Loops, birdtracks and $6j$s — exploring QCD colour structure

Judith M. Alcock-Zeilinger

S. Keppeler, S. Plätzer and M. Sjödahl (arXiv:2209.15013)

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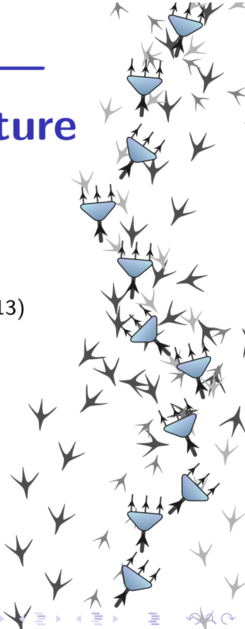
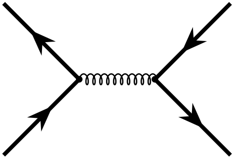
ESI

Erwin Schrödinger International Institute
for Mathematics and Physics



Exchanging partons: SU(N) color structure



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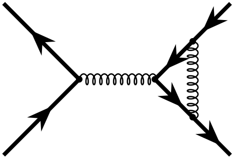

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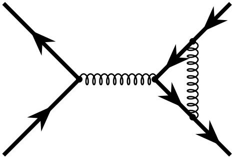
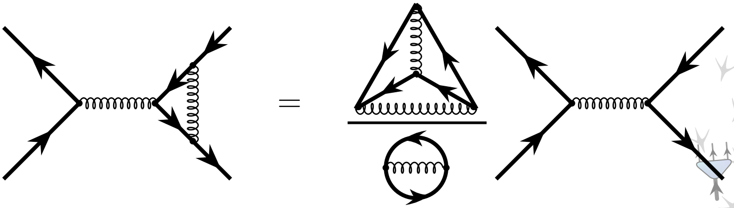


Factorization: consider color structure only

Exchanging partons: SU(N) color structure



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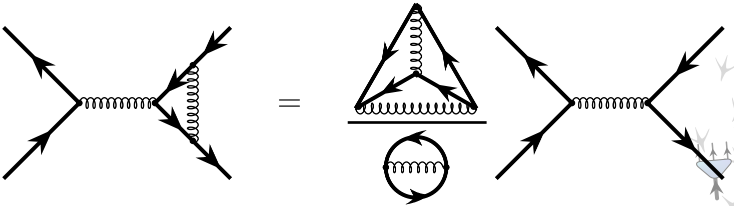

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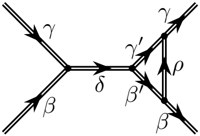

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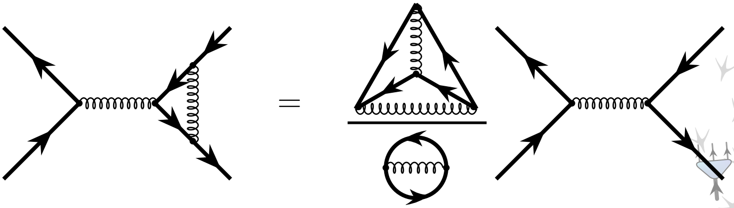

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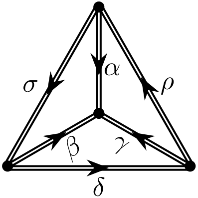
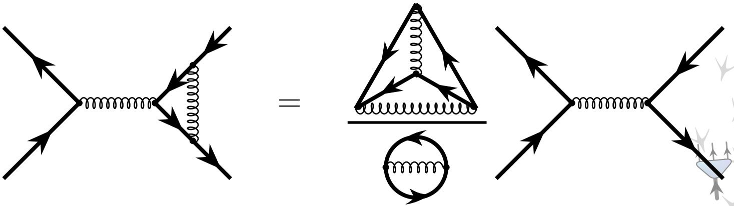


Exchanging partons: SU(N) color structure



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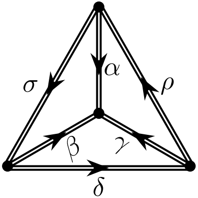
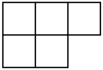


- β, γ, δ, . . . irreps of SU(N)
- labeled by Young diagrams, e.g.



Outline

- Birdtracks $\longrightarrow$ Wigner-6j symbols
- Wigner-6j symbols $\longrightarrow$ 2 quarks
- Calculation strategy:
    1. derive system of equations
    2. solve system

Birdtracks in a 

fundamental representation:

$\delta^b{}_a =$ 

general representation α:

$[\delta_\alpha]^b{}_a =$ 

P. Cvitanović:
"Group Theory - Birdtracks, Lie's and Exceptional Groups"

Birdtracks in a 

fundamental representation:

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$\delta^a{}_a =$  $= N$

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$= d_\alpha$

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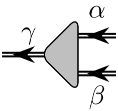
Clebsch-Gordan coefficient (suppressing state labels)

$\alpha \otimes \beta \to \gamma :$ 

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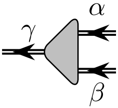
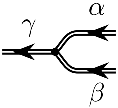
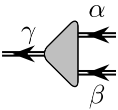
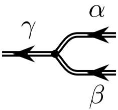
$\alpha \otimes \beta \to \gamma$: 

$=: \frac{1}{\sqrt{a_\gamma}}$ 

Birdtracks in a 

$\alpha \otimes \beta \to \gamma :$ 

$=: \frac{1}{\sqrt{a_\gamma}}$ 

Birdtracks in a 

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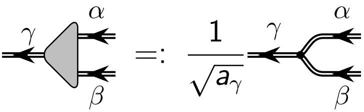
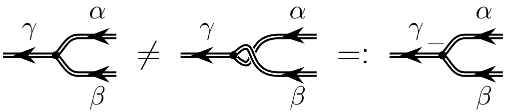
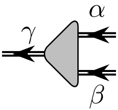
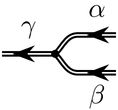


Line ordering:

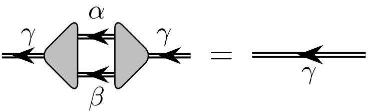


not a problem if $\gamma \neq \alpha \neq \beta \neq \gamma$

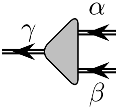
$\alpha \otimes \beta \to \gamma$: 

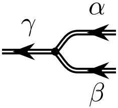
$=: \frac{1}{\sqrt{a_\gamma}}$ 

Normalization:



Birdtracks in a 

$\alpha \otimes \beta \to \gamma :$ 

$=: \dfrac{1}{\sqrt{a_\gamma}}$ 

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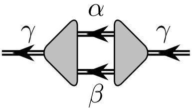
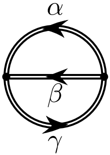


$=$ 

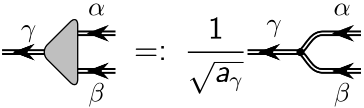
$\Longrightarrow \quad a_\gamma =$ 

$\overline{}$

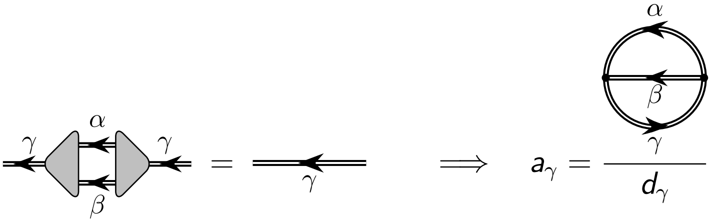
d_γ

Birdtracks in a 

$\alpha \otimes \beta \to \gamma$:



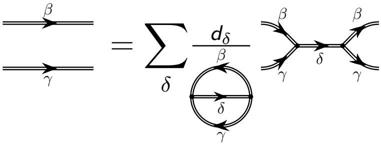
Normalization:



3*j*s can be chosen $\longrightarrow$ convenient: $3j = 1$

Reducing loops to trees

Completeness relation:



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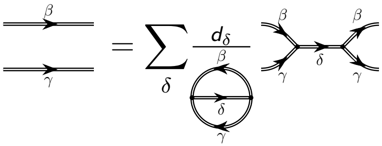
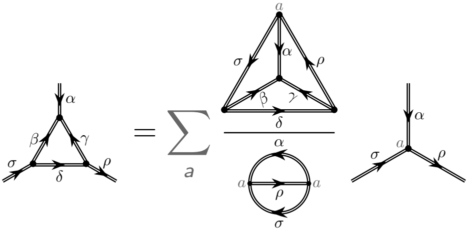


Vertex correction:



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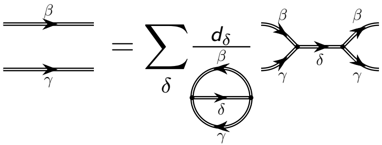

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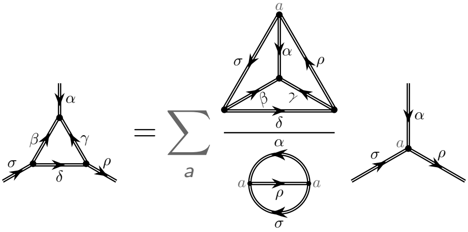

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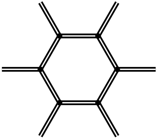

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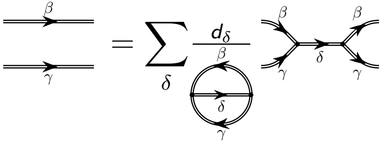

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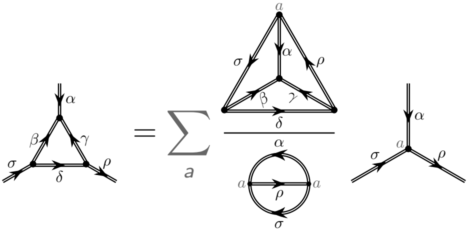

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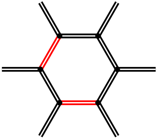

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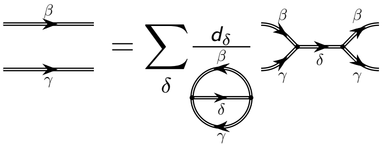

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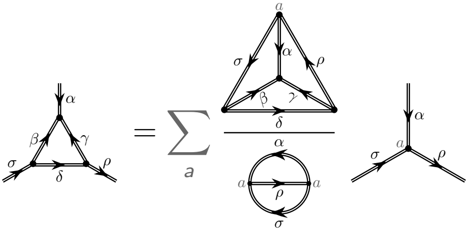

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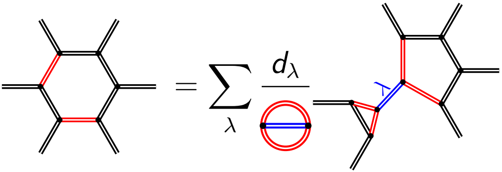

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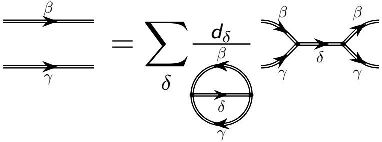

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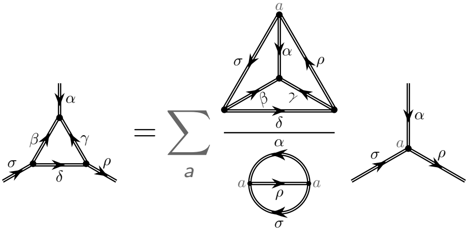

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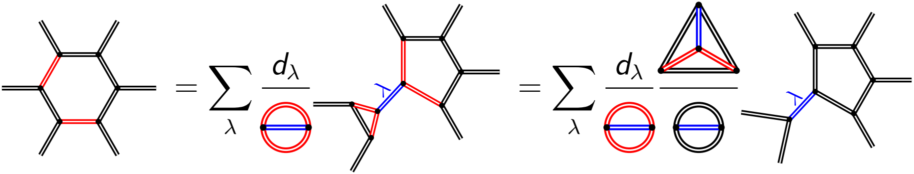
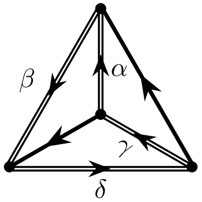
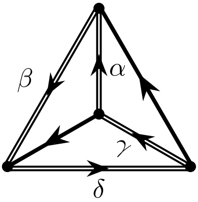


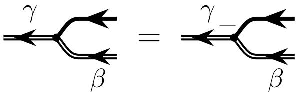
6*j*s with 2 quark lines



6*j*s with 2 quark lines

Line ordering: *all* irreps distinct





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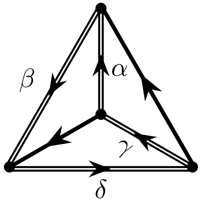

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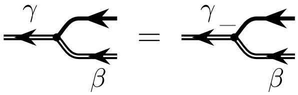
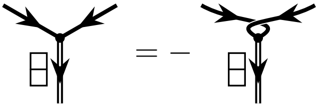
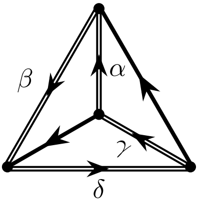


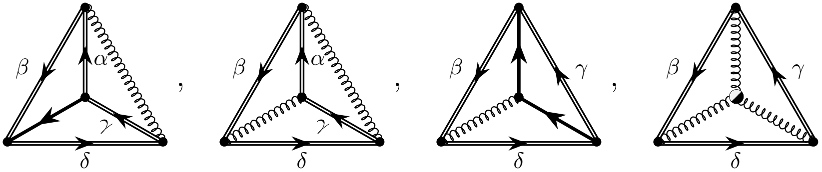
Exception:

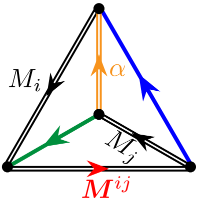


6*j*s with 2 quark lines



M. Sjödahl & J. Thorén (2018):





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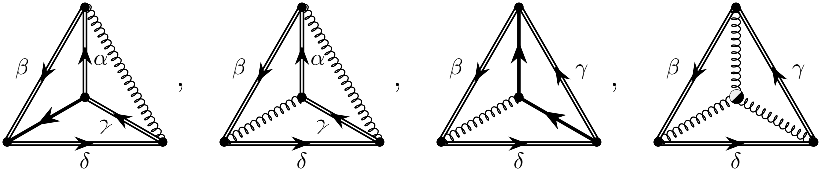

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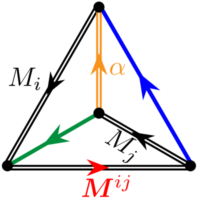
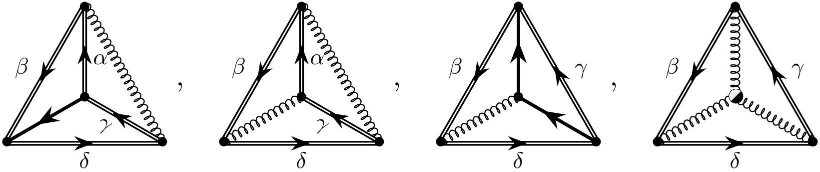
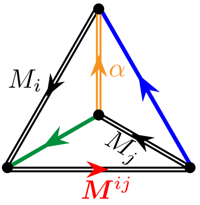


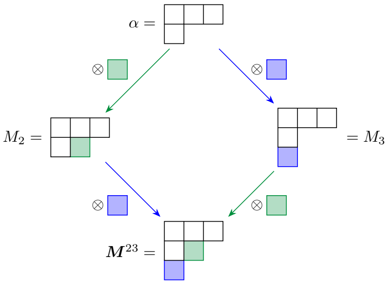
- **fix** α: a Young diagram
- $M_{i(j)}$: add $\square$ in row $i(j)$ of α
- $\boldsymbol{M}^{ij}$: add $\square$ in row j of M_i

M. Sjödahl & J. Thorén (2018):



6js with 2 quark lines





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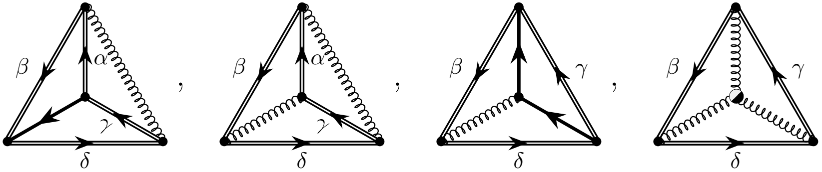
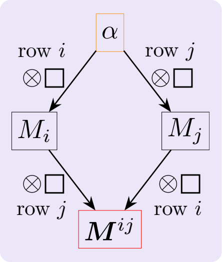
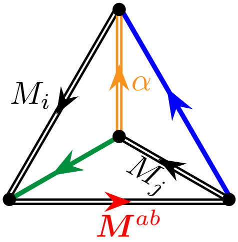


Properties: fixed α, given i, j with $i \neq j$





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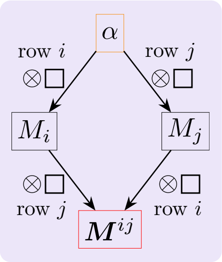
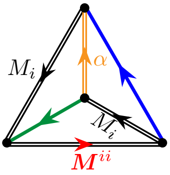


add $\square$ to **same** row:



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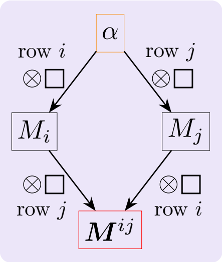
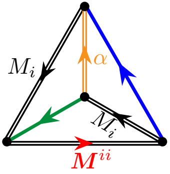
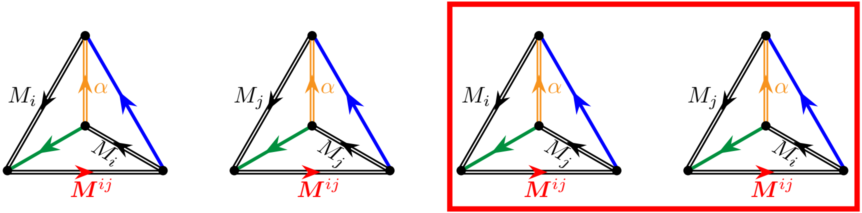


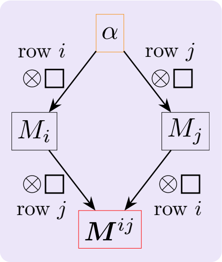
add $\square$ to **same** row:



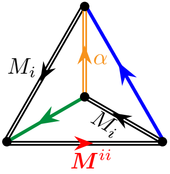
add $\square$ to **different** rows:



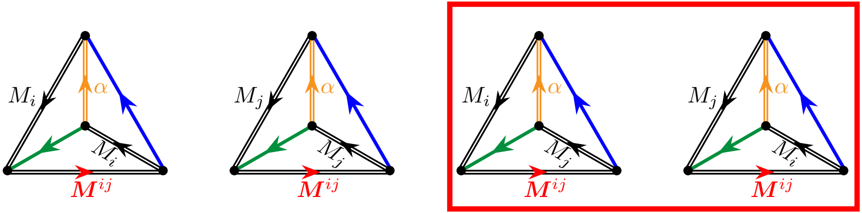
Properties: fixed α, given i, j with $i \neq j$



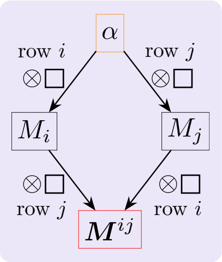
add □ to **same** row:



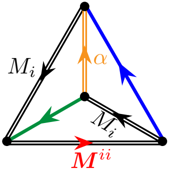
add □ to **different** rows:



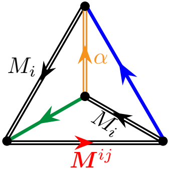
Properties: fixed α, given i, j with $i \neq j$

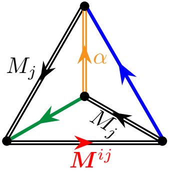


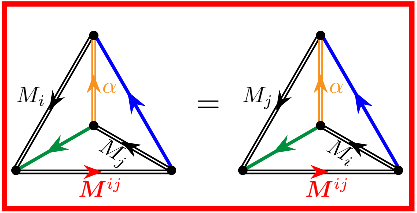
add $\square$ to **same** row:



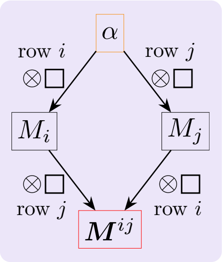
add $\square$ to **different** rows:



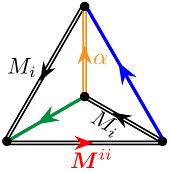




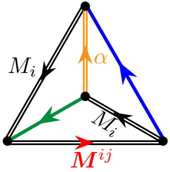
Properties: fixed α, given i, j with $i \neq j$

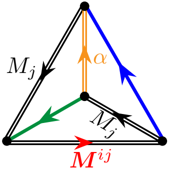


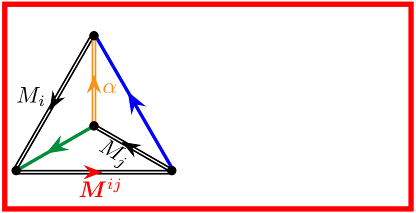
add $\square$ to **same** row:



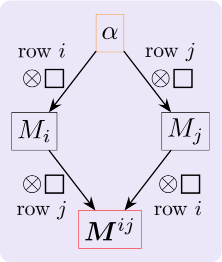
add $\square$ to **different** rows:



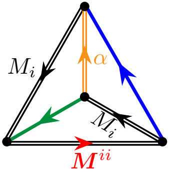




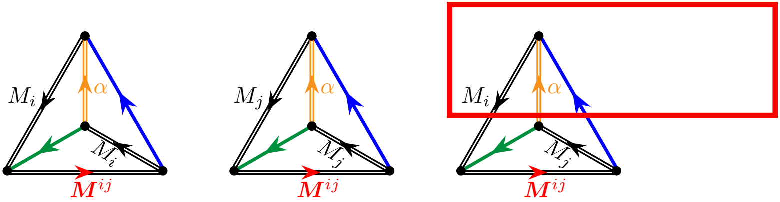
Properties: fixed α, given i, j with $i \neq j$



add □ to **same** row:

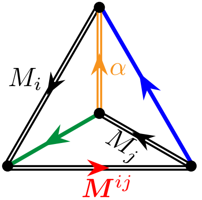


add □ to **different** rows:

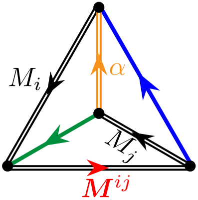


4 distinct $6j$ symbols

Calculating 6j with 2 quark lines

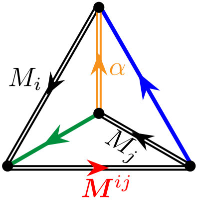


Calculating 6j with 2 quark lines



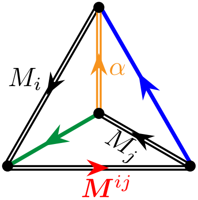
- Clebsches

Calculating 6j with 2 quark lines



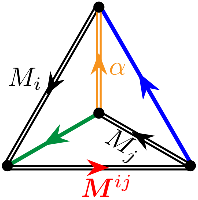
- ~~Clebsches~~

Calculating 6j with 2 quark lines



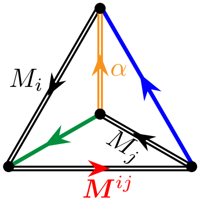
- ~~Clebsches~~
- Projectors

Calculating 6j with 2 quark lines



- ~~Clebsches~~
- ~~Projectors~~

Calculating 6j with 2 quark lines

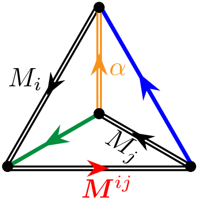


- ~~Clebsches~~
- ~~Projectors~~

Strategy:

- $\alpha \xrightarrow{\otimes \square} M_i \xrightarrow{\otimes \square} \boldsymbol{M}^{ij}$

Calculating 6j with 2 quark lines

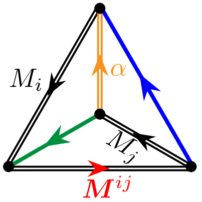


- ~~Clebsches~~
- ~~Projectors~~

Strategy:

- $\alpha \xrightarrow{\otimes \square} M_i \xrightarrow{\otimes \square} \boldsymbol{M}^{ij}$
- Construct birdtrack operators obeying hierarchy

Calculating 6j with 2 quark lines

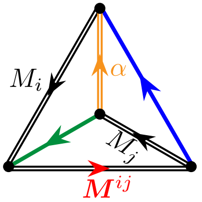


- ~~Clebsches~~
- ~~Projectors~~

Strategy:

- $\alpha \xrightarrow{\otimes \square} M_i \xrightarrow{\otimes \square} \boldsymbol{M}^{ij}$
- Construct birdtrack operators obeying hierarchy
- Completeness relation & vertex correction

Calculating 6j with 2 quark lines

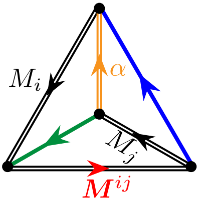


- ~~Clebsches~~
- ~~Projectors~~

Strategy:

- $\alpha \xrightarrow{\otimes \square} M_i \xrightarrow{\otimes \square} \boldsymbol{M}^{ij}$
- Construct birdtrack operators obeying hierarchy
- Completeness relation & vertex correction
- System of equation for $6js$

