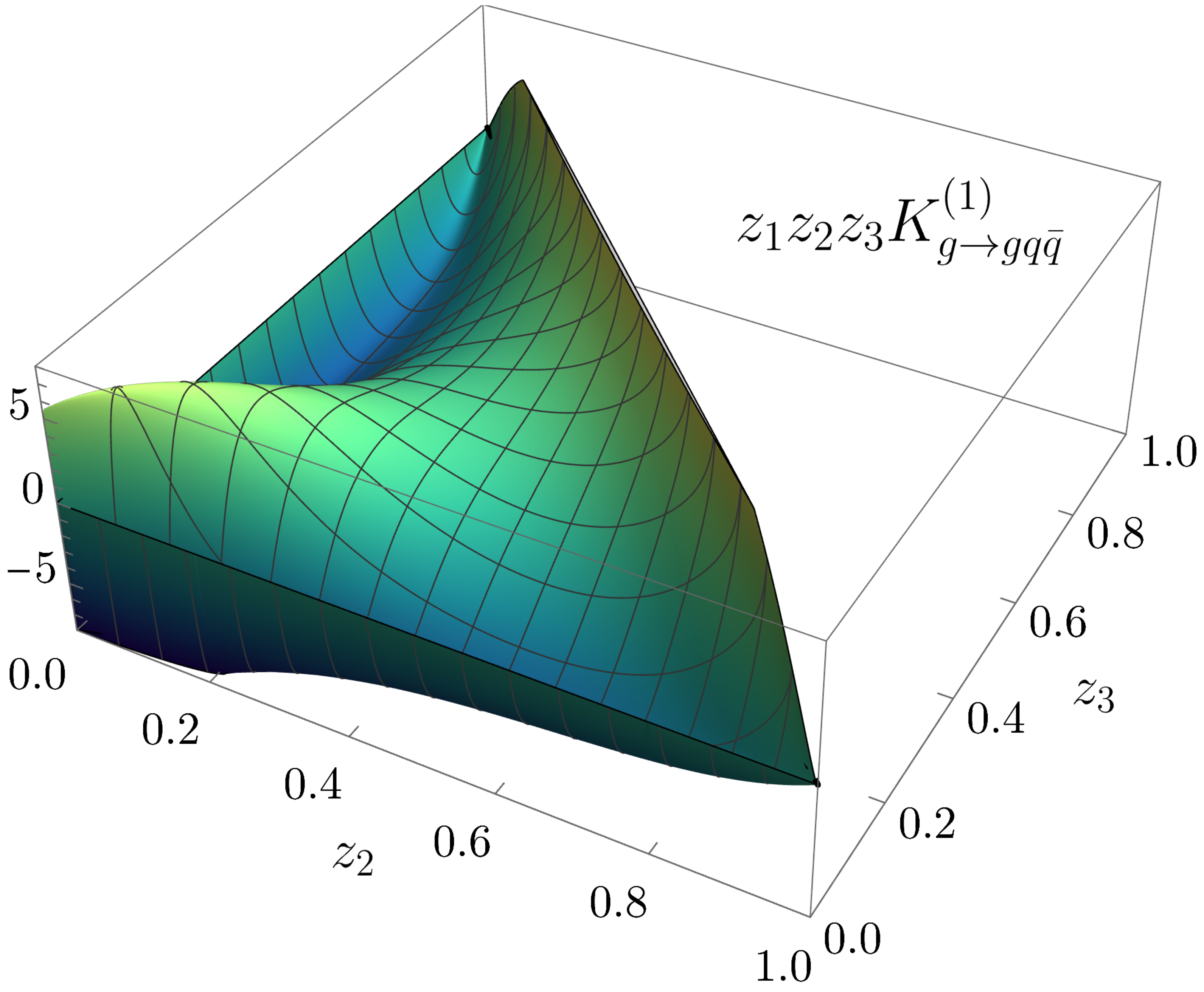
$$\frac{d}{d\ln\mu^2}\begin{bmatrix}\sigma_g(3)\\ \sigma_q(3)\end{bmatrix} = \begin{bmatrix}-\gamma_{gg}(4) & -\gamma_{qg}(4)\\ -\gamma_{gq}(4) & -\gamma_{qq}(4)\end{bmatrix}\begin{bmatrix}\sigma_g(3)\\ \sigma_q(3)\end{bmatrix} + \begin{bmatrix}\gamma_{gg}^{\sigma\Delta} & \gamma_{qg}^{\sigma\Delta}\\ \gamma_{gq}^{\sigma\Delta} & \gamma_{qq}^{\sigma\Delta}\end{bmatrix}\begin{bmatrix}\sigma_g(2)\\ \sigma_q(2)\end{bmatrix}\Delta + \begin{bmatrix}\gamma_g^{\Delta^3}\\ \gamma_q^{\Delta^3}\end{bmatrix}\Delta^3$$

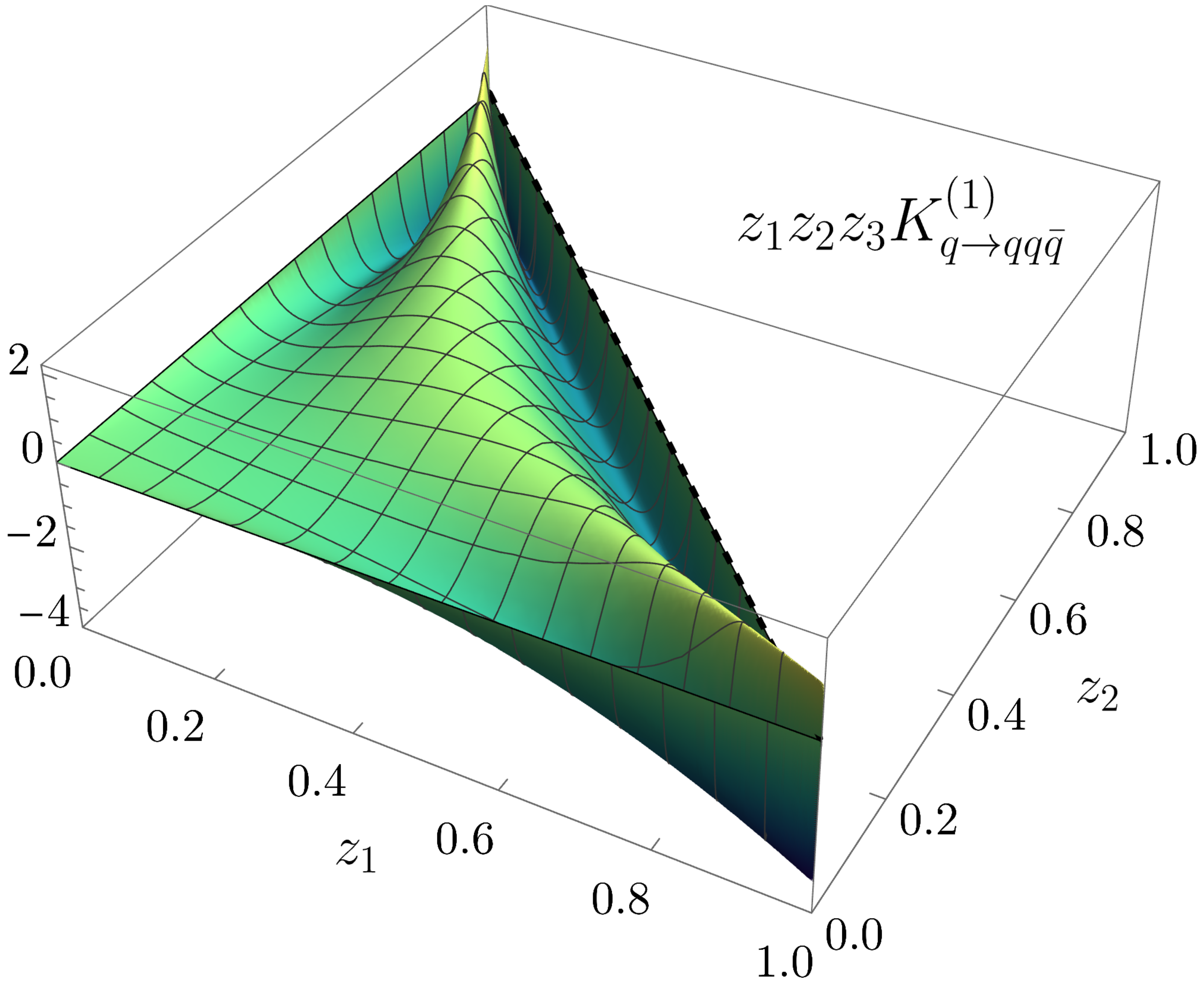
shift-invariant objects:

$$\Delta := T_q(1) - T_g(1)$$
$$\sigma_i(2) := T_i(2) - T_i(1)^2$$
$$\sigma_i(3) := T_i(3) - 3T_i(2)T_i(1) + 2T_i(1)^3$$

Here $\gamma_{ji}(n) = -\int_0^1 dz\, z^{n-1} P_{ji}(z, a_s)$ where P_{ji} denotes the singlet timelike splitting function.

Calculational Techniques & NLO Results





Track Jet Functions

In DR: $T_i^{(0)} = T_i^{\text{bare}}$

LO track jet function: $J_i^{(0)} = \delta(s) T_i^{(0)}$

- We use the jet function to extract the track function evolution.
- The definition for the jet function on tracks is

$$J_{\text{tr},i}^{\text{bare}}(s,x) = \sum_N \sum_{\{i_f\}} \int \mathrm{d}\Phi_N^c \delta(s-s') \sigma_{i\to\{i_f\}}^c(\{i_f\},\{s_{ff'}\},s') \int \left[\prod_{m=1}^N \mathrm{d}x_m T_{i_m}^{(0)}(x_m)\right] \delta\left(x - \sum_{m=1}^N x_m z_m\right)$$

