The Hadronic-Light-By-Light Contribution to the Muon g-2 in Triangle Kinematics at Short Distances

Michael Adam, 21st of March 2023

Particle Physics Seminar

in collaboration with Jan Lüdtke and Massimiliano Procura

universität wien

Outline

- A brief introduction to the muon g-2
- Dispersive approaches for the Hadronic-Light-By-Light contribution
- Kinematic regimes and asymptotic matching
- Summary and Outlook

A brief introduction to the muon g-2

Why the muon is special

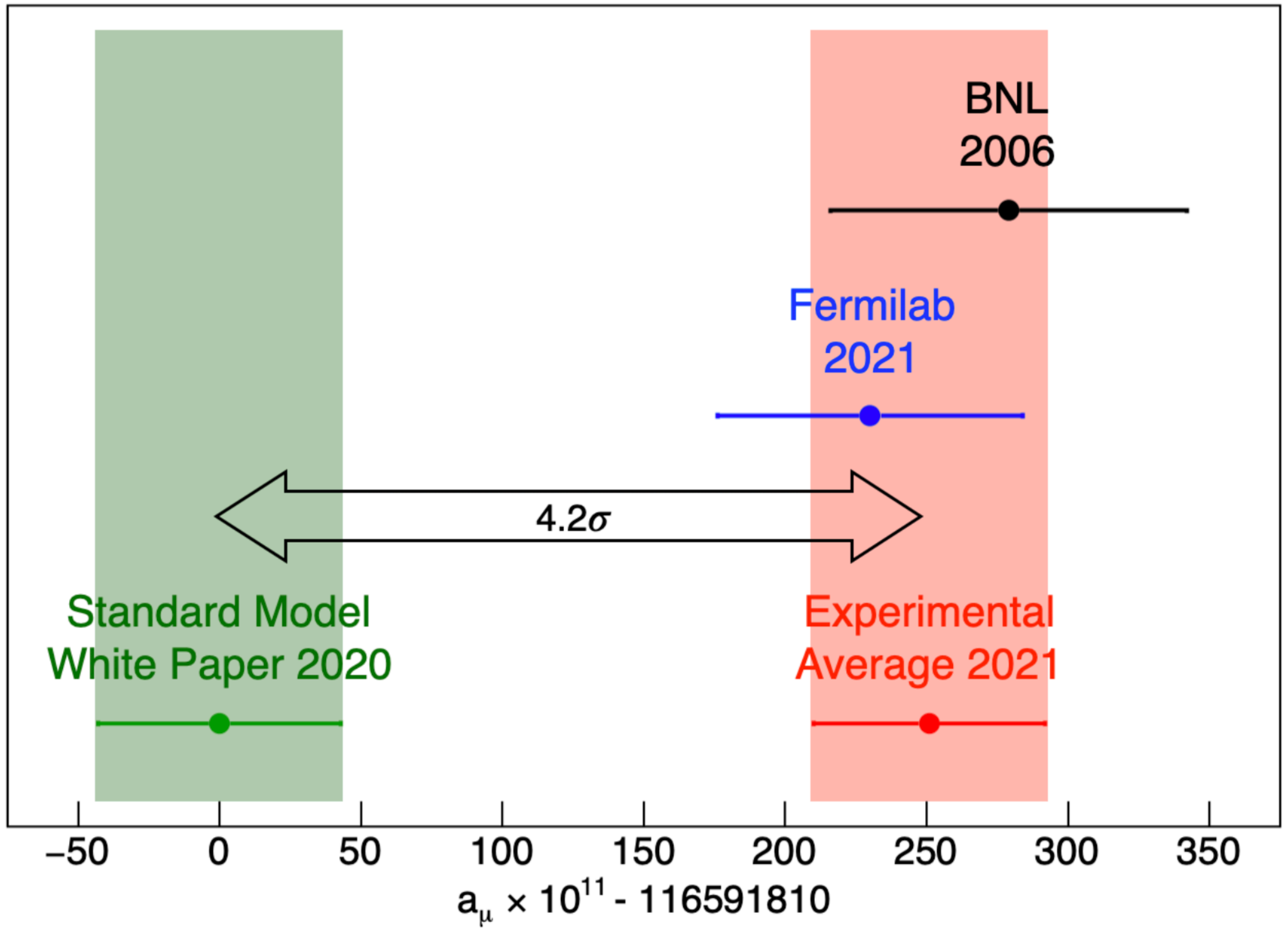
- Dirac predicted $g_e = 2$
- For leptons: radiative corrections lead to small deviations from this: $a_\ell = \frac{g_\ell - 2}{2}$
- Prediction for electron anomaly is limited by measurement of fine-structure constant
- Muon is heavier than electron
  - More sensitive to effects from heavy new particles
  - Suitable for indirect search of beyond SM particle

A brief introduction to the muon g-2

Tension between theory and experiment

Contribution	value$\times 10^{11}$	error$\times 10^{11}$
Experiment	116592061	41
QED	116584718.931	0.104
Electroweak	153.6	1.0
HVP LO	6931	40
HVP NLO	-98.3	0.7
HVP NNLO	12.4	0.1
HLbL LO	90	17
HLbL NLO	2	1
Sum SM	116591810	43

Aoyama et al. (2020)



- Tension calls for further improvements on the theoretical and experimental side

A brief introduction to the muon g-2

QED sector

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Aoyama et al. (2020)

- Fully dominated by one-loop result
- Has been impressively refined by including 5-loop calculation
- Uncertainty negligible

A brief introduction to the muon g-2

Electroweak sector

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Aoyama et al. (2020)

- large masses of W- and Z-bosons suppress contribution
- known up to two loops
- uncertainty dominated by non-perturbative hadronic effects (only enter at two-loop)

A brief introduction to the muon g-2

Hadronic sector (HVP)

Contribution	value×10^{11}	error×10^{11}
Experiment	116592061	41
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Electroweak	153.6	1.0
HVP LO	6931	40
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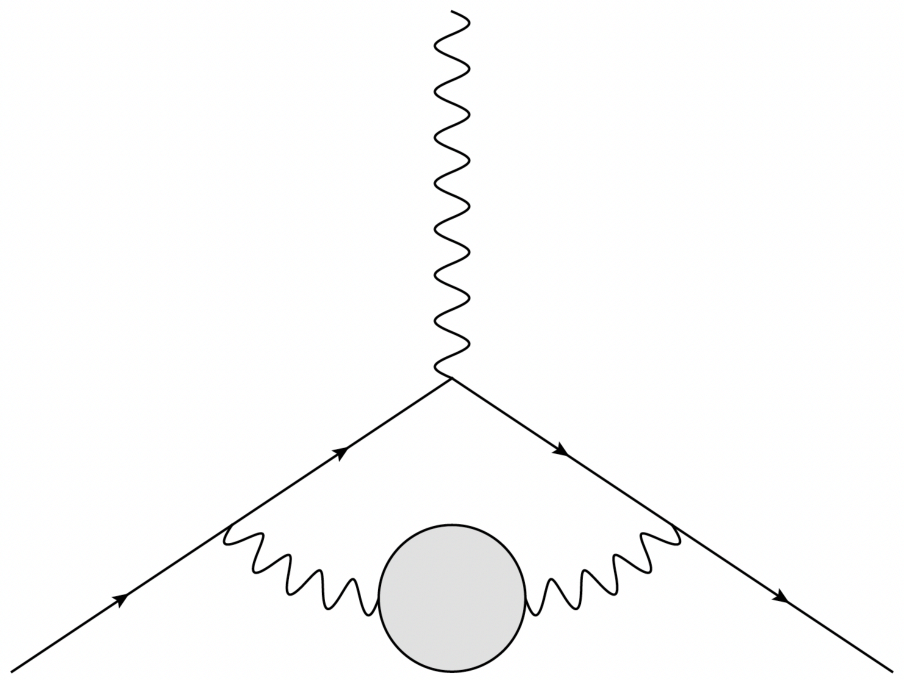
Aoyama et al. (2020)