Calculating 6j with 2 quark lines

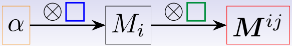


- ~~Clebsches~~
- ~~Projectors~~

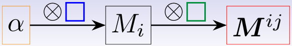
Strategy:

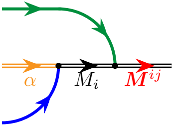
- $\alpha \xrightarrow{\otimes \square} M_i \xrightarrow{\otimes \square} \boldsymbol{M}^{ij}$
- Construct birdtrack operators obeying hierarchy
- Completeness relation & vertex correction
- System of equation for 6js
- Solve system

Consider:

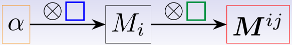


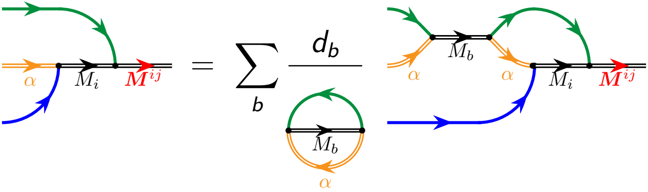
Consider:


α
$\otimes$
M_i
$\otimes$
$\boldsymbol{M}^{ij}$


α
M_i
$\boldsymbol{M}^{ij}$

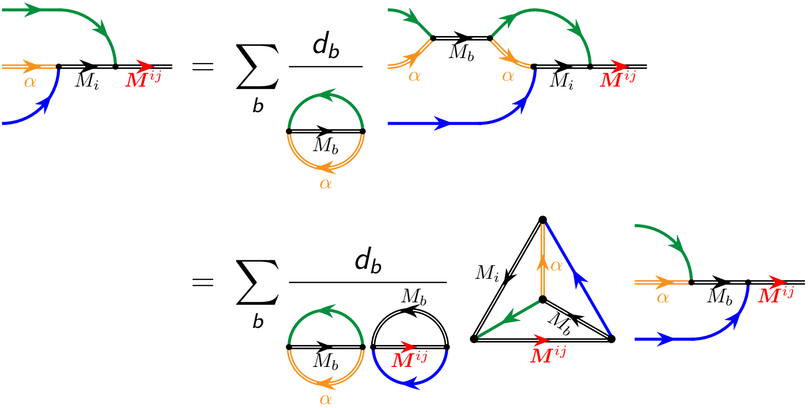
Consider:



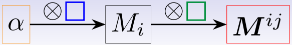


Consider:

$$\alpha \xrightarrow{\otimes \square} M_i \xrightarrow{\otimes \square} \boldsymbol{M}^{ij}$$



Consider:



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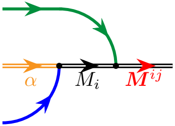

$=$

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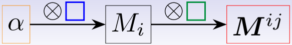
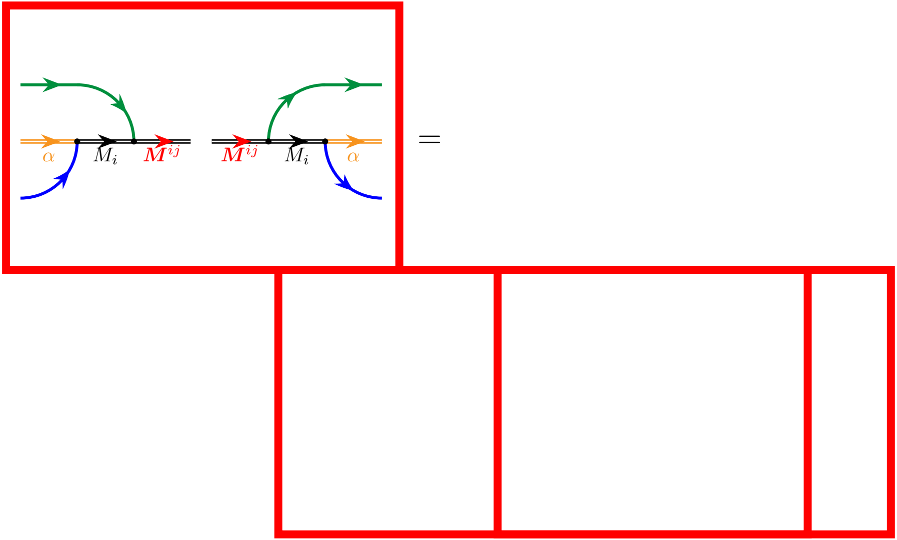
Consider:

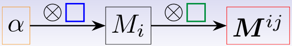

α
$\otimes$
M_i
$\otimes$
$\boldsymbol{M}^{ij}$

.

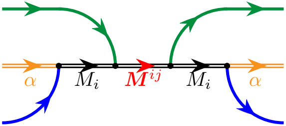

α
M_i
$\boldsymbol{M}^{ij}$
$\boldsymbol{M}^{ij}$
M_i
α
$=$

.

Consider:


α
$\otimes\square$
M_i
$\otimes\square$
$\boldsymbol{M}^{ij}$

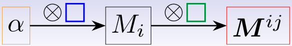
.


α
M_i
$\boldsymbol{M}^{ij}$
M_i
α

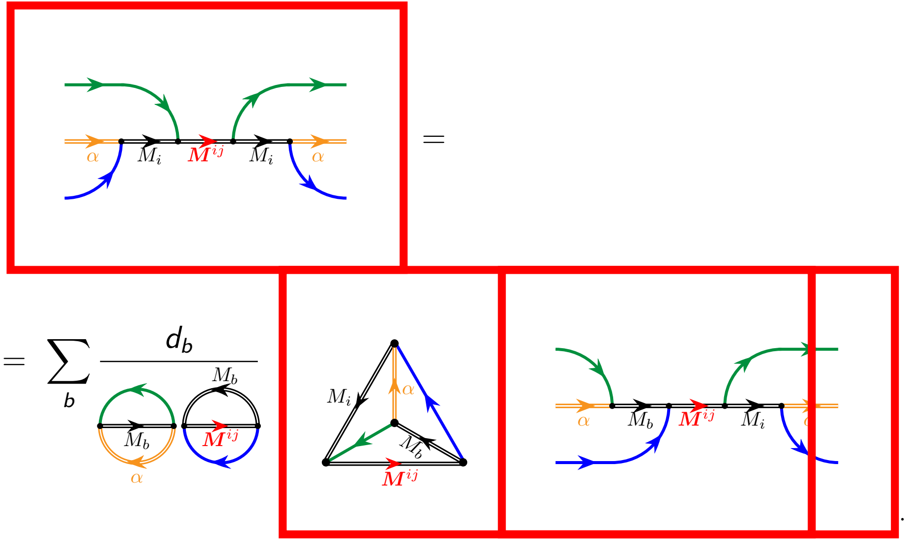
$=$

.

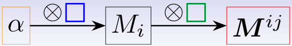
Consider:



α
M_i
M^{ij}



α
M_i
M^{ij}
M_i
α
$=$
$= \sum_b \frac{d_b}{}$
M_b
M_b
α
M^{ij}
M_i
α
M_b
M^{ij}
α
M_b
M^{ij}
M_i

Consider: 

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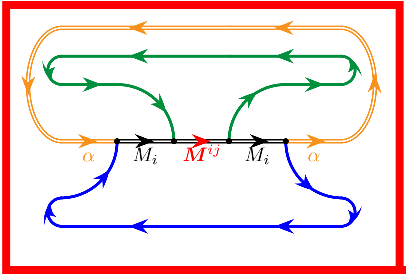
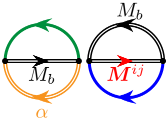
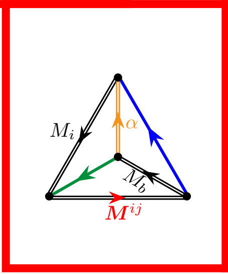
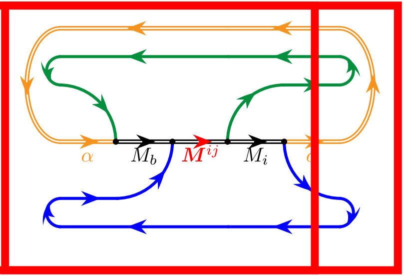


$=$

$$= \sum_b \frac{d_b}{}$$







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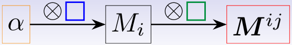
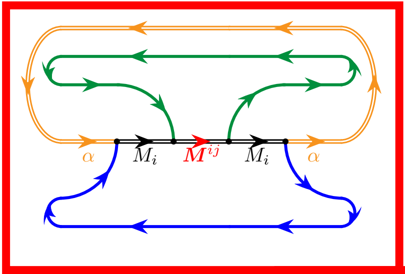
Consider:



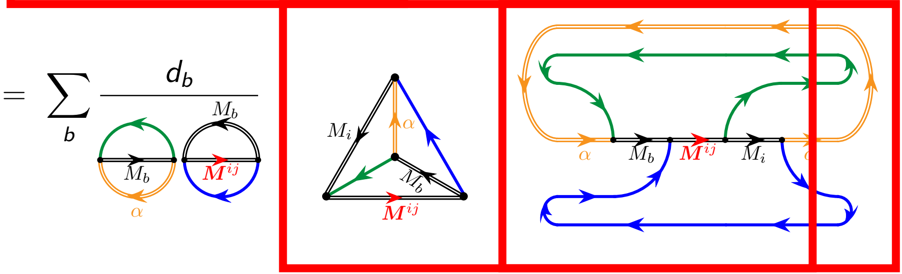
α
$\otimes$
M_i
$\otimes$
M^{ij}

.