- After integration over angular variables,

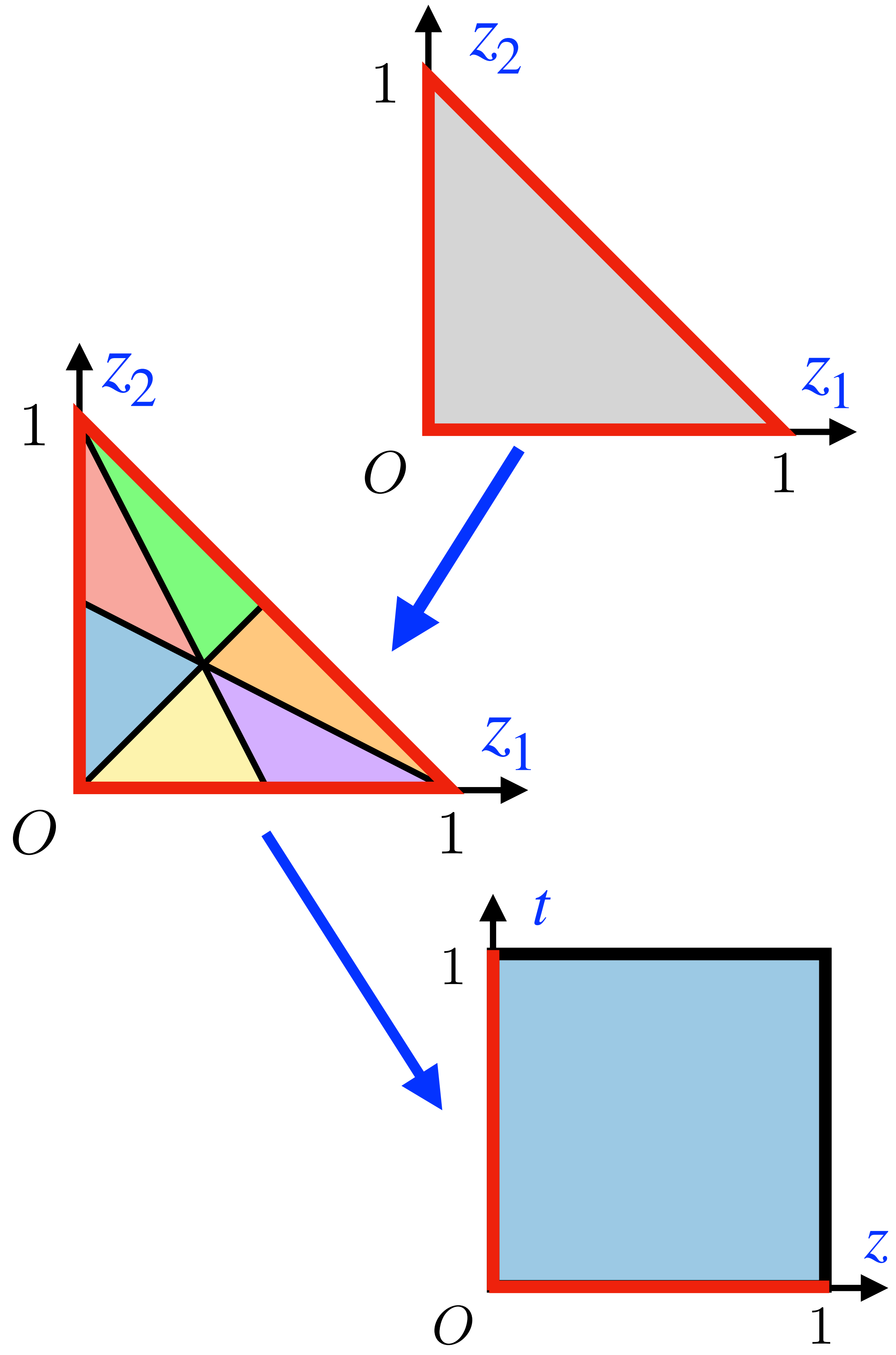
$$J_{\text{tr},i}^{\text{bare}}(s,x) \supset \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \int_0^1 \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 \delta(1-z_1-z_2-z_3) P_{i\to i_1 i_2 i_3}(z_1,z_2,z_3)$$
$$\times T_{i_1}^{(0)}(x_1) T_{i_2}^{(0)}(x_2) T_{i_3}^{(0)}(x_3) \delta(x - z_1 x_1 - z_2 x_2 - z_3 x_3)$$

have not been expanded in ϵ

- For $z_{i_1} < z_{i_2} < z_{i_3}$ $(i_1, i_2, i_3 = 1,2,3)$, do the coordinate transformation

[Sector decomposition (Heinrich, arXiv:0803.4177)]

$$t = \frac{z_{i_1}}{z_{i_2}}, z = \frac{z_{i_2}}{z_{i_3}} \quad \text{, i.e.,} z_{i_1} \to \frac{zt}{1+z+zt}, z_{i_2} \to \frac{z}{1+z+zt}, z_{i_3} \to \frac{1}{1+z+zt}$$



Results in $\mathcal{N} = 4$ SYM

a: t' Hooft coupling constant

$$\frac{d}{d\ln\mu^2}T(x) = a^2 \Bigg\{ K^{(1)}_{1\to1}\, T(x) + \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}z\, K^{(1)}_{1\to2}(z)\, T(x_1)T(x_2)\, \delta\left(x - x_1\frac{1}{1+z} - x_2\frac{z}{1+z}\right)$$

$$+ \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}x_3 \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}t\, K^{(1)}_{1\to3}(z,t)\, T(x_1)T(x_2)T(x_3)$$

$$\times\, \delta\left(x - x_1\frac{1}{1+z+zt} - x_2\frac{z}{1+z+zt} - x_3\frac{zt}{1+z+zt}\right)\Bigg\}$$

where

$$K^{(1)}_{1\to1} = -25\zeta_3 \qquad K^{(1)}_{1\to2}(z) = \frac{8}{3}\pi^2\left[\frac{1}{z}\right]_+ + \frac{32\ln^2(z+1)}{z} - \frac{16\ln(z)\ln(z+1)}{z}$$

$$K^{(1)}_{1\to3}(z,t) = 8\Bigg\{\frac{4\ln(1+z)}{z}\left[\frac{1}{t}\right]_+ + \left[\frac{1}{z}\right]_+\left(4\left[\frac{\ln t}{t}\right]_+ - \frac{\ln t}{1+t} - \frac{7\ln(1+t)}{t}\right)$$

$$+ \frac{2\left[\ln(1+tz) - \ln(1+z+tz)\right]}{(1+t)(1+z)(1+tz)} + \frac{10\left[\ln(1+z+tz) - \ln(1+z)\right]}{tz} + \frac{\ln(1+tz)}{(1+t)z(1+z)}$$

$$- \frac{7\ln(1+tz)}{tz} + \frac{\ln(1+t) - \ln t}{(1+t)(1+tz)} + \frac{\ln(1+z) + \ln(1+t)}{(1+t)(1+z)} - \frac{\ln(1+z)}{(1+t)z} - \frac{z\ln(1+z)}{(1+z)(1+tz)}\Bigg\}$$

Results in QCD

E.g. Gluon case:

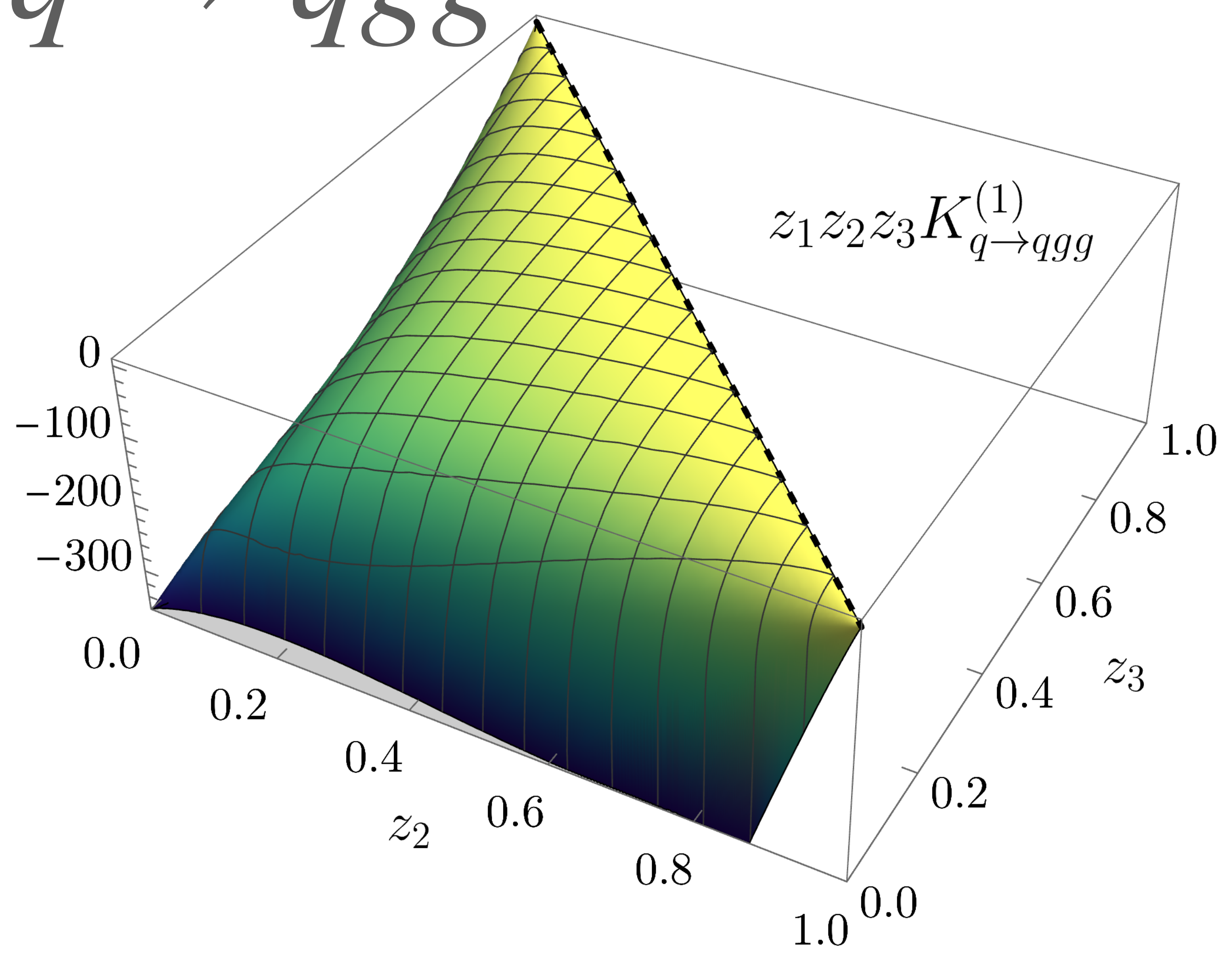
For brevity, $a_s^2 = [\alpha_s(\mu)/(4\pi)]^2$ is suppressed.

$$
\begin{aligned}
\frac{d}{d\ln\mu^2} T_g(x) = {} & T_g(x)\; K_g^{(1)} \\
& + \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}z \delta\left(x - x_1 \frac{1}{1+z} - x_2 \frac{z}{1+z}\right) \Big[T_g(x_1) T_g(x_2)\; K_{gg,1}^{(1)}(z) \\
& + \sum_q \left(T_q(x_1) T_{\bar{q}}(x_2) + T_q(x_2) T_{\bar{q}}(x_1)\right)\; K_{q\bar{q},1}^{(1)}(z) \Big] \\
& + \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}x_3 \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}t\; \delta\left(x - x_1 \frac{1}{1+z+zt} - x_2 \frac{z}{1+z+zt} - x_3 \frac{zt}{1+z+zt}\right) \\
& \times \Big\{ 6\; T_g(x_1) T_g(x_2) T_g(x_3)\; K_{ggg,1}^{(1)}(z,t) \\
& \quad + \sum_q \Big[T_g(x_3) \left(T_q(x_2) T_{\bar{q}}(x_1) + T_q(x_1) T_{\bar{q}}(x_2)\right)\; K_{gq\bar{q},1}^{(1)}(z,t) \\
& \quad + T_g(x_2) \left(T_q(x_3) T_{\bar{q}}(x_1) + T_q(x_1) T_{\bar{q}}(x_3)\right)\; K_{gq\bar{q},2}^{(1)}(z,t) \\
& \quad + T_g(x_1) \left(T_q(x_3) T_{\bar{q}}(x_2) + T_q(x_2) T_{\bar{q}}(x_3)\right)\; K_{gq\bar{q},3}^{(1)}(z,t) \Big] \Big\} .
\end{aligned}
$$