- Largest contribution to uncertainty
- Data-driven methods using dispersion relations
- Lattice QCD
  - Recent result relaxes tension

S. Borsányi et al. (2021)



A brief introduction to the muon g-2

Hadronic Sector (HLbL)

Contribution	value$\times 10^{11}$	error$\times 10^{11}$
Experiment	116592061	41
QED	116584718.931	0.104
Electroweak	153.6	1.0
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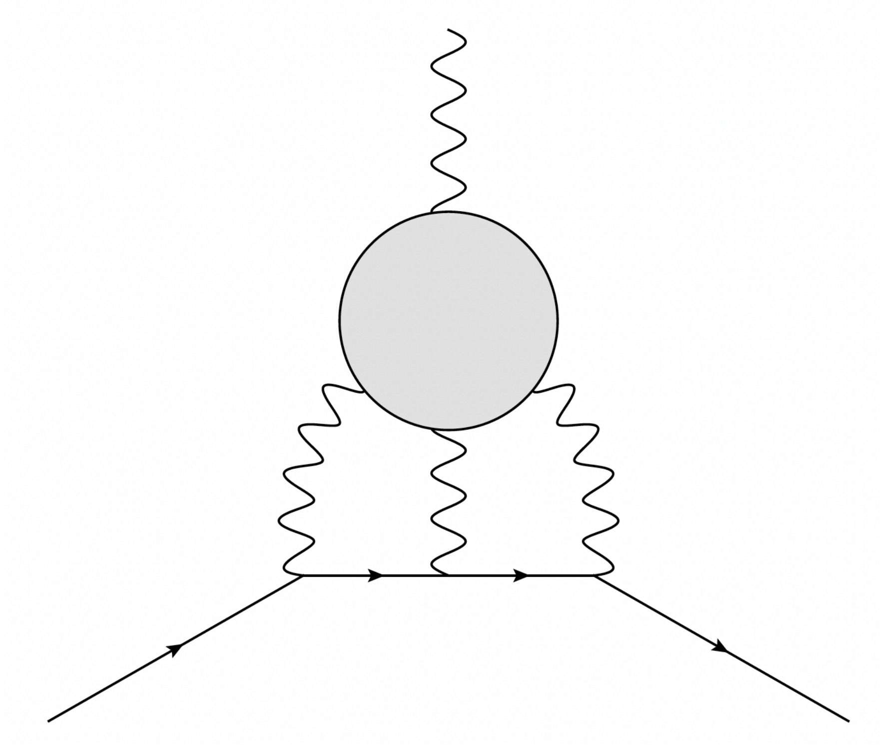
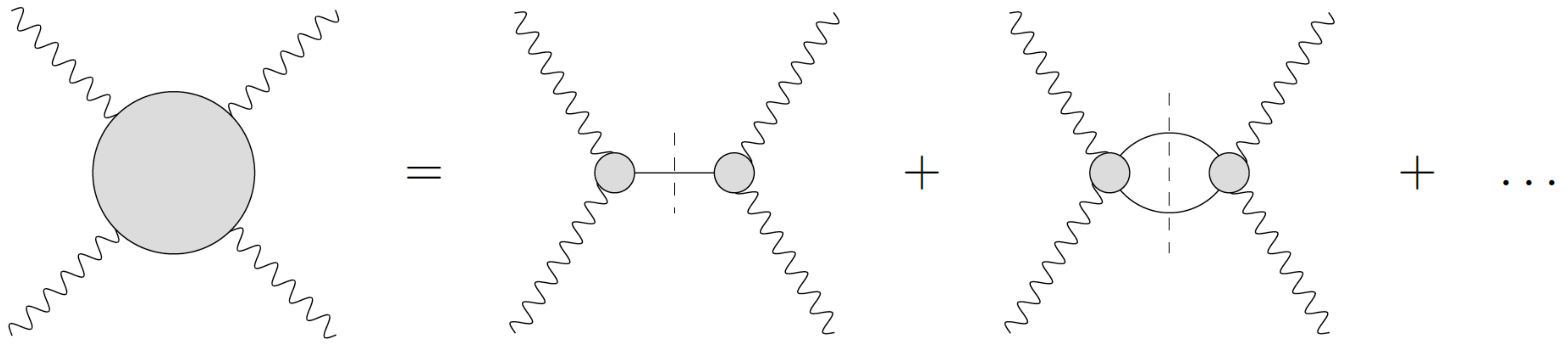
Aoyama et al. (2020)



- More legs, more complicated
- Less precision needed
- two model-independent approaches for dominant low-energy contributions exist
  - Lattice QCD
  - Dispersive approach:



Colangelo, Procura, Stoffer, Hoferichter (2015)

A brief introduction to the muon g-2

Uncertainities from HLbL

Contribution		Our estimate
π^0, η, η'-poles		93.8(4.0)
π, K-loops/boxes		−16.4(2)
S-wave $\pi\pi$ rescattering		−8(1)
subtotal		69.4(4.1)
scalars	2* }	− 1(3)
tensors		
axial vectors		6(6)
u, d, s-loops / short-distance		15(10)
c-loop		3(1)
total		92(19)

Aoyama et al. (2020)

- 1- and 2-pseudoscalar contributions under control through dispersive approach
- Constrain remainder of infinite sum in unitarity relation by OPE and pQCD
- Matching from low- to high-energy regime dominates uncetainty
  - Model-based approaches are used so far
  - Need model-independent description
- Propose such a description based on a dispersive approach
  - Finite number of intermediate states + quark loop
  - Will allow for controlled uncertainites

Dispersive Approaches for HLbL

The HLbL tensor and its tensor structures

- Central object: $\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = -i \int d^4x\, d^4y\, d^4z e^{-i(q_1\cdot x + q_2\cdot y + q_3\cdot z)} \langle 0|T\{j^\mu_{\text{em}}(x) j^\nu_{\text{em}}(y) j^\lambda_{\text{em}}(z) j^\sigma_{\text{em}}(0)\}|0\rangle$
- Naive tensor decomposition leads to: $\Pi_{\mu\nu\lambda\sigma} = \sum_{i=1}^{138} L_i^{\mu\nu\lambda\sigma} \Xi_i$
- Ward identities imply 95 linear relations between scalar functions
    - Set reduces but kinematical singularities in tensor structures
- Use Bardeen-Tung projection technique: singularities gone but set incomplete at certain kinematic points Bardeen & Tung (1968), Tarrach (1975)
- Tarrach: Add redundant structures: $\Pi_{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$
    - Set now overcomplete, but scalar functions free of kinematic singularites

Colangelo, Procura, Stoffer, Hoferichter (2015)

Dispersive Approaches for HLbL

Master formula

- Need to compute loop integral over $\dfrac{\partial}{\partial q_4^\rho}\Pi_{\mu\nu\lambda\sigma}(q_1, q_2, q_4 - q_1 - q_2)$
- 35 tensor structures vanish in this limit, only 19 left
- 5 of 8 integrals can be computed analytically
- Crossing symmetry: only twelve scalar functions enter the Master formula in the limit