α
M_i
M^{ij}
M_i
α

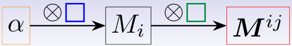
$=$

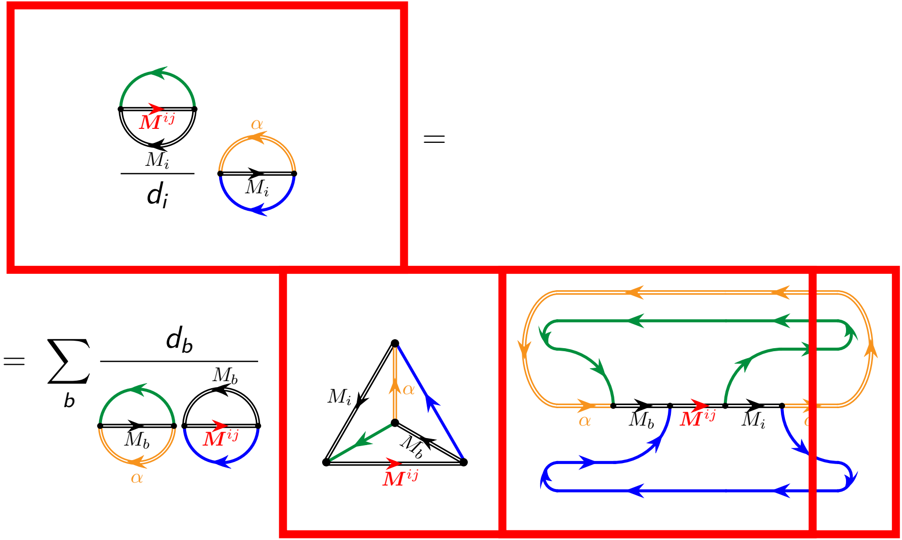


$= \sum_b$
d_b
M_b
M_b
α
M^{ij}
M_i
α
M_b
M^{ij}
α
M_b
M^{ij}
M_i

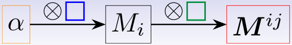
.

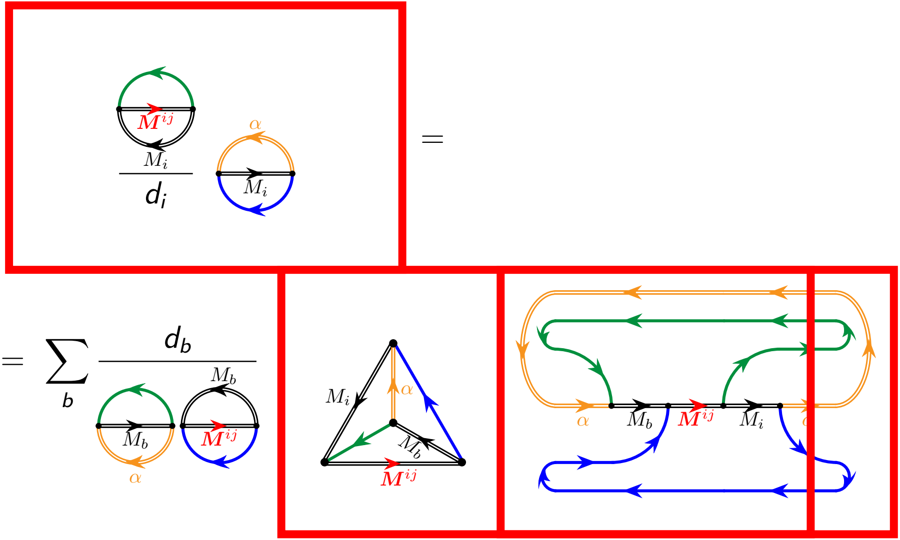
Consider:



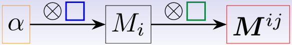


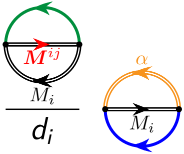
Consider:





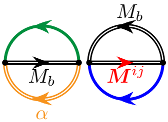
Consider:

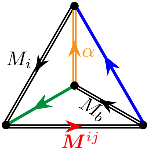


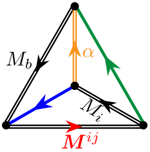


$=$

$= \sum_b \dfrac{d_b}{}$







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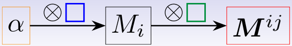
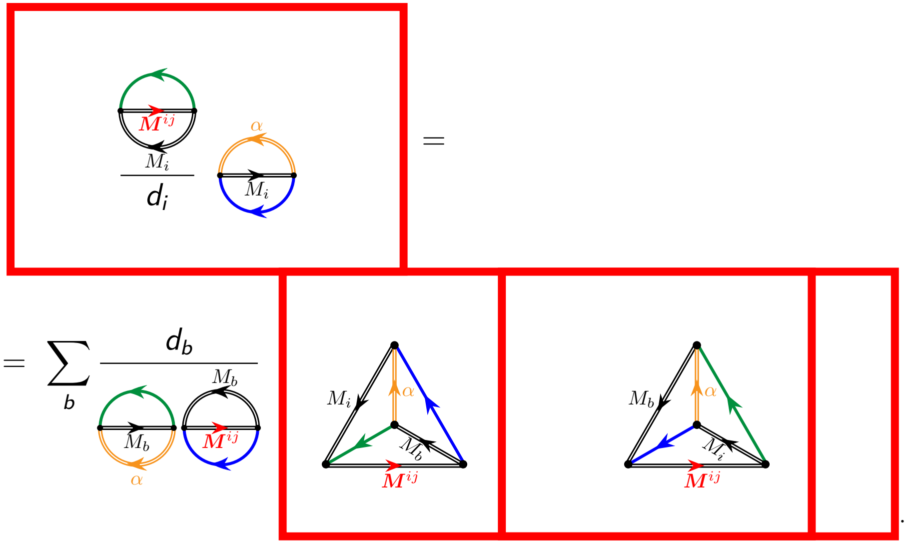
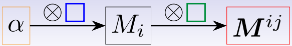
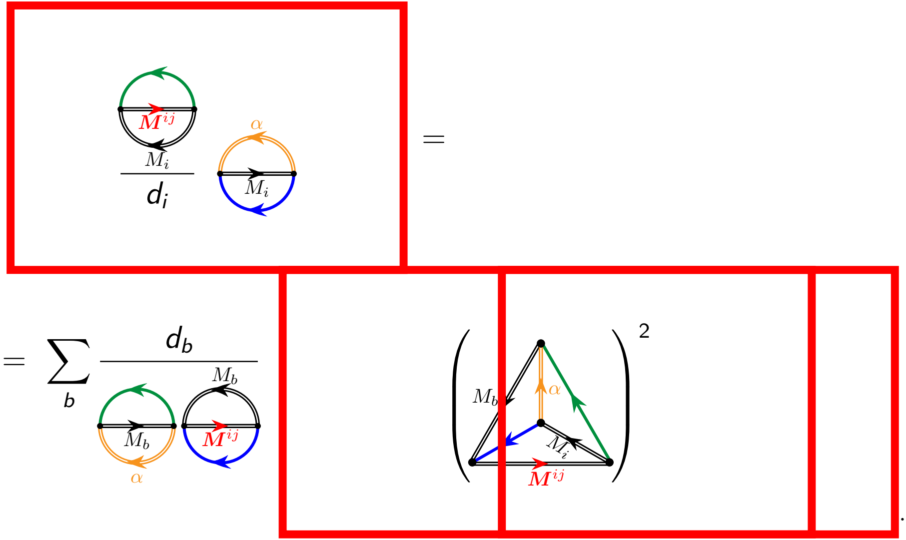
Consider:





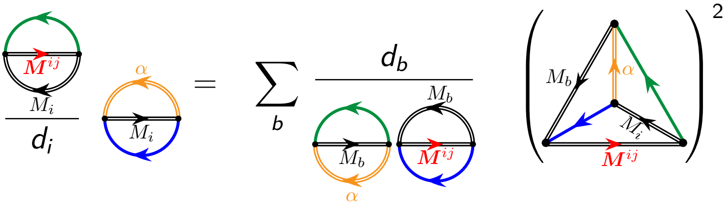
Consider:





Consider:

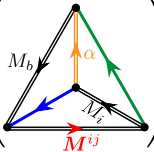
$\alpha \xrightarrow{\otimes \square} M_i \xrightarrow{\otimes \square} \boldsymbol{M}^{ij}$



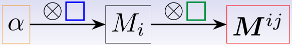
Consider: $\alpha \xrightarrow{\otimes \square} M_i \xrightarrow{\otimes \square} \boldsymbol{M}^{ij}$

.   .

$$1 \quad = \quad d_i \sum_b \quad d_b \left(\right)^2$$



Consider:



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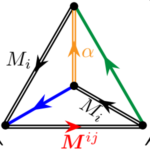
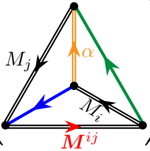
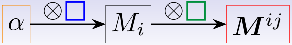
.

$$1 = (d_i)^2 \left(\right)^2 + d_i d_j \left(\right)^2$$

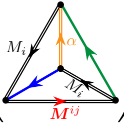




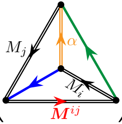
Consider:



$1 = (d_i)^2$ (

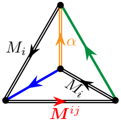


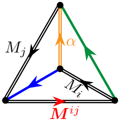
)^2 $+ \, d_i d_j$ (



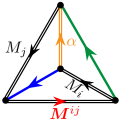
)^2

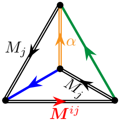
$0 = d_i$



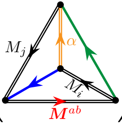


$+ \, d_j$



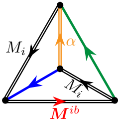


$\frac{1}{d_\alpha} = \sum_{M^{ab}} d_{ab}$ (

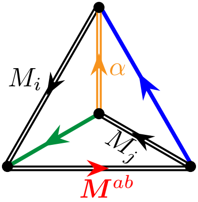


)^2

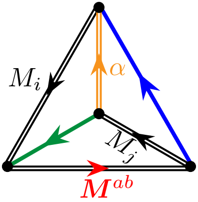
$1 = \sum_b d_{ib}$



Solving for closed form expressions

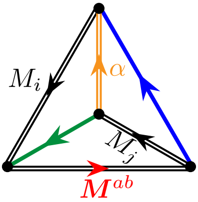


Solving for closed form expressions



$$= S_{i,j}^{ab}$$

Solving for closed form expressions



$$= S^{ab}_{i,j}$$

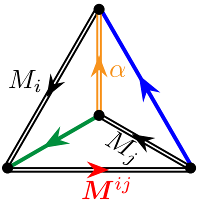
$$1 = (d_i)^2 (S^{ij}_{i,i})^2 + d_i d_j (S^{ij}_{i,j})^2 \quad (1)$$

$$0 = d_i S^{ij}_{i,i} S^{ij}_{i,j} + d_j S^{ij}_{i,j} S^{ij}_{j,j} \quad (2)$$

$$\frac{1}{d_\alpha} = \sum_{\boldsymbol{M}^{ab}} d_{ab} (S^{ab}_{i,j})^2 \quad (3)$$

$$1 = \sum_{b} d_{ib} S^{ib}_{i,i} \quad (4)$$

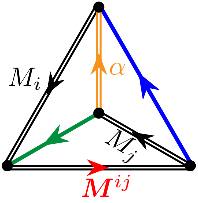
Solving for closed form expressions



$$= S^{ij}_{i,j}$$

eq. (1): $\quad 1 = (d_i)^2 (S^{ij}_{i,i})^2 + d_i d_j (S^{ij}_{i,j})^2$

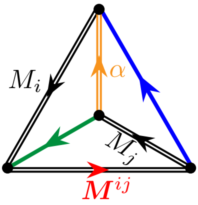
Solving for closed form expressions



$= S^{ij}_{i,j}$

eq. (1): $1 = (d_i)^2 (S^{ij}_{i,i})^2 + d_i d_j (S^{ij}_{i,j})^2 \xrightarrow{M^{ij} \to M^{ii}}$

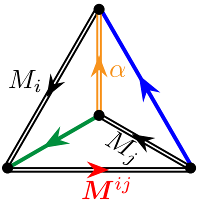
Solving for closed form expressions



$= S^{ij}_{i,j}$

eq. (1): $1 = (d_i)^2 (S^{ij}_{i,i})^2 + d_i d_j (S^{ij}_{i,j})^2 \xrightarrow{\boldsymbol{M}^{ij} \to \boldsymbol{M}^{ii}} 1 = (d_i)^2 (S^{ii}_{i,i})^2$

Solving for closed form expressions

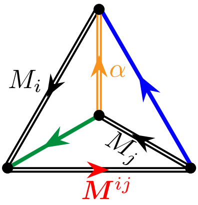


$= S^{ij}_{i,j}$

$$\text{eq. (1):} \quad 1 = (d_i)^2 (S^{ij}_{i,i})^2 + d_i d_j (S^{ij}_{i,j})^2 \quad \xrightarrow{M^{ij} \to M^{ii}} \quad 1 = (d_i)^2 (S^{ii}_{i,i})^2$$

$$\iff \quad S^{ii}_{i,i} = \pm \frac{1}{d_i}$$

Solving for closed form expressions



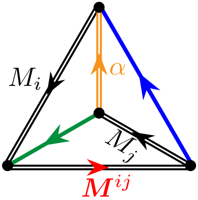
$= S^{ij}_{i,j}$

- $S^{ii}_{i,i} = \pm \frac{1}{d_i}$

eq. (1): $1 = (d_i)^2 (S^{ij}_{i,i})^2 + d_i d_j (S^{ij}_{i,j})^2 \xrightarrow{M^{ij} \to M^{ii}} 1 = (d_i)^2 (S^{ii}_{i,i})^2$

$\iff S^{ii}_{i,i} = \pm \frac{1}{d_i}$

Solving for closed form expressions

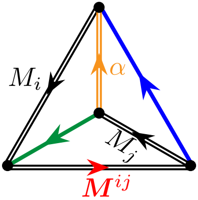


$= S^{ij}_{i,j}$

- $S^{ii}_{i,i} = \pm \frac{1}{d_i}$

eq. (3): $\frac{1}{d_\alpha} = \sum_{M^{ab}} d_{ab} (S^{ab}_{i,j})^2$

Solving for closed form expressions

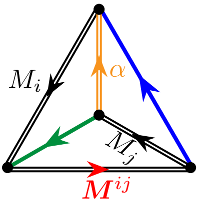


$= S^{ij}_{i,j}$

- $S^{ii}_{i,i} = \pm \frac{1}{d_i}$

eq. (3): $\frac{1}{d_\alpha} = \sum_{M^{ab}} d_{ab} (S^{ab}_{i,j})^2 \xrightarrow{i \neq j}$

Solving for closed form expressions

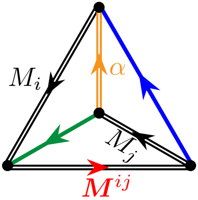


$= S^{ij}_{i,j}$

- $S^{ii}_{i,i} = \pm \frac{1}{d_i}$

eq. (3): $$\frac{1}{d_\alpha} = \sum_{M^{ab}} d_{ab} (S^{ab}_{i,j})^2 \quad \xrightarrow{i \neq j} \quad \frac{1}{d_\alpha} = d_{ij} (S^{ij}_{i,j})^2$$

