
IBP Reduction with Gröbner bases

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In collaboration with:

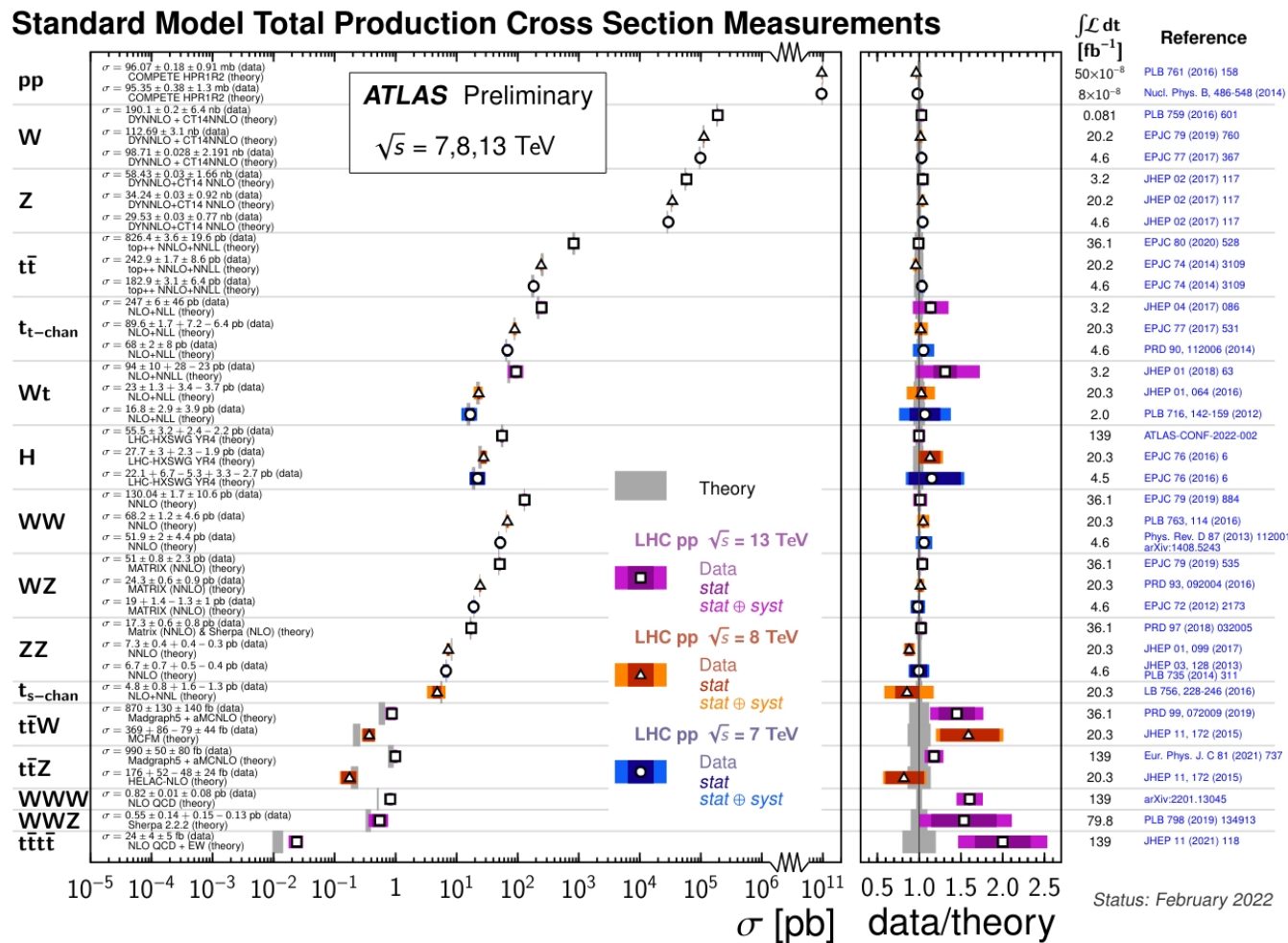
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Precision physics

The standard model of particle physics is extremely successful!



Precision physics

The standard model of particle physics is extremely successful!

However: No new particle beyond the standard model discovered so far!

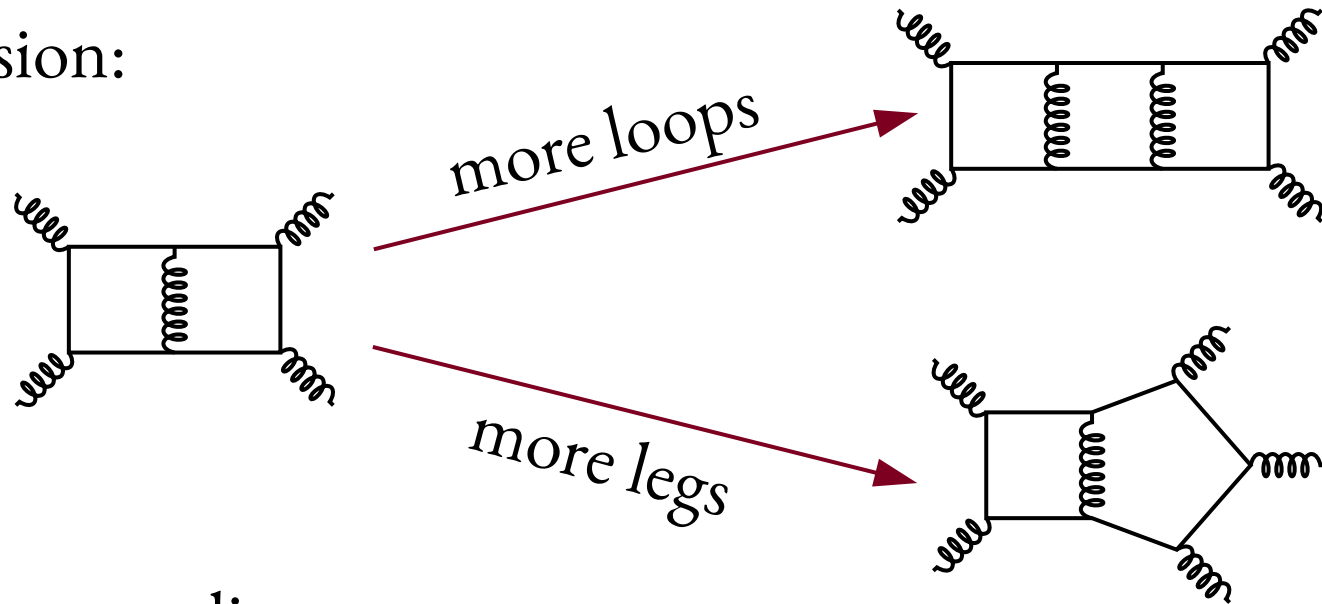
Discovery through precision:
experimental measurements
+
theoretical predictions



search for deviations
from the standard model

Loop calculations

Increasing precision:



Challenges:

- number of Feynman diagrams
- size of intermediate expressions
- complexity of loop integrals
- ...

integration-by parts
(IBP) reduction

This talk:

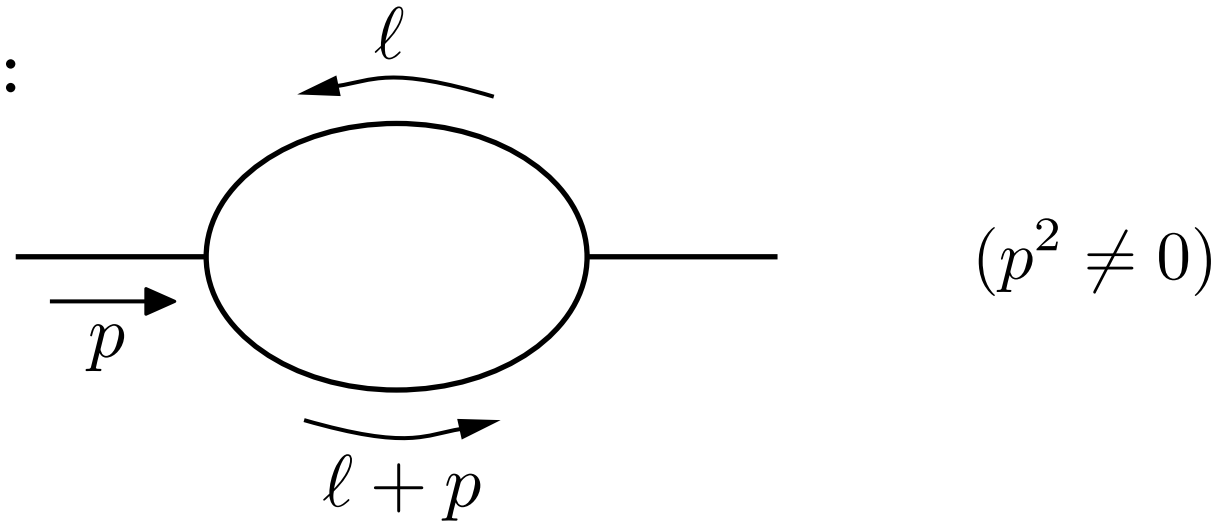
- new conceptually insights into IBP reduction

1. IBP reduction
2. Gröbner bases
3. IBP reduction with Gröbner bases
4. Linear algebra ansatz

IBP Reduction

Integral family

Bubble integrals:



$$F(a_1, a_2) = \int \frac{d^d \ell}{i\pi^{d/2}} \frac{1}{D_1^{a_1} D_2^{a_2}}, \quad a_i \in \mathbb{Z}$$

$$D_1 = -\ell^2$$

$$D_2 = -(\ell + p)^2$$

$$\Longleftrightarrow$$

$$\ell^2 = -D_1$$

$$\ell \cdot p = \frac{1}{2}(D_1 - D_2 - p^2)$$

Vacuum polarization

$$i\Pi^{\mu\nu} = i(p^2 g^{\mu\nu} - p^\mu p^\nu) \Pi(p^2) = \text{diagram with a circle labeled 1PI}$$

One-loop (R_ξ gauge):

$$\Pi_{1\text{-loop}} = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4}$$

$$\propto C_A \left[c_{1,1} F(1, 1) + c_{1,2} F(1, 2) + c_{2,1} F(2, 1) + c_{2,2} F(2, 2) \right] + \frac{2(d-2)}{(d-1)} T_F n_f F(1, 1)$$

IBP reduction

$$\propto \tilde{c}_{1,1} F(1, 1)$$

$\in \mathbb{Q}(d, p^2)$
(rational in d and p^2)
+ polynomial in ξ

Vacuum polarization

$$i\Pi^{\mu\nu} = i(p^2 g^{\mu\nu} - p^\mu p^\nu)\Pi(p^2) = \text{diagram with a circle labeled 1PI}$$

Two-loop (R_ξ gauge):

$$\Pi_{2\text{-loop}} = \sum_{i=1}^{213} c_i \times (\text{scalar integral})_i$$

IBP reduction

$$= \tilde{c}_1 \times \text{diagram 1} + \tilde{c}_2 \times \text{diagram 2}$$

two master integrals

Scaleless Integrals

Consider:

$$F(a_1, 0) = \int \frac{d^d \ell}{i\pi^{d/2}} \frac{1}{[-\ell^2]^{a_1}}$$


Rescale loop momenta: $\ell \rightarrow \lambda \ell$

$$\Rightarrow F(a_1, 0) = \lambda^{d-2a_1} F(a_1, 0) \quad \Rightarrow \quad F(a_1, 0) = 0$$

In general we have:

$$F(a_1, a_2) = 0, \text{ if } a_1 \leq 0 \text{ or } a_2 \leq 0.$$

(Scaleless integrals can efficiently be identified with the help of the Feynman or Lee-Pomeransky representation.)

[Pak, Smirnov, '11 | Lee, '13]

The bubble integral family has the symmetry

$$F(a_1, a_2) = F(a_2, a_1)$$

This symmetry is realized by the following shift of loop momentum: $\ell \rightarrow -\ell - p$

reminder: $D_1 = -\ell^2$

$$D_2 = -(\ell + p)^2$$

(Symmetries can efficiently be found with the help of the Feynman or Lee-Pomeransky representation.) [Pak, '12 | Hoff '15]

IBP equations

Starting point:

[Tkachov, '81 | Chetyrkin, Tkachov, '81]

$$0 = \int \frac{d^d \ell}{i\pi^{d/2}} \frac{\partial}{\partial \ell^\mu} \frac{v^\mu}{D_1^{a_1} D_2^{a_2}}$$

standard IBP equations:

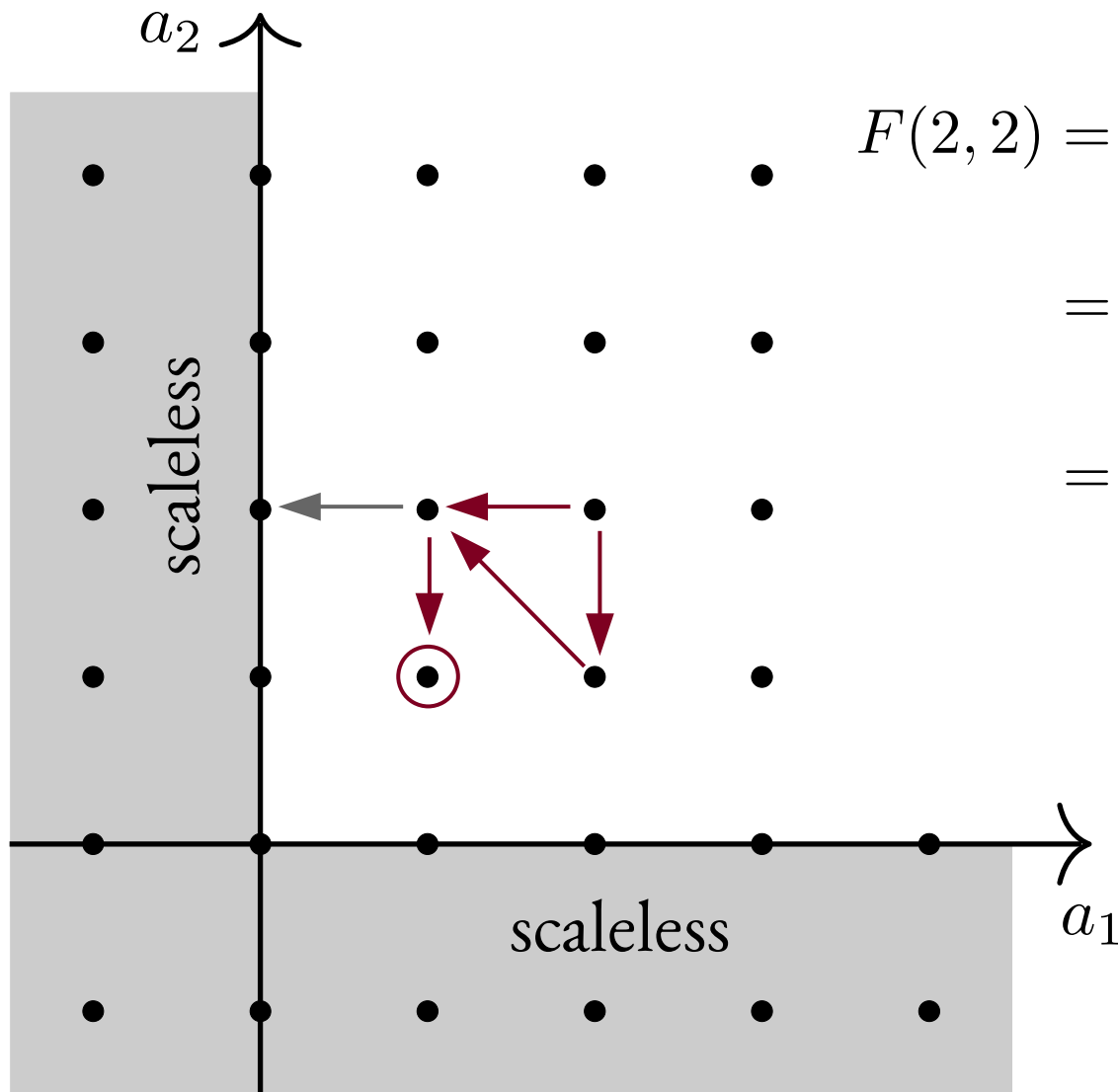
$v = \ell$:

$$0 = (d - a_2 - 2a_1)F(a_1, a_2) - a_2 p^2 F(a_1, a_2 + 1) \\ - a_2 F(a_1 - 1, a_2 + 1)$$

$v = p$:

$$0 = (a_1 - a_2)F(a_1, a_2) + a_2 p^2 F(a_1, a_2 + 1) - a_1 p^2 F(a_1 + 1, a_2) \\ + a_2 F(a_1 - 1, a_2 + 1) - a_1 F(a_1 + 1, a_2 - 1)$$

IBP Reduction by hand

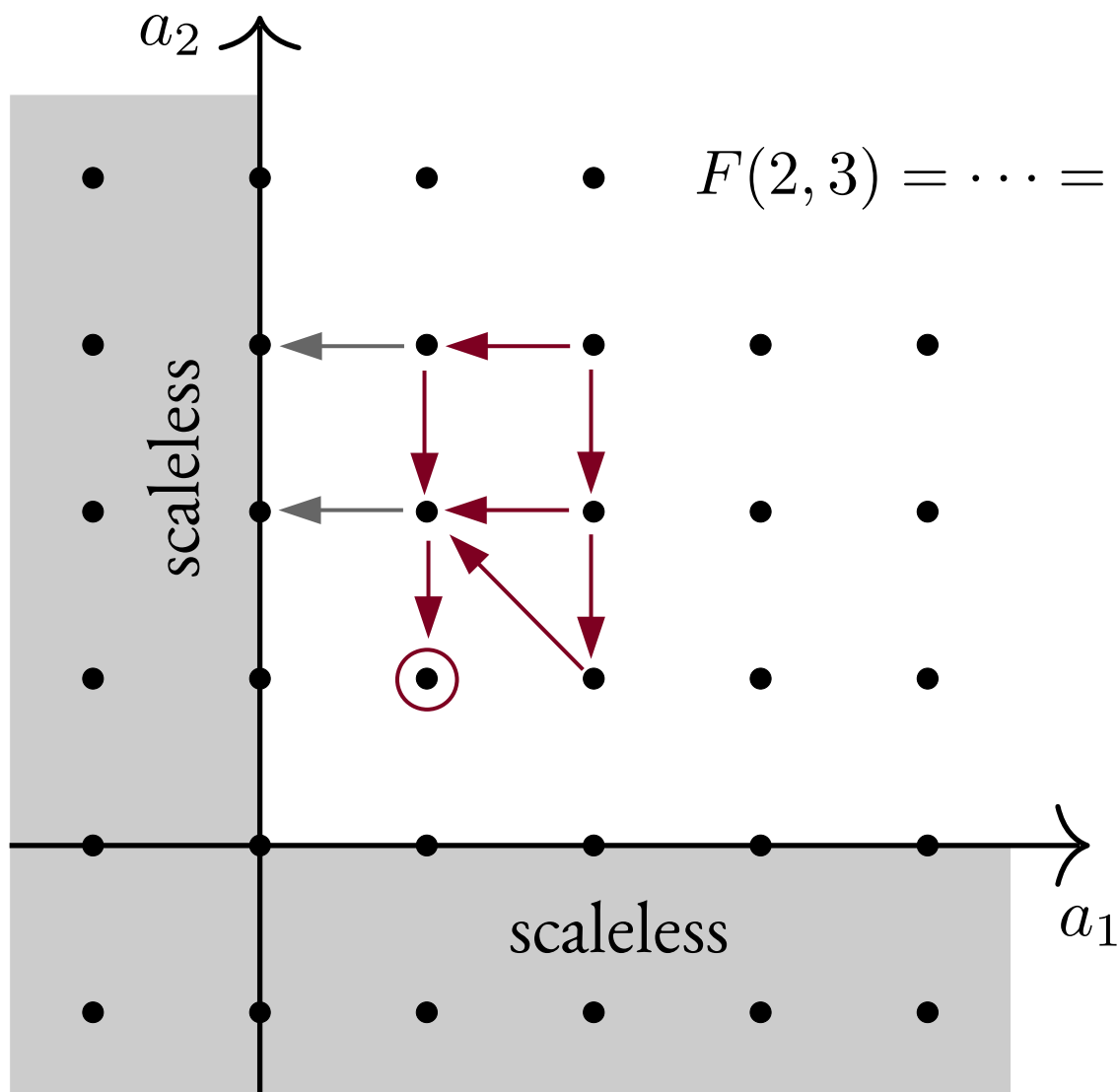


(using standard IBP eq.)

$$\begin{aligned}
 F(2, 2) &= \frac{1}{p^2} [(d - 5)F(1, 2) - F(2, 1)] \\
 &= \frac{(d - 6)}{p^2} F(1, 2) \\
 &= \frac{(d - 6)(d - 3)}{p^4} F(1, 1) \\
 &\quad - \frac{(d - 6)}{p^4} F(0, 2)
 \end{aligned}$$

master integral

IBP Reduction by hand



$$F(2, 3) = \dots = \frac{(d-8)(d-5)(d-3)}{2p^6} F(1, 1)$$

master integral

(the bubble integral family has only one master integral)

(using standard IBP eq.)

Laporta's algorithm

Generate linear system of equations:

→ specialize set of IBP equations to a range of integer indices

$$\{0 = (d - a_2 - 2a_1)F(a_1, a_2) - a_2 p^2 F(a_1, a_2 + 1) - a_2 F(a_1 - 1, a_2 + 1), \dots\}$$

$$(a_1, a_2) \in \{(1, 1), (1, 2), (2, 1), \dots\}$$

Laporta's algorithm

Generate linear system of equations:

$$\begin{aligned}0 &= F(2, 1) - F(1, 2) && \leftarrow \text{include symmetries} \\0 &= (d - 3)F(1, 1) - p^2 F(1, 2) && \text{directly discard} \\0 &= (d - 4)F(1, 2) - 2p^2 F(1, 3) && \text{scaleless integrals} \\0 &= 2p^2 F(1, 3) - p^2 F(2, 2) - F(1, 2) - F(2, 1) \\0 &= (d - 5)F(2, 1) - p^2 F(2, 2) - F(1, 2) \\&\vdots\end{aligned}$$

Solve system by expressing “complicated” integrals in terms of “simpler” integrals.