BTT functions

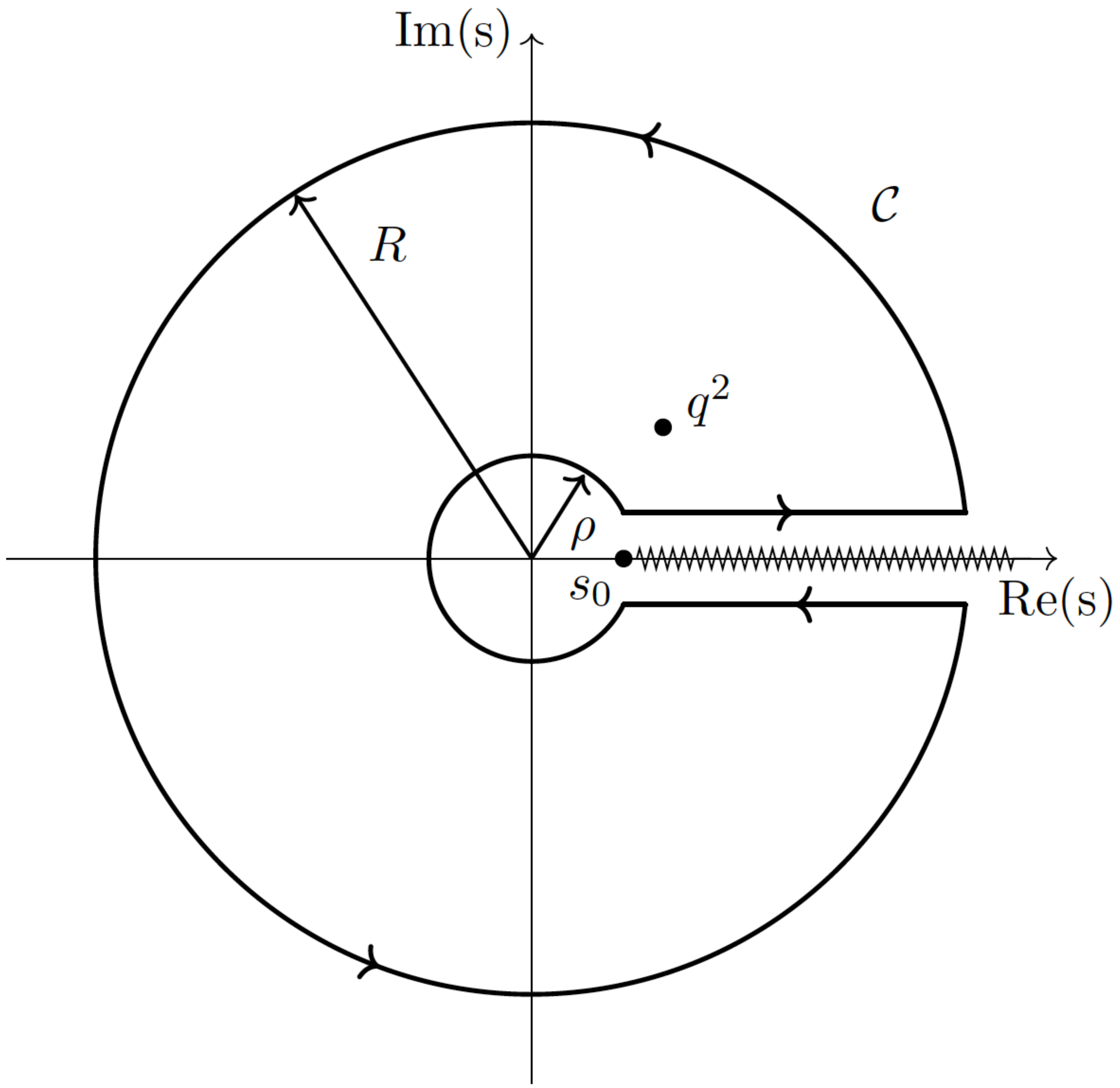
$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2} Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau)\bar{\Pi}_i(Q_1, Q_2, \tau).$$

Colangelo, Procura, Stoffer, Hoferichter (2015, 2017)

Euclidean momenta | Function of momenta | Kernel functions (known)

Dispersive Approaches for HLbL

Dispersion relations for the BTT functions



- BTT function analytic except along branch cut starting at s_0
- Cauchy's formula
- If BTT functions falls off fast along circles, only rims contribute
- Using Schwartz' reflection principle gives

$$F(s) = \frac{1}{\pi} \int_{4m^2}^{\infty} ds \frac{\operatorname{Im} F(s)}{s - q^2 - i\varepsilon}$$

Dispersion relation (DR)

Dispersive Approaches for HLbL

Unitarity relations

- Imaginary part of intermediate-state contributions from S-matrix unitarity
- For the T-matrix one has $T^\dagger T = -i\left(T - T^\dagger\right)$
- Inserting a complete set of states yields

$$\mathrm{Im}\, T_{fi} = \frac{1}{2}\sum_n \left(\prod_{i=1}^{n}\int \widetilde{dp_i}\right)(2\pi)^4\delta^4(q_n - q_i)T^*_{nf}T_{ni}$$

- Can write hadronic matrix element as sum over intermediate-state contributions

Dispersive Approaches for HLbL

Four-point kinematics

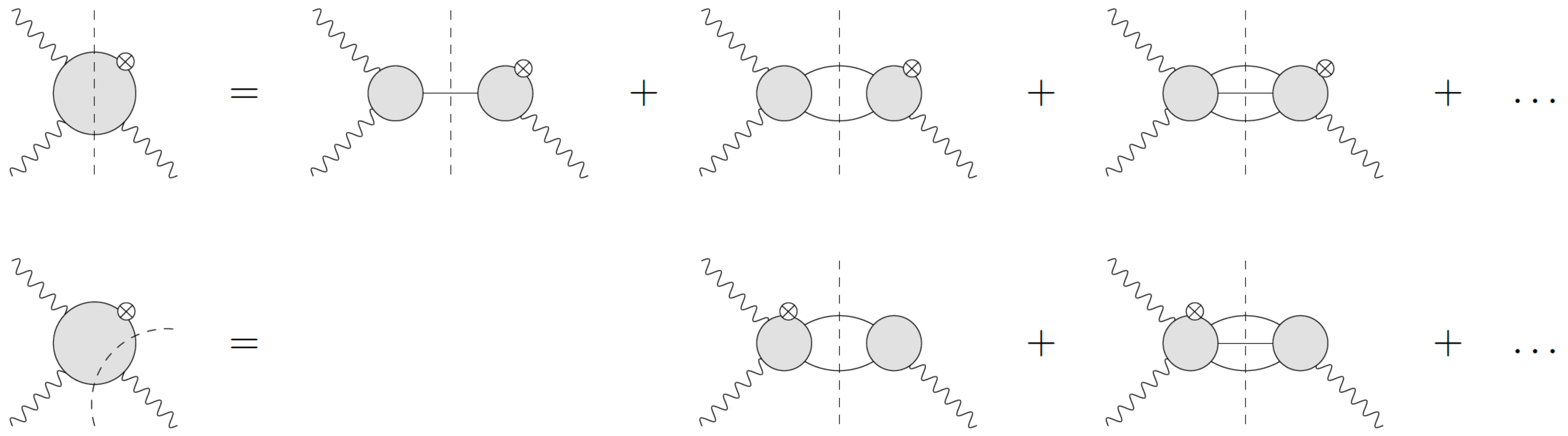
- Use this to write dispersion relations in Mandelstam variables and at fixed q_i^2 for the overcomplete set Π_i
- Take $q_4 \to 0$ only afterwards
- Allows for model-independent evaluation of exclusive hadronic states
- One- and two particle pseudoscalar contributions dominate full result for HLbL and are under good control

  Colangelo, Procura, Stoffer, Hoferichter (2015, 2017)
- One of the problems: intermediate states with spin ≥ 2
  - Become important for asymptotic matching

Dispersive Approaches for HLbL

Triangle kinematics

- Remedy: use dispersion relation in triangle kinematics
  - First take $q_4 \to 0$
  - Afterwards derive DRs in q_i^2



Lüdtke, Procura, Stoffer 2023

  - Resulting DRs are unambiguous and allow for model-independent evaluation of higher intemediate-state contributions
- Focus on single-particle intermediate states now
- will see that a finite number of intermediate states saturates asymptotic constraint

Kinematic regimes and asymptotic matching

Multi-scale dynamics of the HLbL tensor

- External photon always soft
- Low-energy regime (DRs); Short-distance (SD) regimes are under control using OPE and pQCD methods
- Distinguish two SD regimes:
  - Asymptotic regime: $\{Q_1^2, Q_2^2, Q_3^2\} \gg \Lambda_{\text{QCD}}$ — Massless quark loop

