Mathematical aspects of FOPT/CIPT discrepancy in Tau decays

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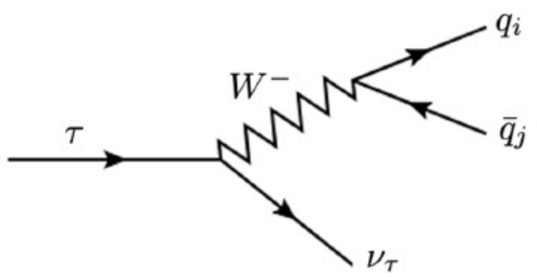
Context and motivation

- $\tau^- \to \text{hadrons}\, \nu_\tau$ presents a good frame to study QCD under clean conditions



- It allows for the extraction of α_s with precision competitive to the world-average

Dominated by the FOPT-CIPT discrepancy

$$\alpha_s(m_\tau) = 0.332 \pm 0005_{\text{exp}} \pm 0.011_{\text{theo}}$$

ALEPH collaboration. *Physics Reports*, 421:191–284, 12 2005.

- α_s is extracted from matching the integrals of the experimentally accessible invariant mass spectral functions, running from threshold to m_τ^2 (LEP), to integrals over the vacuum polarization function, carried out in a closed contour of the complex plane:

$$\delta^{(0)}_{W(x)} = \frac{1}{2i\pi} \oint_{|x|=1} \frac{\mathrm{d}x}{x} W(x) \hat{D}(m_\tau^2 x)$$

$$D(s) = \frac{N_c}{12\pi^2}\left(1 + \hat{D}(s)\right) = -s \frac{\mathrm{d}}{\mathrm{d}s}\Pi(s)$$

- The weight function is a polynomial with $W(1) = 0$ for physical applications and the kinematic weight function $W_\tau(x) = 1 - 2x + 2x^3 - x^4$ recovers the inclusive hadronic τ decay width

Fixed-Order vs Contour Improved Perturbation Theory

Context and motivation

$$\delta^{(0)}_{W(x)} = \frac{1}{2i\pi} \oint_{|x|=1} \frac{\mathrm{d}x}{x} W(x) \hat{D}(m_\tau^2 x)$$

- $\hat{D}(s)$ is μ-independent

$$\begin{aligned}
\hat{D}(s) &= \sum_{n=1}^{\infty} \left[\frac{\alpha_s(\mu^2)}{\pi}\right]^n \sum_{k=1}^{n} k\, c_{n,k} \ln^{k-1}\!\left(-\frac{s}{\mu^2}\right) \\
&= \sum_{n=1}^{\infty} \left[\frac{\alpha_s(m_\tau^2)}{\pi}\right]^n \sum_{k=1}^{n} k\, c_{n,k} \ln^{k-1}\!\left(-\frac{s}{m_\tau^2}\right) \\
&= \sum_{n=1}^{\infty} \left[\frac{\alpha_s(-s)}{\pi}\right]^n c_{n,1}
\end{aligned}$$

FOPT: $\mu^2 = m_\tau^2$

CIPT: $\mu^2 = -s$

- Considering weight functions that are monomials $(-x)^\ell$

$$\delta^{(0)}_{\mathrm{FOPT},\ell} = \frac{1}{2i\pi} \sum_{n=1}^{\infty} \left[\frac{\alpha_s(m_\tau^2)}{\pi}\right]^n \sum_{k=1}^{n} k\, c_{n,k} \oint_{|x|=1} \frac{\mathrm{d}x}{x} (-x)^\ell \ln^{k-1}(-x)$$

$$\delta^{(0)}_{\mathrm{CIPT},\ell} = \frac{1}{2i\pi} \sum_{n=1}^{\infty} \frac{c_{n,1}}{\pi} \oint_{|x|=1} \frac{\mathrm{d}x}{x} (-x)^\ell\, \alpha_s^n(-x m_\tau^2)$$

Fixed-Order vs Contour Improved Perturbation Theory

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FOPT: $\mu^2 = m_\tau^2$

CIPT: $\mu^2 = -s$

$$\left(\begin{aligned} a(\mu^2) &= \frac{\beta_0 \alpha_s(\mu^2)}{4\pi} \\ a &= a(m_\tau^2) \\ \bar{c}_{n,k} &= (4/\beta_0)^n c_{n,k}\end{aligned}\right)$$

- Considering weight functions that are monomials $(-x)^\ell$

$$\delta^{(0)}_{\mathrm{FOPT},\ell} = \frac{1}{2i\pi} \sum_{n=1}^{\infty}\left[\frac{\alpha_s(m_\tau^2)}{\pi}\right]^n \sum_{k=1}^{n} k\, c_{n,k} \oint_{|x|=1} \frac{\mathrm{d}x}{x} (-x)^\ell \ln^{k-1}(-x)$$