IBP reduction is a cornerstone
of multi-loop calculations

Public IBP programs:

- FIRE6 [Smirnov, Chukharev, '19]
- Kira 2 [Klappert, Lange, Maierhöfer, Usovitsch, '20]
- Reduce 2 [Manteuffel, Studerus, '19]
- FiniteFlow [Peraro, '19]
- LiteRed [Lee, '13]

Gröbner bases

Univariate polynomial division:

$$(x^3 + 2x)/(x + 1) = x^2 - x + 3 + \frac{-3}{x + 1}$$

remainder

Extend to multivariate case with several polynomials:

$$p = x_1^3 x_2^2 + x_2^2, \quad q_1 = x_1^2 x_2 + 1, \quad q_2 = x_1 x_2^2 + 1$$

→ decomposition: $p = p_1 q_1 + p_2 q_2 + h$

remainder

polynomials



→ Ideal membership problem

Let R be a ring over the field $\mathbb{K}$. A subset $I \subset R$ is an ideal if:

- I is an additive subgroup of R
- $y \in R \wedge z \in I \Rightarrow yz \in I$

In our example we have a polynomial ring $R = \mathbb{Q}[x_1, x_2]$ with $p, q_1, q_2 \in \mathbb{Q}[x_1, x_2]$ and the ideal is given by:

$$I = \langle q_1, q_2 \rangle = \{f_1 q_1 + f_2 q_2 \mid f_1, f_2 \in \mathbb{Q}[x_1, x_2]\}$$

Question: $p \in I$?

Some Definitions

Multi-index notation: $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

A monomial order is a total order: $x^\alpha > x^\beta \Rightarrow x^\alpha x^\gamma > x^\beta x^\gamma$

Lexicographic order:

$$x^\alpha >_{\text{lex}} x^\beta \Leftrightarrow \text{first nonzero entry of } \alpha - \beta > 0$$

Degree lexicographic order:

$$x^\alpha >_{\text{dlex}} x^\beta \Leftrightarrow (\deg x^\alpha > \deg x^\beta) \text{ or } (\deg x^\alpha = \deg x^\beta \text{ and the first nonzero entry of } \alpha - \beta > 0)$$

$$\text{Example: } x_1^2 x_2 >_{\text{lex}} x_1 x_2^5 >_{\text{lex}} x_1 x_2^2$$

$$x_1 x_2^5 >_{\text{dlex}} x_1^2 x_2 >_{\text{dlex}} x_1 x_2^2$$

Some Definitions

Multi-index notation: $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

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For $p \in R$, the lead term $\text{LT}(p)$ is the largest term of p w.r.t. $>$

Back to our example:

$$p = x_1^3 x_2^2 + x_2^2, \quad q_1 = x_1^2 x_2 + 1, \quad q_2 = x_1 x_2^2 + 1$$

First possibility:

$$p = (x_1 x_2) \cdot q_1 - x_1 x_2 + x_2^2$$


LT(p)/LT(q_1)


not divisible by LT(q_2)

(using $>_{\text{dlex}}$)

Back to our example:

$$p = x_1^3 x_2^2 + x_2^2, \quad q_1 = x_1^2 x_2 + 1, \quad q_2 = x_1 x_2^2 + 1$$

First possibility:

$$p = (x_1 x_2) \cdot q_1 + 0 \cdot q_2 - x_1 x_2 + x_2^2$$

(using $>_{\text{dlex}}$)

Back to our example:

$$p = x_1^3 x_2^2 + x_2^2, \quad q_1 = x_1^2 x_2 + 1, \quad q_2 = x_1 x_2^2 + 1$$

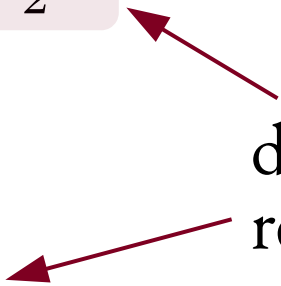
First possibility:

$$p = (x_1 x_2) \cdot q_1 + 0 \cdot q_2 - x_1 x_2 + x_2^2$$

Second possibility:

$$p = 0 \cdot q_1 + x_1^2 \cdot q_2 - x_1^2 + x_2^2$$

different non-vanishing
remainders!



→ Naively $p \notin I$ (wrong!)

(using $>_{\text{dlex}}$)

Gröbner bases

Let I be an Ideal. A finite subset $G = \{g_1, \dots, g_n\} \subset I$ is a Gröbner basis for I if $\text{LT}(G) = \text{LT}(I)$.

Properties:

- G generates I

- unique remainder $p = \sum_k p_k g_k + h$

$$\rightarrow h = 0 \Leftrightarrow p \in I$$

Remainder is called normal form $\text{NF}_G(p) = h$

Gröbner basis can be computed with the Buchberger algorithm.

Back to our example:

$$p = x_1^3 x_2^2 + x_2^2, \quad q_1 = x_1^2 x_2 + 1, \quad q_2 = x_1 x_2^2 + 1$$

Ideal: $I = \langle q_1, q_2 \rangle$

Gröbner basis: $G = \{g_1, g_2\} = \{x_1 - x_2, x_2^3 + 1\}$

Decomposition: $p = (x_1^2 x_2^2 + x_1 x_2^3 + x_2^4) \cdot g_1 + x_2^2 \cdot g_2 + 0$

$$\rightarrow \boxed{p \in I}$$

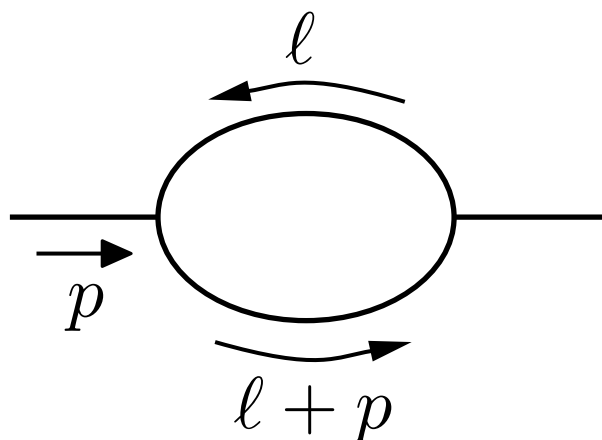
(using $>_{\text{dlex}}$)

Gröbner bases and IBP reduction

We are not the first to consider Gröbner bases for IBP reductions:

- Gröbner bases in combination with partial differential equation and dimension shift [Tarasov, '98, '04]
- Gröbner bases with shift algebras [Gerdt '05 (but chosen example is scaleless)]
- Maple implementation of IBP reduction with shift algebras [Gerdt, Robertz, '06]
- Sector bases [Smirnov, Smirnov, '05 - '08]
- IBPs and scaleless integrals as left and right ideal [Lee, '08]

Consider again the bubble integrals



$$F(a_1, a_2) = \int \frac{d^d \ell}{i\pi^{d/2}} \frac{1}{D_1^{a_1} D_2^{a_2}}$$

Introduce operators with partial right action

$$F(a_1, a_2) \bullet \hat{a}_1 = a_1 F(a_1, a_2)$$

$$F(a_1, a_2) \bullet \frac{1}{\hat{a}_1} = \frac{1}{a_1} F(a_1, a_2)$$

$a_1 \neq 0$

$$F(a_1, a_2) \bullet \hat{D}_1 = F(a_1 - 1, a_2)$$

$$F(a_1, a_2) \bullet \hat{D}_1^- = F(a_1 + 1, a_2)$$

not scaleless

We have a **non-commutative rational double-shift algebra**

$$Y := \mathbb{Q}(d, p^2)(\hat{a}_1, \hat{a}_2) \langle \hat{D}_1, \hat{D}_1^-, \hat{D}_2, \hat{D}_2^- \rangle$$

with relations:

$$[\hat{a}_i, \hat{D}_j] = \delta_{ij} \hat{D}_i, \quad [\hat{a}_i, \hat{D}_j^-] = -\delta_{ij} \hat{D}_i^-, \quad \hat{D}_i \hat{D}_i^- = \hat{D}_i^- \hat{D}_i = 1,$$

$$[\hat{a}_i, \hat{a}_j] = [\hat{D}_i, \hat{D}_j] = [\hat{D}_i^-, \hat{D}_j^-] = [\hat{D}_i, \hat{D}_j^-] = 0$$