    Bijnens (2019, 2020, 2021)
  - Mixed regimes: $Q_1^2 \sim Q_2^2 \gg \{Q_3^2, \Lambda_{\text{QCD}}\}$ $\quad Q_2^2 \sim Q_3^2 \gg \{Q_1^2, \Lambda_{\text{QCD}}\}$

    $Q_1^2 \sim Q_3^2 \gg \{Q_2^2, \Lambda_{\text{QCD}}\}$

    Melnikov & Vainshtein (2004), Bijnens (2022)
- Problem: matching between long- and short-distance regimes
  - So far: Regge model, holographic models, interpolants

    Colangelo et al. (2020) Leutgeb, Rebhan (2020, 2021) Cappiello et al. (2020, 2022) Lüdtke, Procura (2020)

Kinematic regimes and asymptotic matching

Mixed regimes

- OPE for HLbL tensor in $Q_1^2 \sim Q_2^2 \gg \{Q_3^2, \Lambda_{\text{QCD}}\}$

Melnikov, Vainshtein 2004

- Define $\hat{q} = \frac{1}{2}(q_1 - q_2)$

- The leading-order OPE is given by the vector-vector-axial (VVA) correlator

$$\Pi_{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \frac{2}{\hat{q}^2} \epsilon_{\mu\nu\alpha\beta} \hat{q}^{\alpha} W_{\lambda\sigma}{}^{\beta}(-q_3, q_4).$$

where

$$W_{\mu\nu\rho}(q_1, q_2) = i \int d^4x \, d^4y \, e^{i(q_1 \cdot x + q_2 \cdot y)} \langle 0 | T\{j_\mu(x) j_\nu(y) j_\rho^5(0)\} | 0 \rangle.$$

Kinematic regimes and asymptotic matching

Mixed regimes

- At the level of BTT functions the OPE gives the constraints

Longitudinal function

Transversal functions …

$$
\begin{aligned}
\hat{\Pi}_1 &= 2w_L(q_3^2, 0, q_3^2) f(\hat{q}^2) + \mathcal{O}(\hat{q}^{-3}), \\
\hat{\Pi}_5 &= \frac{4}{3}\left(w_T^+ + \tilde{w}_T^-\right)(q_3^2, 0, q_3^2) f(\hat{q}^2) + c_5^{(2)} + c_5^{(3)} + \mathcal{O}(\hat{q}^{-4}), \\
\hat{\Pi}_6 &= \frac{4}{3}\left(w_T^+ + \tilde{w}_T^-\right)(q_3^2, 0, q_3^2) f(\hat{q}^2) + c_6^{(2)} + c_6^{(3)} + \mathcal{O}(\hat{q}^{-4}), \ldots
\end{aligned}
$$

Colangelo et al. (2020)

where $f(\hat{q}^2) = -\dfrac{1}{18\pi^2 \hat{q}^2}$ and $c_i^{(n)} \sim \hat{q}^{-n}$

- OPE constrains only linear combinations of transversal BTT functions
- Constraints in other mixed regions follow from crossing-symmetry

Kinematic regimes and asymptotic matching

Our approach

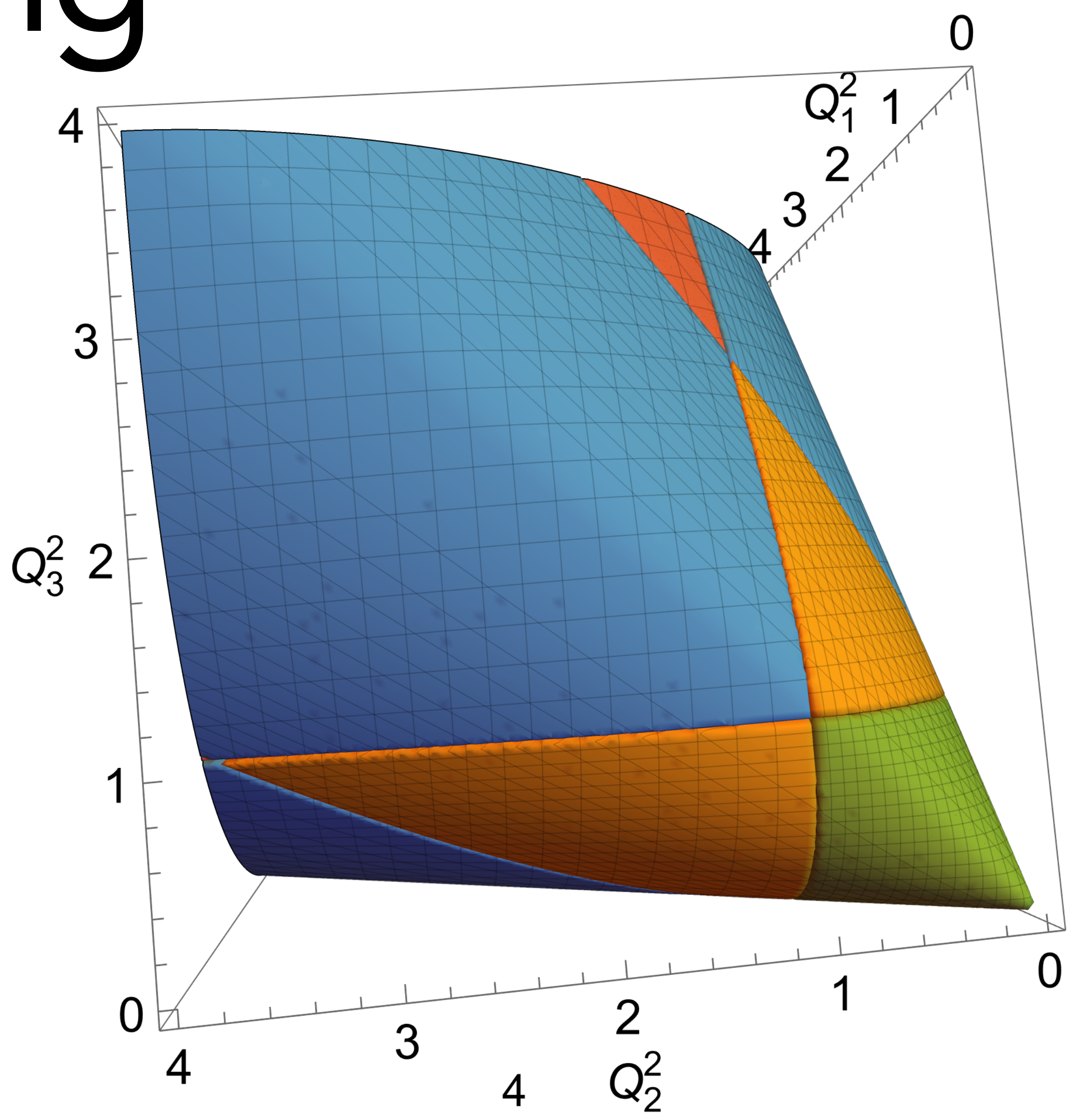
- Propose strategy for the matching between low and mixed regimes without using underlying models
- Achieve this by using dispersion relations in triangle kinematics for
  - single-particle intermediate states
  - the quark loop
- Short-distance constraints for the HLbL tensor (derived by means of an OPE)

