

Loops, birdtracks and $6js$ — exploring QCD colour structure

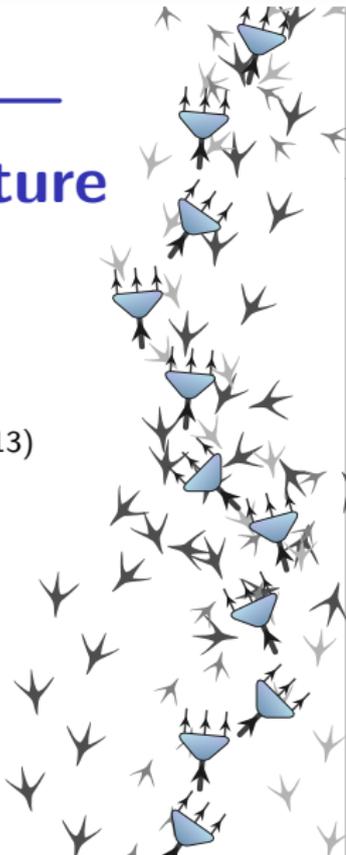
Judith M. Alcock-Zeilinger

S. Keppeler, S. Plätzer and M. Sjödaahl (arXiv:2209.15013)

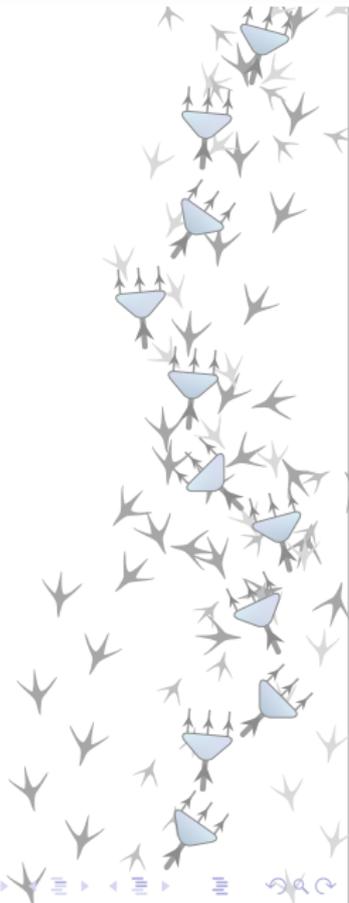
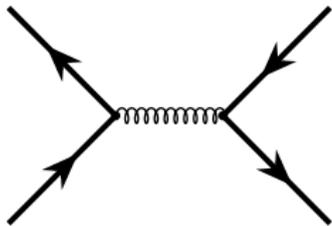
October 28th, 2022

ESI

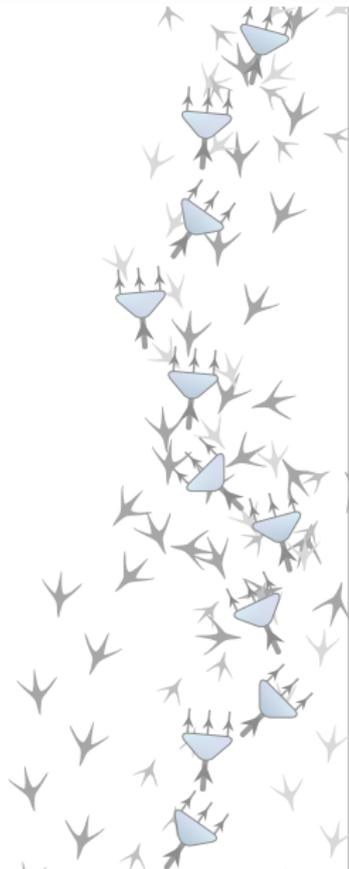
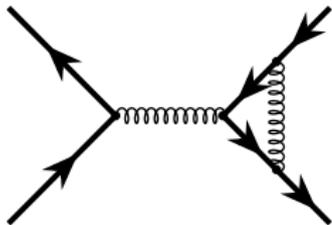
Erwin Schrödinger International Institute
for Mathematics and Physics



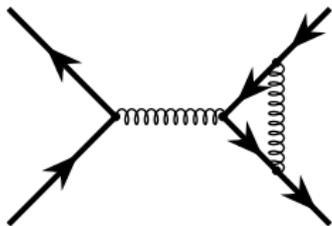
Exchanging partons: $SU(N)$ color structure



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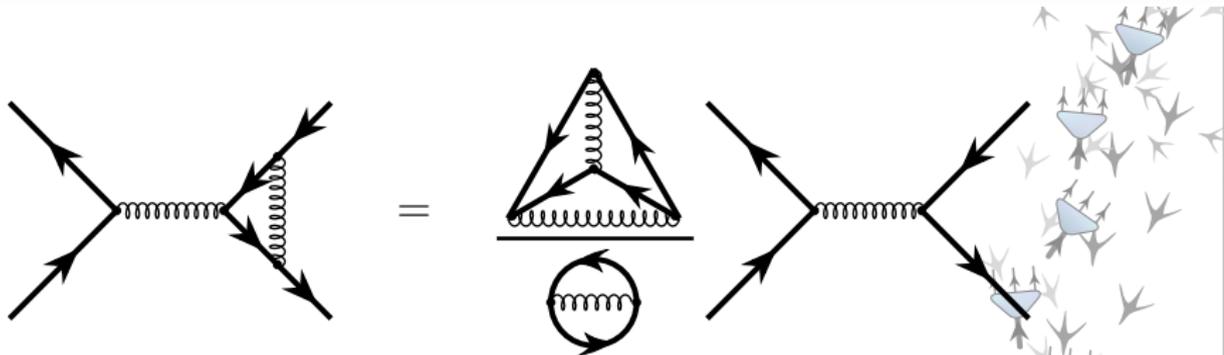


Exchanging partons: $SU(N)$ color structure



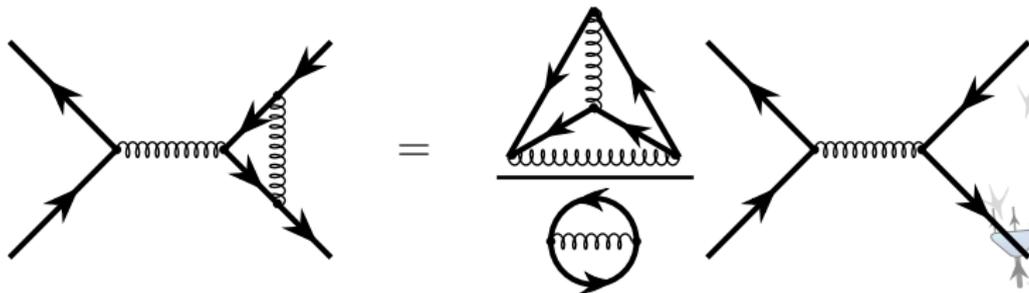
Factorization: consider color structure only

Exchanging partons: $SU(N)$ color structure

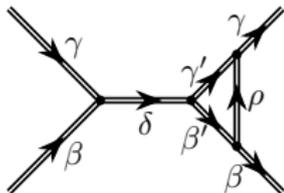


Factorization: consider color structure only

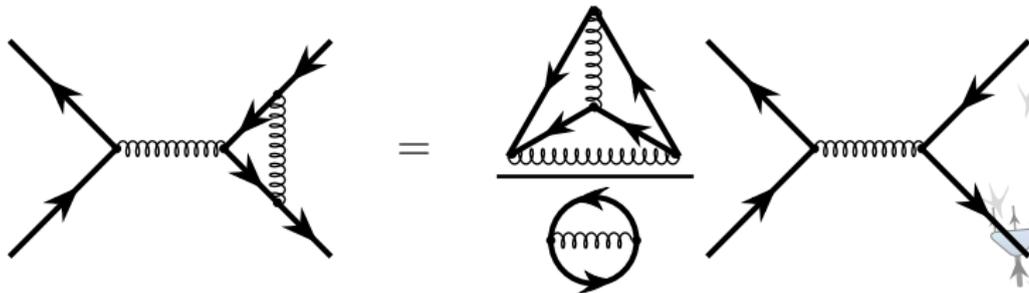
Exchanging partons: $SU(N)$ color structure



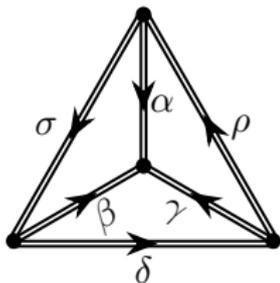
Factorization: consider color structure only



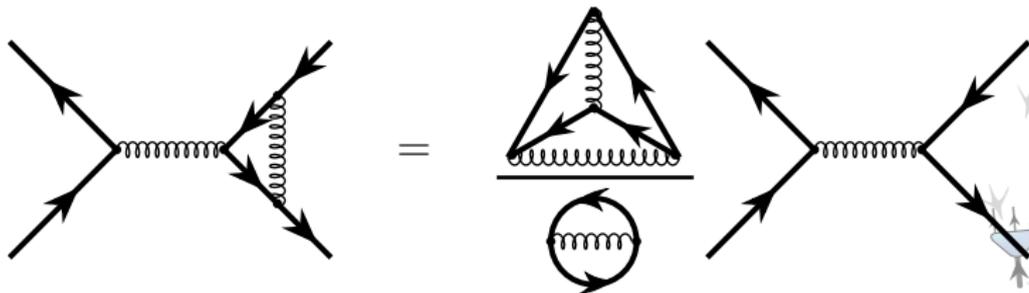
Exchanging partons: $SU(N)$ color structure



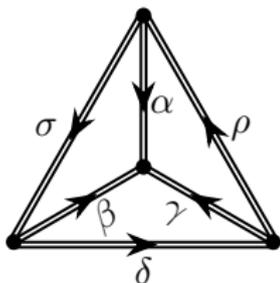
Factorization: consider color structure only



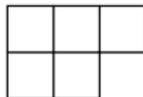
Exchanging partons: $SU(N)$ color structure



Factorization: consider color structure only

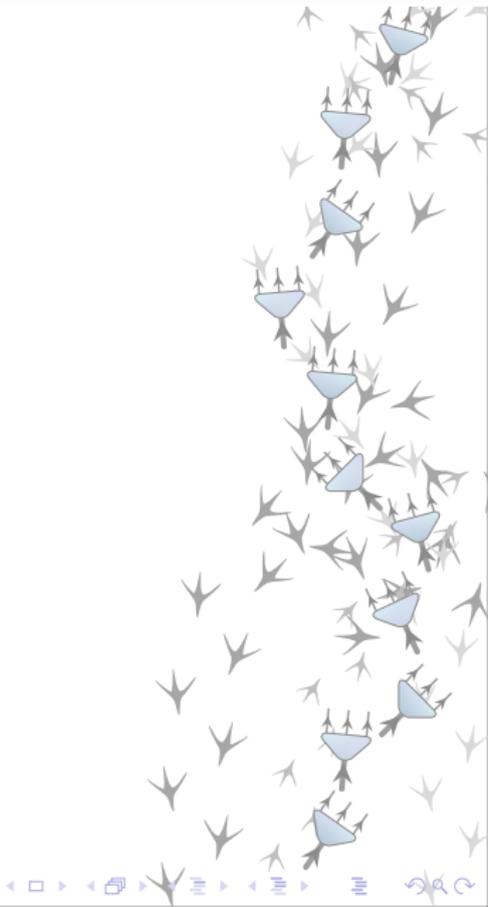


- $\beta, \gamma, \delta, \dots$ irreps of $SU(N)$
- labeled by Young diagrams, e.g.