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Power expansion in a

$$\delta^{(0)}_{\mathrm{FOPT},\ell} = \sum_{n=1}^{\infty} a^n \sum_{k=1}^{n} k\, \bar{c}_{n,k} I_{k-1,\ell}$$

$$\delta^{(0)}_{\mathrm{CIPT},\ell} = \sum_{n=1}^{\infty} \bar{c}_{n,1} H_{n,\ell}(a)$$

Expansion in functions $H_{n,\ell}(a)$

- The Adler function is known in the large-β_0

$$\hat{D}(s) = \int_0^\infty \mathrm{d}u \big[B(u)\big]_{\text{Taylor}} e^{-\frac{u}{a(-s)}}$$

$$B(u) = \frac{128}{3\beta_0} e^{\frac{5u}{3}} \left\{ \frac{3}{16(2-u)} + \sum_{p=3}^{\infty} \left[\frac{d_2(p)}{(p-u)^2} - \frac{d_1(p)}{p-u} \right] - \sum_{p=-1}^{-\infty} \left[\frac{d_2(p)}{(u-p)^2} + \frac{d_1(p)}{u-p} \right] \right\},$$

- For single pole IR renormalons $1/(p-u)$ and moments $W(x) = (-x)^\ell$ the OPE corrections ($\text{const} \times \langle \mathcal{O}_p \rangle / m_\tau^{2p}$) vanish in the conotur integration for $p \neq \ell$

- So we consider series arising from the contribution $B_p(u) = 1/(p-u)$ and expect a convergent series and start by reconfirming

  - FOPT is convergent
  - CIPT is not

This fixes

$$\bar{c}_{n,k,p} = \frac{(-1)^{k+1}\Gamma(n)}{p^{n-k+1}\Gamma(k+1)}$$

Fixed-Order and Contour Improved as series

$$\bar{c}_{n,k,p} = \frac{(-1)^{k+1}\Gamma(n)}{p^{n-k+1}\Gamma(k+1)}$$

- Both series are fully known analytically

$$\delta^{(0)}_{\mathrm{FOPT},\ell} = \sum_{n=1}^{\infty} d^{\mathrm{FOPT}}_{n,\ell}\, a^n$$

$$d^{\mathrm{FOPT}}_{n,\ell} = \sum_{k=1}^{n} k\, \bar{c}_{n,k} I_{k-1,\ell}$$

$$\delta^{(0)}_{\mathrm{CIPT},\ell} = \sum_{n=1}^{\infty} \bar{c}_{n,1} H_{n,\ell}(a)$$

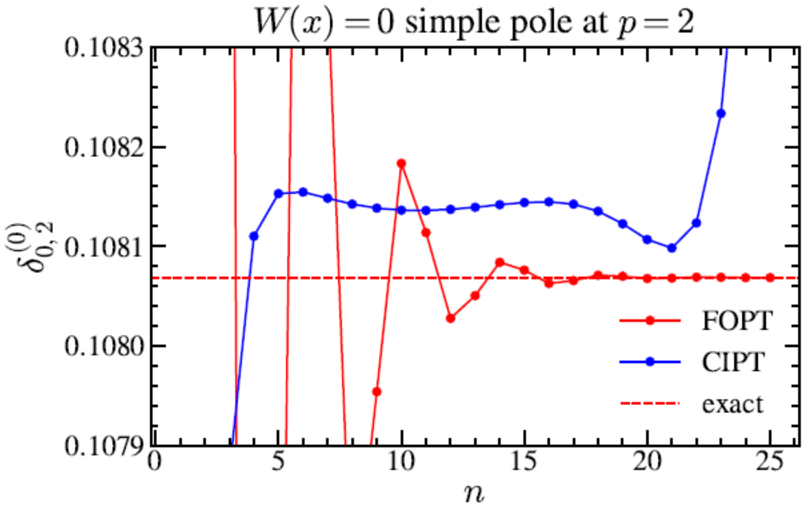
$$H_{n,\ell}(a) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \mathrm{d}\phi\, e^{i\ell\phi} \boxed{\left[\frac{a}{1+ia\phi}\right]}^n$$