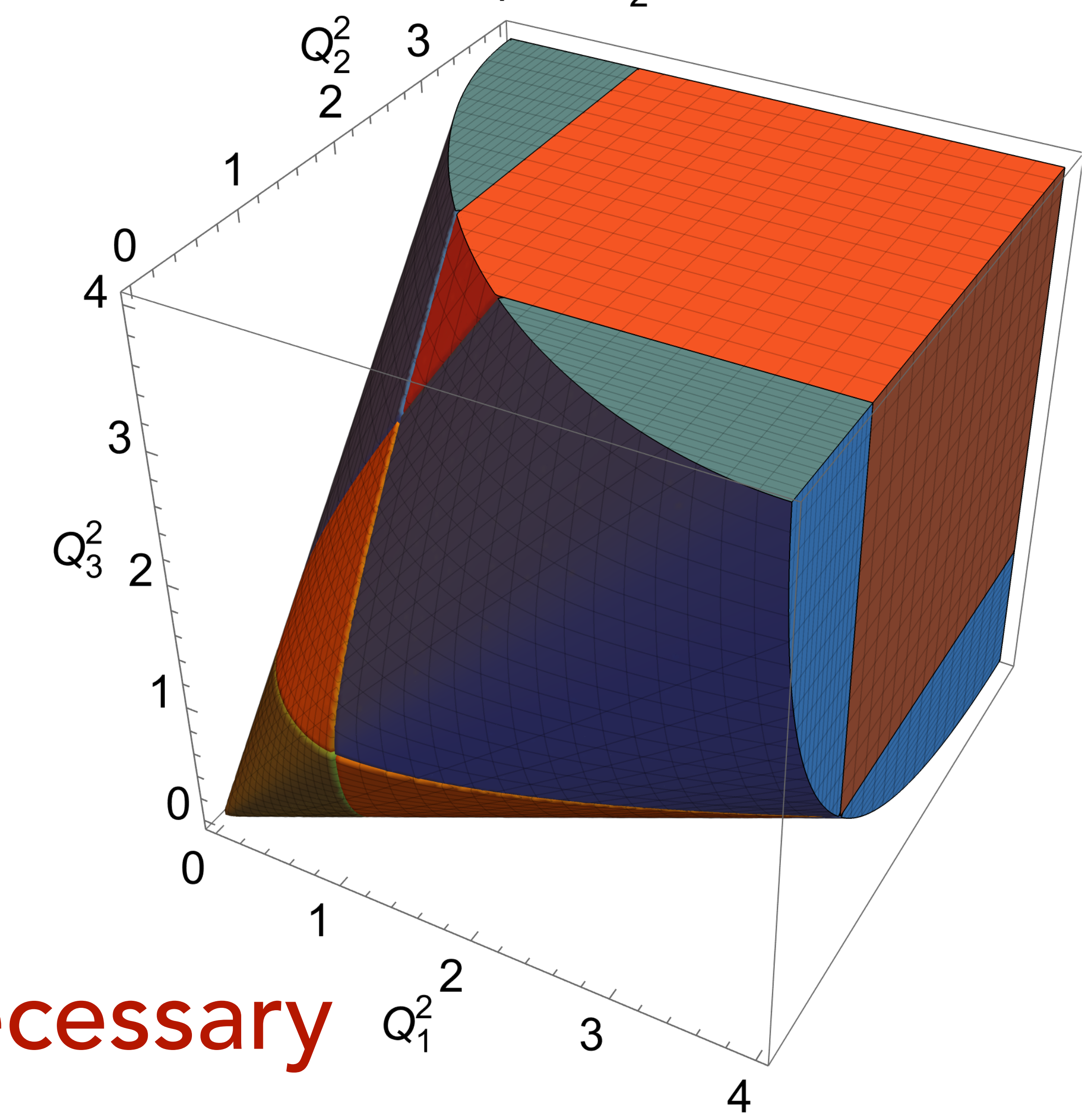
Kinematic regimes and asymptotic matching

BTT functions in kinematic regions

- By introducing Q_{cut}, we define

$$\hat{\Pi}_i(-Q_1^2,-Q_2^2,-Q_3^2) = \begin{cases} \hat{\Pi}_i^{\text{low-energy}}(-Q_1^2,-Q_2^2,-Q_3^2), & \text{for } \{Q_1^2,Q_2^2,Q_3^2\} < Q_{\text{cut}}^2, \\ \hat{\Pi}_i^{\text{mix},\,1}(-Q_1^2,-Q_2^2,-Q_3^2), & \text{for } \{Q_2^2,Q_3^2\} > Q_{\text{cut}}^2 > Q_1^2, \\ \hat{\Pi}_i^{\text{mix},\,2}(-Q_1^2,-Q_2^2,-Q_3^2), & \text{for } \{Q_1^2,Q_3^2\} > Q_{\text{cut}}^2 > Q_2^2, \\ \hat{\Pi}_i^{\text{mix},\,3}(-Q_1^2,-Q_2^2,-Q_3^2), & \text{for } \{Q_1^2,Q_2^2\}, > Q_{\text{cut}}^2 > Q_3^2 \\ \hat{\Pi}_i^{\text{inv.mix},\,1}(-Q_1^2,-Q_2^2,-Q_3^2), & \text{for } Q_1^2 > Q_{\text{cut}}^2 > \{Q_2^2,Q_3^2\}, \\ \hat{\Pi}_i^{\text{inv.mix},\,2}(-Q_1^2,-Q_2^2,-Q_3^2), & \text{for } Q_2^2 > Q_{\text{cut}}^2 > \{Q_1^2,Q_3^2\}, \\ \hat{\Pi}_i^{\text{inv.mix},\,3}(-Q_1^2,-Q_2^2,-Q_3^2), & \text{for } Q_3^2 > Q_{\text{cut}}^2 > \{Q_1^2,Q_2^2\}, \\ \hat{\Pi}_i^{\text{asymptotic}}(-Q_1^2,-Q_2^2,-Q_3^2), & \text{for } \{Q_1^2,Q_2^2,Q_3^2\} > Q_{\text{cut}}^2, \end{cases}$$





- Evaluate every contribution in a model-independent way

- Constraints are strictly valid only in parts of the regions: matching is necessary

Kinematic regimes and asymptotic matching

Leitmotif

- Aim: compare scaling behaviour in short-distance regimes for BTT functions of
  - HLbL tensor
  - Single-particle intermediate states
  - High- and low-energy part of dispersion relation for quark loop
- Based on this, we separate perturbative and non-perturbative effects in $\hat{\Pi}_i^{\mathrm{mix}}$

Kinematic regimes and asymptotic matching

Single-particle intermediate states

- Pion-pole only contributes to longitudinal BTT functions

$$\hat{\Pi}_1^{\pi}(q_1^2, q_2^2, q_3^2) = \frac{\mathcal{F}_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2)\mathcal{F}_{\pi\gamma^*\gamma^*}(M_\pi^2, 0)}{q_3^2 - M_\pi^2}$$

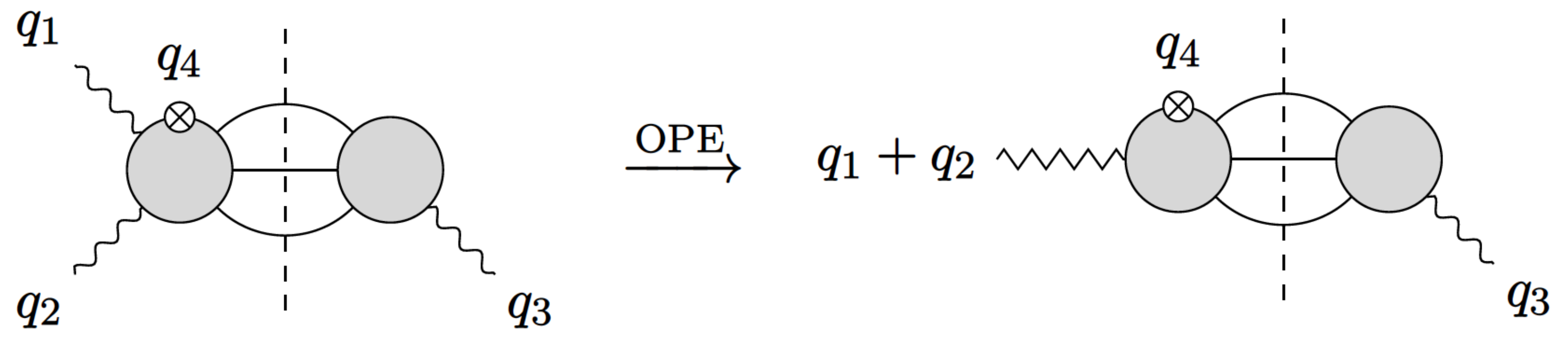
- Short-distance expansion for transition form factors

$$\mathcal{F}_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{4\sum_a C_a F_\pi^a}{q^2} f^\pi(w)$$