Solving for closed form expressions



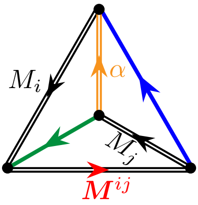
$= S^{ij}_{i,j}$

- $S^{ii}_{i,i} = \pm \frac{1}{d_i}$

eq. (3): $$\frac{1}{d_\alpha} = \sum_{M^{ab}} d_{ab}(S^{ab}_{i,j})^2 \xrightarrow{i \neq j} \frac{1}{d_\alpha} = d_{ij}(S^{ij}_{i,j})^2$$

$$\iff S^{ij}_{i,j} = \pm \frac{1}{\sqrt{d_\alpha d_{ij}}}$$

Solving for closed form expressions



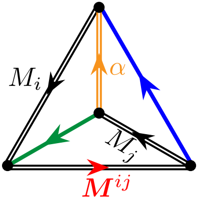
$= S^{ij}_{i,j}$

- $S^{ii}_{i,i} = \pm \frac{1}{d_i}$
- $S^{ij}_{i,j} = \pm \frac{1}{\sqrt{d_\alpha d_{ij}}}$

eq. (3): $$\frac{1}{d_\alpha} = \sum_{M^{ab}} d_{ab} (S^{ab}_{i,j})^2 \quad \xrightarrow{i \neq j} \quad \frac{1}{d_\alpha} = d_{ij} (S^{ij}_{i,j})^2$$

$$\iff \quad S^{ij}_{i,j} = \pm \frac{1}{\sqrt{d_\alpha d_{ij}}}$$

Solving for closed form expressions

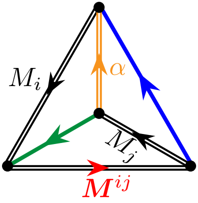


$= S^{ij}_{i,j}$

- $S^{ii}_{i,i} = \pm\dfrac{1}{d_i}$
- $S^{ij}_{i,j} = \pm\dfrac{1}{\sqrt{d_\alpha d_{ij}}}$

eq. (1): $$1 = (d_i)^2 (S^{ij}_{i,i})^2 + d_i d_j (S^{ij}_{i,j})^2$$

Solving for closed form expressions



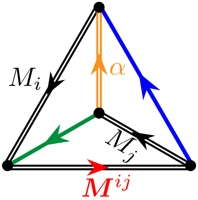
$= S^{ij}_{i,j}$

- $S^{ii}_{i,i} = \pm \dfrac{1}{d_i}$
- $S^{ij}_{i,j} = \pm \dfrac{1}{\sqrt{d_\alpha d_{ij}}}$

$$\text{eq. (1):} \qquad 1 = (d_i)^2 (S^{ij}_{i,i})^2 + d_i d_j (S^{ij}_{i,j})^2$$

$$\xrightarrow{(S^{ij}_{i,j})^2 = \frac{1}{d_\alpha d_{ij}}} \qquad 1 = (d_i)^2 (S^{ij}_{i,i})^2 + \frac{d_i d_j}{d_\alpha d_{ij}}$$

Solving for closed form expressions



$= S^{ij}_{i,j}$

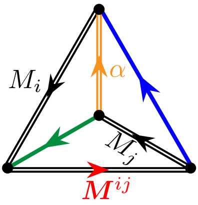
- $S^{ii}_{i,i} = \pm \dfrac{1}{d_i}$
- $S^{ij}_{i,j} = \pm \dfrac{1}{\sqrt{d_\alpha d_{ij}}}$

$$\text{eq. (1):} \qquad 1 = (d_i)^2 (S^{ij}_{i,i})^2 + d_i d_j (S^{ij}_{i,j})^2$$

$$\xrightarrow{(S^{ij}_{i,j})^2 = \frac{1}{d_\alpha d_{ij}}} \qquad 1 = (d_i)^2 (S^{ij}_{i,i})^2 + \frac{d_i d_j}{d_\alpha d_{ij}}$$

$$\Longleftrightarrow \qquad S^{ij}_{i,i} = \pm \frac{1}{d_i} \sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}}$$

Solving for closed form expressions



$= S^{ij}_{i,j}$

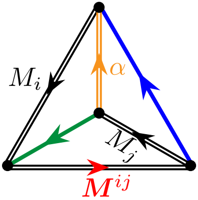
- $S^{ii}_{i,i} = \pm \frac{1}{d_i}$
- $S^{ij}_{i,j} = \pm \frac{1}{\sqrt{d_\alpha d_{ij}}}$
- $S^{ij}_{i,i} = \pm \frac{1}{d_i} \sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}}$

eq. (1): $$1 = (d_i)^2 (S^{ij}_{i,i})^2 + d_i d_j (S^{ij}_{i,j})^2$$

$$\xrightarrow{(S^{ij}_{i,j})^2 = \frac{1}{d_\alpha d_{ij}}} \quad 1 = (d_i)^2 (S^{ij}_{i,i})^2 + \frac{d_i d_j}{d_\alpha d_{ij}}$$

$$\Longleftrightarrow \quad S^{ij}_{i,i} = \pm \frac{1}{d_i} \sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}}$$

Solving for closed form expressions

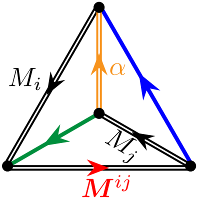


$= S^{ij}_{i,j}$

- $S^{ii}_{i,i} = \pm \dfrac{1}{d_i}$
- $S^{ij}_{i,j} = \pm \dfrac{1}{\sqrt{d_\alpha d_{ij}}}$
- $S^{ij}_{i,i} = \pm \dfrac{1}{d_i} \sqrt{1 - \dfrac{d_i d_j}{d_\alpha d_{ij}}}$

eq. (2): $0 = d_i S^{ij}_{i,i} S^{ij}_{i,j} + d_j S^{ij}_{j,j} S^{ij}_{i,j} = (d_i S^{ij}_{i,i} + d_j S^{ij}_{j,j}) S^{ij}_{i,j}$

Solving for closed form expressions



$= S^{ij}_{i,j}$

- $S^{ii}_{i,i} = \pm \frac{1}{d_i}$
- $S^{ij}_{i,j} = \pm \frac{1}{\sqrt{d_\alpha d_{ij}}}$
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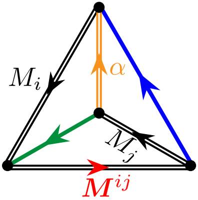
eq. (2): $$0 = d_i S^{ij}_{i,i} S^{ij}_{i,j} + d_j S^{ij}_{j,j} S^{ij}_{i,j} = (d_i S^{ij}_{i,i} + d_j S^{ij}_{j,j}) S^{ij}_{i,j}$$

$$\xrightarrow{S^{ij}_{i,j} \neq 0} \quad d_j S^{ij}_{j,j} = -\, d_i S^{ij}_{i,i}$$

Solving for closed form expressions



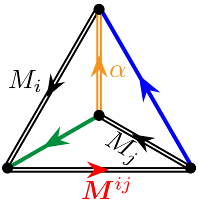
$= S^{ij}_{i,j}$

- $S^{ii}_{i,i} = \pm \frac{1}{d_i}$
- $S^{ij}_{i,j} = \pm \frac{1}{\sqrt{d_\alpha d_{ij}}}$
- $S^{ij}_{i,i} = \pm \frac{1}{d_i} \sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}}$

eq. (2): $$0 = d_i S^{ij}_{i,i} S^{ij}_{i,j} + d_j S^{ij}_{j,j} S^{ij}_{i,j} = (d_i S^{ij}_{i,i} + d_j S^{ij}_{j,j}) S^{ij}_{i,j}$$

$$\xRightarrow{S^{ij}_{i,j} \neq 0} \quad d_j S^{ij}_{j,j} = -d_i S^{ij}_{i,i} = \mp \sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}}$$

Solving for closed form expressions



$= S^{ij}_{i,j}$

- $S^{ii}_{i,i} = \pm \frac{1}{d_i}$
- $S^{ij}_{i,j} = \pm \frac{1}{\sqrt{d_\alpha d_{ij}}}$
- $S^{ij}_{i,i} = \pm \frac{1}{d_i} \sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}}$
- $d_j S^{ij}_{j,j} = -d_i S^{ij}_{i,i}$

eq. (2): $0 = d_i S^{ij}_{i,i} S^{ij}_{i,j} + d_j S^{ij}_{j,j} S^{ij}_{i,j} = (d_i S^{ij}_{i,i} + d_j S^{ij}_{j,j}) S^{ij}_{i,j}$

$$\xrightarrow{S^{ij}_{i,j} \neq 0} \quad d_j S^{ij}_{j,j} = -d_i S^{ij}_{i,i} = \mp \sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}}$$

Closed form expressions

$$S^{ii}_{i,i} = \pm\frac{1}{d_i} \qquad d_i S^{ij}_{i,i} = \pm\sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}} = -d_j S^{ij}_{j,j} \qquad S^{ij}_{i,j} = \pm\frac{1}{\sqrt{d_\alpha d_{ij}}}$$