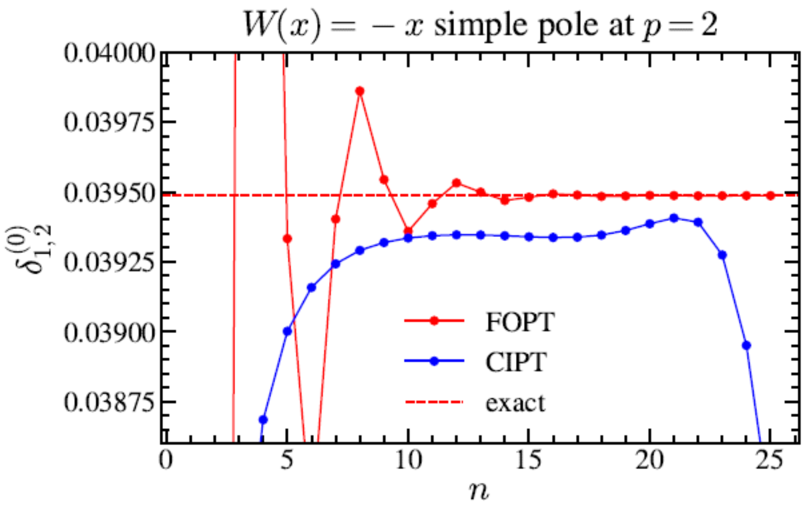
Large-β_0 running

$$\begin{aligned}
d^{\mathrm{FOPT}}_{n,0,p} &= \frac{(-1)^p\Gamma(n+1, i\pi p, -i\pi p)}{2i\pi n p^{n+1}}, \\
d^{\mathrm{FOPT}}_{n,\ell>0,p\neq\ell} &= \frac{\ell^{-n}\Gamma(n, i\pi\ell, -i\pi\ell) - (-1)^{\ell+p} p^{-n}\Gamma(n, i\pi p, -i\pi p)}{2i\pi(\ell-p)}, \\
d^{\mathrm{FOPT}}_{n,p,p} &= \frac{\Gamma(n, i\pi p) + \Gamma(n, -i\pi p)}{2p^n} + \frac{\Gamma(n+1, -i\pi p, i\pi p)}{2\pi i p^{n+1}}.
\end{aligned}$$

$$\begin{aligned}
H_{1,0}(a) &= \frac{1}{\pi}\mathrm{arctan}(a\pi) \\
H_{n\geq 2,0} &= \left(\frac{a}{\sqrt{1+a^2\pi^2}}\right)^n \frac{\sin[(n-1)\mathrm{arctan}(a\pi)]}{\pi(n-1)} \\
H_{n,\ell\geq 1} &= \frac{(-1)^n}{2i\pi} e^{-\ell/a} \ell^{n-1} \Gamma\left(1-n, -\frac{\ell}{a} - i\pi\ell, -\frac{\ell}{a} + i\pi\ell\right)
\end{aligned}$$

Convergence and large order behavior





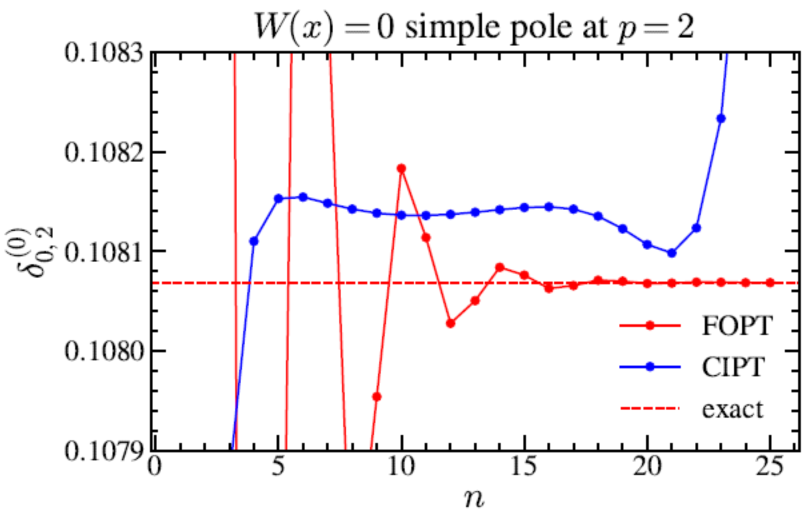
- FOPT is an absolute convergent series within the circle $|a| < 1/\pi$ in the complex plane of a: seen from large n behavior and the root test

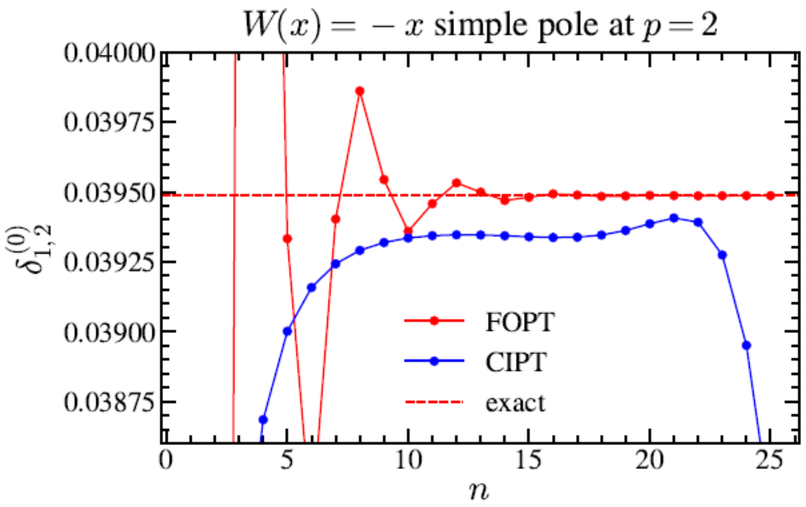
$$d^{\mathrm{FOPT}}_{n,\ell,p} - \frac{\Gamma(n)}{p^n}\delta_{\ell p} = (-1)^{\ell+1}\pi^n \sum_{k=1}^{\infty} \frac{\sin\left[\frac{\pi}{2}(k+n)\right](p\pi)^{k-1}}{(n)_{k+1}} \sum_{j=0}^{k-1}\left(\frac{\ell}{p}\right)^j, \qquad (a)_b = \Gamma(a+b)/\Gamma(a)$$