Hoferichter, Stoffer 2020

  - Similar expressions for other single-particle intermediate states
  - Based on BTT projection and light-cone expansion

- Compare this to single-particle intermediate states for VVA



Kinematic regimes and asymptotic matching

Single-particle intermediate states

- Show that solution for $c_i^{(n)}$ in

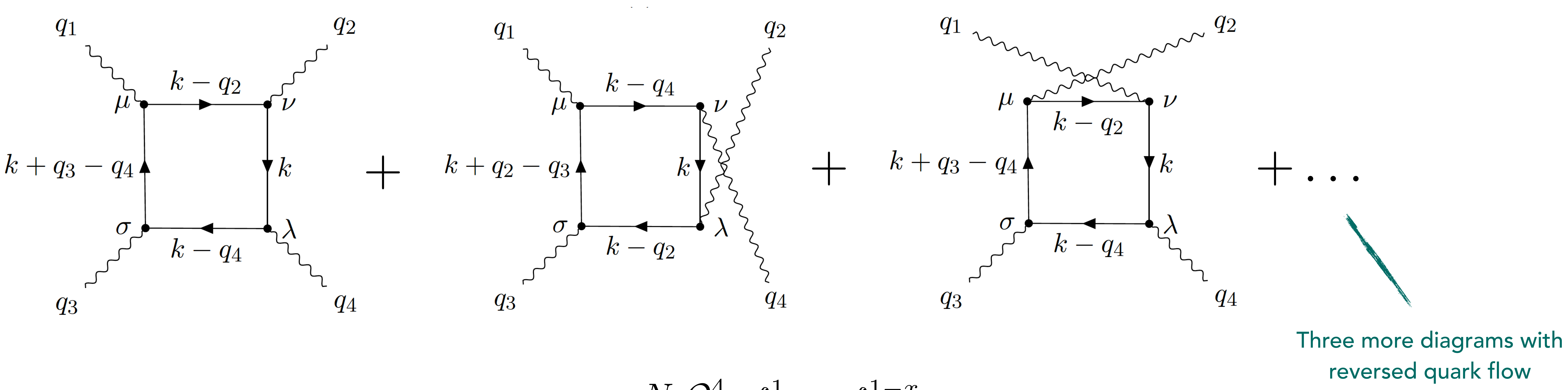
$$\hat{\Pi}_1 = 2w_L(q_3^2, 0, q_3^2) f(\hat{q}^2) + \mathcal{O}(\hat{q}^{-3}),$$
$$\hat{\Pi}_5 = \frac{4}{3}(w_T^+ + \tilde{w}_T^-)(q_3^2, 0, q_3^2) f(\hat{q}^2) + c_5^{(2)} + c_5^{(3)} + \mathcal{O}(\hat{q}^{-4}),$$
$$\hat{\Pi}_6 = \frac{4}{3}(w_T^+ + \tilde{w}_T^-)(q_3^2, 0, q_3^2) f(\hat{q}^2) + c_6^{(2)} + c_6^{(3)} + \mathcal{O}(\hat{q}^{-4}), \ldots$$

  exists

- single-particle intermediate states for HLbL correspond to the ones for VVA in the short-distance regimes
- Exclusive hadronic states for HLbL provide improved description of $\hat{\Pi}_i^{\text{mix}}$ compared to OPE together with VVA
- Based on this study scaling behaviour

Dispersion relations for the quark loop

Quark loop



- BTT functions $$\hat{\Pi}_i^{\mathrm{ql}}(q_1^2, q_2^2, q_3^2) = \sum_{q=u,d,s} \frac{N_c Q_q^4}{\pi^2} \int_0^1 dx \int_0^{1-x} dy I_i(x,y)$$

Colangelo et al. (2020)

$$I_1(x,y) = -\frac{x(-x-y+1)}{\Delta_{132}^2} - \frac{(1-2x)x(1-2y)y}{\Delta_{32}\Delta_{132}}, \ldots$$

Functions of momenta and Feyman parameters

Kinematic regimes and asymptotic matching

Quark loop

- Aim: derive single-dispersion relations in all kinematic variables

$$\hat{\Pi}_1^{ql}(q_1^2, q_2^2, q_3^2) = \frac{1}{\pi}\int_{s_0}^{\infty} ds \frac{\mathrm{Im}\,\hat{\Pi}_1^{ql}(s, q_2^2, q_3^2)}{s - q_1^2 - i\epsilon} \qquad \hat{\Pi}_1^{ql}(q_1^2, q_2^2, q_3^2) = \frac{1}{\pi}\int_{4m^2}^{\infty} du \frac{\mathrm{Im}_3\,\hat{\Pi}_1^{\mathrm{ql}}(q_1^2, q_2^2, u)}{u - q_3^2 - i\epsilon}$$

- Method: Split up the scalar function (suppress arguments on the right-hand side)

$$\hat{\Pi}_1^{ql}(q_1^2, q_2^2, q_3^2) \sim \int_0^1 dx \left(I_1^A(x) + I_1^B(x) + I_1^C(x) + I_1^D(x) + I_1^E(x) + I_1^F(x)\right)$$

  - Determine imaginary part of each term in the integration region (technical step)
  - Carry out the remaining integral analytically

- In the massless limit delta-functions appear in some imaginary parts

Kinematic regimes and asymptotic matching

Quark loop

- Split DR into low- and high-energy part

$$\frac{1}{\pi}\int_{4m^2}^{\infty} du \frac{\mathrm{Im}_3\,\hat{\Pi}_1(\hat{q}^2,\hat{q}\cdot q_3,u)}{u-q_3^2-i\epsilon} = \frac{1}{\pi}\left(\int_{4m^2}^{u_1}+\int_{u_1}^{u_2}+\int_{u_2}^{\infty}\right) du \frac{\mathrm{Im}_3\,\hat{\Pi}_1(\hat{q}^2,\hat{q}\cdot q_3,u)}{u-q_3^2-i\epsilon}$$

- Impose hierarchy between cut-offs, momenta and integration variables
- Expand the integrands
- Carry out the dispersive integrals
- Determine scaling behaviour in $\hat{q}$ for low-, intermediate- and high-energy part of quark loop DR