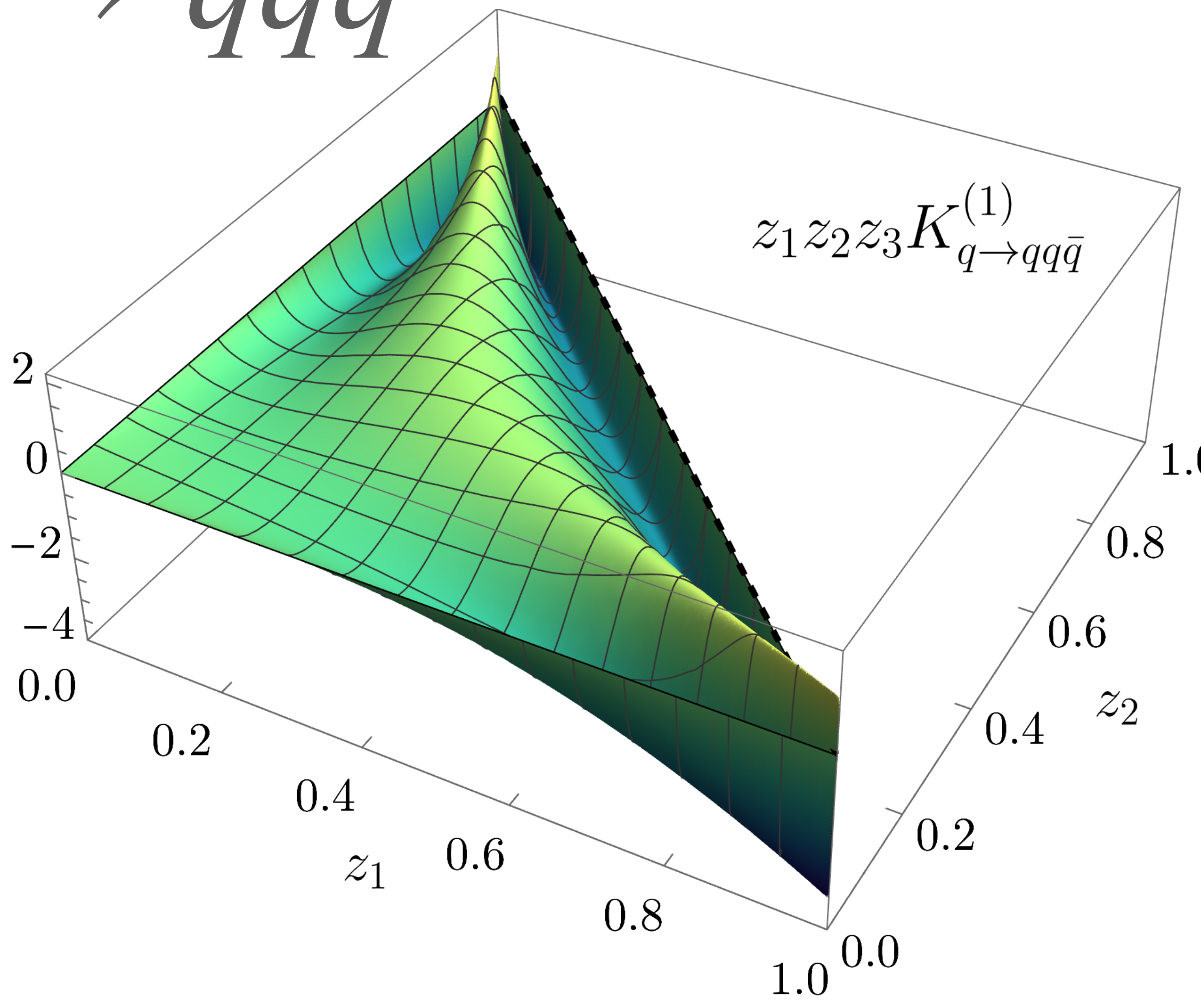
Results in QCD, Pictorially

the 1→3 Kernels

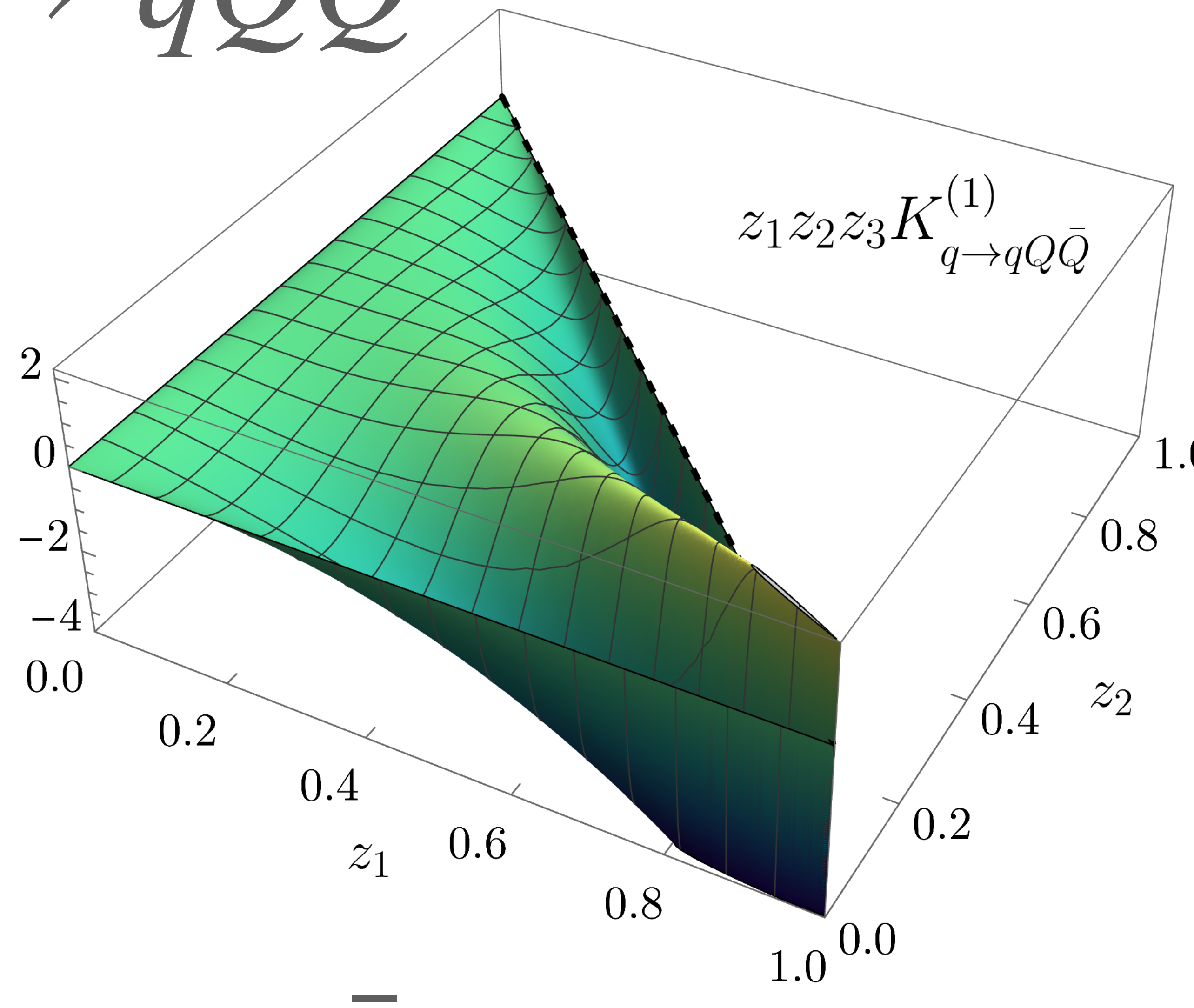
$q \to qgg$



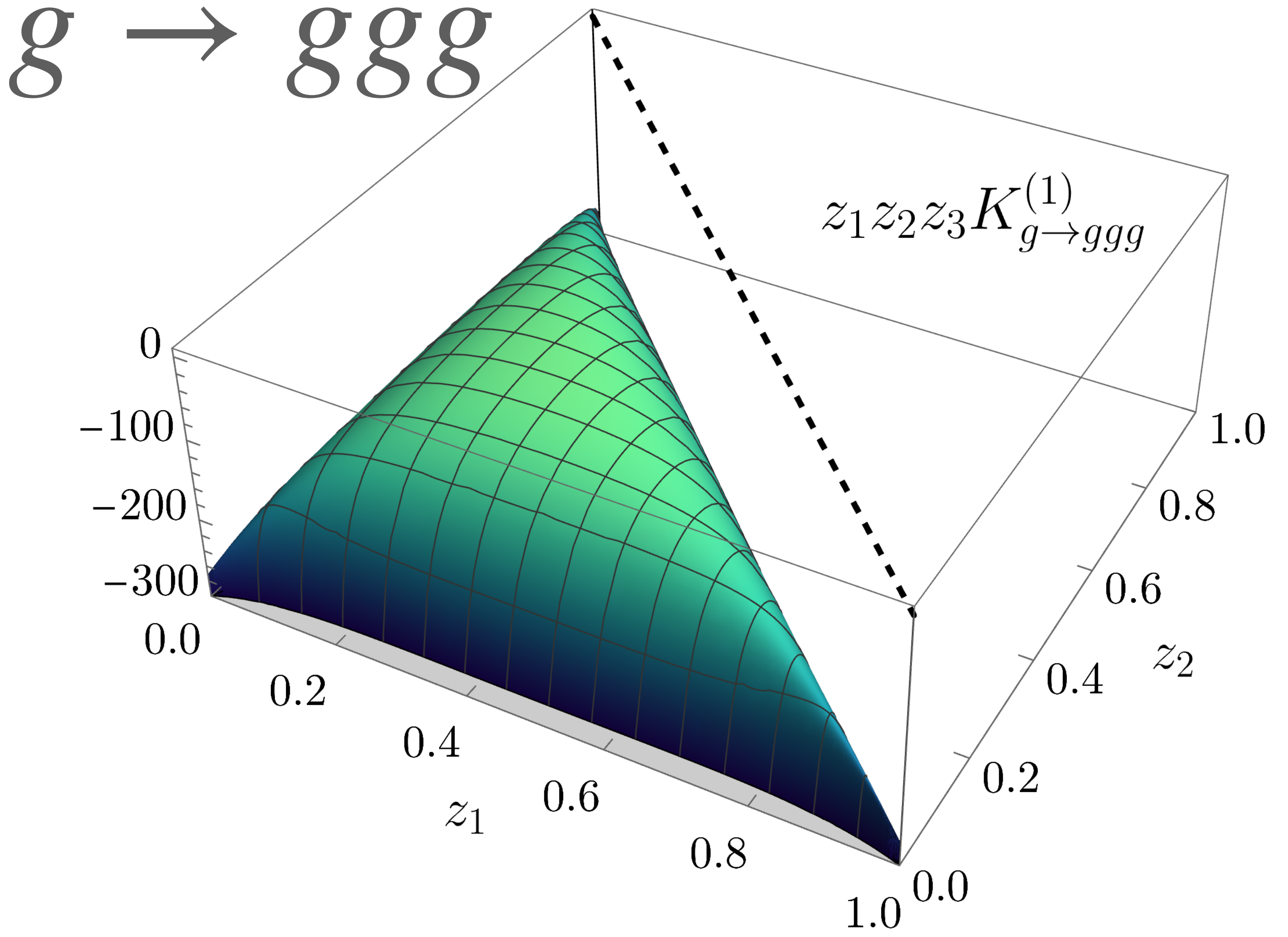
$q \to qq\bar{q}$



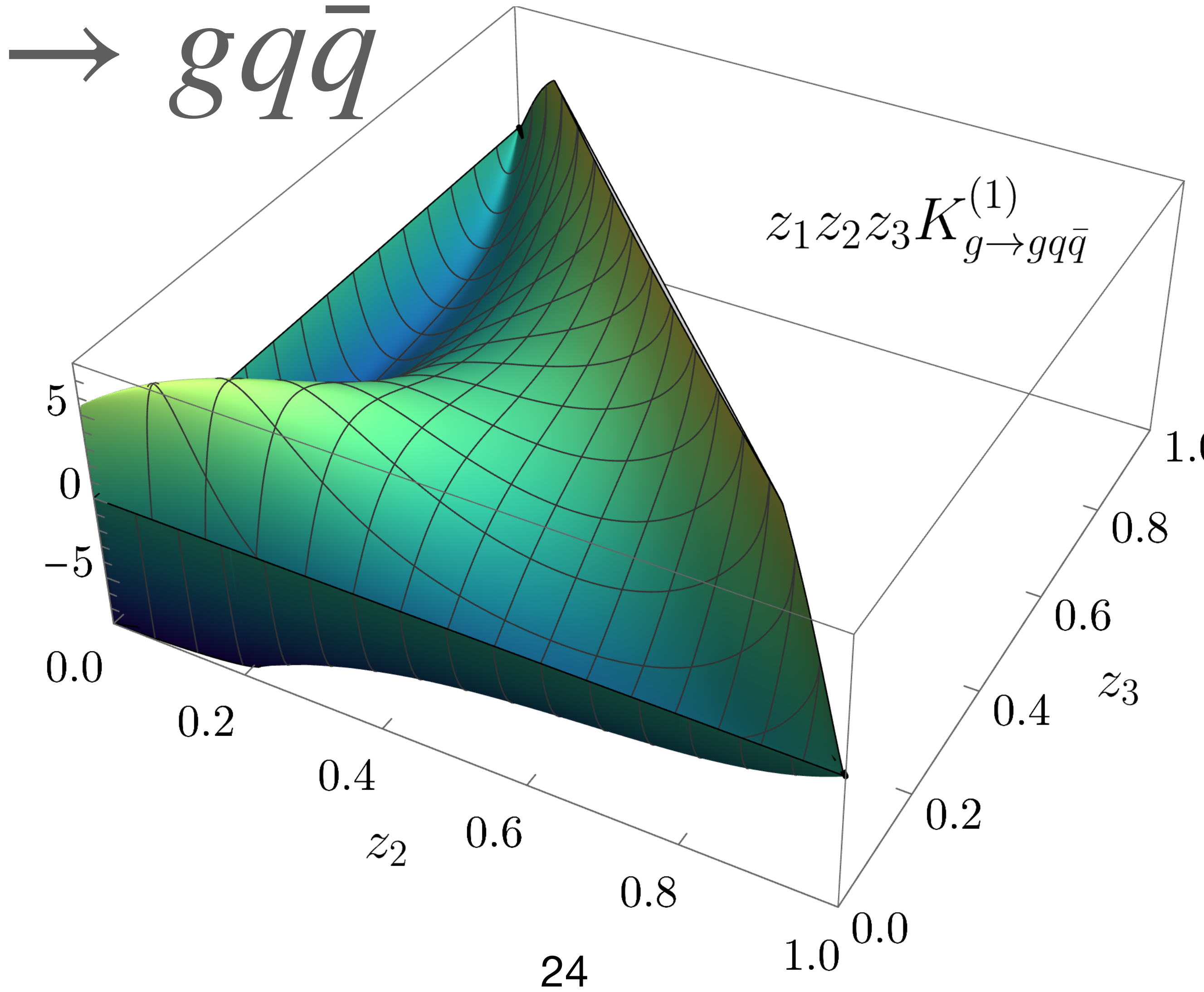
$q \to qQ\bar{Q}$



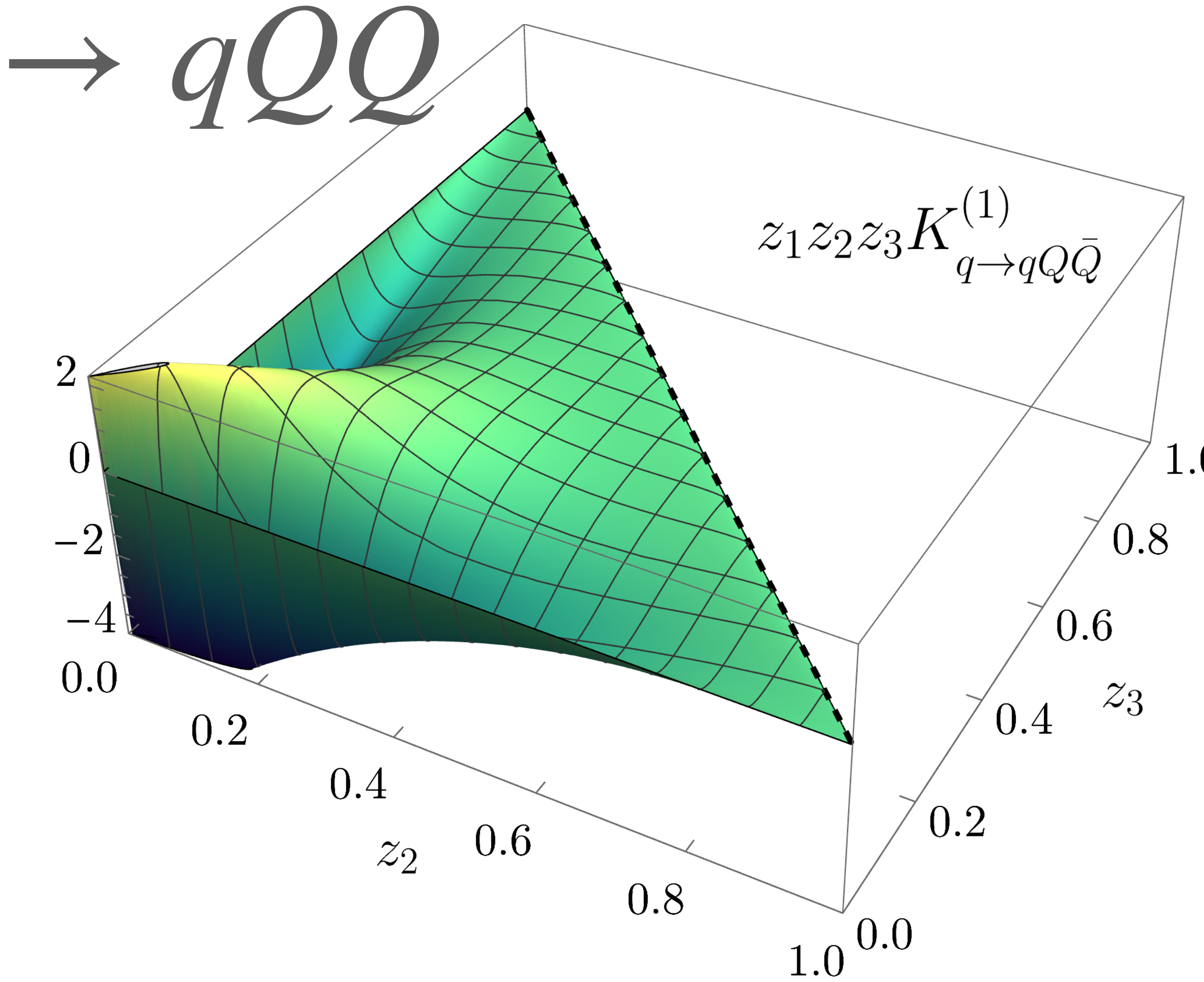
$g \to ggg$



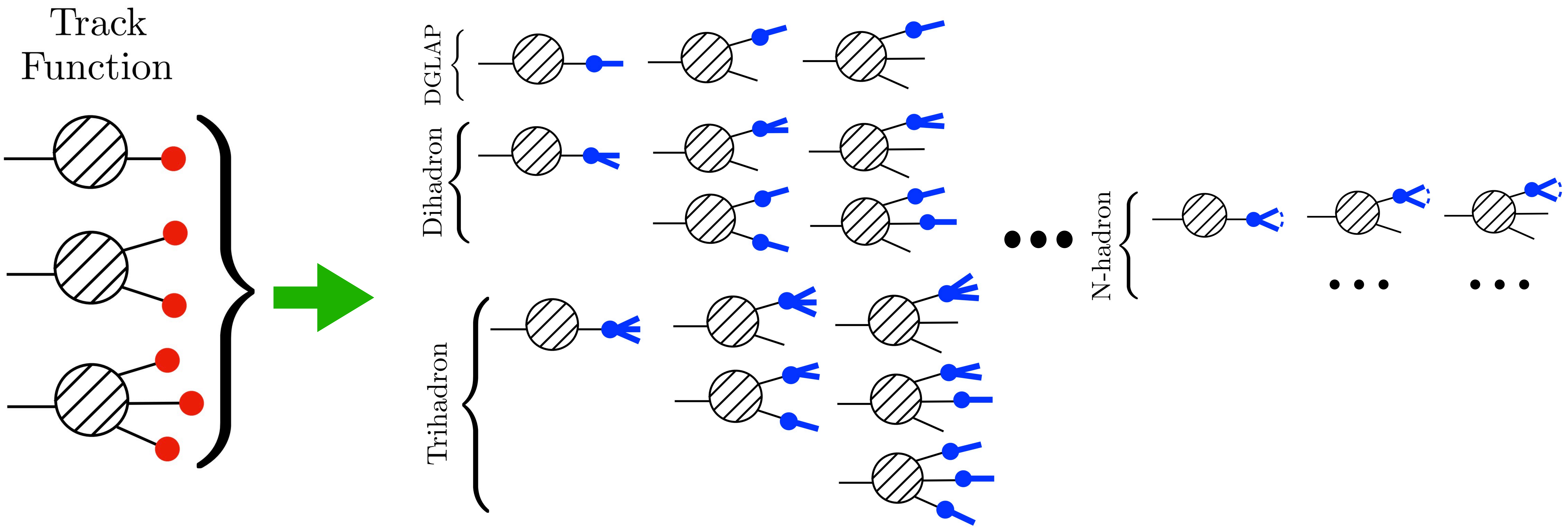
$g \to gq\bar{q}$



$q \to qQ\bar{Q}$



Reduction to Single- & Multi-Hadron Fragmentation



Fragmentation Functions

Single- and Multi-hadron cases

[U. P. Sukhatme and K. E. Lassila, *Phys.Rev.D* 22 (1980) 1184]
[D. de Florian, L. Vanni: arXiv:0310196]

- **The single-hadron fragmentation function** $D_{i\to h}(y)$ gives the probability of finding in a jet a single hadron h with momentum fraction y of that possessed by the jet-initiating parton i (a quark, antiquark or gluon).