Closed form expressions

$$S^{ii}_{i,i} = \pm\frac{1}{d_i} \qquad d_i S^{ij}_{i,i} = \pm\sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}} = -d_j S^{ij}_{j,j} \qquad S^{ij}_{i,j} = \pm\frac{1}{\sqrt{d_\alpha d_{ij}}}$$

Closed form expressions

$$S^{ii}_{i,i} = +\frac{1}{d_i} \qquad d_i S^{ij}_{i,i} = \pm\sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}} = -d_j S^{ij}_{j,j} \qquad S^{ij}_{i,j} = \pm\frac{1}{\sqrt{d_\alpha d_{ij}}}$$

symmetric

Closed form expressions

$$S^{ii}_{i,i} = +\frac{1}{d_i} \qquad d_i S^{ij}_{i,i} = \pm\sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}} = -d_j S^{ij}_{j,j} \qquad S^{ij}_{i,j} = \pm\frac{1}{\sqrt{d_\alpha d_{ij}}}$$

symmetric

Closed form expressions

$$S^{ii}_{i,i} = +\frac{1}{d_i} \qquad d_i S^{ij}_{i,i} = \pm\sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}} = -d_j S^{ij}_{j,j} \qquad S^{ij}_{i,j} = \pm\frac{1}{\sqrt{d_\alpha d_{ij}}}$$

symmetric

lin. relation ($N \leq 3$)

Closed form expressions

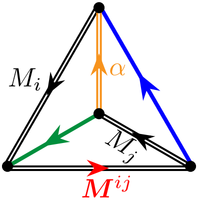
$$S^{ii}_{i,i} = +\frac{1}{d_i} \qquad d_i S^{ij}_{i,i} = \pm\sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}} = -d_j S^{ij}_{j,j} \qquad S^{ij}_{i,j} = \pm\frac{1}{\sqrt{d_\alpha d_{ij}}}$$

symmetric lin. relation ($N \leq 3$)

Closed form expressions

$$S^{ii}_{i,i} = +\frac{1}{d_i} \qquad d_i S^{ij}_{i,i} = \pm\sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}} = -d_j S^{ij}_{j,j} \qquad S^{ij}_{i,j} = \pm\frac{1}{\sqrt{d_\alpha d_{ij}}}$$

symmetric lin. relation ($N \leq 3$)



Closed form expressions

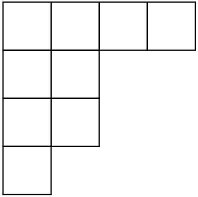
$$S^{ii}_{i,i} = +\frac{1}{d_i} \qquad d_i S^{ij}_{i,i} = \pm\sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}} = -d_j S^{ij}_{j,j} \qquad S^{ij}_{i,j} = \pm\frac{1}{\sqrt{d_\alpha d_{ij}}}$$

symmetric — lin. relation ($N \leq 3$) — vertex def.

Closed form expressions

$$S^{ii}_{i,i} = +\frac{1}{d_i} \qquad d_i S^{ij}_{i,i} = \pm\sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}} = -d_j S^{ij}_{j,j} \qquad S^{ij}_{i,j} = \pm\frac{1}{\sqrt{d_\alpha d_{ij}}}$$

symmetric — lin. relation ($N \leq 3$) — vertex def.



Closed form expressions

$$S^{ii}_{i,i} = +\frac{1}{d_i} \qquad d_i S^{ij}_{i,i} = \pm\sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}} = -d_j S^{ij}_{j,j} \qquad S^{ij}_{i,j} = \pm\frac{1}{\sqrt{d_\alpha d_{ij}}}$$

symmetric lin. relation ($N \leq 3$) vertex def.

factors: (add N)

0	1	2	3
-1	0		
-2	-1		
-3			

Closed form expressions

$$S^{ii}_{i,i} = +\frac{1}{d_i} \qquad d_i S^{ij}_{i,i} = \pm\sqrt{1-\frac{d_i d_j}{d_\alpha d_{ij}}} = -d_j S^{ij}_{j,j} \qquad S^{ij}_{i,j} = \pm\frac{1}{\sqrt{d_\alpha d_{ij}}}$$

symmetric — lin. relation ($N \leq 3$) — vertex def.

factors: (add N)

0	1	2	3
−1	0		
−2	−1		
−3			

hook lengths:

7	5	2	1
4	2		
3	1		
1			

Closed form expressions

$$S^{ii}_{i,i} = +\frac{1}{d_i} \qquad d_i S^{ij}_{i,i} = \pm\sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}} = -d_j S^{ij}_{j,j} \qquad S^{ij}_{i,j} = \pm\frac{1}{\sqrt{d_\alpha d_{ij}}}$$

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0	1	2	3
-1	0		
-2	-1		
-3			

hook lengths:

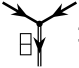
7	5	2	1
4	2		
3	1		
1			

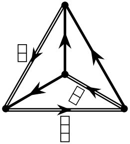
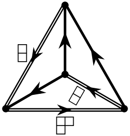
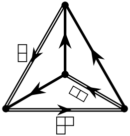
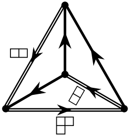
$$\dim = \left[\tfrac{N}{7}\tfrac{(N+1)}{5}\tfrac{(N+2)}{2}(N+3)\right]\left[\tfrac{(N-1)}{4}\tfrac{N}{2}\right]\left[\tfrac{(N-2)}{3}(N-1)\right][N-3]$$

Closed form expressions

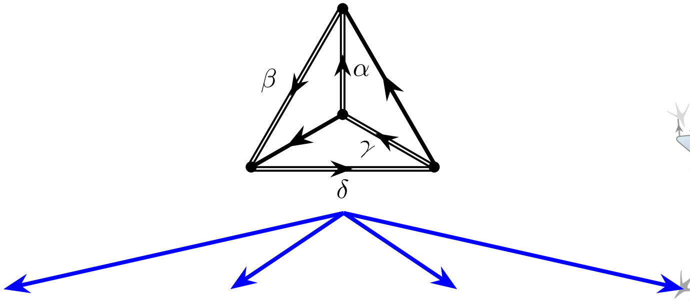
$$S^{ii}_{i,i} = +\frac{1}{d_i} \qquad d_i S^{ij}_{i,i} = \pm\sqrt{1 - \frac{d_i d_j}{d_\alpha d_{ij}}} = -d_j S^{ij}_{j,j} \qquad S^{ij}_{i,j} = \pm\frac{1}{\sqrt{d_\alpha d_{ij}}}$$

symmetric — lin. relation ($N \leq 3$) — vertex def.

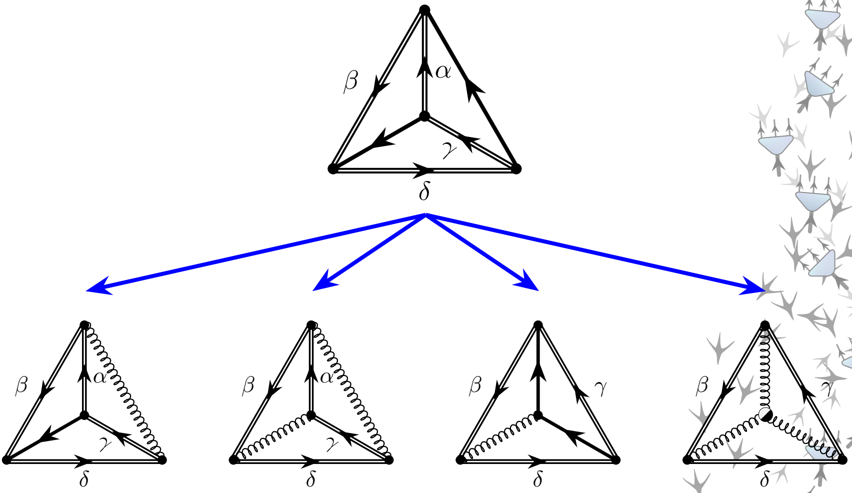
Exceptions containing  : [M. Sjödahl & J. Thorén (2018)]

 ,  ,  and 

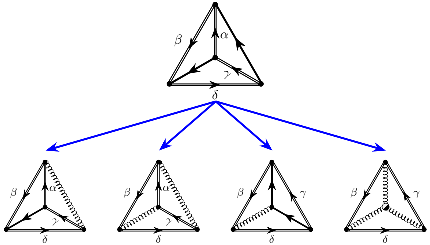
What's next?



What's next?



What's next?



What’s next?