Kinematic regimes and asymptotic matching

Comparison in $Q_1^2 \sim Q_2^2 \gg \{Q_3^2, \Lambda_{\rm QCD}\}$

	SD $c_i^{(n)}=0$	SD $c_i^{(n)}\neq 0$	Pion	Scalars	Axials	Tensors	$\int_{u_0}^{u_1}$	$\int_{u_1}^{u_2}$	$\int_{u_2}^{\infty}$
$\hat\Pi_1$	$\hat q^{-2}$	$\hat q^{-2}$	$\hat q^{-2}$	0	0	0	$\hat q^{-2}$	$\mathcal{O}(\hat q^{-4})$	$\mathcal{O}(\hat q^{-4})$
$\hat\Pi_2$	$\mathcal{O}(\hat q^{-4})$	$\mathcal{O}(\hat q^{-4})$	0	0	0	0	$\mathcal{O}(\hat q^{-4})$	$\mathcal{O}(\hat q^{-4})$	$\mathcal{O}(\hat q^{-4})$
$\hat\Pi_3$	$\mathcal{O}(\hat q^{-4})$	$\mathcal{O}(\hat q^{-4})$	0	0	0	0	$\mathcal{O}(\hat q^{-4})$	$\mathcal{O}(\hat q^{-4})$	$\mathcal{O}(\hat q^{-4})$
$\hat\Pi_4$	$\mathcal{O}(\hat q^{-4})$	$\mathcal{O}(\hat q^{-4})$	0	$\hat q^{-4}$	0	$\hat q^{-4}$	$\mathcal{O}(\hat q^{-4})$	$\mathcal{O}(\hat q^{-4})$	$\mathcal{O}(\hat q^{-4})$
$\hat\Pi_5$	$\hat q^{-2}$	$\hat q^{-2}$	0	0	$\hat q^{-2}$	$\hat q^{-2}$	$\hat q^{-2}$	$\mathcal{O}(\hat q^{-4})$	$\mathcal{O}(\hat q^{-4})$
$\hat\Pi_6$	$\hat q^{-2}$	$\hat q^{-2}$	0	0	$\hat q^{-2}$	$\hat q^{-2}$	$\hat q^{-2}$	$\mathcal{O}(\hat q^{-4})$	$\mathcal{O}(\hat q^{-4})$
$\hat\Pi_7$	$\mathcal{O}(\hat q^{-6})$	$\hat q^{-5}$	0	0	0	$\mathcal{O}(\hat q^{-7})$	$\mathcal{O}(\hat q^{-6})$	$\mathcal{O}(\hat q^{-6})$	$\mathcal{O}(\hat q^{-6})$
$\hat\Pi_8$	$\mathcal{O}(\hat q^{-6})$	$\hat q^{-5}$	0	0	0	$\mathcal{O}(\hat q^{-7})$	$\mathcal{O}(\hat q^{-6})$	$\mathcal{O}(\hat q^{-6})$	$\mathcal{O}(\hat q^{-6})$
$\hat\Pi_9$	$\mathcal{O}(\hat q^{-4})$	$\hat q^{-3}$	0	0	$\hat q^{-5}$	$\hat q^{-5}$	$\hat q^{-5}$	$\mathcal{O}(\hat q^{-6})$	$\mathcal{O}(\hat q^{-6})$
$\hat\Pi_{10}$	$\hat q^{-4}$	$\hat q^{-4}$	0	0	$\hat q^{-4}$	$\hat q^{-4}$	$\hat q^{-4}$	$\mathcal{O}(\hat q^{-6})$	$\mathcal{O}(\hat q^{-6})$
$\hat\Pi_{11}$	$\mathcal{O}(\hat q^{-6})$	$\hat q^{-4}$	0	0	$\hat q^{-5}$	$\hat q^{-5}$	$\mathcal{O}(\hat q^{-6})$	$\mathcal{O}(\hat q^{-6})$	$\mathcal{O}(\hat q^{-6})$
$\hat\Pi_{13}$	$\mathcal{O}(\hat q^{-4})$	$\hat q^{-3}$	0	0	$\hat q^{-5}$	$\hat q^{-5}$	$\hat q^{-5}$	$\mathcal{O}(\hat q^{-6})$	$\mathcal{O}(\hat q^{-6})$
$\hat\Pi_{14}$	$\hat q^{-4}$	$\hat q^{-4}$	0	0	$\hat q^{-4}$	$\hat q^{-4}$	$\hat q^{-4}$	$\mathcal{O}(\hat q^{-6})$	$\mathcal{O}(\hat q^{-6})$
$\hat\Pi_{16}$	$\mathcal{O}(\hat q^{-6})$	$\hat q^{-4}$	0	0	$\hat q^{-5}$	$\hat q^{-5}$	$\mathcal{O}(\hat q^{-6})$	$\mathcal{O}(\hat q^{-6})$	$\mathcal{O}(\hat q^{-6})$
$\hat\Pi_{17}$	$\mathcal{O}(\hat q^{-4})$	$\mathcal{O}(\hat q^{-4})$	0	$\hat q^{-4}$	$\hat q^{-4}$	$\hat q^{-4}$	$\hat q^{-4}$	$\mathcal{O}(\hat q^{-6})$	$\mathcal{O}(\hat q^{-6})$
$\hat\Pi_{39}$	$\hat q^{-4}$	$\hat q^{-4}$	0	0	$\hat q^{-4}$	$\hat q^{-4}$	$\hat q^{-4}$	$\mathcal{O}(\hat q^{-6})$	$\mathcal{O}(\hat q^{-6})$
$\hat\Pi_{50}$	$\hat q^{-4}$	$\hat q^{-4}$	0	0	$\hat q^{-4}$	$\hat q^{-4}$	$\hat q^{-4}$	$\mathcal{O}(\hat q^{-6})$	$\mathcal{O}(\hat q^{-6})$
$\hat\Pi_{51}$	$\hat q^{-4}$	$\hat q^{-4}$	0	0	$\hat q^{-4}$	$\hat q^{-4}$	$\hat q^{-4}$	$\mathcal{O}(\hat q^{-6})$	$\mathcal{O}(\hat q^{-6})$
$\hat\Pi_{54}$	$\mathcal{O}(\hat q^{-6})$	$\hat q^{-4}$	0	0	$\hat q^{-5}$	$\hat q^{-5}$	$\mathcal{O}(\hat q^{-6})$	$\mathcal{O}(\hat q^{-6})$	$\mathcal{O}(\hat q^{-6})$

- Use DRs in the small variable here
- Single-particle intermediate states and low-energy part of the quark loop dispersion relation satisfy the leading-order SDC in the mixed regions
- Choosing DR in other variable leads to a different picture

Kinematic regimes and asymptotic matching

Conclusions

- One obtains an equivalent picture in other mixed regions by choosing a DR in the small variable

- Based on this, we propose a separation of perturbative and non-perturbative effects in the BTT functions

$$\hat{\Pi}_i^{\text{mix},3}(q_1^2, q_2^2, q_3^2) = \frac{1}{\pi}\int_{s_0}^{s_1} \frac{\text{Im}\,\hat{\Pi}_i(q_1^2, q_2^2, s)}{s - q_3^2 - i\epsilon} + \frac{1}{\pi}\int_{s_1}^{\infty} \frac{\text{Im}\,\hat{\Pi}_i(q_1^2, q_2^2, s)}{s - q_3^2 - i\epsilon}$$