Outline

- Birdtracks \longrightarrow Wigner- $6j$ symbols
- Wigner- $6j$ symbols \longrightarrow 2 quarks
- Calculation strategy:
 1. derive system of equations
 2. solve system



Birdtracks in a

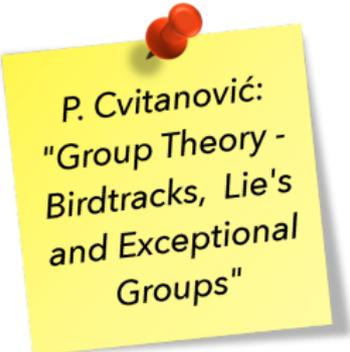


fundamental representation:

$$\delta_a^b = b \longleftarrow a$$

general representation α :

$$[\delta_\alpha]_a^b = b \xleftarrow{\alpha} a$$



P. Cvitanović:
"Group Theory -
Birdtracks, Lie's
and Exceptional
Groups"

Birdtracks in a



fundamental representation:

$$\delta_a^b = b \longleftarrow a$$

$$\delta_a^a = \text{circle with arrow} = N$$

general representation α :

$$[\delta_\alpha]_a^b = b \xleftarrow{\alpha} a$$

$$[\delta_\alpha]_a^a = \text{circle with arrow and } \alpha = d_\alpha$$

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Birdtracks in a



fundamental representation:

$$\delta_a^b = b \longleftarrow a \quad \delta_a^a = \text{circle with arrow} = N$$

general representation α :

$$[\delta_\alpha]_a^b = b \xleftarrow[\alpha]{} a \quad [\delta_\alpha]_a^a = \text{double circle with arrow} = d_\alpha$$

P. Cvitanović:
"Group Theory -
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Groups"

Clebsch-Gordan coefficient (suppressing state labels)

$$\alpha \otimes \beta \rightarrow \gamma : \quad \text{triangle diagram with arrows}$$

Birdtracks in a



fundamental representation:

$$\delta_a^b = b \longleftarrow a \quad \delta_a^a = \text{circle with arrow} = N$$

general representation α :

$$[\delta_\alpha]_a^b = b \xleftarrow{\alpha} a \quad [\delta_\alpha]_a^a = \text{double circle with arrow } \alpha = d_\alpha$$

P. Cvitanović:
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Clebsch-Gordan coefficient (suppressing state labels)

$$\alpha \otimes \beta \rightarrow \gamma : \quad \text{triangle with } \alpha, \beta \text{ in, } \gamma \text{ out} =: \frac{1}{\sqrt{a_\gamma}} \text{ fork with } \alpha, \beta \text{ in, } \gamma \text{ out}$$

Birdtracks in a



$$\alpha \otimes \beta \rightarrow \gamma : \quad \begin{array}{c} \alpha \\ \leftarrow \text{triangle} \leftarrow \\ \beta \end{array} \quad =: \quad \frac{1}{\sqrt{a_\gamma}} \begin{array}{c} \alpha \\ \leftarrow \text{fork} \leftarrow \\ \beta \end{array}$$

Birdtracks in a



$$\alpha \otimes \beta \rightarrow \gamma : \quad \begin{array}{c} \alpha \\ \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \\ \beta \end{array} \quad \equiv : \quad \frac{1}{\sqrt{a_\gamma}} \begin{array}{c} \alpha \\ \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \\ \beta \end{array}$$



Line ordering:

$$\begin{array}{c} \alpha \\ \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \\ \beta \end{array} \neq \begin{array}{c} \alpha \\ \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \\ \beta \end{array} \equiv : \begin{array}{c} \alpha \\ \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \\ \beta \end{array}$$

not a problem if $\gamma \neq \alpha \neq \beta \neq \gamma$

Birdtracks in a



$$\alpha \otimes \beta \rightarrow \gamma : \quad \begin{array}{c} \alpha \\ \leftarrow \text{triangle} \\ \beta \end{array} \quad \equiv \quad \frac{1}{\sqrt{a_\gamma}} \begin{array}{c} \alpha \\ \leftarrow \text{cup} \\ \beta \end{array}$$

Normalization:

$$\begin{array}{c} \alpha \\ \leftarrow \text{triangle} \\ \beta \end{array} \quad \begin{array}{c} \alpha \\ \leftarrow \text{triangle} \\ \beta \end{array} \quad \equiv \quad \begin{array}{c} \gamma \\ \leftarrow \text{double line} \end{array}$$

Birdtracks in a



$$\alpha \otimes \beta \rightarrow \gamma : \quad \begin{array}{c} \alpha \\ \beta \end{array} \begin{array}{c} \leftarrow \\ \leftarrow \end{array} \begin{array}{c} \gamma \end{array} \quad =: \quad \frac{1}{\sqrt{a_\gamma}} \begin{array}{c} \alpha \\ \beta \end{array} \begin{array}{c} \leftarrow \\ \leftarrow \end{array} \begin{array}{c} \gamma \end{array}$$

Normalization:

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Birdtracks in a



$$\alpha \otimes \beta \rightarrow \gamma : \quad \begin{array}{c} \alpha \\ \beta \end{array} \begin{array}{c} \leftarrow \\ \leftarrow \end{array} \begin{array}{c} \gamma \end{array} \quad =: \quad \frac{1}{\sqrt{a_\gamma}} \begin{array}{c} \alpha \\ \beta \end{array} \begin{array}{c} \leftarrow \\ \leftarrow \end{array} \begin{array}{c} \gamma \end{array}$$

Normalization:

$$\begin{array}{c} \alpha \\ \beta \end{array} \begin{array}{c} \leftarrow \\ \leftarrow \end{array} \begin{array}{c} \gamma \end{array} \begin{array}{c} \leftarrow \\ \leftarrow \end{array} \begin{array}{c} \alpha \\ \beta \end{array} \begin{array}{c} \leftarrow \\ \leftarrow \end{array} \begin{array}{c} \gamma \end{array} = \begin{array}{c} \leftarrow \\ \leftarrow \end{array} \begin{array}{c} \gamma \end{array} \quad \Rightarrow \quad a_\gamma = \frac{d_\gamma}{d_\gamma}$$

3js can be chosen \longrightarrow convenient: $3j = 1$

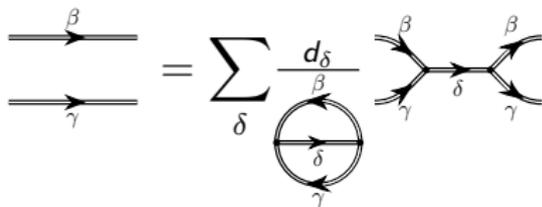
Reducing loops to trees

Completeness relation:

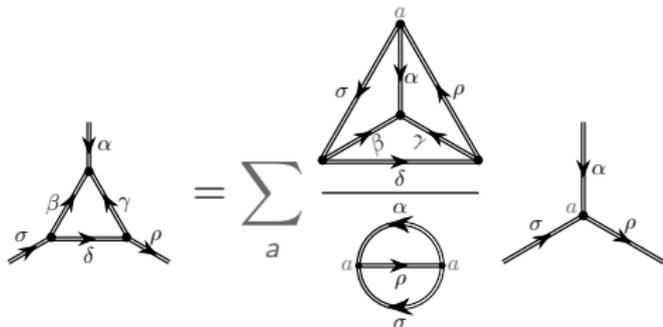
$$\begin{array}{c} \beta \\ \longrightarrow \\ \hline \hline \\ \gamma \\ \longrightarrow \end{array} = \sum_{\delta} \frac{d_{\delta}}{\beta} \begin{array}{c} \beta \\ \nearrow \\ \gamma \\ \nearrow \end{array} \xrightarrow{\delta} \begin{array}{c} \beta \\ \nwarrow \\ \gamma \\ \nwarrow \end{array}$$

Reducing loops to trees

Completeness relation:

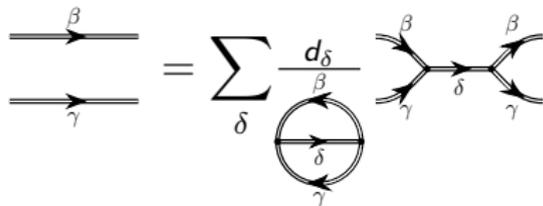


Vertex correction:

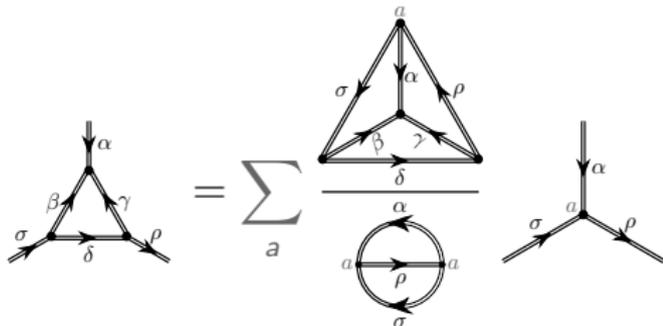


Reducing loops to trees

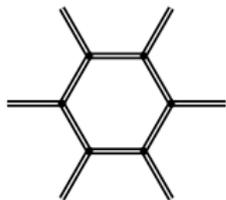
Completeness relation:



Vertex correction:

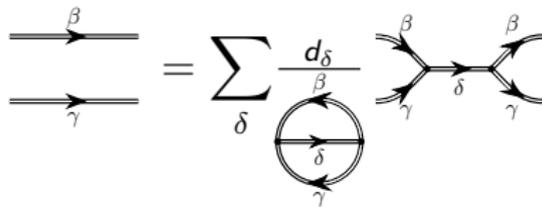


Reduce loops:

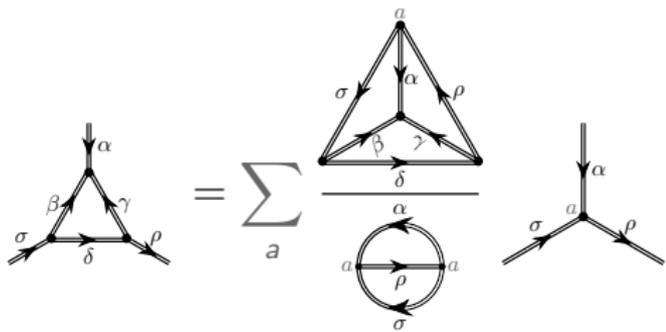


Reducing loops to trees

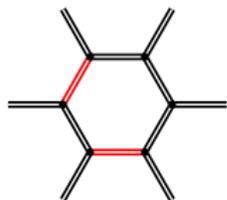
Completeness relation:



Vertex correction:

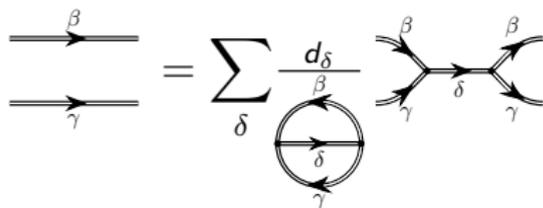


Reduce loops:

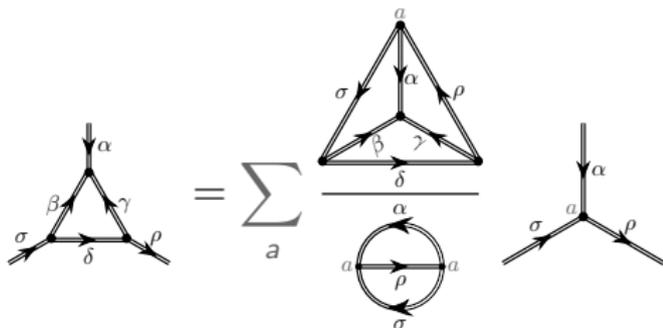


Reducing loops to trees

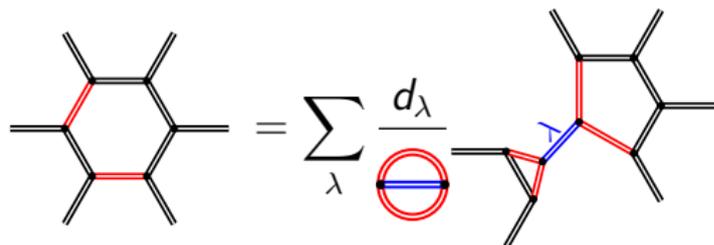
Completeness relation:



Vertex correction:



Reduce loops:



Reducing loops to trees

Completeness relation:

Diagram illustrating the completeness relation for a loop. On the left, two external lines labeled β and γ meet at a vertex. This is equal to a sum over δ of a loop diagram (a circle with a horizontal line through it, labeled δ and γ) multiplied by a tree diagram where the loop is cut, resulting in two vertices connected by a line labeled δ . The external lines are labeled β and γ .