- **The** N**-hadron fragmentation function** $D_{i\to h_1h_2\cdots h_N}(y_1, y_2, \cdots, y_N)$ for fragmentation of parton i into N hadrons which carry fractions $y_1, y_2, \cdots, y_N$ of the momentum carried by the initial parton.

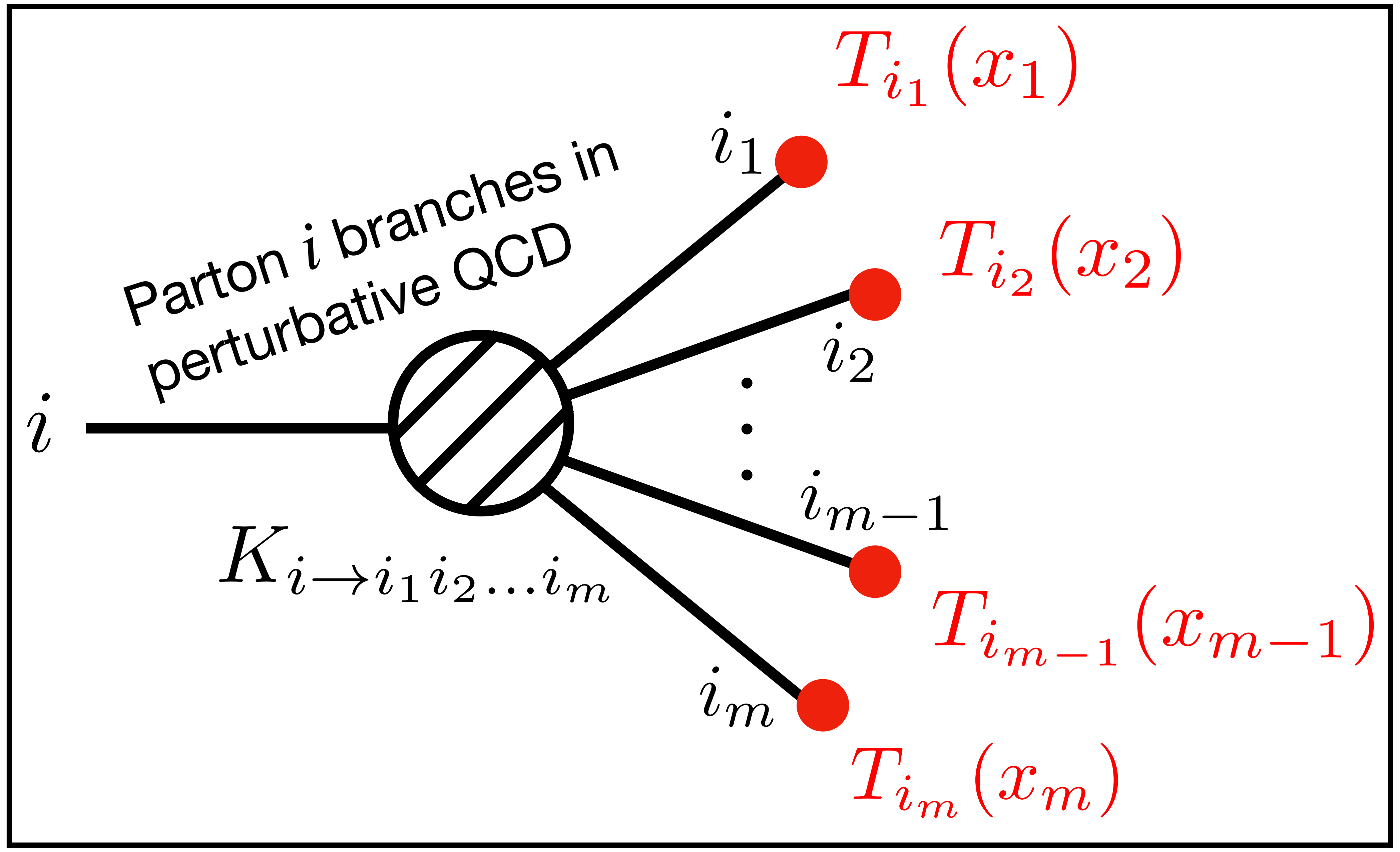
- $N=2$: **Di-hadron fragmentation function** $D_{i\to h_1h_2}(y_1, y_2)$.

Notation

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}T_i(x) = \sum_M \sum_{\{i_f\}} \left[\prod_{m=1}^{M}\int_0^1 \mathrm{d}z_m\right]\delta\left(1-\sum_{m=1}^{M} z_m\right)K_{i\to\{i_f\}}(\{z_f\})$$

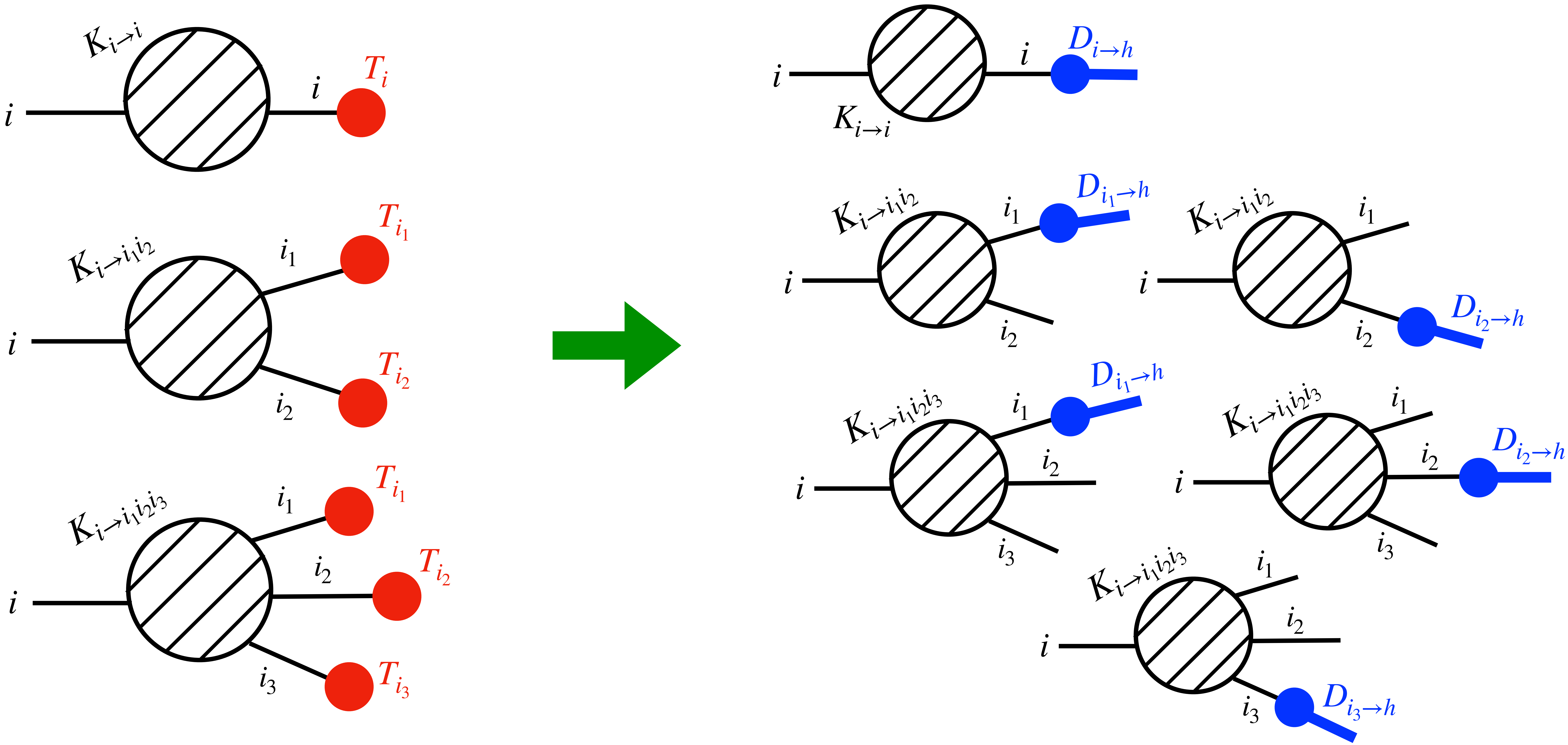
$$\times \left[\prod_{m=1}^{M}\int_0^1 \mathrm{d}x_m\, T_{i_m}(x_m)\right]\delta\left(x-\sum_{m=1}^{M} z_m x_m\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}T_i(x) = \sum_M \sum_{\{i_f\}} K_{i\to i_1 i_2\cdots i_M}\otimes T_{i_1}T_{i_2}\cdots T_{i_M}(x)$$



- For notational simplicity, set $M \leq 3$.