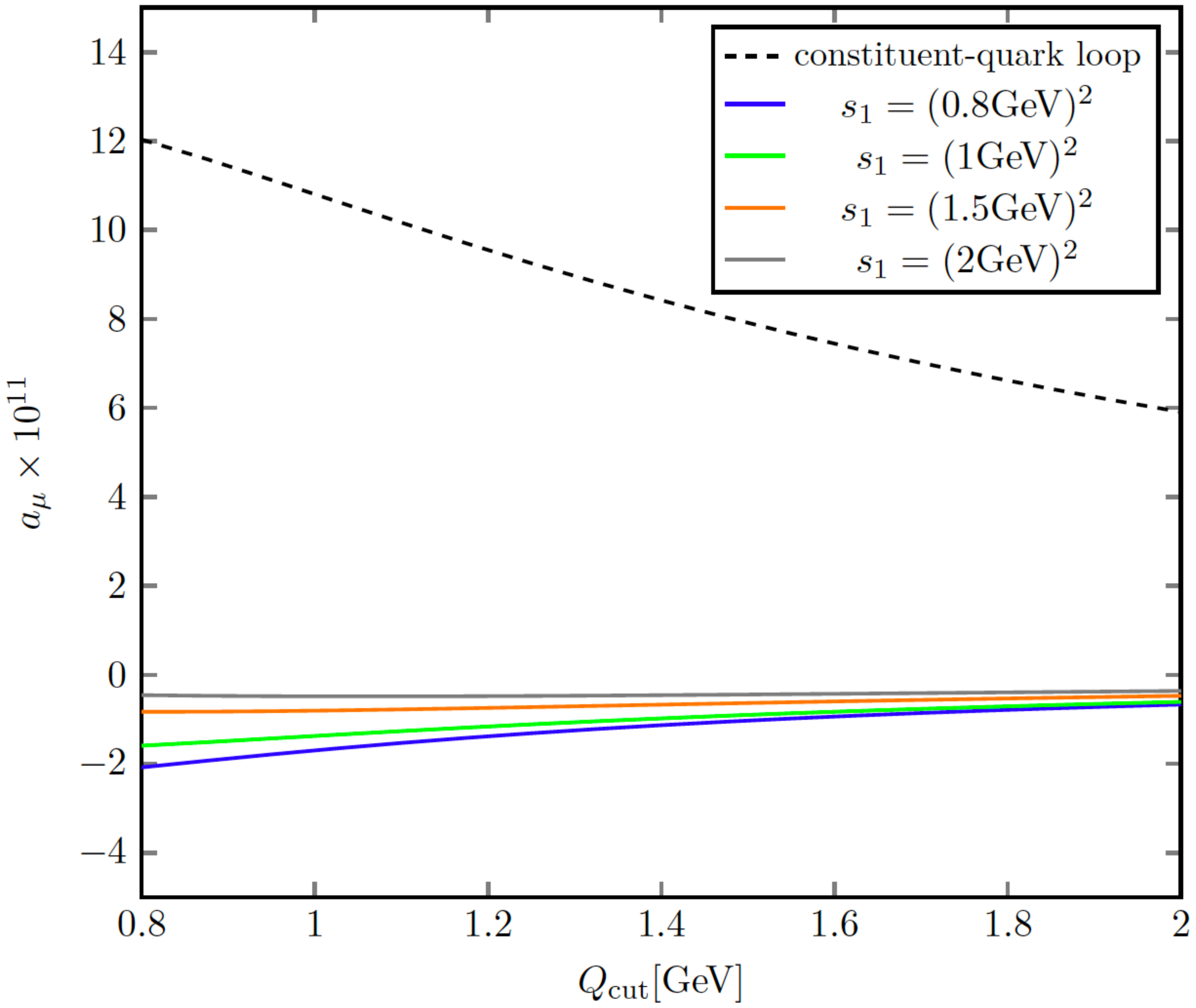
- The first term is replaced by a finite number of exclusive hadronic-state contributions and the second one by the quark loop

- Infinite tower of intermediate states for asymptotic region (quark loop)

Kinematic regimes and asymptotic matching

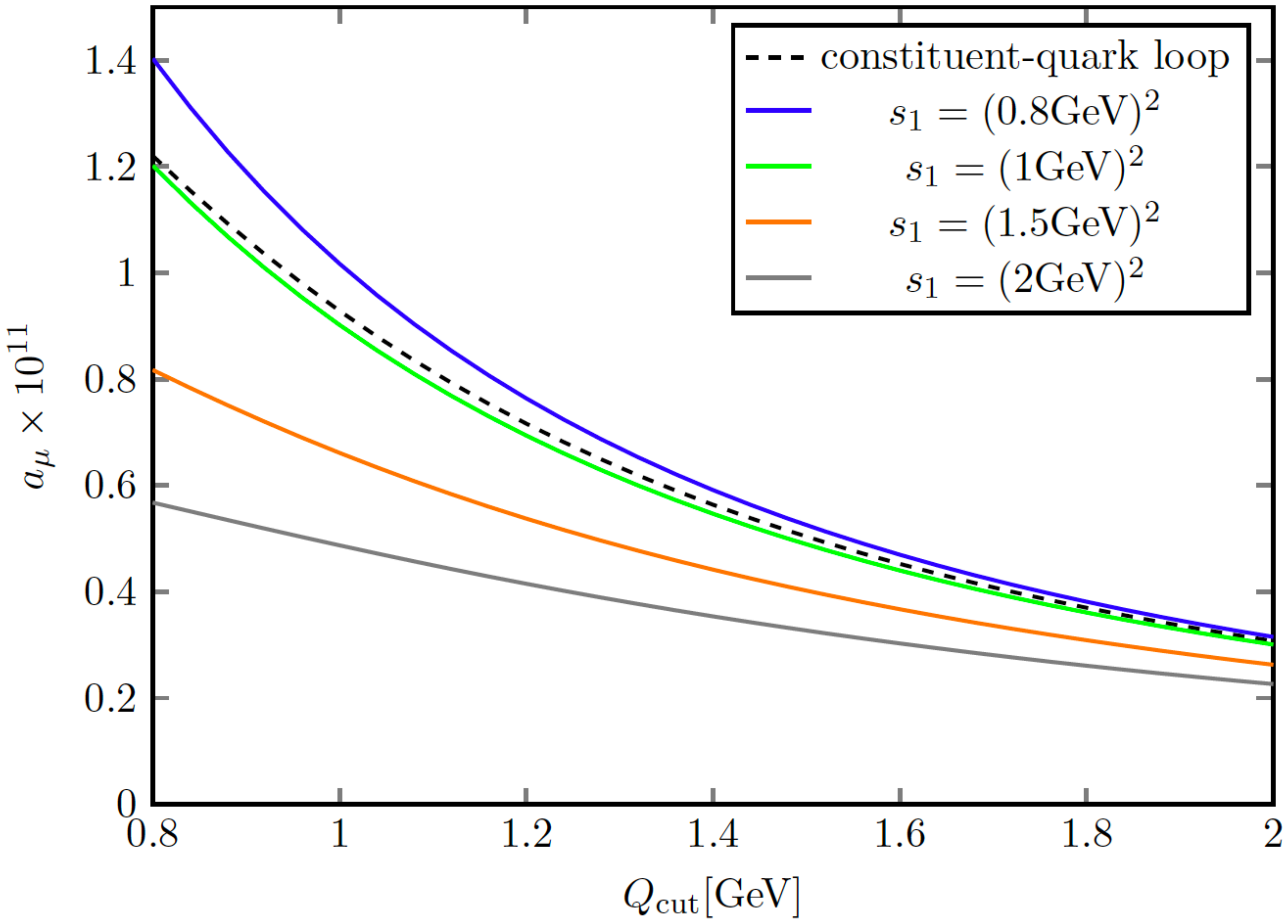
Numerics for longitudinal contribution

$Q_1^2 \sim Q_2^2 \gg \{Q_3^2, \Lambda_{\mathrm{QCD}}\}$



Pion pole dominates

$Q_1^2 \sim Q_3^2 \gg \{Q_2^2, \Lambda_{\mathrm{QCD}}\}$



Non-perturbative effects much smaller

Outlook

Reliable Uncertainty Estimates

- Once contributions from higher exclusive states are available, the splitting parameter s_1 can be used as an uncertainty estimate

- Since the left-hand-side of the equation

$$\hat{\Pi}_i^{\mathrm{mix},3}(q_1^2, q_2^2, q_3^2) = \frac{1}{\pi}\int_{s_0}^{s_1} \frac{\mathrm{Im}\,\hat{\Pi}_i^{\mathrm{exclusive}}(q_1^2, q_2^2, s)}{s - q_3^2 - i\epsilon} + \frac{1}{\pi}\int_{s_1}^{\infty} \frac{\mathrm{Im}\,\hat{\Pi}_i^{\mathrm{ql}}(q_1^2, q_2^2, s)}{s - q_3^2 - i\epsilon}.$$

must be independent of s_1, the dependence of the muon anomaly on this splitting parameter is a measure for the uncertainty

Summary

- HLbL is important to improve theoretical prediction and find out if muon anomaly is hint to new physics
- Asymptotic matching to reduce uncertainty on the HLbL contribution
- Derived DRs for the quark loop in triangle kinematics
- Studied the relation between single-particle intermediate states for HLbL and VVA
- Separated perturbative and non-perturbative effects in $\hat{\Pi}_i^{\mathrm{mix}}$
- Based on this we proposed strategy for a model-independent matching from the low-energy to the mixed regimes based on DR