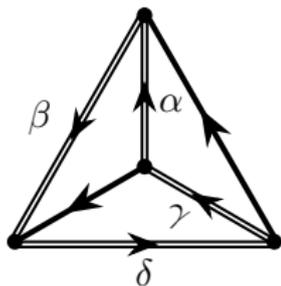
Vertex correction:

Diagram illustrating a vertex correction. On the left, a triangle loop with vertices labeled σ , β , and γ , and an external line labeled α meeting at a vertex. This is equal to a sum over a of a loop diagram (a circle with a horizontal line through it, labeled a and ρ) multiplied by a tree diagram where the loop is cut, resulting in a vertex labeled a with external lines labeled σ , α , and ρ .

Reduce loops:

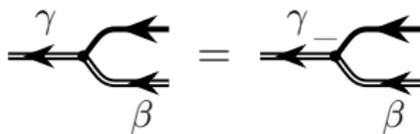
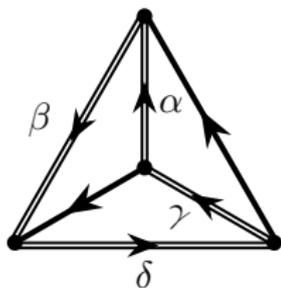
Diagram illustrating the reduction of a loop in a complex structure. On the left, a hexagonal loop with two red edges. This is equal to a sum over λ of a loop diagram (a circle with a horizontal line through it, labeled λ) multiplied by a tree diagram where the loop is cut, resulting in a vertex labeled λ connected to the rest of the structure. This is further equal to a sum over λ of a loop diagram (a circle with a horizontal line through it, labeled λ) multiplied by a loop diagram (a triangle with a horizontal line through it, labeled λ) multiplied by a tree diagram where the loop is cut, resulting in a vertex labeled λ connected to the rest of the structure.

6js with 2 quark lines



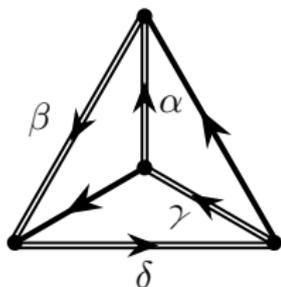
6js with 2 quark lines

Line ordering: *all* irreps distinct



$6js$ with 2 quark lines

Line ordering: *all* irreps distinct

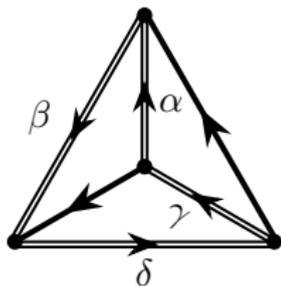


$$\begin{array}{c} \gamma \\ \leftarrow \quad \rightarrow \\ \quad \quad \quad \rightarrow \\ \quad \quad \quad \leftarrow \\ \beta \end{array} = \begin{array}{c} \gamma \\ \leftarrow \quad \rightarrow \\ \quad \quad \quad \rightarrow \\ \quad \quad \quad \leftarrow \\ \beta \end{array}$$

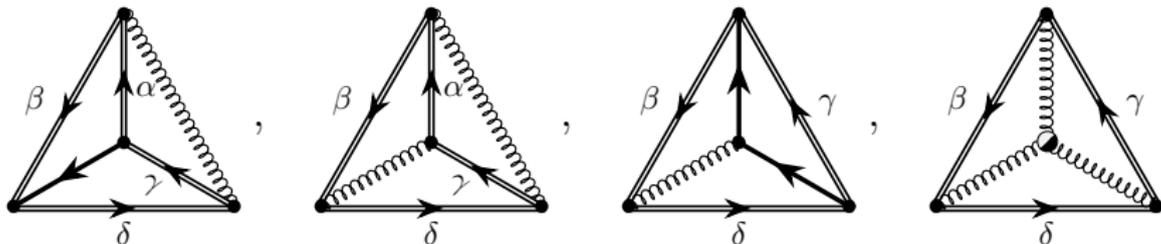
Exception:

$$\begin{array}{c} \rightarrow \quad \rightarrow \\ \quad \quad \downarrow \\ \square \end{array} = - \begin{array}{c} \rightarrow \quad \rightarrow \\ \quad \quad \downarrow \\ \square \end{array}$$

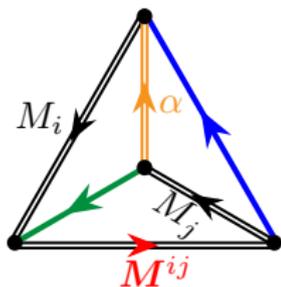
6js with 2 quark lines



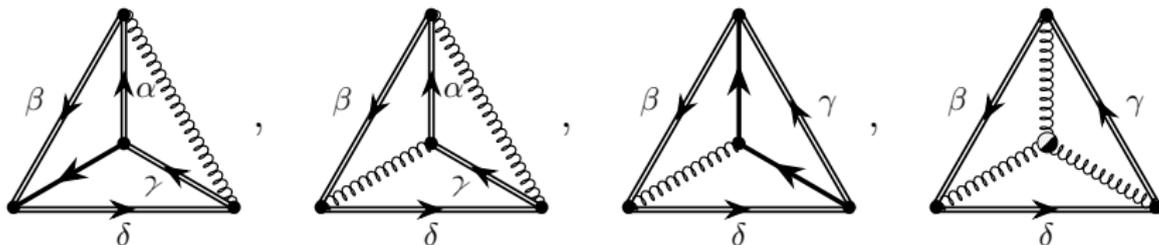
M. Sjö Dahl & J. Thorén (2018):



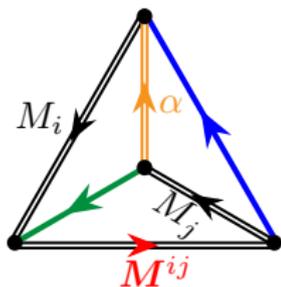
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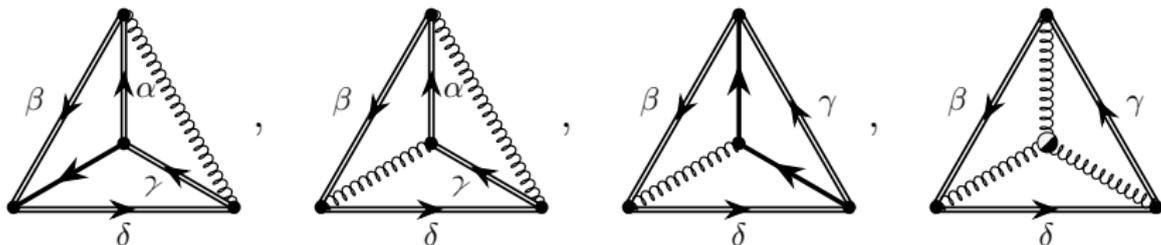


6js with 2 quark lines

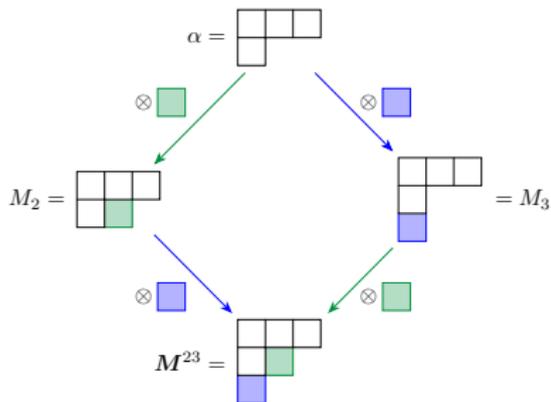
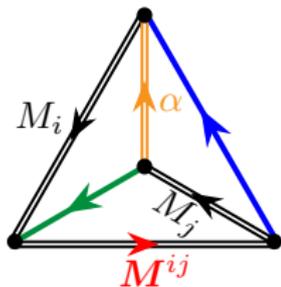


- **fix** α : a Young diagram
- $M_{i(j)}$: add \square in row $i(j)$ of α
- M^{ij} : add \square in row j of M_i

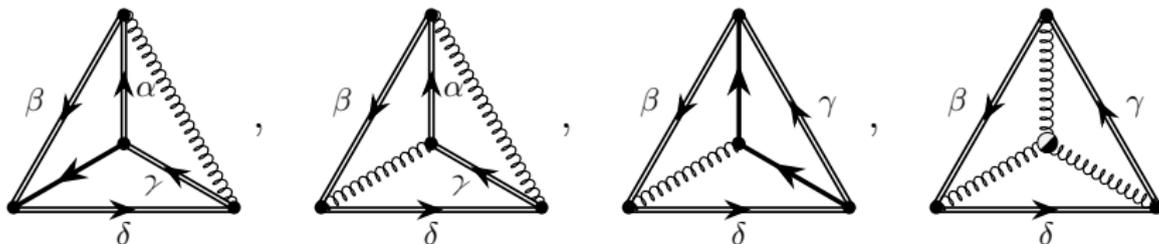
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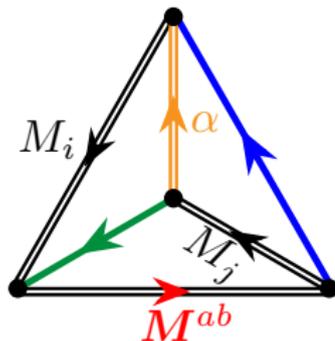
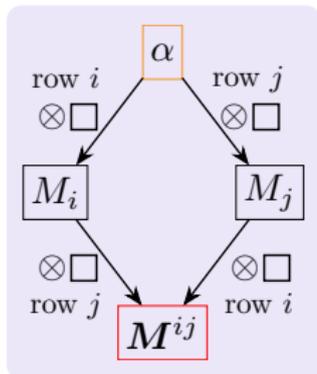
6js with 2 quark lines



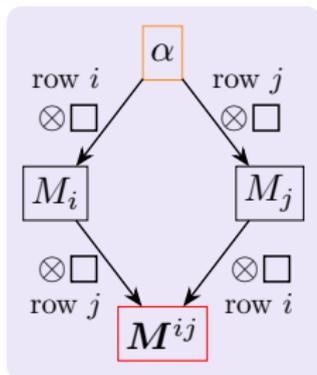
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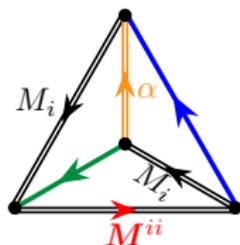
Properties: fixed α , given i, j with $i \neq j$



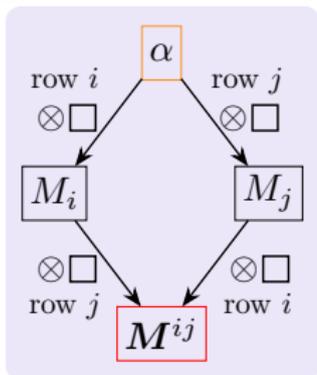
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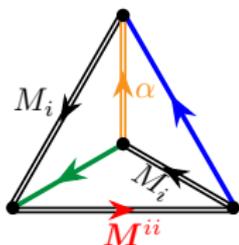
add to **same** row:



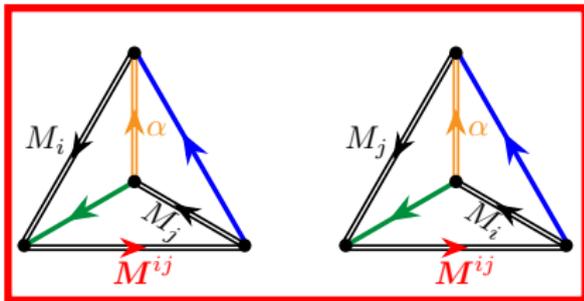
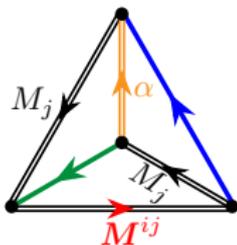
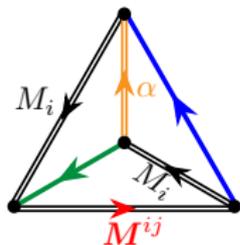
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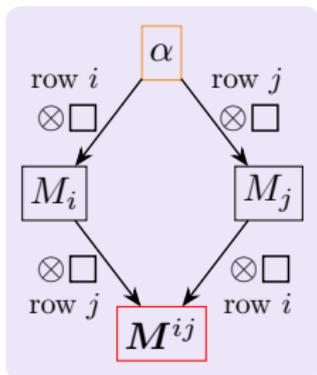
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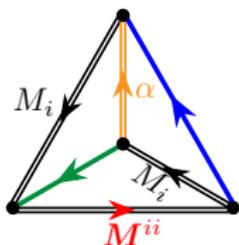
add to **different** rows:



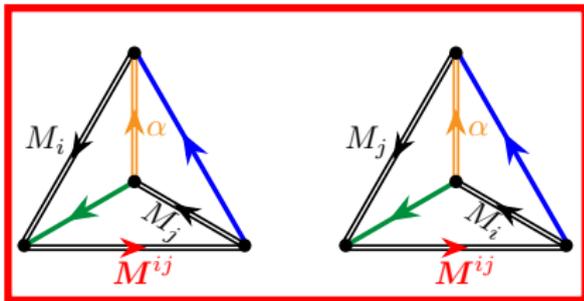
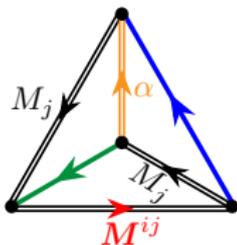
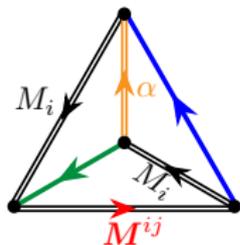
Properties: fixed α , given i, j with $i \neq j$



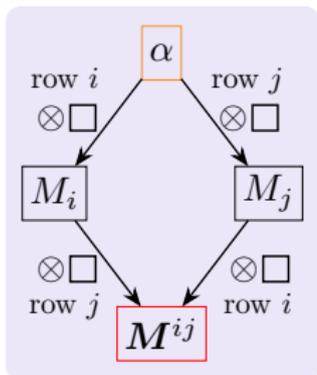
add to **same** row:



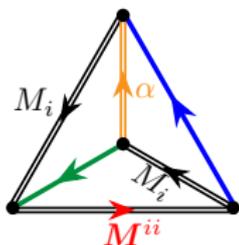
add to **different** rows:



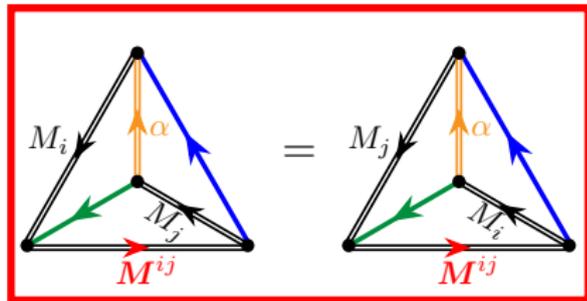
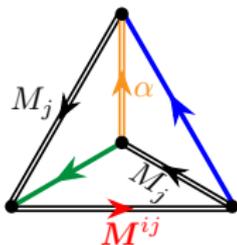
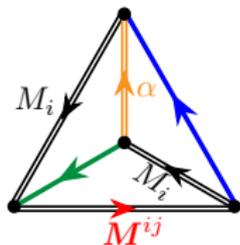
Properties: fixed α , given i, j with $i \neq j$



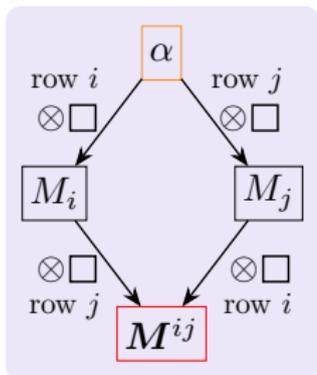
add to **same** row:



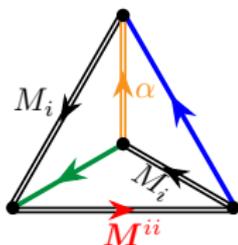
add to **different** rows:



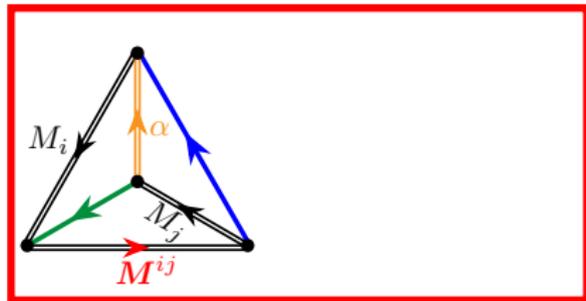
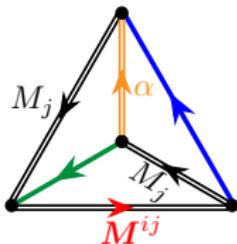
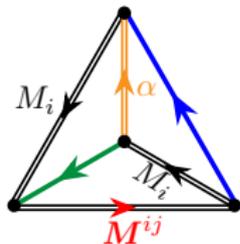
Properties: fixed α , given i, j with $i \neq j$



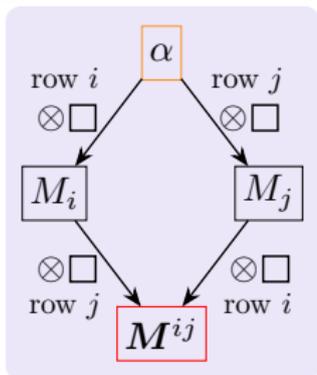
add to **same** row:



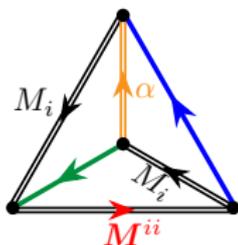
add to **different** rows:



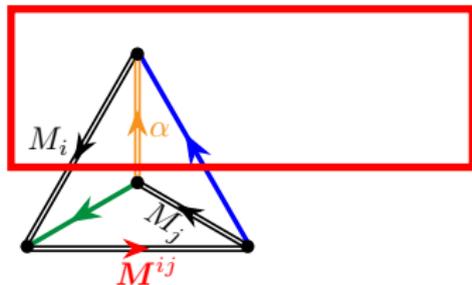
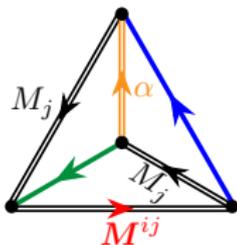
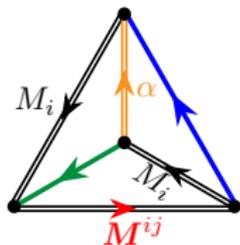
Properties: fixed α , given i, j with $i \neq j$



add to **same** row:

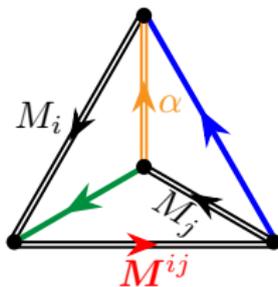


add to **different** rows:

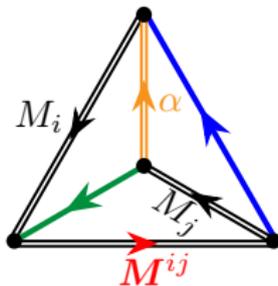


4 distinct $6j$ symbols

Calculating $6j$ with 2 quark lines

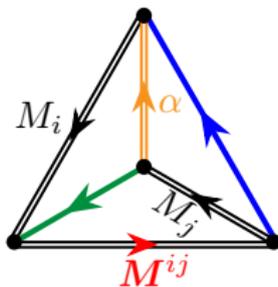


Calculating $6j$ with 2 quark lines



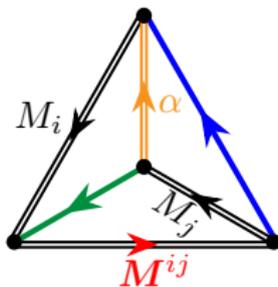
- Clebsches

Calculating $6j$ with 2 quark lines



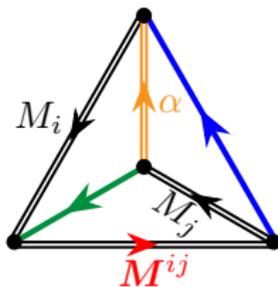
- Clebsches

Calculating $6j$ with 2 quark lines



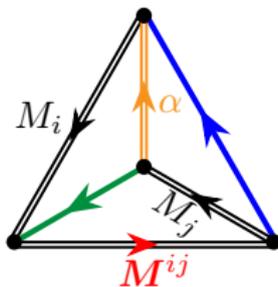
- Clebsches
- Projectors

Calculating $6j$ with 2 quark lines



- Clebsches
- Projectors

Calculating 6_j with 2 quark lines

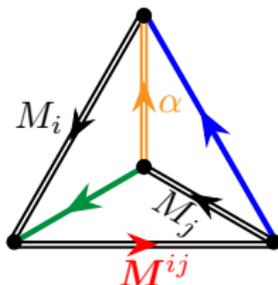


- Clebsches
- Projectors

Strategy:



Calculating 6_j with 2 quark lines



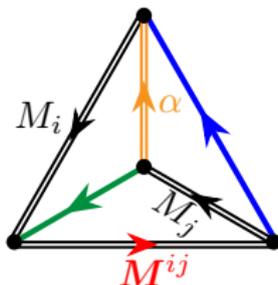
- Clebsches
- Projectors

Strategy:

$$\alpha \otimes \square \rightarrow M_i \otimes \square \rightarrow M^{ij}$$

- Construct birdtrack operators obeying hierarchy

Calculating 6_j with 2 quark lines

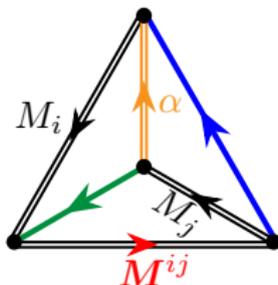


- Clebsches
- Projectors

Strategy:

- $\alpha \otimes \square \rightarrow M_i \otimes \square \rightarrow M^{ij}$
- Construct birdtrack operators obeying hierarchy
- Completeness relation & vertex correction

Calculating $6j$ with 2 quark lines

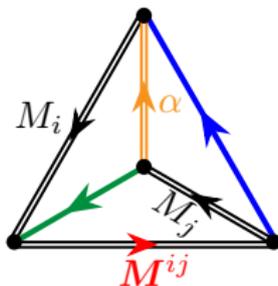


- Clebsches
- Projectors

Strategy:

- $\alpha \otimes \square \rightarrow M_i \otimes \square \rightarrow M^{ij}$
- Construct birdtrack operators obeying hierarchy
- Completeness relation & vertex correction
- System of equation for $6js$

Calculating $6j$ with 2 quark lines

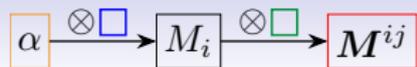


- Clebsches
- Projectors

Strategy:

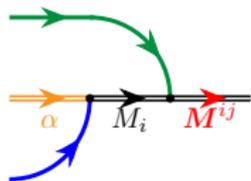
- $\alpha \otimes \square \rightarrow M_i \otimes \square \rightarrow M^{ij}$
- Construct birdtrack operators obeying hierarchy
- Completeness relation & vertex correction
- System of equation for $6js$
- Solve system