Reduction to DGLAP, Pictorially



Reduction to DGLAP

- At NLO,

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}T_i(x)$$
$$= K^{(1)}_{i\to i}T_i(x)$$
$$+\sum_{\{i_f\}}K^{(1)}_{i\to i_1 i_2}\otimes T_{i_1}(x_1)T_{i_2}(x_2)$$
$$+\sum_{\{i_f\}}K^{(1)}_{i\to i_1 i_2 i_3}\otimes T_{i_1}(x_1)T_{i_2}(x_2)T_{i_3}(x_3)$$

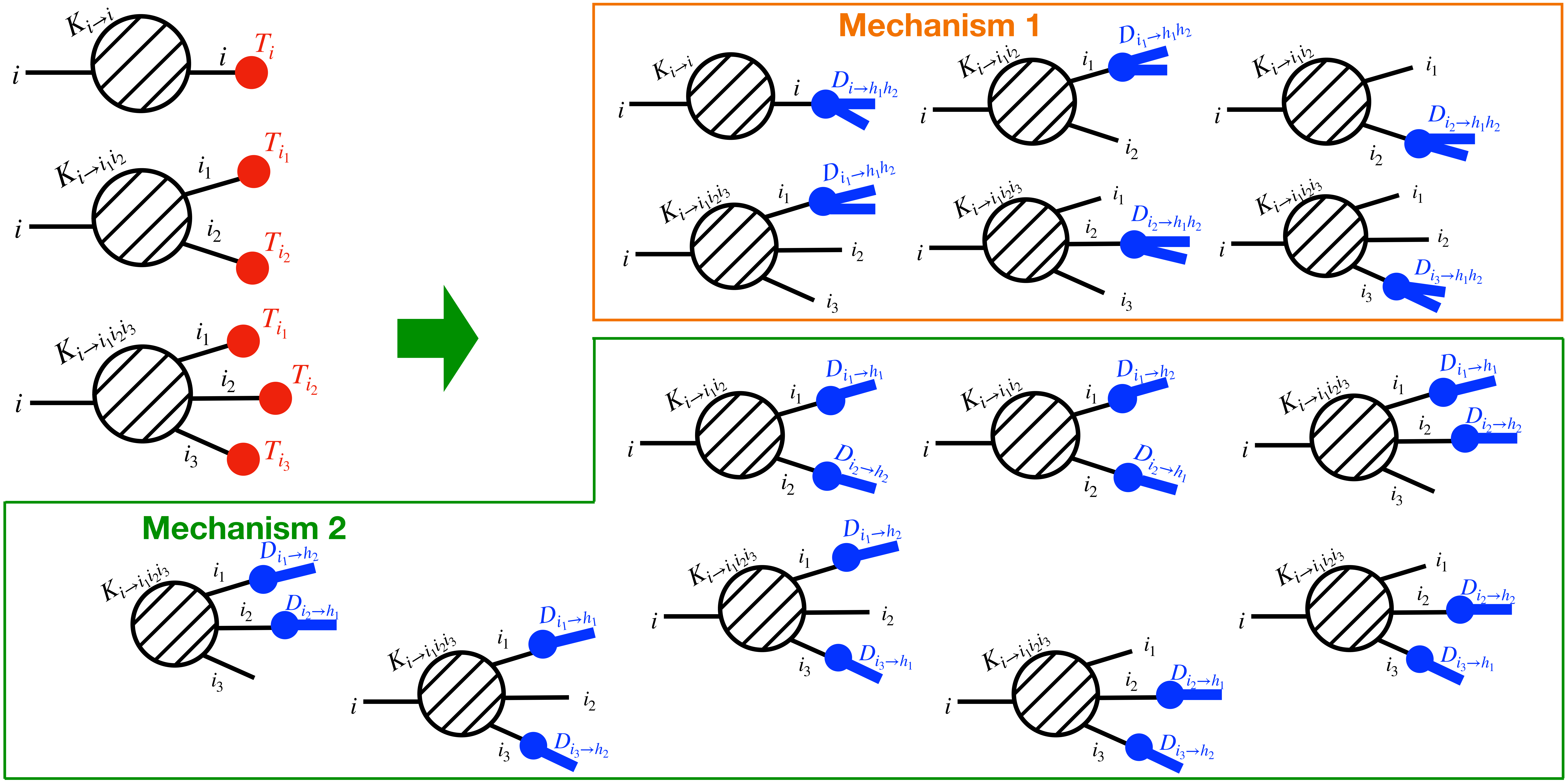
→

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}D_{i\to h}(x)$$
$$= K^{(1)}_{i\to i}D_{i\to h}(x)$$
$$+\sum_{\{i_f\}}K^{(1)}_{i\to i_1 i_2}\otimes [D_{i_1\to h}(x_1)+D_{i_2\to h}(x_2)]$$
$$+\sum_{\{i_f\}}K^{(1)}_{i\to i_1 i_2 i_3}\otimes [D_{i_1\to h}(x_1)+D_{i_2\to h}(x_2)$$
$$+D_{i_3\to h}(x_3)]$$

↕ equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}D_{i\to h}(x)=\sum_j D_{j\to h}\otimes P^T_{ji}(x)$$

Reduction to Di-hadron Fragmentation, Pictorially



Reduction to Di-hadron Fragmentation

- **First** NLO ($\mathcal{O}(\alpha_s^2)$) equation for the di-hadron fragmentation function evolution:

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2} D_{i\to h_1 h_2}(y_1,y_2) = \left\{ K^{(1)}_{i\to i} D_{i\to h_1 h_2}(y_1,y_2) + \sum_{\{i_f\}} K^{(1)}_{i\to i_1 i_2} \otimes \left[D_{i_1\to h_1 h_2} + D_{i_2\to h_1 h_2}\right] \right.$$

$$\left. + \sum_{\{i_f\}} K^{(1)}_{i\to i_1 i_2 i_3} \otimes \left[D_{i_1\to h_1 h_2} + D_{i_2\to h_1 h_2} + D_{i_3\to h_1 h_2}\right] \right\}$$

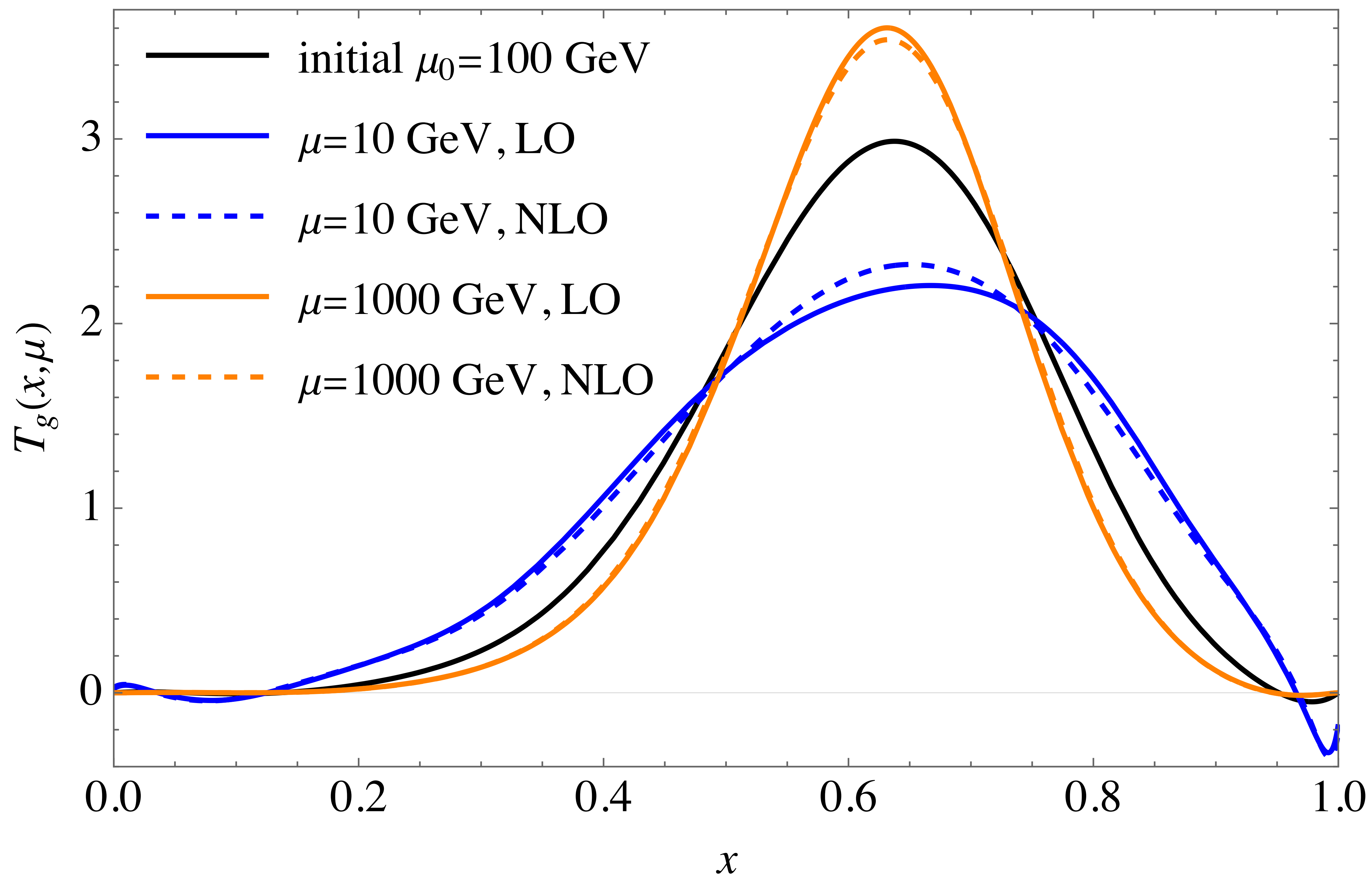
$$+ \left\{ \sum_{\{i_f\}} K^{(1)}_{i\to i_1 i_2} \otimes \left[D_{i_1\to h_1} D_{i_2\to h_2} + D_{i_1\to h_2} D_{i_2\to h_1}\right] \right.$$

$$+ \sum_{\{i_f\}} K^{(1)}_{i\to i_1 i_2 i_3} \otimes \left[D_{i_1\to h_1} D_{i_2\to h_2} + D_{i_1\to h_2} D_{i_2\to h_1}\right.$$

$$+ D_{i_1\to h_1} D_{i_3\to h_2} + D_{i_1\to h_2} D_{i_3\to h_1}$$

$$\left.\left. + D_{i_2\to h_1} D_{i_3\to h_2} + D_{i_2\to h_2} D_{i_3\to h_1}\right] \right\}$$