$$\limsup_{n\to\infty} \left|d^{\mathrm{FOPT}}_{n,\ell,p=\ell}\right|^{1/n} = \infty \qquad \limsup_{n\to\infty} \left|d^{\mathrm{FOPT}}_{n,\ell,p\neq\ell}\right|^{1/n} = \pi$$

True for linear combinations of $(-x)^\ell$

Convergence and large order behavior





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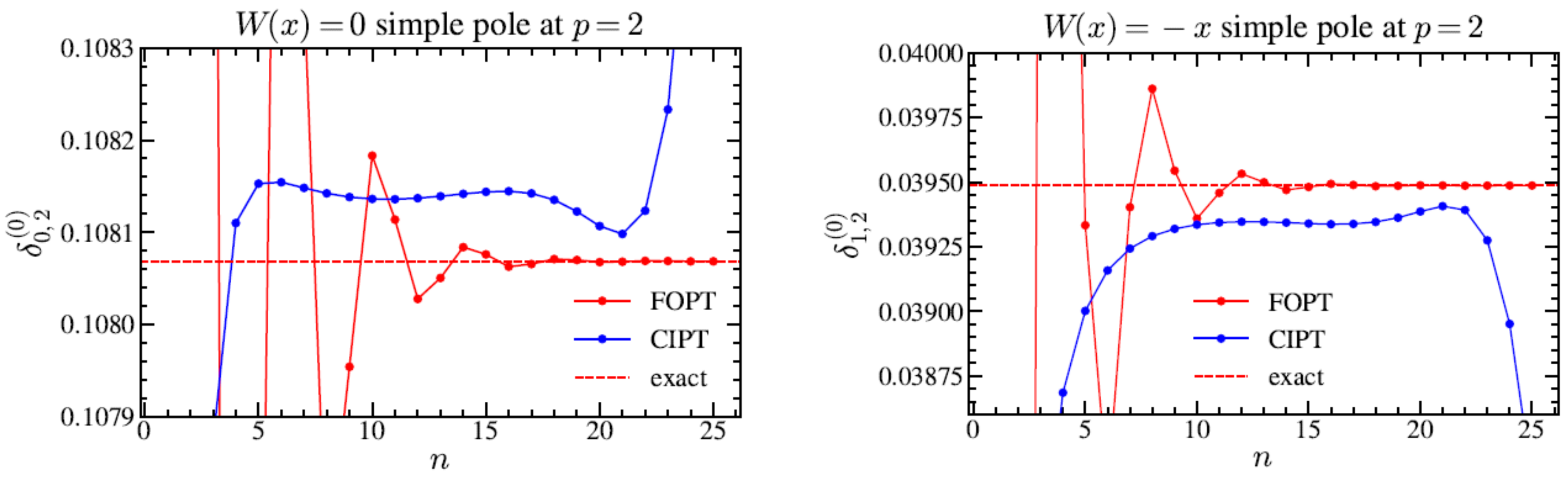
Bonus observation: for $W(x)$ a linear combination of $(-x)^\ell$ such that $W(1) = 0$ and $W'(1) = W''(1) = \ldots = W^{(k)}(1) = 0$ the first k terms in the sum cancel