(no summation over repeated indices)

First IBP operator ($v = \ell$):

$$\begin{aligned} 0 &= (d - a_2 - 2a_1)F(a_1, a_2) - a_2 p^2 F(a_1, a_2 + 1) \\ &\quad - a_2 F(a_1 - 1, a_2 + 1) \\ &= F(a_1, a_2) \bullet \underbrace{\left[(d - \hat{a}_2 - 2\hat{a}_1) - p^2 \hat{a}_2 \hat{D}_2^- - \hat{a}_2 \hat{D}_1 \hat{D}_2^- \right]}_{= \hat{r}_1} \end{aligned}$$

Second IBP operator ($v = p$):

$$\hat{r}_2 = -\hat{a}_1 \hat{D}_1^- \hat{D}_2 + \hat{a}_2 \hat{D}_1 \hat{D}_2^- - p^2 \hat{a}_1 \hat{D}_2^- + (\hat{a}_1 - \hat{a}_2)$$

IBPs as a left ideal

IBP operators form a **left** ideal:

$$I_{\text{IBP}} = \langle \hat{r}_1, \hat{r}_2 \rangle_Y = \{ \hat{f}_1 \hat{r}_1 + \hat{f}_2 \hat{r}_2 \mid \hat{f}_1, \hat{f}_2 \in Y \}$$


$$f \in Y, r \in I_{\text{IBP}} \Rightarrow fr \in I_{\text{IBP}} \quad \text{but} \quad rf \notin I_{\text{IBP}}$$

IBPs as a left ideal

IBP operators form a **left** ideal:

$$I_{\text{IBP}} = \langle \hat{r}_1, \hat{r}_2 \rangle = \{ \hat{f}_1 \hat{r}_1 + \hat{f}_2 \hat{r}_2 \mid \hat{f}_1, \hat{f}_2 \in Y \}$$

By construction we have

$$F(a_1, a_2) \bullet \hat{r} = 0 \quad \text{for all} \quad \hat{r} \in I_{\text{IBP}}$$

Reduced rational Gröbner basis (4 elements)

$$\begin{aligned} G = \{ & (d - \hat{a}_1 - \hat{a}_2)(d - 2\hat{a}_1 - 2\hat{a}_2 + 2)\hat{D}_2 - p^2(\hat{a}_2 - 1)(d - 2\hat{a}_2), \\ & (d - \hat{a}_1 - \hat{a}_2)(d - 2\hat{a}_1 - 2\hat{a}_2 + 2)\hat{D}_1 - p^2(\hat{a}_1 - 1)(d - 2\hat{a}_1), \\ & p^2\hat{a}_2(d - 2\hat{a}_2 - 2)\hat{D}_2^- - (d - \hat{a}_1 - \hat{a}_2 - 1)(d - 2\hat{a}_1 - 2\hat{a}_2), \\ & p^2\hat{a}_1(d - 2\hat{a}_1 - 2)\hat{D}_1^- - (d - \hat{a}_1 - \hat{a}_2 - 1)(d - 2\hat{a}_1 - 2\hat{a}_2) \} \end{aligned}$$

$$F(2, 2) = F(1, 1) \bullet [\hat{D}_1^- \hat{D}_2^-]$$

reference integral

$$\begin{aligned} F(2, 2) &= F(1, 1) \bullet [\hat{D}_1^- \hat{D}_2^-] \\ &= F(1, 1) \bullet \left[\sum_{i=1}^4 f_i g_i + \text{NF}_G \left(\hat{D}_1^- \hat{D}_2^- \right) \right] \end{aligned}$$


annihilates any integral

$$\begin{aligned} F(2, 2) &= F(1, 1) \bullet [\hat{D}_1^- \hat{D}_2^-] \\ &= F(1, 1) \bullet \left[\sum_{i=1}^4 f_i g_i + \text{NF}_G \left(\hat{D}_1^- \hat{D}_2^- \right) \right] \\ &= F(1, 1) \bullet \text{NF}_G \left(\hat{D}_1^- \hat{D}_2^- \right) \end{aligned}$$

only need to compute normal form



$$\begin{aligned} F(2, 2) &= F(1, 1) \bullet [\hat{D}_1^- \hat{D}_2^-] \\ &= F(1, 1) \bullet \left[\sum_{i=1}^4 f_i g_i + \text{NF}_G \left(\hat{D}_1^- \hat{D}_2^- \right) \right] \\ &= F(1, 1) \bullet \text{NF}_G \left(\hat{D}_1^- \hat{D}_2^- \right) \\ &= F(1, 1) \bullet \frac{(d-\hat{a}_1-\hat{a}_2-1)(d-\hat{a}_1-\hat{a}_2-2)(d-2\hat{a}_1-2\hat{a}_2)(d-2\hat{a}_1-2\hat{a}_2-2)}{p^4 \hat{a}_1 \hat{a}_2 (d-2\hat{a}_1-2)(d-2\hat{a}_2-2)} \\ &= \frac{(d-6)(d-3)}{p^4} F(1, 1) \end{aligned}$$

Note: Can compute IBP reduction w.r.t. any reference integral (e.g. $F(11, 42)$) $\rightarrow$ different master integral

Scaleless integrals are correctly identified:

$$F(1, 1) \bullet \hat{D}_1 = F(1, 1) \bullet \text{NF}_G(\hat{D}_1) = F(1, 1) \bullet (\hat{a}_1 - 1)(\dots) = 0$$



$$F(0, 1) = 0$$

Particular interesting are IBP operators of the form

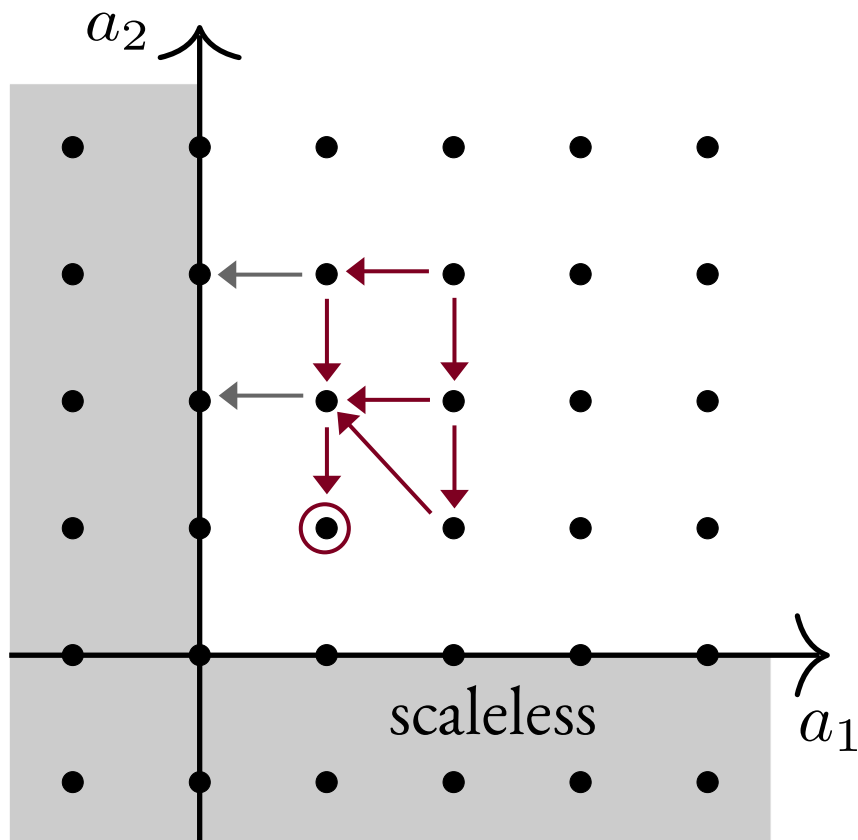
$$\hat{R}_i = \hat{a}_i \hat{D}_i^- - \text{NF}_G(\hat{a}_i \hat{D}_i^-) \in I_{\text{IBP}}$$