Numerical Implementation

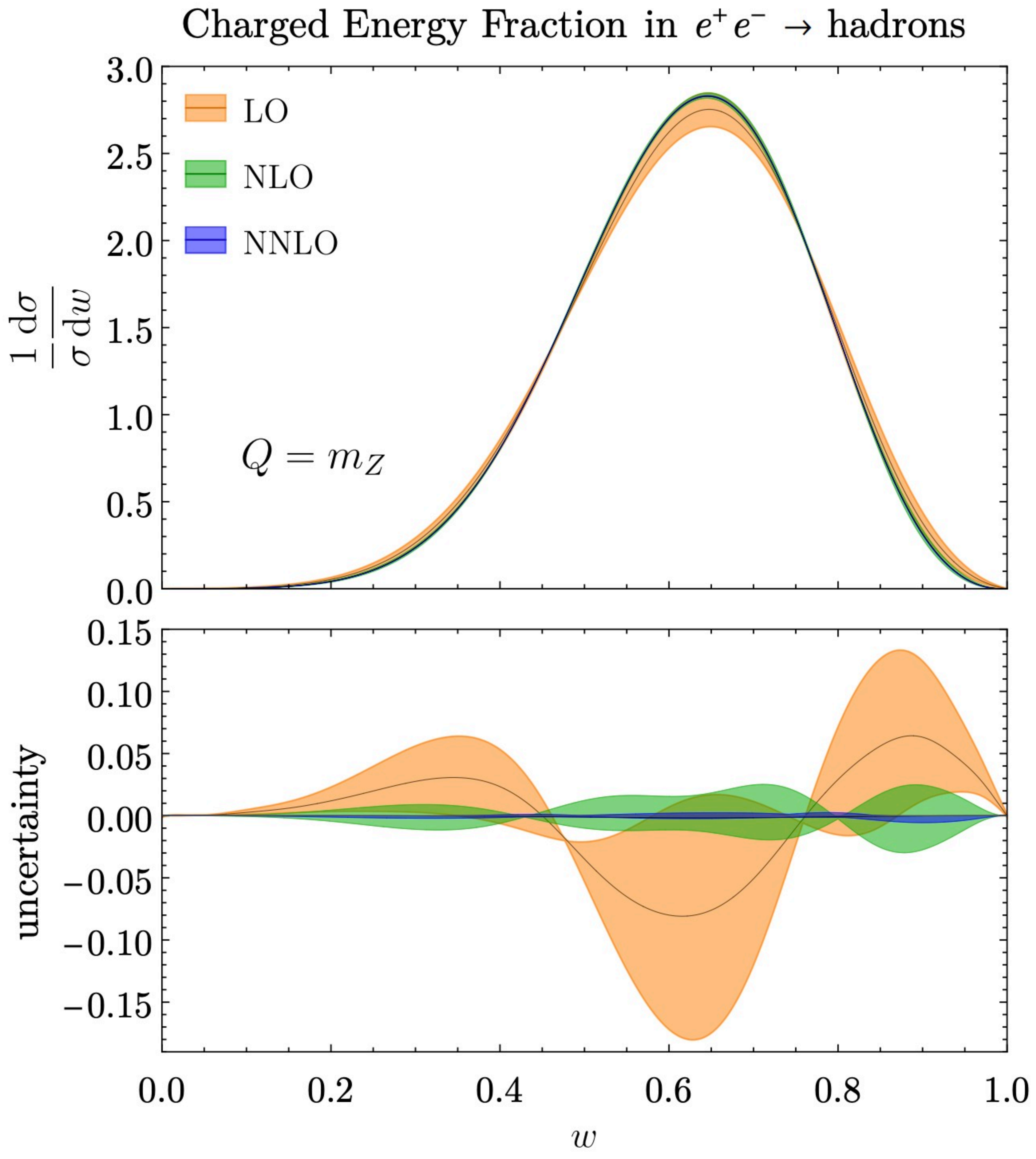


- Numerical methods developed: The moment method, Fourier series, …
  - Initial condition: The NLO track functions extracted from Pythia.

Ready for phenomenology!

Fraction of Charged Hadrons

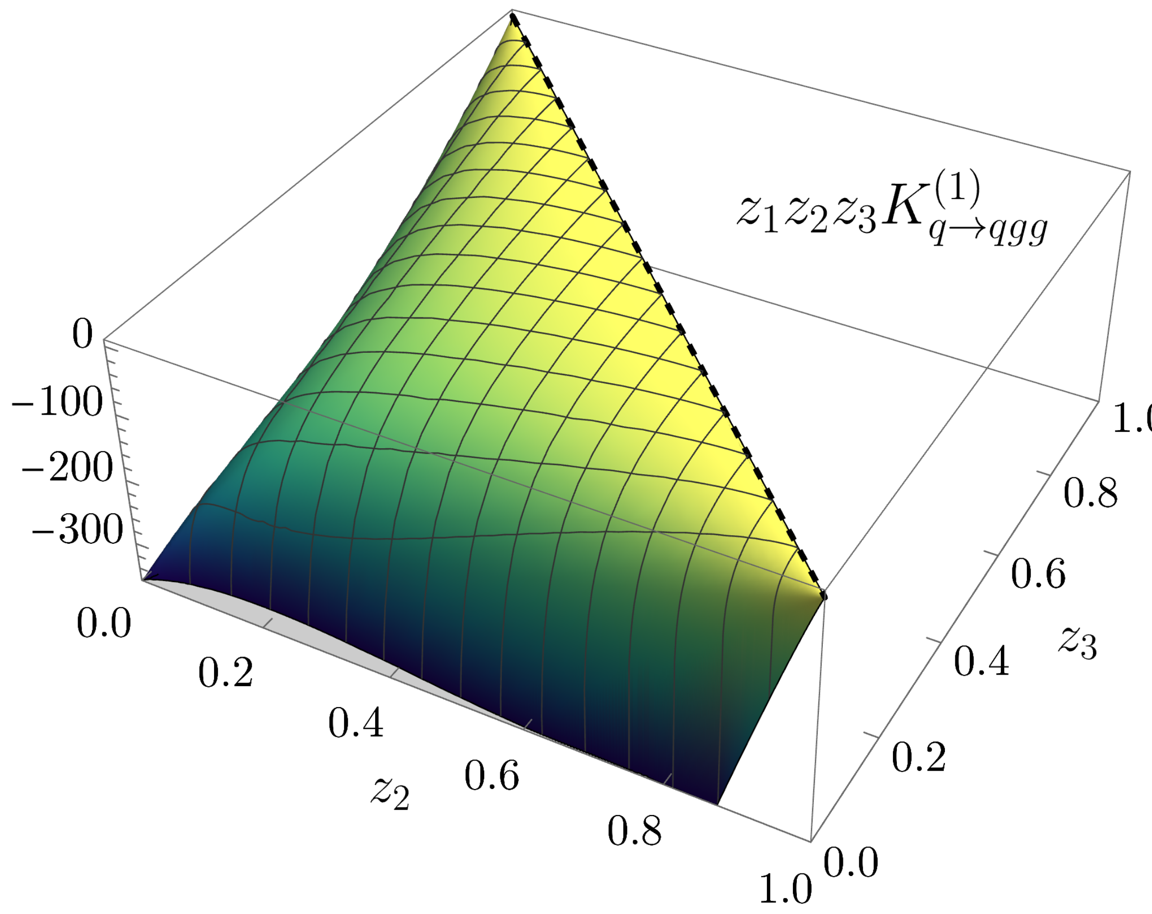
- Significant reduction in scale uncertainties with the NLO evolution of track functions included.