Consider:



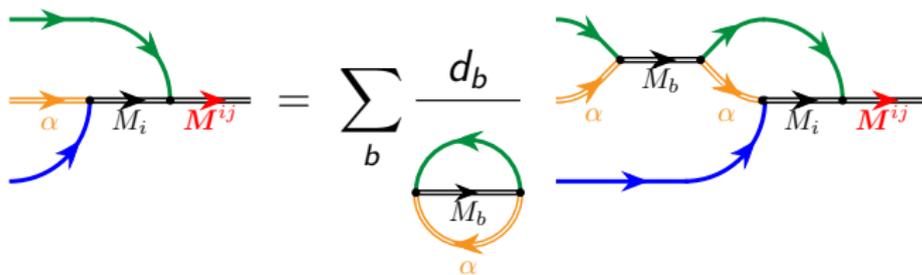
Consider:

$$\alpha \xrightarrow{\otimes \square} M_i \xrightarrow{\otimes \square} M^{ij}$$



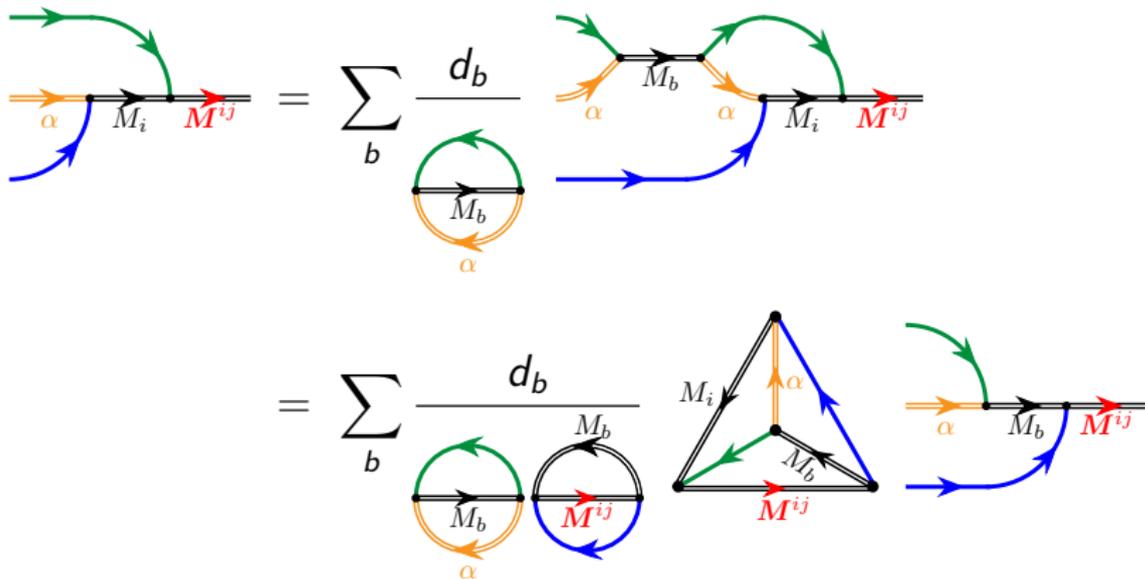
Consider:

$$\alpha \otimes \square \rightarrow M_i \otimes \square \rightarrow M^{ij}$$



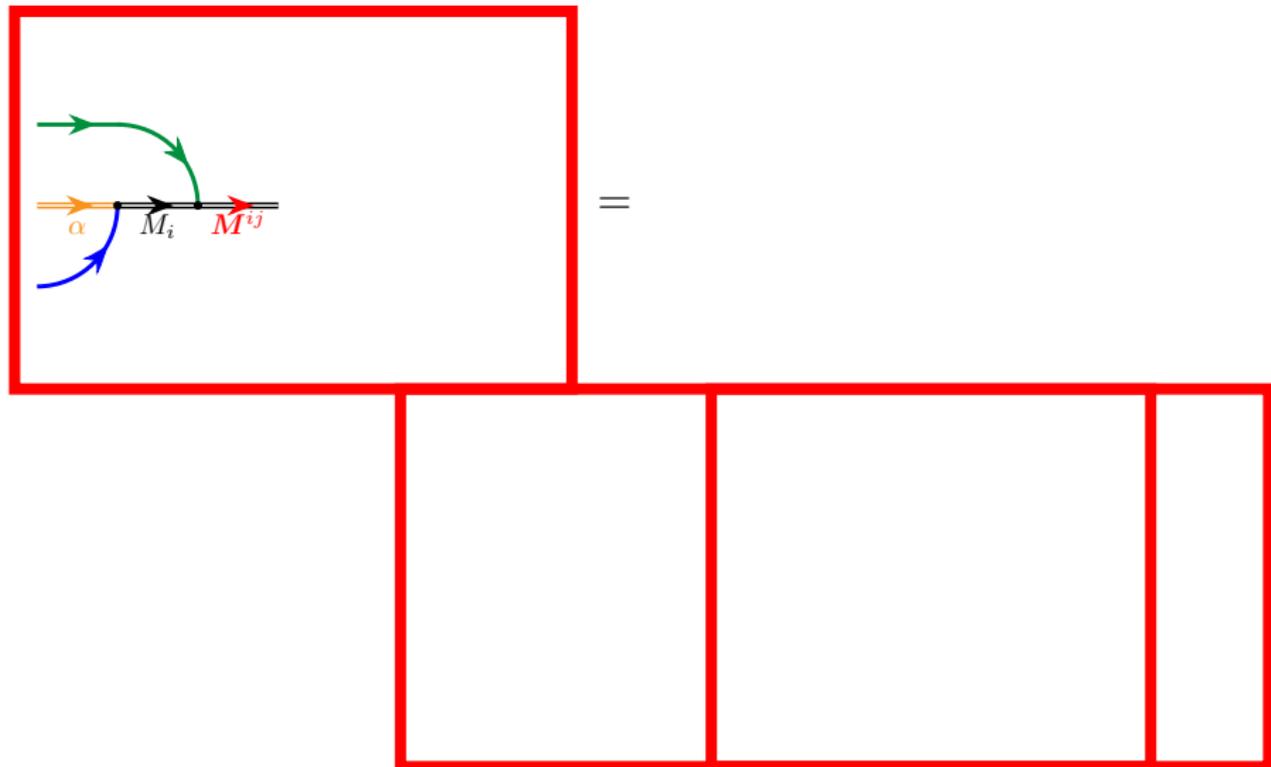
Consider:

$$\alpha \otimes \square \rightarrow M_i \otimes \square \rightarrow M^{ij}$$



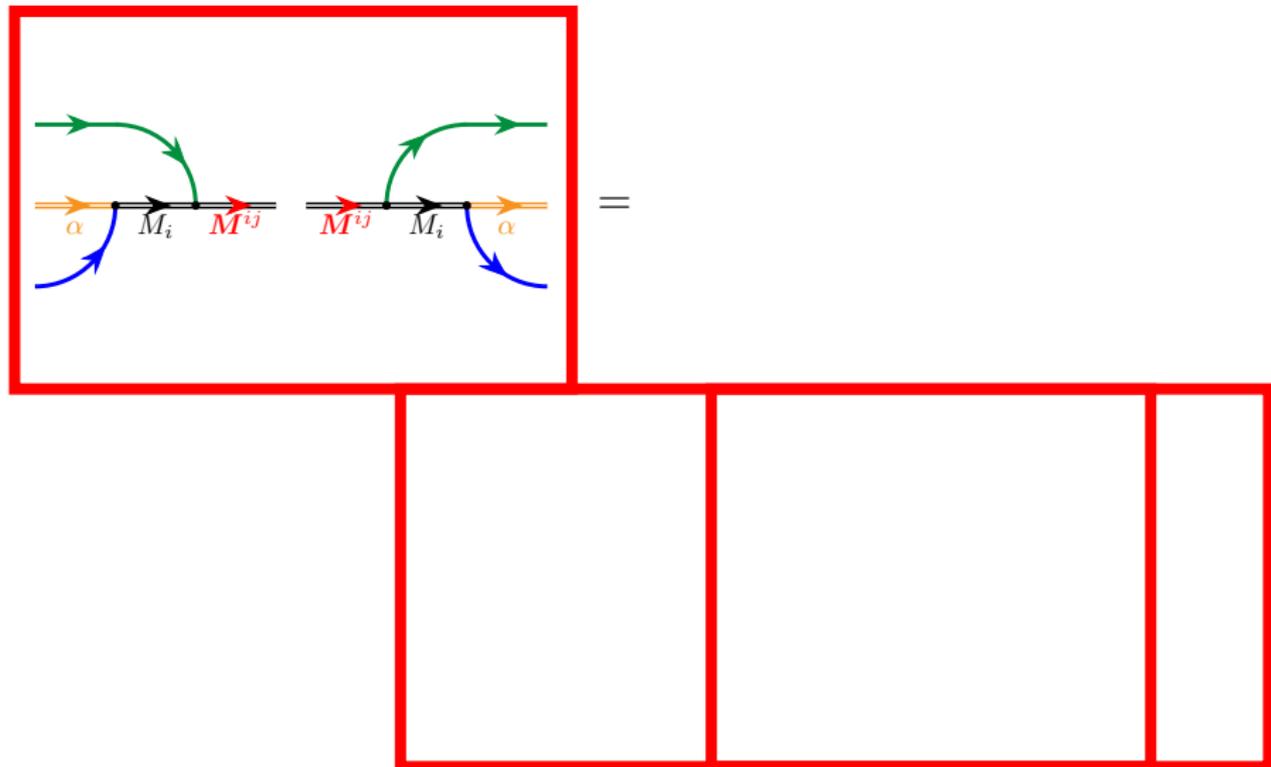
Consider:

$$\alpha \otimes \square \rightarrow M_i \otimes \square \rightarrow M^{ij}$$



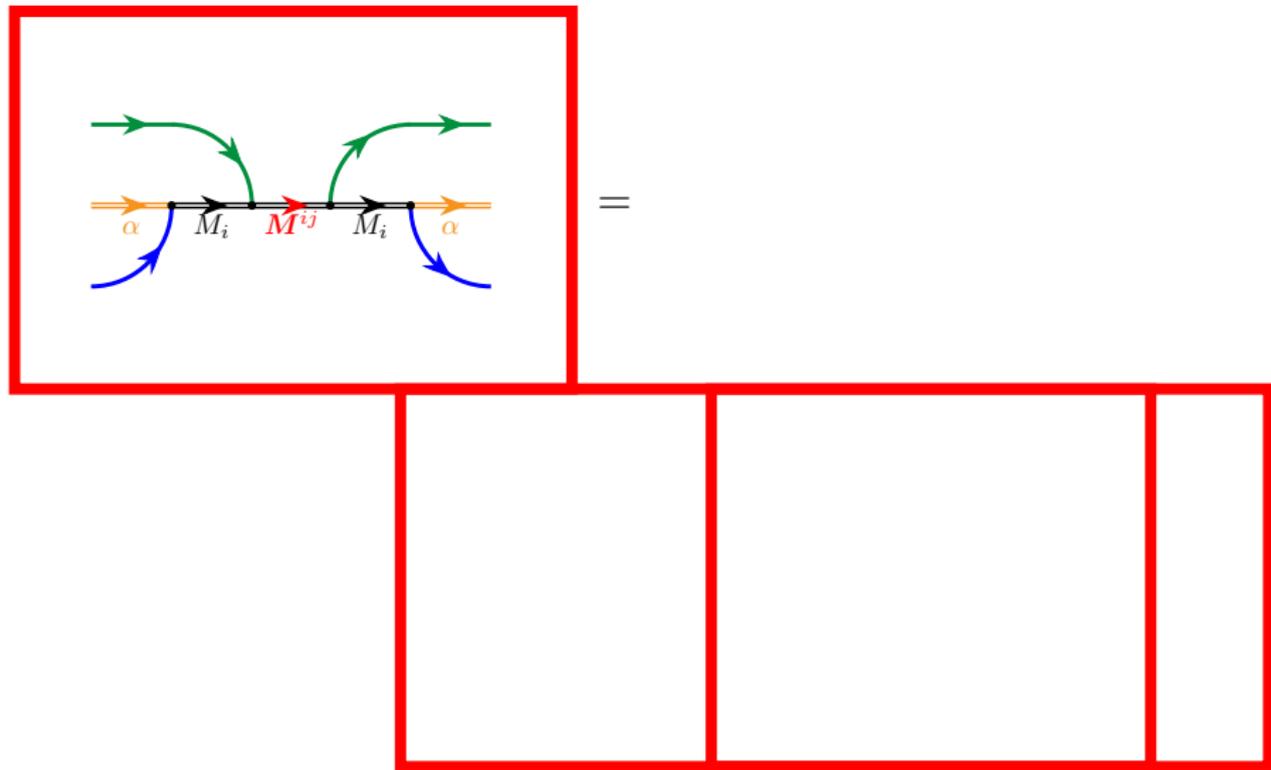
Consider:

$$\alpha \otimes \square \rightarrow M_i \otimes \square \rightarrow M^{ij}$$



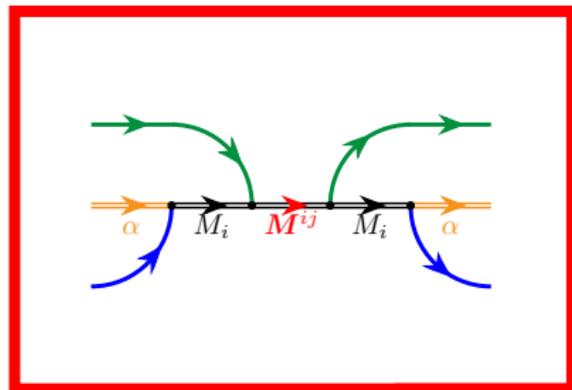
Consider:

$$\alpha \otimes \square \rightarrow M_i \otimes \square \rightarrow M^{ij}$$



Consider:

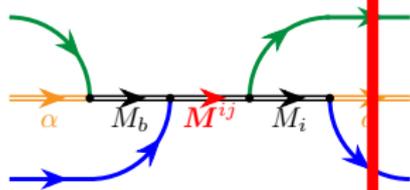
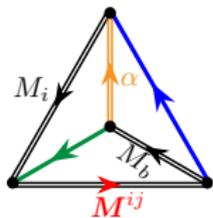
$$\alpha \otimes \square \rightarrow M_i \otimes \square \rightarrow M^{ij}$$



=

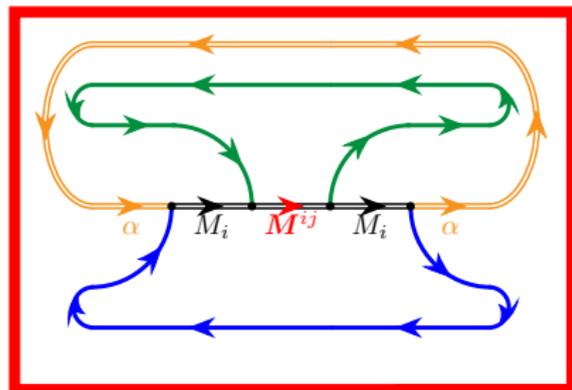
$$= \sum_b \frac{d_b}{d_b} \begin{array}{c} \text{Diagram with } M_b \text{ and } \alpha \end{array}$$

The diagram shows two circles. The left circle has an orange arrow labeled alpha at the bottom and a green arrow labeled M_b at the top. The right circle has a red arrow labeled M^{ij} at the bottom and a blue arrow labeled M_b at the top.



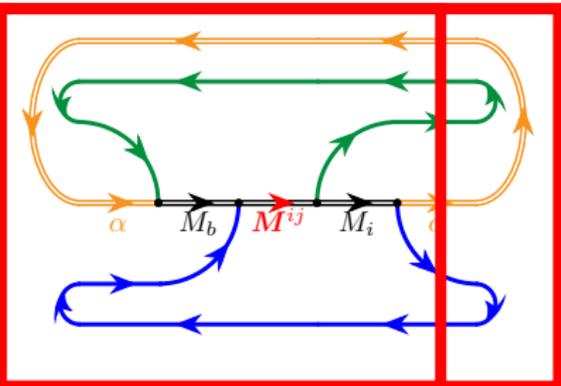
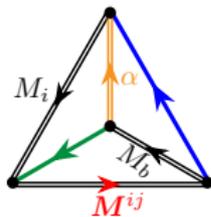
Consider:

$$\alpha \otimes \square \rightarrow M_i \otimes \square \rightarrow M^{ij}$$



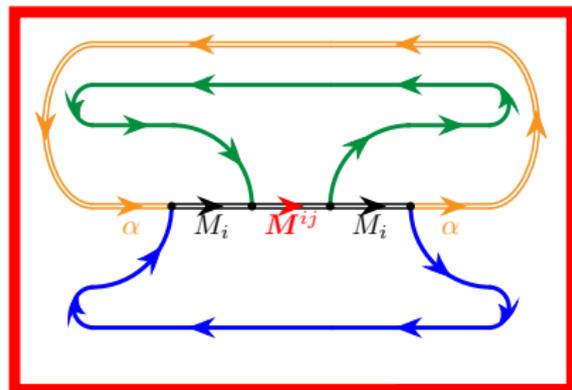
=

$$= \sum_b \frac{d_b}{d_b} \begin{array}{c} \text{Diagram with } M_b \text{ and } M^{ij} \end{array}$$



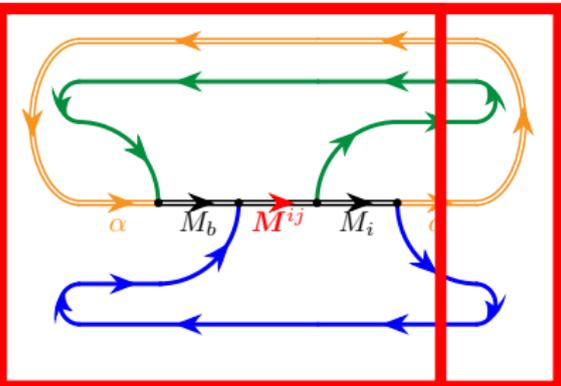
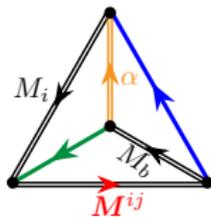
Consider:

$$\alpha \otimes \square \rightarrow M_i \otimes \square \rightarrow M^{ij}$$



=

$$= \sum_b \frac{d_b}{d_b} \begin{array}{c} \text{Diagram with two circles: the left one has an orange arrow labeled alpha and a blue arrow labeled M_b; the right one has a red arrow labeled M^{ij} and a blue arrow labeled M_b.} \end{array}$$

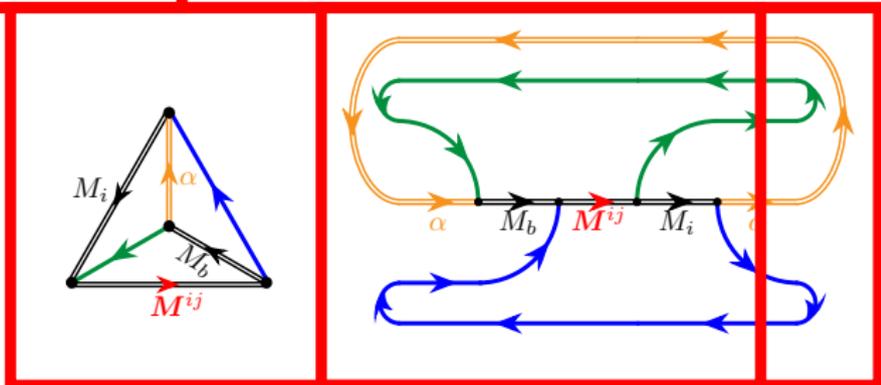


Consider:

$$\alpha \otimes \square \rightarrow M_i \otimes \square \rightarrow M^{ij}$$

$$\frac{\text{Diagram of } M^{ij} \text{ and } M_i}{d_i} = \text{Diagram of } \alpha \text{ and } M_i$$

$$= \sum_b \frac{d_b}{\text{Diagram of } M_b \text{ and } M^{ij}}$$

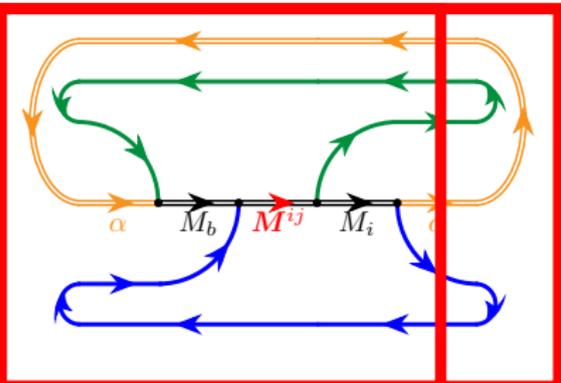
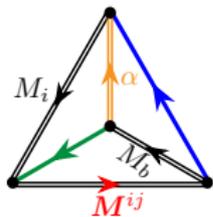


Consider:

$$\alpha \otimes \square \rightarrow M_i \otimes \square \rightarrow M^{ij}$$

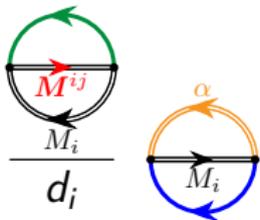
$$\frac{\begin{array}{c} \text{Green circle with } M^{ij} \\ \text{Blue circle with } \alpha \\ \hline M_i \end{array}}{d_i} = \begin{array}{c} \text{Blue circle with } \alpha \\ \hline M_i \end{array}$$

$$= \sum_b \frac{d_b}{\begin{array}{c} \text{Green circle with } M^{ij} \\ \text{Blue circle with } \alpha \\ \hline M_b \end{array}}$$



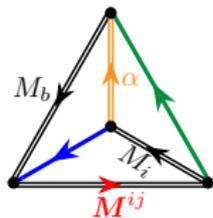
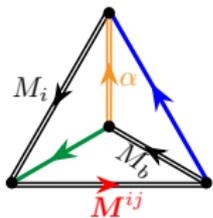
Consider:

$$\alpha \otimes \square \rightarrow M_i \otimes \square \rightarrow M^{ij}$$



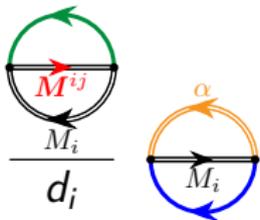
=

$$= \sum_b \frac{d_b}{\alpha}$$



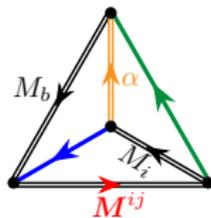
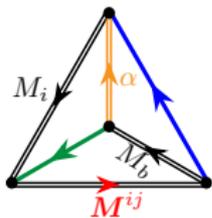
Consider:

$$\alpha \otimes \square \rightarrow M_i \otimes \square \rightarrow M^{ij}$$



=

$$= \sum_b \frac{d_b}{\alpha}$$



Consider:

$$\alpha \otimes \square \rightarrow M_i \otimes \square \rightarrow M^{ij}$$

$$\frac{\begin{array}{c} \text{Green circle with } M^{ij} \text{ arrow} \\ \text{Black circle with } M_i \text{ arrow} \end{array}}{d_i} = \begin{array}{c} \text{Orange circle with } \alpha \text{ arrow} \\ \text{Blue circle with } M_i \text{ arrow} \end{array}$$

$$= \sum_b \frac{d_b}{\begin{array}{c} \text{Green circle with } M^{ij} \text{ arrow} \\ \text{Black circle with } M_b \text{ arrow} \end{array} \begin{array}{c} \text{Orange circle with } \alpha \text{ arrow} \\ \text{Blue circle with } M_b \text{ arrow} \end{array}}$$