Convergence and large order behavior

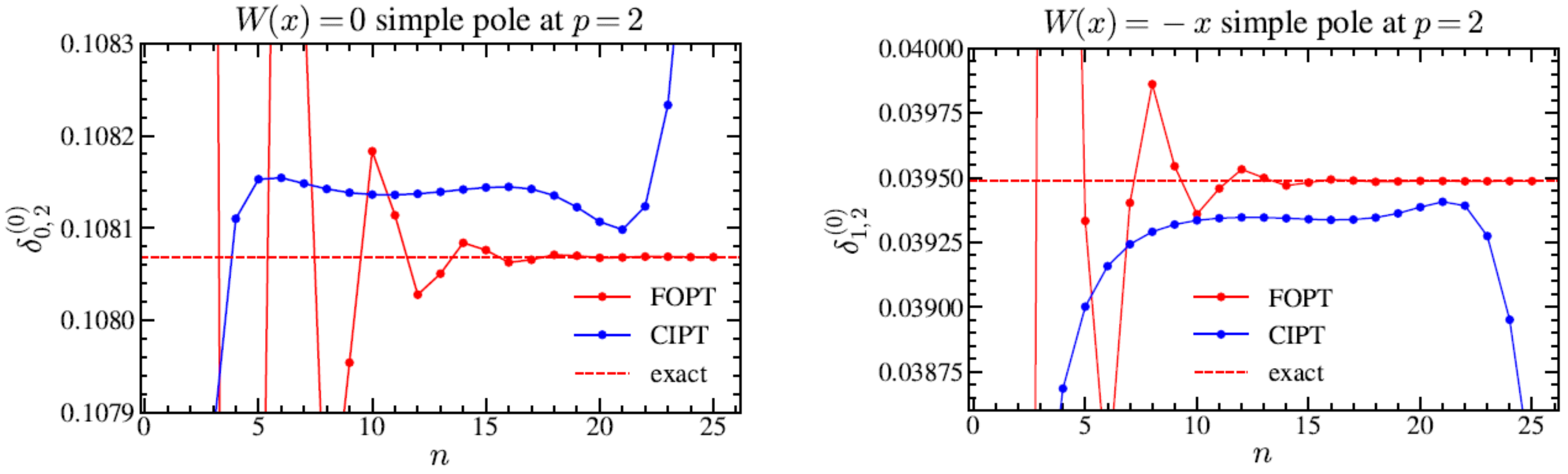


- CIPT has zero convergence radius: seen from large-n behavior and the root test

$$\limsup_{n\to\infty} |H_{n,\ell}(a)|^{1/n} = \frac{|a|}{\sqrt{1+|a|^2\pi^2 - 2\pi|\mathrm{Im}(a)|}}$$

$$\delta^{(0)}_{\mathrm{CIPT},\ell} = \sum_{n=1}^{\infty} \underbrace{\bar{c}_{n,1}}_{\text{Divergent}} \overbrace{H_{n,\ell}(a)}^{\text{Bounded}} \qquad \bar{c}_{n,1} = \Gamma(n)/p^n$$

Convergence and large order behavior



- CIPT has zero convergence radius: seen from large-n behavior and the root test

Bonus observation 1: for physical weight functions ($W(1) = 0$), the leading term in the large n expansion also cancels

Bonus observation 2: for real a, $H_{n,\ell}(a)$ scales in powers of $\frac{|a|}{\sqrt{1+a^2\pi^2}} < \min(|a|, \frac{1}{\pi})$
Better convergence than FOPT's expansion in a^n at low orders.

A deeper look: asymptotic sequences

- The sequence $\{a^n\}_{n=1}^{\infty}$ is an asymptotic sequence as $a \to 0$

Definition (o-relation). *It is said that $f = o(g)$ as $x \to x_0$, if for any $\epsilon > 0$ there exists a neighborhood U_ϵ of x_0 such that $|f(x)| < \epsilon|g(x)|$ for all $x \in R \cap U_\epsilon$. When $g(x) \neq 0$ in some neighborhood of x_0, the condition is equivalent to $\lim_{x \to x_0} f(x)/g(x) = 0$.*

Definition (Asymptotic Sequence). *A sequence of functions $\{\phi_n(x)\}$ is an asymptotic sequence as $x \to x_0$ in R if $\phi_{n+1} = o(\phi_n)$ for all n.*

$$\lim_{a \to 0} a^{n+1}/a^n = 0$$

It is also uniform in n, in the sense that U_ϵ is the same for all n

A deeper look: asymptotic sequences

- The sequence $\{a^n\}_{n=1}^{\infty}$ is an asymptotic sequence as $a \to 0$

  It is also uniform in n, in the sense that U_ϵ is the same for all n

- The functions $H_{n,\ell}(a)$ are analytic in the circle $|a| < 1/\pi$ and vanish as a^{n+1} (a^n if $\ell = 0$) as $a \to 0$

  As sequence, $\{H_{n,\ell}(a)\}_{n=1}^{\infty}$ is an asymptotic sequence as $a \to 0$