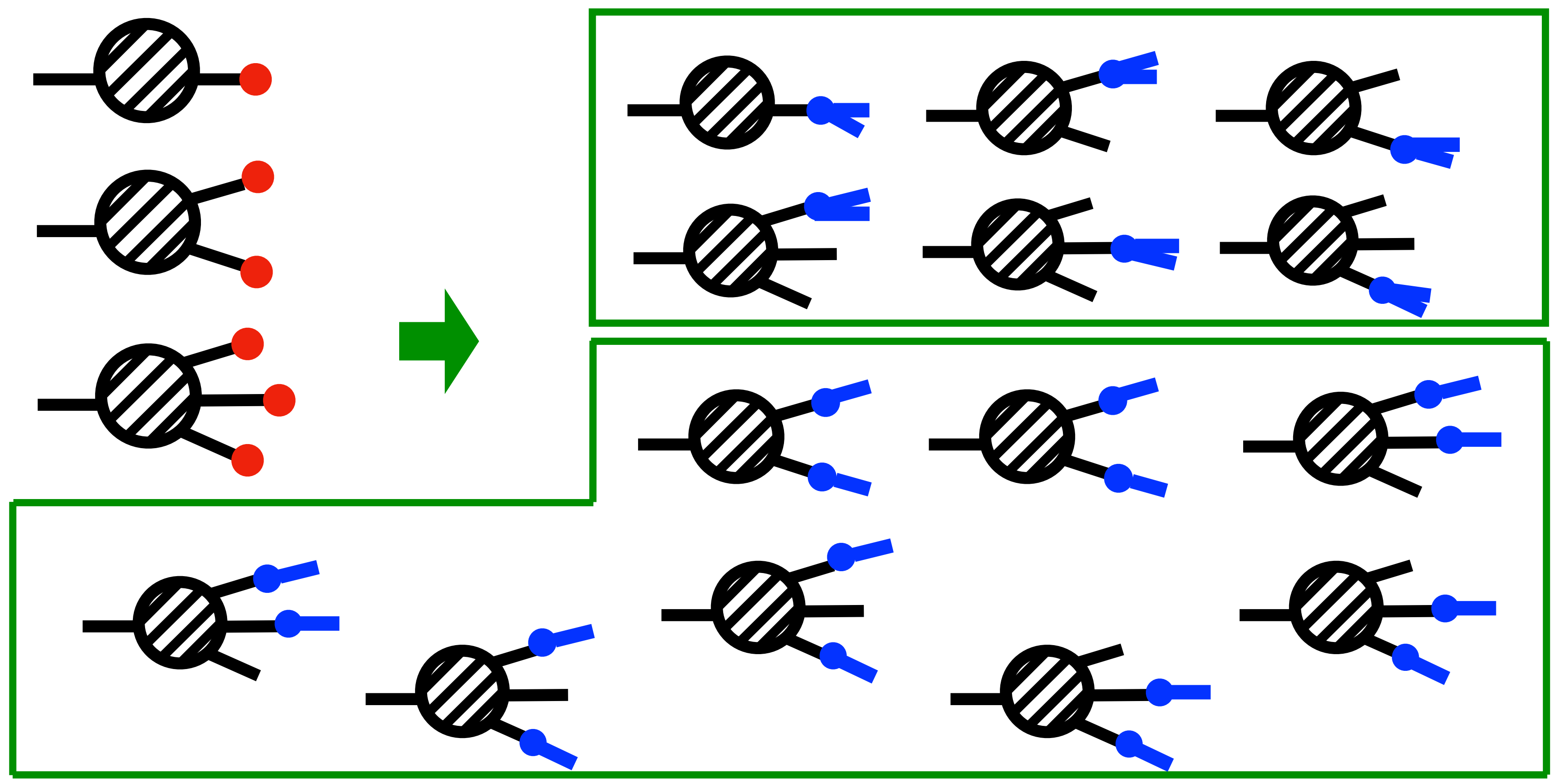


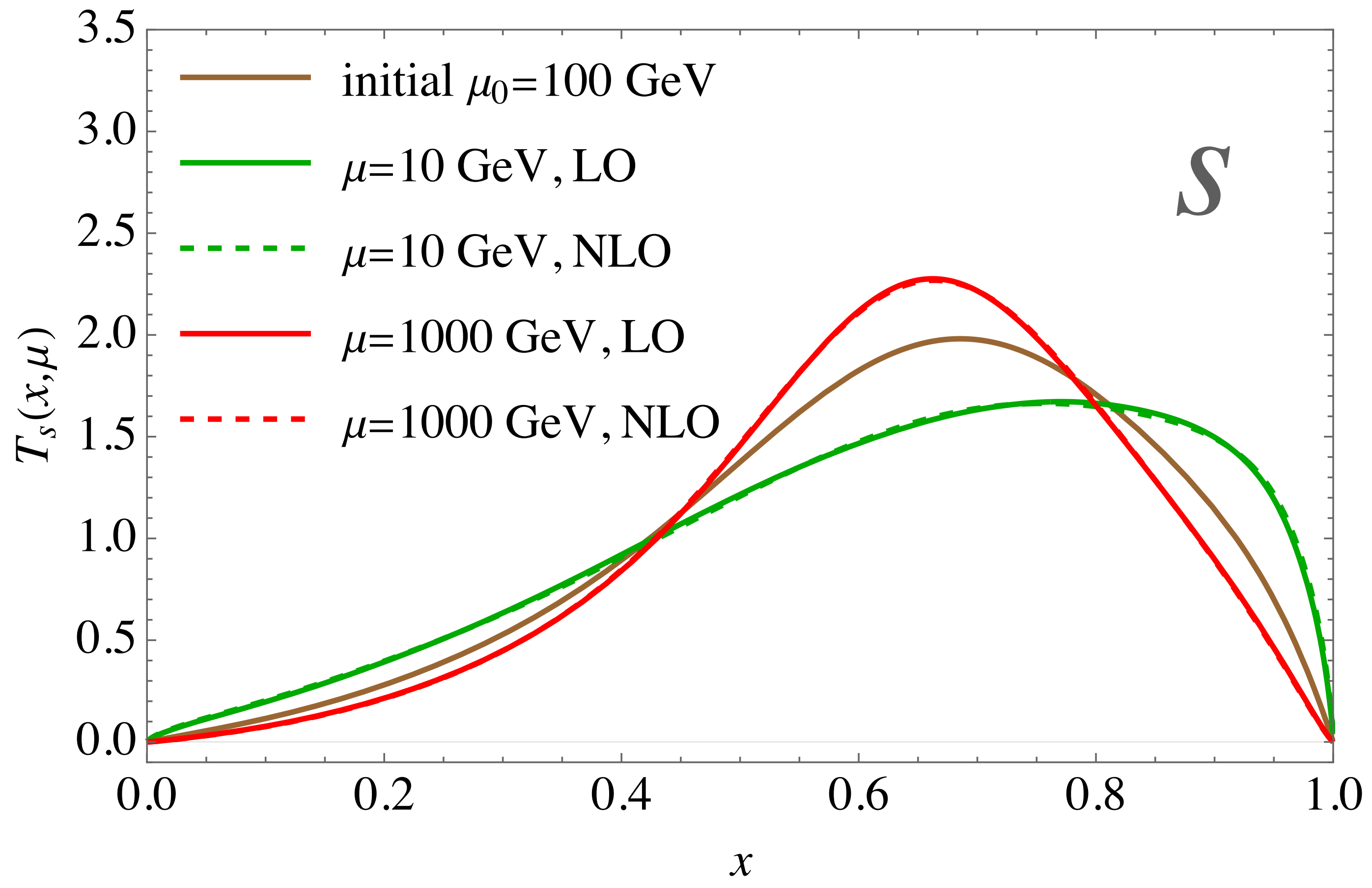
Summary

- Track functions offer a QFT approach to calculating track-based observables.
- Track energy correlators involve the integer moments for which the track function evolution simplifies.
- Full results of the nonlinear x-space evolution at $\mathcal{O}(\alpha_s^2)$:
  - Most general equation for collinear evolution at NLO;

    ➡ the NLO corrections to any N-hadron fragmentation function evolution derived.
  - Numerical implementation.







Outlook

- A benchmark for triple collinear evolution in parton showers.
- Precision phenomenology with tracks.
- Applications of multi-hadron fragmentation functions.

Backup

A Surprising Symmetry:

- $x \to x + a$ leads to $T_j(n) \to \sum_{k=0}^{n} \binom{n}{k} (-a)^{n-k} T_j(k)$ in Mellin space, e.g.,

$$T_j(1) \to T_j(1) - a \ ,$$

$$T_j(2) \to T_j(2) - 2aT_j(1) + a^2 \ ,$$

$$T_j(3) \to T_j(3) - 3aT_j(2) + 3a^2 T_j(1) - a^3$$

- The **shift symmetry** requires that the evolution for moments of track function should *still* hold after the above transformation, which constrains the form of the evolution up to all loop orders: e.g.,

$$\frac{d}{d\ln\mu^2}\left[T_g(1) - a\right] = c_1 \left[T_g(1) - a\right] + c_2 \left[T_q(1) - a\right] \ ,$$

$$\frac{d}{d\ln\mu^2}\left[T_g(2) - 2aT_g(1) - a^2\right] = b_1 \left[T_g(2) - 2aT_g(1) - a^2\right] + b_2 \left[T_q(2) - 2aT_q(1) - a^2\right]$$

$$+ y_1 \left[T_g(1) - a\right]^2 + y_2 \left[T_g(1) - a\right]\left[T_q(1) - a\right] + y_3 \left[T_q(1) - a\right]^2$$

A Surprising Symmetry:

- $x \to x + a$ leads to $T_j(n) \to \sum_{k=0}^{n} \binom{n}{k} (-a)^{n-k} T_j(k)$ in Mellin space, e.g.,

$$T_j(1) \to T_j(1) - a \ ,$$
$$T_j(2) \to T_j(2) - 2aT_j(1) + a^2 \ ,$$
$$T_j(3) \to T_j(3) - 3aT_j(2) + 3a^2T_j(1) - a^3$$

- The **shift symmetry** requires that the evolution for moments of track function should *still* hold after the above transformation, which constrains the form of the evolution up to all loop orders: e.g.,

$$\frac{d}{d\ln\mu^2} T_g(2) = b_1 T_g(2) + b_2 T_q(2) + y_1 T_g(1)T_g(1) + y_2 T_g(1)T_q(1) + y_3 T_q(1)T_q(1)$$
$$+ a\left[(-2b_1 - 2y_1 - y_2 + 2c_1)T_g(1) + (-2b_2 - y_2 - 2y_3 + 2c_2)T_q(1)\right]$$
$$+ a^2(b_1 + b_2 + y_1 + y_2 + y_3) = 0$$

→ $c_1 + c_2 = 0 \ ,$

A Surprising Symmetry:

- $x \to x + a$ leads to $T_j(n) \to \sum_{k=0}^{n} \binom{n}{k} (-a)^{n-k} T_j(k)$ in Mellin space, e.g.,

$$T_j(1) \to T_j(1) - a \ ,$$
$$T_j(2) \to T_j(2) - 2aT_j(1) + a^2 \ ,$$
$$T_j(3) \to T_j(3) - 3aT_j(2) + 3a^2T_j(1) - a^3$$

- The **shift symmetry** requires that the evolution for moments of track function should *still* hold after the above transformation, which constrains the form of the evolution up to all loop orders: e.g.,

→ This implies there is redundancy for these evolution kernels.

- We can use shift invariant objects to reorganize the form of the evolution to avoid this redundancy.

Up to this stage we haven't used the Feynman diagram approach.

Calculational Techniques

$T_i^{(0)}(x) = T_i^{\text{bare}}(x)$

$T_i(x,\mu)$ in a_s-series: $$T_i(x,\mu) = T_i^{(0)}(x) + a_s\, T_i^{(1)}(x) + a_s^2\, T_i^{(2)}(x) + \mathcal{O}(a_s^3)$$

LO track function | UV-renormalized NLO track function | UV-renormalized NNLO track function

$$T_i^{(1)}(x) = -\frac{1}{\epsilon_{\text{IR}}} K_{i\to jk}^{(0)} \otimes T_j^{(0)} T_k^{(0)}(x)\,,$$

$$\begin{aligned}T_i^{(2)}(x) =& \frac{1}{2}\Bigg\{ -\frac{1}{\epsilon_{\text{IR}}}\Big[K_{i\to jk}^{(1)} \otimes T_j^{(0)} T_k^{(0)}(x) + K_{i\to jmn}^{(1)} \otimes T_j^{(0)} T_m^{(0)} T_n^{(0)}(x) \Big] \\ &+ \frac{1}{\epsilon_{\text{IR}}^2}\Big\{ K_{i\to jk}^{(0)} \otimes \Big[T_j^{(0)} \Big(K_{k\to mn}^{(0)} \otimes T_m^{(0)} T_n^{(0)} \Big) \Big](x) + \beta_0\, K_{i\to jk}^{(0)} \otimes T_j^{(0)} T_k^{(0)}(x) \\ &+ K_{i\to jk}^{(0)} \otimes \Big[T_k^{(0)} \Big(K_{j\to mn}^{(0)} \otimes T_m^{(0)} T_n^{(0)} \Big) \Big](x) \Big\} \Bigg\}\,.\end{aligned}$$

Track Jet Functions

$$J_{\mathrm{tr},i}(s,x,\mu) = \delta(s)\sum_{\ell=0}^{\infty}\sum_{\{i_f\}}\left[a_s^\ell \mathcal{J}^{(\ell)}_{i\to\{i_f\}} \otimes \prod_{j\in\{i_f\}} T_j\right](x,\mu)$$

$$\begin{aligned}
&J_{\mathrm{tr},i}(s,x,\mu)\\
&= \delta(s)\,T_i^{(0)}(x) + a_s(\mu)\,\delta(s)\left\{\left(\mathcal{J}^{(1)}_{i\to i} - \frac{1}{\epsilon}K^{(0)}_{i\to i}\right)\otimes T_i^{(0)}(x) + \left(\mathcal{J}^{(1)}_{i\to i_1 i_2} - \frac{1}{\epsilon}K^{(0)}_{i\to i_1 i_2}\right)\otimes T_{i_1}^{(0)}T_{i_2}^{(0)}(x)\right\}\\
&\quad + a_s^2(\mu)\,\delta(s)\left\{\left(\mathcal{J}^{(2)}_{i\to i} - \frac{1}{2\epsilon}K^{(1)}_{i\to i} + \frac{\beta_0}{2\epsilon^2}K^{(0)}_{i\to i}\right)\otimes T_i^{(0)}\right.\\
&\qquad + \left(\mathcal{J}^{(2)}_{i\to i_1 i_2} - \frac{1}{2\epsilon}K^{(1)}_{i\to i_1 i_2} + \frac{\beta_0}{2\epsilon^2}K^{(0)}_{i\to i_1 i_2}\right)\otimes T_{i_1}^{(0)}T_{i_2}^{(0)}\\
&\qquad + \left(\mathcal{J}^{(2)}_{i\to i_1 i_2 i_3} - \frac{1}{2\epsilon}K^{(1)}_{i\to i_1 i_2 i_3}\right)\otimes T_{i_1}^{(0)}T_{i_2}^{(0)}T_{i_3}^{(0)}\\
&\qquad + \left(-\frac{1}{\epsilon}\mathcal{J}^{(1)}_{i\to i} + \frac{1}{2\epsilon^2}K^{(0)}_{i\to i}\right)\otimes\left(K^{(0)}_{i\to i}\otimes T_i^{(0)} + K^{(0)}_{i\to i_1 i_2}\otimes T_{i_1}^{(0)}T_{i_2}^{(0)}\right)\\
&\qquad + \left(-\frac{1}{\epsilon}\mathcal{J}^{(1)}_{i\to i_1 i_2} + \frac{1}{2\epsilon^2}K^{(0)}_{i\to i_1 i_2}\right)\otimes\left[T_{i_1}^{(0)}\left(K^{(0)}_{i_2\to i_2}\otimes T_{i_2}^{(0)} + K^{(0)}_{i_2\to j_1 j_2}\otimes T_{j_1}^{(0)}T_{j_2}^{(0)}\right)\right.\\
&\qquad\left.\left. + \left(K^{(0)}_{i_1\to i_1}\otimes T_{i_1}^{(0)} + K^{(0)}_{i_1\to k_1 k_2}\otimes T_{k_1}^{(0)}T_{k_2}^{(0)}\right)T_{i_2}^{(0)}\right]\right\} + \mathcal{O}(a_s^3)\,,
\end{aligned}$$

LO evolution kernel

NLO evolution kernel