with:
$$\text{NF}_G(\hat{a}_1 \hat{D}_1^-) = \frac{(d - \hat{a}_1 - \hat{a}_2 - 1)(d - 2\hat{a}_1 - 2\hat{a}_2)}{(d - 2\hat{a}_1 - 2)p^2}$$

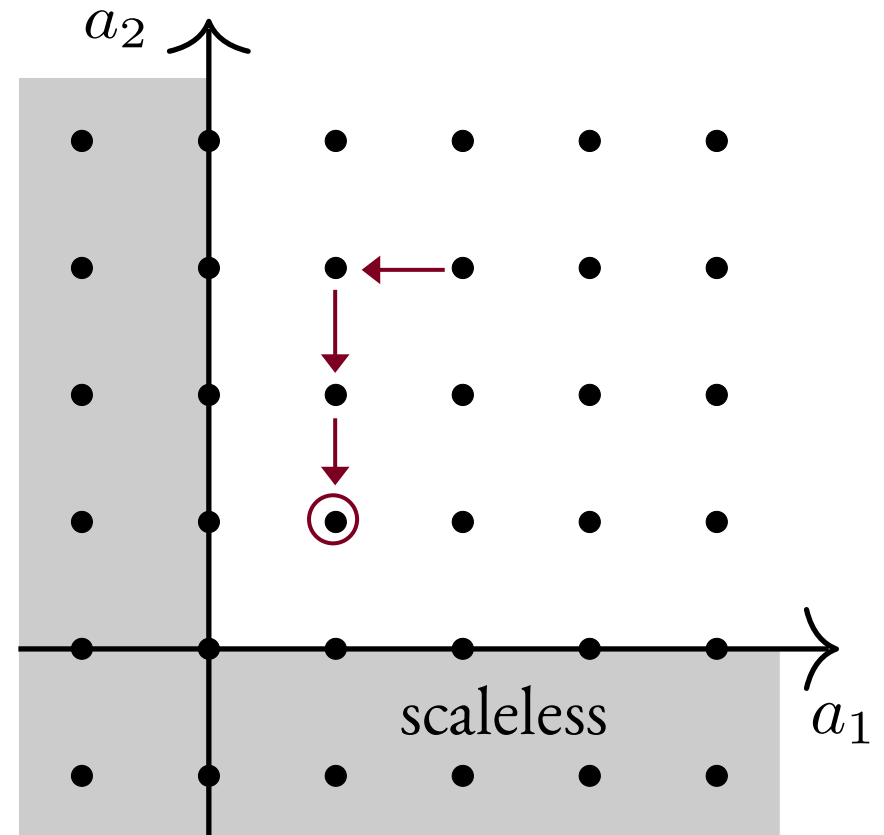
Comparison of IBPs

Reduction of $F(3, 2)$:

using standard IBPs

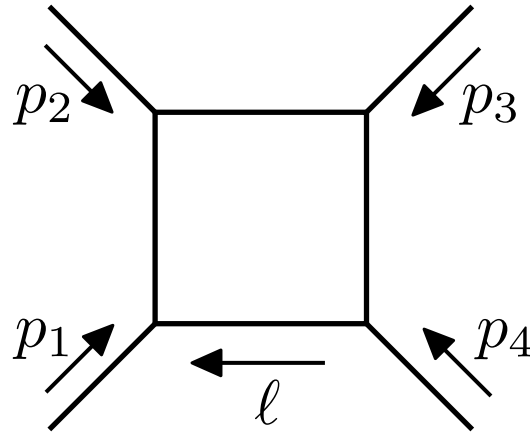


using $\hat{R}_i = \hat{a}_i \hat{D}_i^- - \text{NF}_G(\hat{a}_i \hat{D}_i^-)$



Box integrals

$$F(a_1, a_2, a_3, a_4) =$$



$$D_1 = -\ell^2$$

$$D_2 = -(\ell - p_1)^2$$

$$D_3 = -(\ell - p_1 - p_2)^2$$

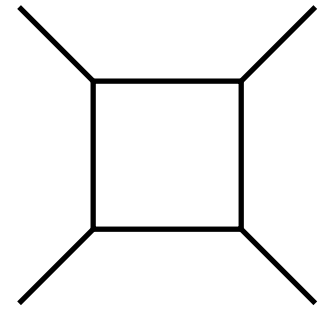
$$D_4 = -(\ell + p_4)^2$$

- kinematics: $p_i^2 = 0$, $p_1 \cdot p_2 = s_{12}$, $p_1 \cdot p_4 = s_{14}$
- 4 standard IBP equations
- algebra: $Y := \mathbb{Q}(d, s_{12}, s_{14})(\hat{a}_1, \dots, \hat{a}_4) \langle \hat{D}_i, \hat{D}_i^- | i = 1, \dots, 4 \rangle$

Box integrals

The Gröbner basis has 9 elements:

$$G = \left\{ \hat{D}_4 - \hat{D}_2 + \frac{(\hat{a}_2 - \hat{a}_4)s_{12}}{d - \hat{a}_{1234}}, \dots \right\}$$



IBP reduction with Gröbner basis easy to implement in Mathematica or FORM (+ fast and parallelization trivial)

$F(10, 10, 10, 10)$, ~ 5 sec.

```
FORM 4.2.1 (Jul  7 2022, v4.2.1-40-g982111a) 64-bits
#-
Local Expr = Int(10,10,10,10);
#call ReductionBox

.sort
On Statistics;
.end

Time =      4.51 sec    Generated terms =      3
Expr      Terms in output =      3
          Bytes used   =    202756
```

$F(10, 10, 10, -10)$, ~ 3 sec.

```
FORM 4.2.1 (Jul  7 2022, v4.2.1-40-g982111a) 64-bits
#-
Local Expr = Int(10,10,10,-10);
#call ReductionBox

.sort
On Statistics;
.end

Time =      2.86 sec    Generated terms =      1
Expr      Terms in output =      1
          Bytes used   =    18044
```

Standard monomials

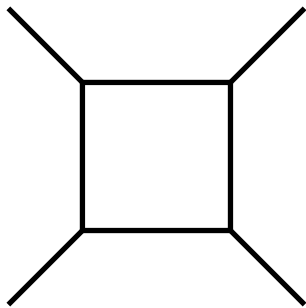
A standard monomial f is a monomial in $\hat{D}_i, \hat{D}_i^-$, such that:

$$\text{NF}_G(f) = f$$

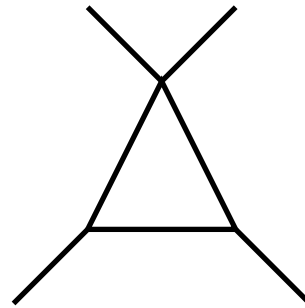
→ corresponds to master integral

Example box integrals:

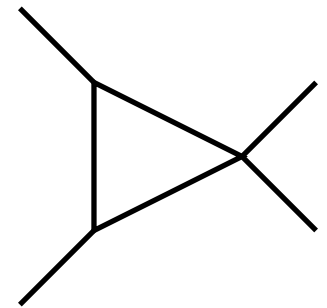
$$\text{NF}_G(1) = 1$$



$$\text{NF}_G(\hat{D}_3) = \hat{D}_3$$



$$\text{NF}_G(\hat{D}_4) = \hat{D}_4$$



(reference integral: $F(1, 1, 1, 1)$)

Computing a Gröbner bases

The computations were done with the GAP package LoopIntegrals

<https://homalg-project.github.io/pkg/LoopIntegrals>

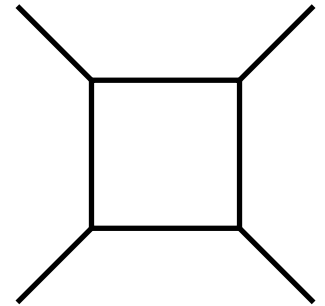
Dependencies:

[Decker, Greuel, Pfister, Schönemann, '19]

- SINGULAR for commutative Gröbner bases in polynomial rings
- PLURAL for non-commutative Gröbner bases in the double-shift algebra with polynomial coefficients ($\rightarrow \hat{a}_i$) [Levandovskyy, Schönemann, '03]
- Chyzak's Maple package Ore_algebra for non-commutative Gröbner bases in the double-shift algebra with rational coefficients ($\rightarrow \hat{a}_i$) [Chyzak, '98]
- The Julia package HECKE for simulating the Gröbner bases computation with linear algebra to obtain IBPs of the form $\hat{R}_i = \hat{a}_i \hat{D}_i^- - \text{NF}_G(\hat{a}_i \hat{D}_i^-)$ (last part of this talk)

LoopIntegrals

Box integrals



```
In [17]: LoadPackage( "LoopIntegrals" )
```

```
In [18]: R = RingOfLoopDiagram( LD )
```

```
Out[18]: GAP: Q[D,s12,s14][D1,D2,D3,D4]
```

```
In [19]: Ypol = DoubleShiftAlgebra( R )
```

```
Out[19]: GAP: Q[D,s12,s14][a1,a2,a3,a4]<D1,D1_,D2,D2_,D3,D3_,D4,D4_>/ ( D1*D1_-1, D2*D2_-1, D3*D3_-1, D4  
*D4_-1 )
```

```
In [20]: ibps = MatrixOfIBPRelations( LD )
```

```
Out[20]: GAP: <A 4 x 1 matrix over a residue class ring>
```

```
In [21]: r1 = ibps[1,1]
```

```
Out[21]: GAP: |[ -a2*D1*D2_-s12*a3*D3_-a3*D1*D3_-a4*D1*D4_+D-2*a1-a2-a3-a4 ]|
```

```
In [22]: bas_pol = BasisOfRows( ibps )
```

```
Out[22]: GAP: <A non-zero 28 x 1 matrix over a residue class ring>
```

```
In [23]: NormalForm( "a1*D1_" / Ypol, bas_pol )
```

```
Out[23]: GAP: |[ a1*D1_ ]|
```

polynomial $\hat{a}_i$

reduction does not work!

LoopIntegrals

The following command needs Chyzak's Maple package `Ore_algebra` for the noncommutative Gröbner bases of the rational double-shift algebra:

```
In [24]: Y = RationalDoubleShiftAlgebra( R )
```

```
Out[24]: GAP: Q(D,s12,s14)(a1,a2,a3,a4)<D1,D1_,D2,D2_,D3,D3_,D4,D4_>/ ( D1*D1_-1, D2*D2_-1, D3*D3_-1, D4*D4_-1 )
```

```
In [25]: ribps = Y * ibps
```

```
Out[25]: GAP: <A 4 x 1 matrix over a residue class ring>
```

```
In [26]: bas = BasisOfRows( ribps )
```

```
Out[26]: GAP: <A non-zero 9 x 1 matrix over a residue class ring>
```

```
In [27]: NormalForm( "a1*D1_" / Y, bas )
```

```
Out[27]: GAP: |[ -2*(D-2*a1-2*a2-2*a4)*(D-a1-a2-a3-a4)*(-a4-1+D-a1-a2-a3)*D3/(D-2*a1-2*a4-2)/(D-2*a1-2*a2-2)/s12/s14+4*(a3-1)*(D-a1-a2-a3-a4)*(-a4-1+D-a1-a2-a3)*D4/(D-2*a1-2*a4-2)/(D-2*a1-2*a2-2)/s12/s14+(-a4-1+D-a1-a2-a3)*(D-2*a1-2*a3-2*a4)/(D-2*a1-2*a4-2)/s12 ]|
```

rational $\hat{a}_i$

reduction works!

LoopIntegrals

```
In [28]: NormalFormWrtInitialIntegral( "D1_" / Y, bas )
```

```
Out[28]: GAP: | [ -2/(D-6)*(D-4)*(-5+D)*D3/s12/s14+(-5+D)/s12 ] |
```

```
In [29]: NormalFormWrtInitialIntegral( "D3_" / Y, bas )
```

```
Out[29]: GAP: | [ -2/(D-6)*(D-4)*(-5+D)*D3/s12/s14+(-5+D)/s12 ] |
```

```
In [30]: NormalFormWrtInitialIntegral( "D1*D2" / Y, bas )
```

```
Out[30]: GAP: | [ 0 ] |
```

reduction w.r.t.
 $F(1, 1, 1, 1)$

scaleless integral

The Jupyter notebook can be found online:

<https://homalg-project.github.io/nb/1LoopBox/>

Linear algebra ansatz

Linear algebra ansatz

Especially interested in the IBPs:

$$\hat{R}_i = \hat{a}_i \hat{D}_i^- - \text{NF}_G(\hat{a}_i \hat{D}_i^-)$$

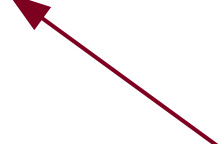
- allow for easy reduction (at least of the top-level sector)
- in all considered problems these generate the left ideal:

$$\langle \hat{R}_i | i = 1, \dots, n \rangle_Y = I_{\text{IBP}}$$

number of propagators



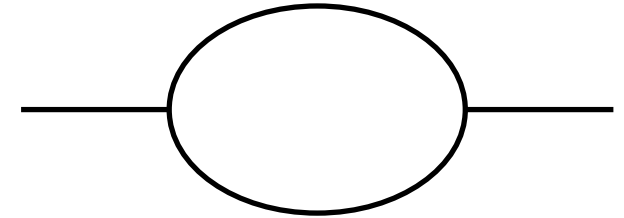
do NOT expect to be true
for all integral families!



Idea: Use linear algebra to compute $\hat{R}_i$ when Gröbner basis is not available.