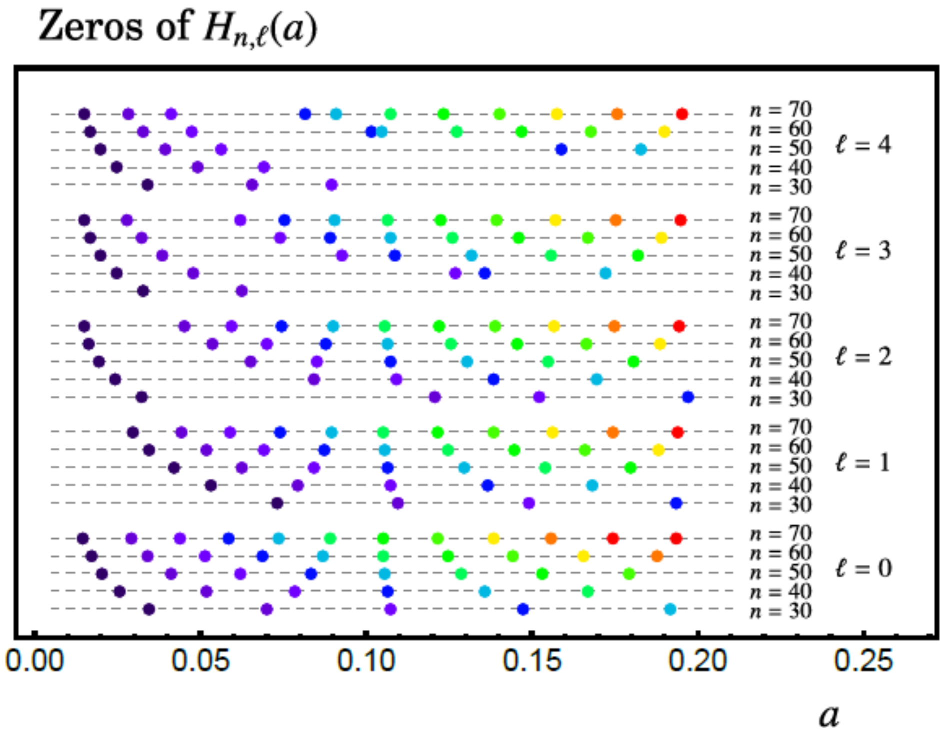
  But it is not uniform in n: $\lim_{a\to 0} H_{n+1,\ell}(a)/H_{n,\ell}(a) = 0$ only in regions $U_\epsilon^{(n)}$ that shrink to zero with n

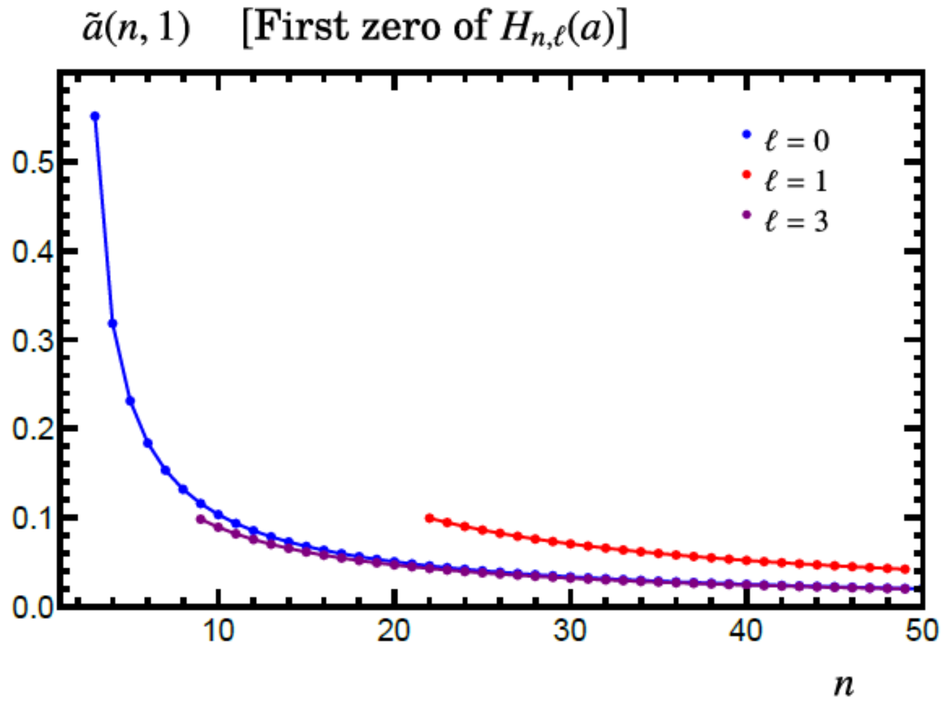
$$H_{n\geq 2,0} = \left(\frac{a}{\sqrt{1+a^2\pi^2}}\right)^n \frac{\sin[(n-1)\arctan(a\pi)]}{\pi(n-1)}$$

Zeros that approach $a = 0$ with n

Zeros at $\tilde{a}(n,k) = \frac{1}{\pi}\tan\left(\frac{k\pi}{n-1}\right), \quad -\frac{n-1}{2} < k < \frac{n-1}{2}$

A deeper look: asymptotic sequences





- The number of zeros grows with n
- As n increases, each zero moves closer to $a = 0$
- The zeros of $H_{n,\ell}(a)$ may provide another reason for the good convergence of CIPT at intermediate orders