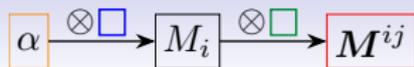
$$\left(\begin{array}{c} \text{Triangle with } M_b \text{ arrow} \\ \text{Triangle with } M_i \text{ arrow} \\ \text{Triangle with } M^{ij} \text{ arrow} \\ \text{Orange arrow } \alpha \end{array} \right)^2$$

Consider:

$$\alpha \otimes \square \rightarrow M_i \otimes \square \rightarrow M^{ij}$$

$$\frac{\begin{array}{c} \text{Green circle with } M^{ij} \text{ arrow} \\ \text{Black circle with } M_i \text{ arrow} \end{array}}{d_i} \otimes \begin{array}{c} \text{Orange circle with } \alpha \text{ arrow} \\ \text{Blue circle with } M_i \text{ arrow} \end{array} = \sum_b \frac{d_b}{\begin{array}{c} \text{Green circle with } M_b \text{ arrow} \\ \text{Orange circle with } \alpha \text{ arrow} \end{array} \otimes \begin{array}{c} \text{Black circle with } M_b \text{ arrow} \\ \text{Blue circle with } M^{ij} \text{ arrow} \end{array}}{\begin{array}{c} \text{Triangle with } M_b, M_i, M^{ij} \text{ arrows} \\ \text{Orange arrow } \alpha \end{array}}^2$$

Consider:



$$1 = d_i \sum_b d_b \left(\begin{array}{c} \text{Diagram} \end{array} \right)^2$$

A diagram of a triangle with three vertices. Three vectors originate from the vertices: a blue vector labeled M_b pointing from the top vertex to the bottom-left vertex, a red vector labeled M_i pointing from the bottom-left vertex to the bottom-right vertex, and a green vector labeled M^{ij} pointing from the bottom-right vertex to the top vertex. An orange vector labeled α points from the top vertex to the bottom-right vertex. The entire diagram is enclosed in large parentheses with a superscript 2.

Consider:

$$\alpha \otimes \square \rightarrow M_i \otimes \square \rightarrow M^{ij}$$

$$1 = (d_i)^2 \left(\begin{array}{c} \text{Triangle with } \alpha \text{ (orange), } M_i \text{ (black), } M_i \text{ (black), } M^{ij} \text{ (red)} \end{array} \right)^2 + d_i d_j \left(\begin{array}{c} \text{Triangle with } \alpha \text{ (orange), } M_j \text{ (black), } M_i \text{ (black), } M^{ij} \text{ (red)} \end{array} \right)^2$$

Consider:

$$\alpha \otimes \square \rightarrow M_i \otimes \square \rightarrow M^{ij}$$

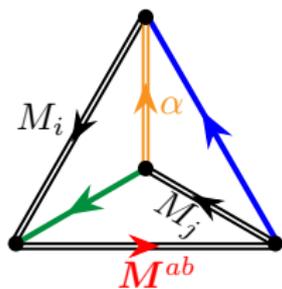
$$1 = (d_i)^2 \left(\begin{array}{c} \text{triangle with } \alpha \text{ (orange), } M_i \text{ (black), } M_i \text{ (black), } M^{ij} \text{ (red)} \end{array} \right)^2 + d_i d_j \left(\begin{array}{c} \text{triangle with } \alpha \text{ (orange), } M_j \text{ (black), } M_i \text{ (black), } M^{ij} \text{ (red)} \end{array} \right)^2$$

$$0 = d_i \begin{array}{c} \text{triangle with } \alpha \text{ (orange), } M_i \text{ (black), } M_i \text{ (black), } M^{ij} \text{ (red)} \end{array} + d_j \begin{array}{c} \text{triangle with } \alpha \text{ (orange), } M_j \text{ (black), } M_j \text{ (black), } M^{ij} \text{ (red)} \end{array}$$

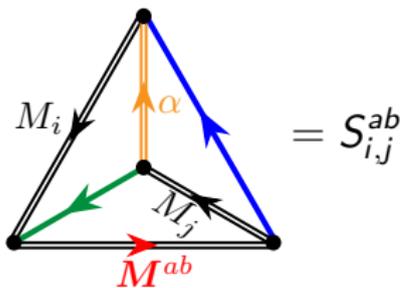
$$\frac{1}{d_\alpha} = \sum_{M^{ab}} d_{ab} \left(\begin{array}{c} \text{triangle with } \alpha \text{ (orange), } M_j \text{ (black), } M_i \text{ (black), } M^{ab} \text{ (red)} \end{array} \right)^2$$

$$1 = \sum_b d_{ib} \begin{array}{c} \text{triangle with } \alpha \text{ (orange), } M_i \text{ (black), } M_i \text{ (black), } M^{ib} \text{ (red)} \end{array}$$

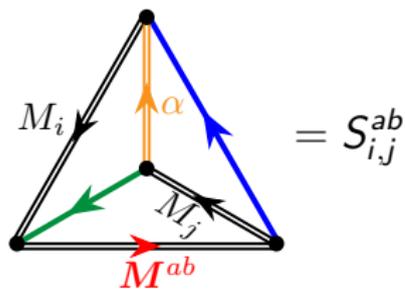
Solving for closed form expressions



Solving for closed form expressions



Solving for closed form expressions



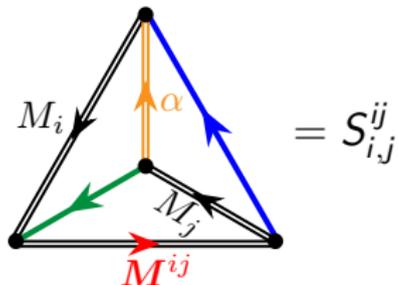
$$1 = (d_i)^2 (S_{i,i}^{ij})^2 + d_i d_j (S_{i,j}^{ij})^2 \quad (1)$$

$$0 = d_i S_{i,i}^{ij} S_{i,j}^{ij} + d_j S_{i,j}^{ij} S_{j,j}^{ij} \quad (2)$$

$$\frac{1}{d_\alpha} = \sum_{M^{ab}} d_{ab} (S_{i,j}^{ab})^2 \quad (3)$$

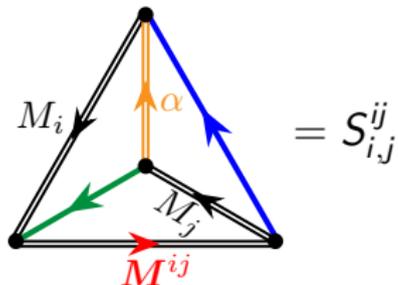
$$1 = \sum_b d_{ib} S_{i,i}^{ib} \quad (4)$$

Solving for closed form expressions



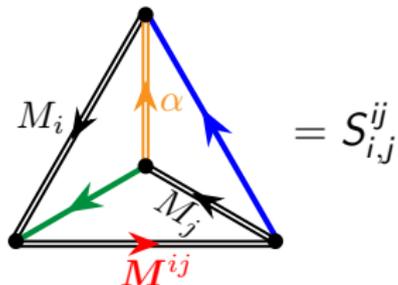
eq. (1): $1 = (d_i)^2(S_{i,i}^{ij})^2 + d_i d_j (S_{i,j}^{ij})^2$

Solving for closed form expressions



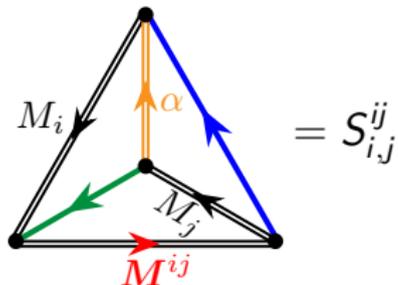
eq. (1): $1 = (d_i)^2 (S_{i,i}^{ij})^2 + d_i d_j (S_{i,j}^{ij})^2 \xrightarrow{M^{ij} \rightarrow M^{ii}}$

Solving for closed form expressions



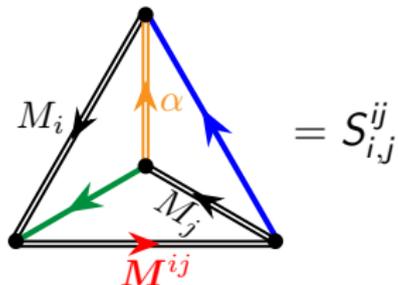
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Solving for closed form expressions



$$\text{eq. (1): } 1 = (d_i)^2 (S_{i,i}^{ij})^2 + d_i d_j (S_{i,j}^{ij})^2 \xrightarrow{M^{ij} \rightarrow M^{ii}} 1 = (d_i)^2 (S_{i,i}^{ii})^2$$
$$\iff S_{i,i}^{ii} = \pm \frac{1}{d_i}$$

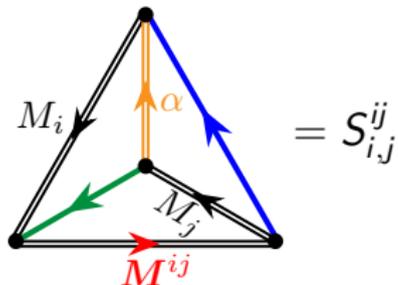
Solving for closed form expressions



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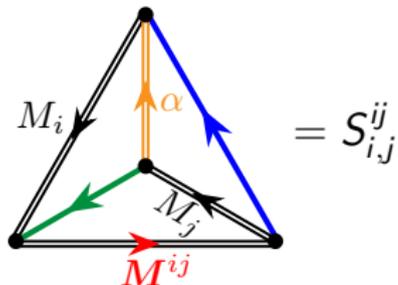
Solving for closed form expressions



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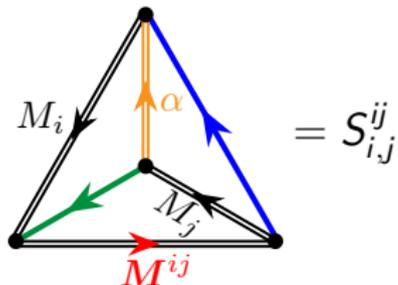
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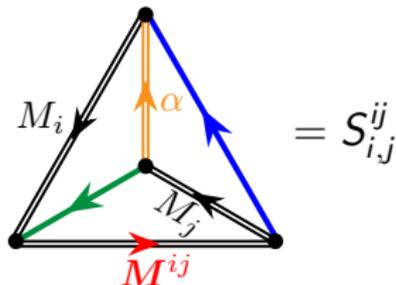
Solving for closed form expressions



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Solving for closed form expressions

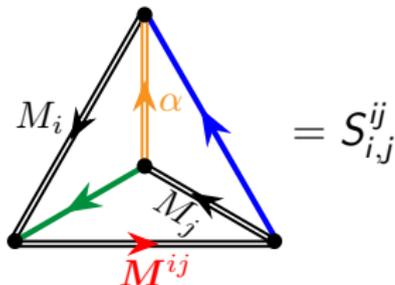


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Solving for closed form expressions

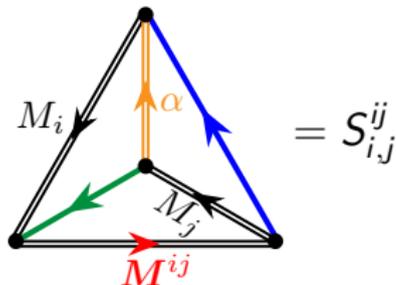


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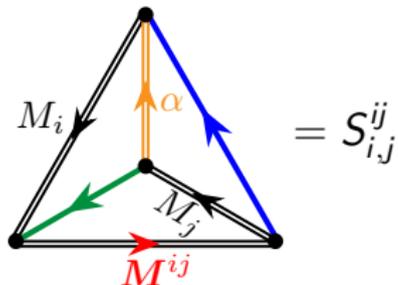
Solving for closed form expressions