Linear algebra ansatz

Consider again the bubble integrals



Start with a generating set of I_{IBP} , e.g. the standard IBPs

$$\begin{aligned}
 \hat{r}_1 &= (d - \hat{a}_2 - 2\hat{a}_1) - p^2 \hat{a}_2 \hat{D}_2^- - \hat{a}_2 \hat{D}_1^- \hat{D}_2^- & " = 0" \\
 \hat{r}_2 &= -\hat{a}_1 \hat{D}_1^- \hat{D}_2^- + \hat{a}_2 \hat{D}_1^- \hat{D}_2^- - p^2 \hat{a}_1 \hat{D}_2^- + (\hat{a}_1 - \hat{a}_2) & " = 0" \\
 \hat{D}_1^- \hat{r}_1 &= (d - \hat{a}_2 - 2\hat{a}_1 - 2) \hat{D}_1^- - p^2 \hat{a}_2 \hat{D}_1^- \hat{D}_2^- - \hat{a}_2 \hat{D}_2^- & " = 0" \\
 \vdots & \leftarrow (\hat{D}_1^-)^{j_1} (\hat{D}_2^-)^{j_2} \hat{r}_i
 \end{aligned}$$

↑ treat as unknowns
↑ schematically

Solve for $\hat{D}_1^-$ and $\hat{D}_2^-$

→ reproduce result of Gröbner basis calculation

Special IBPs

Choice on generating set for I_{IBP} has impact on required max. values for j_1 and j_2 in $(\widehat{D}_1^-)^{j_1} (\widehat{D}_2^-)^{j_2} \widehat{r}_i$

Observation: Special IBPs (unitarity compatible) need lower max. values than standard IBPs

$$0 = \int \frac{d^d \ell}{i\pi^{d/2}} \frac{\partial}{\partial \ell^\mu} \frac{v^\mu}{D_1^{a_1} D_2^{a_2}}$$

consider

$$v^\mu \frac{\partial}{\partial \ell^\mu} \frac{1}{D_1^{a_1} D_2^{a_2}} = \sum_k v^\mu \frac{\partial D_k}{\partial \ell^\mu} \frac{\partial}{\partial D_k} \frac{1}{D_1^{a_1} D_2^{a_2}}$$

$$v^\mu = C_1(D_1, D_2) \ell^\mu + C_2(D_1, D_2) p^\mu$$

polynomial dependence

increases
exponent
 $\rightarrow \widehat{D}_k^-$

Special IBPs

Choose $C_i(D_1, D_2)$ such that $v^\mu \frac{\partial D_k}{\partial \ell^\mu} \propto D_k$


$$C_1(D_1, D_2)\ell^\mu + C_2(D_1, D_2)p^\mu$$

→ IBPs free of $\hat{D}_i^-$

→ special or unitarity compatible IBPs

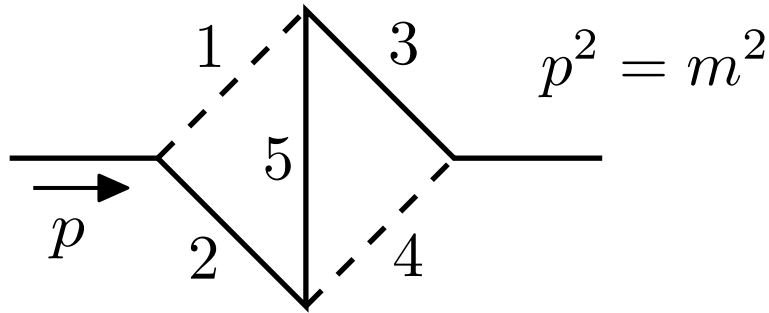
[Kosower et al. '10, '18 | Schabinger et al. '11, '20 | Ita '15 | Böhm et al. '17]

To find $C_i(D_1, D_2)$ we need to compute **syzygies**. However, the computation takes place in a polynomial ring.

(Public programs: LoopIntegrals and NeatIBP)

[Boehm, Ma, Wu, Xu, Zahng '23]

On-shell kite



$$D_1 = -\ell_1^2$$

$$D_2 = -(\ell_1 + p)^2 + m^2$$

$$D_3 = -(\ell_2 + p)^2 + m^2$$

$$D_4 = -\ell_2^2$$

$$D_5 = -(\ell_1 + \ell_2 + p)^2 + m^2$$

Gröbner basis not available

→ use linear algebra ansatz to compute $\hat{R}_i = \hat{a}_i \hat{D}_i^- - \text{NF}(\hat{a}_i \hat{D}_i^-)$

$$\text{NF}(\hat{a}_1 \hat{D}_1^-) = \frac{p_{10}}{4d_1 d_2 d_3 d_4 d_7 d_8 s} + \frac{p_{12} \hat{D}_2 + p_{13} \hat{D}_3 + p_{14} \hat{D}_4 + p_{15} \hat{D}_5}{16d_1 d_2 d_3 d_4 d_7 d_8 d_9 m^4}$$

- p_{ij} too large for printing (polynomial in $\hat{a}_i, d$)
- d_i simple, e.g. $d_3 = 2\hat{a}_1 + \hat{a}_2 - d + 1$
- Allows for reduction of the top level sector

Conclusion

Gröbner basis theory can be used to perform IBP reductions of loop integrals:

- IBPs form a left ideal I_{IBP} in a rational double-shift algebra $\mathcal{Y}$
- Reduction via computing normal forms w.r.t. Gröbner basis
→ fast + parallelization trivial
- Problem: Gröbner basis is computationally expensive

Linear algebra ansatz to compute normal forms when Gröbner basis is not available.

Backup

Buchberger algorithm

Ideal: $I = \langle q_1, q_2 \rangle$ with $q_1 = x_1^2 x_2 + 1$, $q_2 = x_1 x_2^2 + 1$

Idea: The lead term of any element of the ideal should be divisible by the lead term of at least one generator

Consider: $-x_1 + x_2$

- division algorithm: $-x_1 + x_2 = 0 \cdot q_1 + 0 \cdot q_2 - x_1 + x_2$
- however: $-x_1 + x_2 = x_2 \cdot q_1 - x_1 \cdot q_2 \in I$

lin. comb. removes leading
terms of q_1, q_2



(using $>_{\text{dlex}}$)

Buchberger algorithm

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$$\frac{\text{LT}(q_2)}{\text{gcd}(\text{LT}(q_1), \text{LT}(q_2))} = \frac{x_1 x_2^2}{x_1 x_2}$$

(using $>_{\text{dlex}}$)

Buchberger algorithm

S-polynomial:

$$S(q_i, q_j) = \frac{\text{LT}(q_j)}{\text{gcd}(\text{LT}(q_i), \text{LT}(q_j))} q_i - \frac{\text{LT}(q_i)}{\text{gcd}(\text{LT}(q_i), \text{LT}(q_j))} q_j = -S(q_j, q_i)$$

Buchberger's criterion:

$G = \{g_1, \dots, g_n\} \subset I$ is a Gröbner basis **iff** the remainders of all S-polynomials $S(g_i, g_j)$ are zero.

→ use for algorithm

Buchberger algorithm

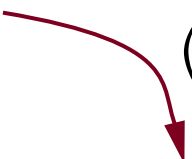
Start: $S(q_1, q_2) = 0 \cdot q_1 + 0 \cdot q_2 \underbrace{- x_1 + x_2}_{=: q_3}$ (add to basis)

Next: $S(q_1, q_3) = 0$ (nothing to do)

Next: $S(q_2, q_3) = 0 \cdot q_1 + 0 \cdot q_2 + 0 \cdot q_3 = \underbrace{-x_2^3 - 1}_{=: q_4}$ (add to basis)

Done: $S(q_i, q_j) = 0$, $i, j = 1, \dots, 4$ (Buchberger's criterion)

Gröbner basis: $G' = \{q_1, q_2, q_3, q_4\}$ (simplify)



Reduced Gröbner basis: $G = \{x_1 - x_2, x_2^3 + 1\}$

(using $>_{\text{dlex}}$)