A deeper look: double series transformations

- By analyticity, $H_{n,\ell}(a)$ can be Taylor-expanded around $a = 0$

$$H_{n,\ell} = \sum_{k=n}^{\infty} s_{n,k}^{\ell}\, a^k \qquad \begin{cases} s_{n,k}^{0} = \dfrac{\Gamma(k)\cos[\frac{\pi}{2}(k-n)]}{\Gamma(n)\Gamma(k-n+2)} \\ s_{n,k}^{\ell} = \dfrac{\Gamma(k)\Gamma(k-n+1,-i\pi\ell,i\pi\ell)}{2\pi i\,\ell^{k+n-1}\,\Gamma(n)\Gamma(k-n+1)} \end{cases}$$

$$\begin{pmatrix} s_{1,1} & s_{1,2} & s_{1,3} & \cdots \\ 0 & s_{2,2} & s_{2,3} & \cdots \\ 0 & 0 & s_{3,3} & \cdots \\ 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

The expansion is absolutely convergent with radius $|a| < 1/\pi$

$$s_{n,k}^{\ell} = \frac{(-1)^{\ell}\pi^{k-n}}{\Gamma(n)} \sum_{j=0}^{\infty} (\ell\pi)^j \frac{\cos[\frac{\pi}{2}(k-n+j)]}{(k)_{2-n+j}}$$

Holds for combinations in $W(x)$ that cancel leading terms

A deeper look: double series transformations

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$$H_{n,\ell} = \sum_{k=n}^{\infty} s_{n,k}^{\ell}\, a^k$$

The expansion provides a transformation into a double series (with no particular summation prescription)

$$\sum_{n=1}^{\infty} c_n H_{n,\ell}(a) \longrightarrow \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} c_n s_{n,k}^{\ell} a^k$$

If the original CIPT series is absolutely convergent, then, because

- $H_{n,\ell}(a)$ are bounded
- $H_{n,\ell} = \sum_{k=n}^{\infty} s_{n,k}^{\ell}\, a^k$ are absolutely convergent

the Weierstrass theorem for double series ensures both prescriptions are convergent and equal

$$\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty} c_n s_{n,k}^{\ell} a^k\right) = \sum_{k=1}^{\infty}\left(\sum_{n=1}^{k} c_n s_{n,k}^{\ell} a^k\right)$$

An absolutely convergent CIPT will turn into a convergent FOPT

A deeper look: double series transformations

- By analyticity, $H_{n,\ell}(a)$ can be Taylor-expanded around $a = 0$

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Even if the original CIPT series is divergent, it is posible that expansion and FOPT reorganization leads lead to a convergent series (as shown by the examples $W(x) = 1, -x$)

A deeper look: double series transformations

- Let us now consider the inverse transformation

$$a^n = \sum_{k=n}^{\infty} t^{\ell}_{n,k} H_{n,\ell}(a)$$

It is posible to determine the coefficients

$$t^0_{n,k} = \frac{(i\pi)^{k-n}(2-2^{k-n})\Gamma(k)}{\Gamma(k-n+1)\Gamma(n)} B_{k-n},$$

Bernoulli numbers

$$t^{\ell>0}_{n>1,k} = \begin{cases} 0, & k < n-1 \\ (-1)^{\ell}\frac{k-1}{n-1} t^0_{n-1,k-1}, & n+k \quad \text{even} \\ (-1)^{\ell+1}\frac{\ell}{n-1} t^0_{n-1,k}, & n+k \quad \text{odd} \end{cases}$$

The expansion has zero radius of convergence

An absolute convergent FOPT will in general turn into a divergent CIPT

$$\sum_{n=1}^{\infty} d_n a^n \quad \longrightarrow \quad \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} d_n t^{\ell}_{n,k} H_{k,\ell}(a)$$

A deeper look: double series transformations

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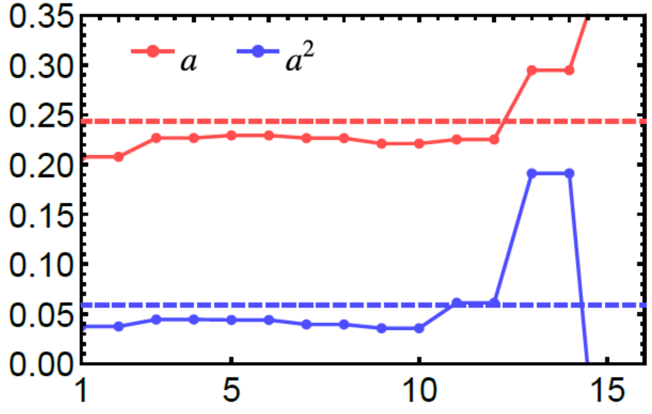
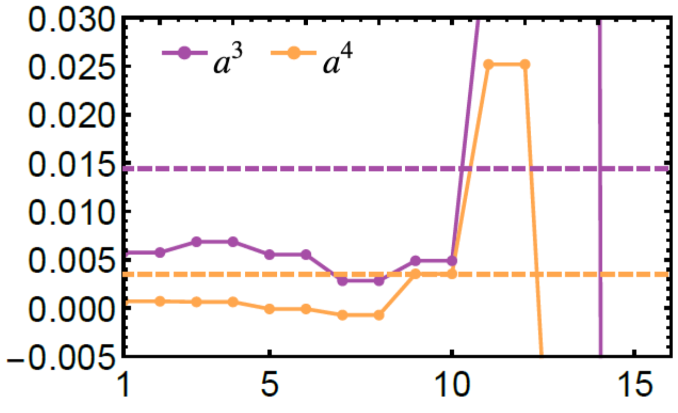
Example: a^n