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Solving for closed form expressions



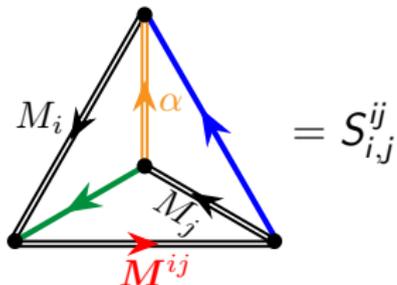
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$(S_{i,j}^{ij})^2 = \frac{1}{d_\alpha d_{ij}}$

$\longrightarrow 1 = (d_i)^2 (S_{i,i}^{ij})^2 + \frac{d_i d_j}{d_\alpha d_{ij}}$

Solving for closed form expressions



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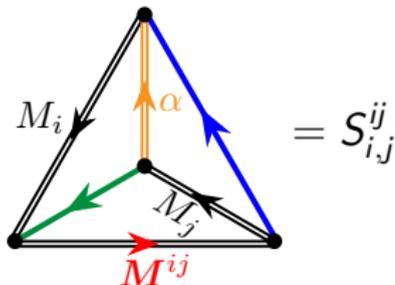
$$(S_{i,j}^{ij})^2 = \frac{1}{d_\alpha d_{ij}} \rightarrow$$

$$1 = (d_i)^2 (S_{i,i}^{ij})^2 + \frac{d_i d_j}{d_\alpha d_{ij}}$$

\Leftrightarrow

$$S_{i,i}^{ij} = \pm \frac{1}{d_i} \sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}}$$

Solving for closed form expressions



- $S_{i,i}^{ij} = \pm \frac{1}{d_i}$
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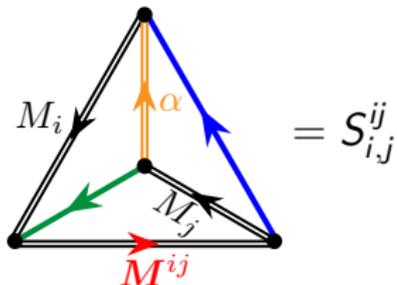
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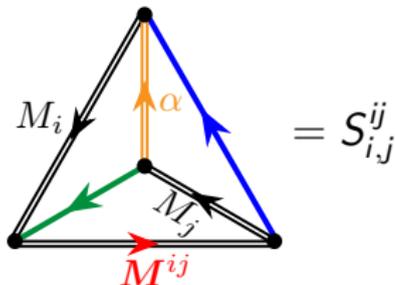
Solving for closed form expressions



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eq. (2): $0 = d_i S_{i,i}^{ij} S_{i,j}^{ij} + d_j S_{j,j}^{ij} S_{i,j}^{ij} = (d_i S_{i,i}^{ij} + d_j S_{j,j}^{ij}) S_{i,j}^{ij}$

Solving for closed form expressions

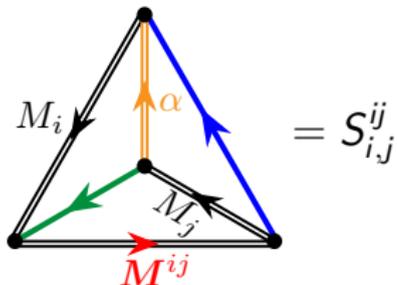


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$$\xrightarrow{S_{i,j}^{ij} \neq 0} d_j S_{j,j}^{ij} = -d_i S_{i,i}^{ij}$$

Solving for closed form expressions

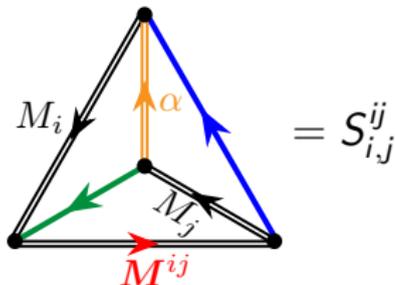


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Solving for closed form expressions



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Closed form expressions

$$S_{i,i}^{ii} = \pm \frac{1}{d_i} \quad d_i S_{i,i}^{ij} = \pm \sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}} = -d_j S_{j,j}^{ij} \quad S_{i,j}^{ij} = \pm \frac{1}{\sqrt{d_\alpha d_{ij}}}$$

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symmetric

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lin. relation ($N \leq 3$)

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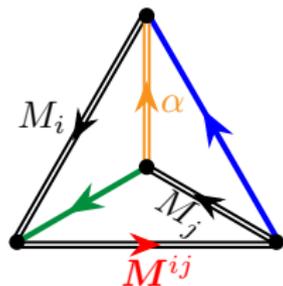
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vertex def.

Closed form expressions

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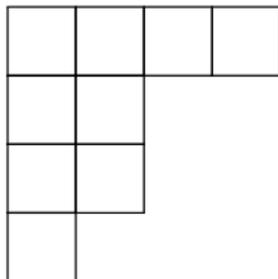
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vertex def.

factors: (add N)

0	1	2	3
-1	0		
-2	-1		
-3			

Closed form expressions

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hook lengths:

7	5	2	1
4	2		
3	1		
1			

Closed form expressions

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7	5	2	1
4	2		
3	1		
1			

$$\dim = \left[\frac{N}{7} \frac{(N+1)}{5} \frac{(N+2)}{2} (N+3) \right] \left[\frac{(N-1)}{4} \frac{N}{2} \right] \left[\frac{(N-2)}{3} (N-1) \right] [N-3]$$

Closed form expressions

$$S_{i,i}^{ii} = +\frac{1}{d_i}$$

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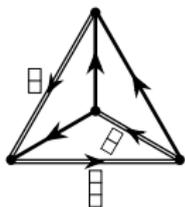
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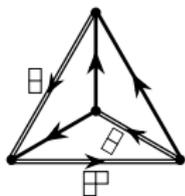
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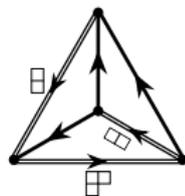
Exceptions containing  : [M. Sjö Dahl & J. Thorén (2018)]



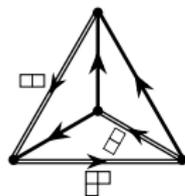
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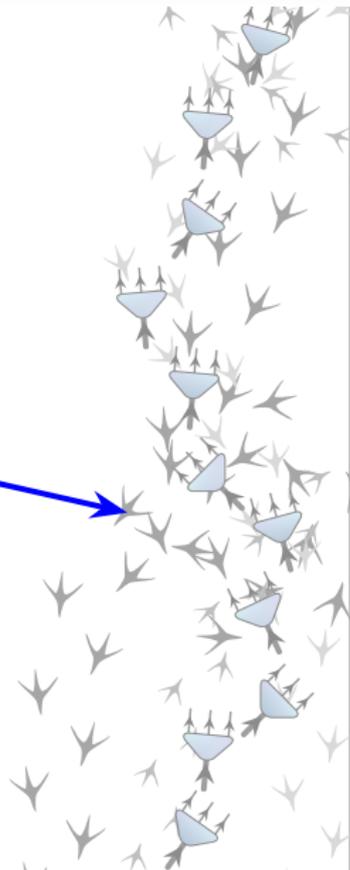
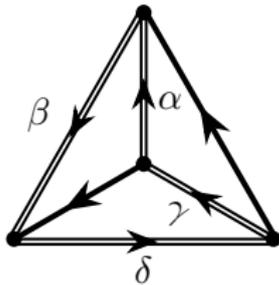
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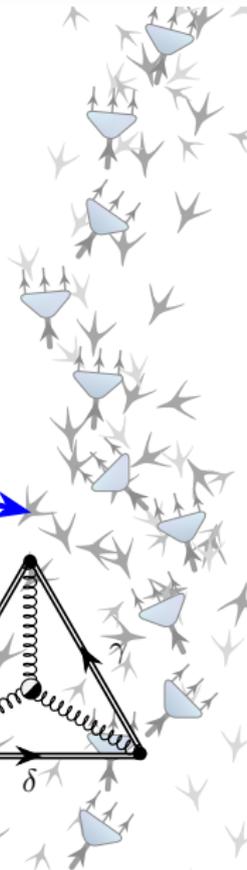
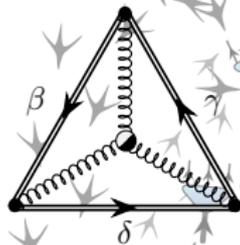
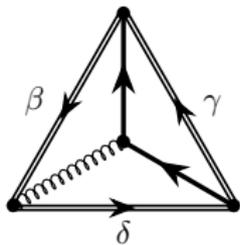
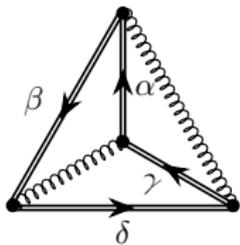
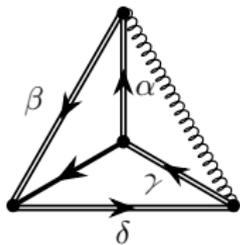
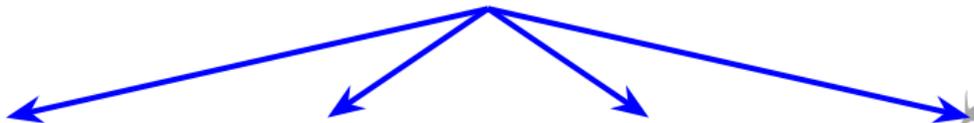
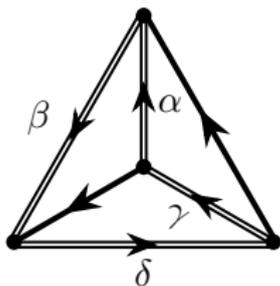
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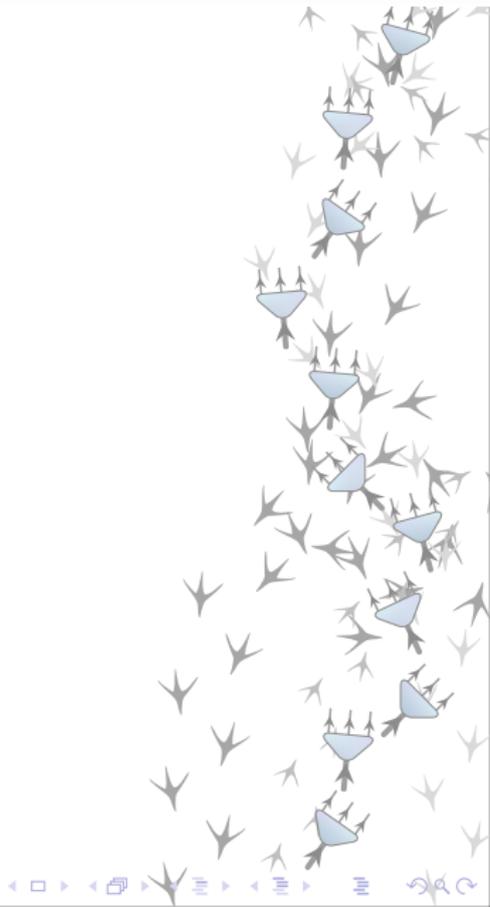
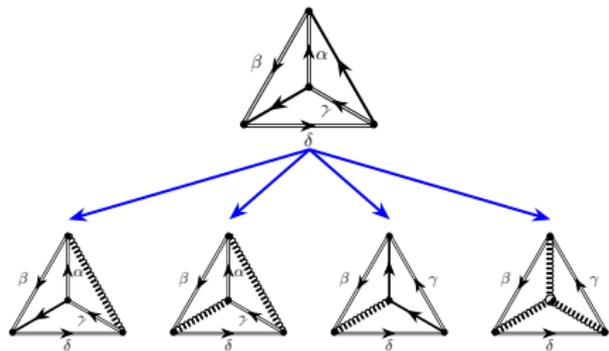
What's next?



What's next?



What's next?



What's next?