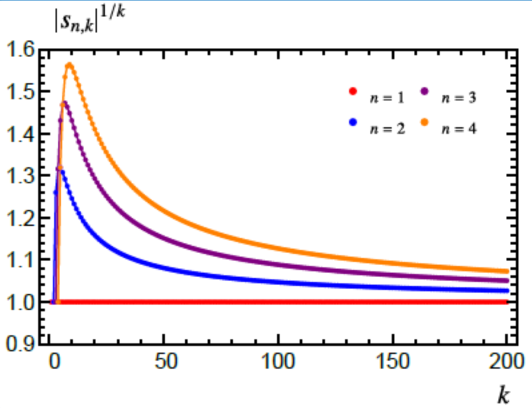


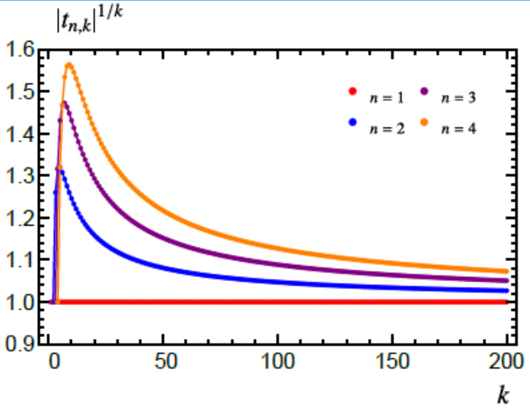


Large-β_0 approximation

A deeper look: double series transformations

- The convergence properties of $a^n = \sum_{k=n}^{\infty} t^{\ell}_{n,k} H_{n,\ell}(a)$ provides a diagnostic tool to test expansions in functions of the running coupling
- Remaining question: connetion to the non-uniformity of $\{H_{n,\ell}(a)\}_{n=1}^{\infty}$ induced by the zeros



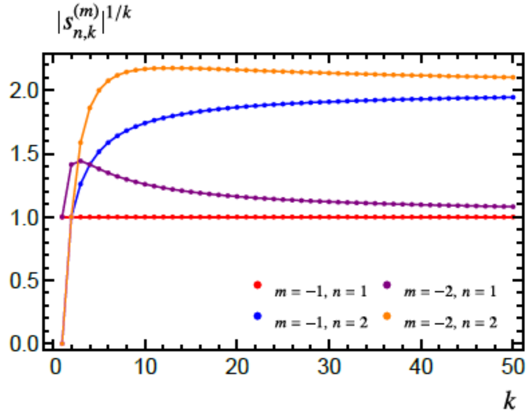


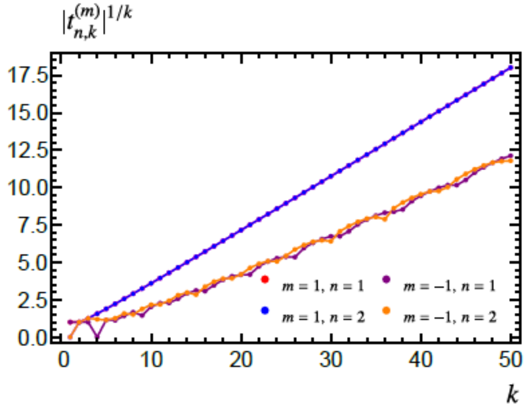
Model with n independent zeros

$$h_n(a) = \frac{a^n}{(1+aL)^n}$$

Model with zeros growing closer to the origin with n

$$\hat{h}^{(m)}_n(a) = a^n(1-\xi n a)^m$$





Conclussions

- While FOPT is an expansion in powers of the strong coupling, a^n, which form a asymptotic as $a \to 0$ uniform in n, CIPT is an expansion in non-trivial functions of the strong coupling, $H_{n,\ell}(a)$, which form also n asymptotic as $a \to 0$ that is <u>not uniform</u> in n
- The <u>non-uniformity</u> of CIPT is due to <u>zeros</u> of the $H_{n,\ell}(a)$ functions approaching $a = 0$ with n
- The CIPT expansion of a single power a^n is factorially divergent, a fact that arises for every studied H-model with zeros aproaching the origin. Ultimately this makes CIPT series divergent and the apparent convergence may yield an unphysical value inconistent with OPE
- These claims are also valid for physical combinations of $(-x)^\ell$ and in particular for $W_\tau(x)$
- … and also in full QCD, where single renormalon poles become cuts. With a combination of analytic formulas in the C-scheme for the strong coupling and numerical studies we provide evidence that our findigs are also valid.

Thank you
for your attention