Dispersive Approach(es) to Hadronic Light-by-Light Scattering for the Muon $g - 2$

Jan Lüdtke

University of Vienna

Seminar on Particle Physics, June 03, 2022
Outline

1. Experiment vs. Standard Model determination of the muon $g - 2$: hadronic contributions

2. Dispersive approaches to hadronic light-by-light
   - Dispersion relations in four-point kinematics (present approach)
   - Dispersion relations in triangle kinematics (new approach)
     → The sub-process $\pi\pi \rightarrow \gamma\pi\pi$

3. Conclusions and outlook

in collaboration with Massimiliano Procura and Peter Stoffer
Introduction
The anomalous magnetic moment of the muon

- **Dirac equation** gives $g_f = 2$ for fermions

- for leptons: permille-level deviations due to **radiative corrections**
  $\rightarrow$ define $a_l = \frac{g_l - 2}{2}$

- **high accuracy** in experiments and calculations (for electron and muon) allows for **strong tests** of the SM

- $a_e$ prediction limited by knowledge of $\alpha_{\text{QED}}$ and no clear tension with experiment

- **different** for $a_\mu$
### Introduction

Measurement and Standard Model prediction for $a_\mu$

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- combination of final BNL E821 result and run 1 of new FNAL E989
- experimental error expected to reduce to $16 \times 10^{-11}$ in near future

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- dominated by
- known up to 5 loops
- negligible uncertainty

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- much smaller due to approximate scaling
  $\sim \frac{m_\mu^2}{\Lambda^2}$
- 2 loop calculation + RGE estimate of 3 loop
- model estimate for mixed EW/QCD
- small uncertainty

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- related to $\sigma(e^+e^- \rightarrow \text{hadrons})$ by **unitarity**
- this data-driven evaluation is in tension with one lattice calculation: $7075(55)$

BMW collab., Science 2020

- **largest uncertainty**
# Introduction

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- diagrams like
- non-negligible at current precision, but well known
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*more complicated than HVP due to more legs attached to blob*

*but:* 10% precision sufficient

number results from average between lattice and phenomenology (agree within uncertainties)

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- sum **differs** by $251(59) \times 10^{-11}$ (4.2 $\sigma$) **from experiment**
- poorly understood effect on experimental or theory side or **new physics**?

Introduction

Current status in HLbL

• WP number is **combination** of

\[ a_{\mu, \text{phen}}^{\text{HLbL}} = 92(19) \times 10^{-11}, \quad a_{\mu, \text{lat}}^{\text{HLbL}} = 79(35) \times 10^{-11} + c\text{-loop} \]

• both **compatible** with latest lattice result

\[ a_{\mu, \text{lat}}^{\text{HLbL}} = 106.8(14.7) \times 10^{-11} + c\text{-loop} \]

• **data-driven approach** allowed for the first time to model-independently define individual contributions and assign numbers with small and reliable uncertainties

  Colangelo, Hoferichter, Procura, Stoffer (CHPS) 2015, 2017

• sub-dominant contributions from **heavier resonances** can currently only be estimated and have **larger uncertainties** due to
  ▶ lack of (precise) **data input**
  ▶ **conceptual difficulties** in the present framework → will be addressed in this talk
Review of present approach

- 4-point function, tensor structures and master formula
- Dispersion relations in general kinematics
- Results and open questions
Tensor structures
Naive decomposition and Ward identities

• have to describe **hadronic correlator** of 4 photons/em-currents

\[
\Pi_{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = -i \int d^4x \, d^4y \, d^4z \, e^{-i(q_1 \cdot x + q_2 \cdot y + q_3 \cdot z)} \times \langle 0 | T\{j_{em}^\mu(x)j_{em}^\nu(y)j_{em}^\lambda(z)j_{em}^\sigma(0)\} | 0 \rangle
\]

• **in general** there are 138 tensor structures consisting of \( q_i^\alpha \) and \( g^{\alpha\beta} \)

\[
\Pi_{\mu\nu\lambda\sigma} = \sum_{i=1}^{138} L_i^{\mu\nu\lambda\sigma} \Xi_i
\]

• **Ward identities** put 95 linear constraints on scalar functions \( \Xi_i \)

\[
\{ q_1^\mu, q_2^\nu, q_3^\lambda, q_4^\sigma \} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = 0
\]
Tensor structures
BTT recipe and ambiguities

• **projection** gives basis for subspace fulfilling the Ward identities, but with **singularities** in the tensor structures → problematic for dispersion relations

• singularities can be **removed**
  but: set becomes **incomplete** at specific kinematic points

• **add** non-singular tensor structures to obtain generating set at all kinematic points

\[ \Pi_{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_{i}^{\mu\nu\lambda\sigma} \Pi_{i} \]

but: BTT set \( T_{i}^{\mu\nu\lambda\sigma} \) is **overcomplete**, which implies **ambiguities** in the scalar coefficient functions \( \Pi_{i} \)
Tensor structures

Limit $q \to 0$ and Master formula

- for $a^{\text{HLbL}}_{\mu}$ we need **two-loop integral** over

\[
\lim_{q_4 \to 0} \frac{\partial}{\partial q_4} \Pi_{\mu\nu\lambda\sigma}
\]

- 35 linear combinations of the 54 structures **vanish** in this limit

- 5 of the 8 integrals can be performed in **full generality**

- due to **symmetry** only 12 linear combinations of scalar functions in the limit $q_4 \to 0$ enter the master formula

\[
a^{\text{HLbL}}_{\mu} = \frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1 - \tau^2} Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)
\]

- $Q_i = \sqrt{-q_i^2}$, $\tau = \tau(Q_1, Q_2, Q_3)$

- kernel functions $T_i$ **known** analytically

→ aim of dispersive approach(es) is to **reconstruct** $\bar{\Pi}_i$
Consider scalar function $F(s)$ that is analytic apart from a branch cut on real axis.

\[ F(s) = \frac{1}{2\pi i} \oint_{C} ds' s'^{-1} F(s') \]

where $C$ is a contour that encloses the branch cut from $s_{\text{thr}}$. If $F(s)$ falls off sufficiently fast, only $C^+$ and $C^-$ contribute.

\[ F(s) = \frac{1}{2\pi i} \int_{s_{\text{thr}}}^{s_{\text{thr}}} ds' \text{disc} F(s') \]

\[ \text{disc} F(s) = F(s + i\epsilon) - F(s - i\epsilon) = 2i \text{Im} F(s) \]
Dispersion relations for low-energy hadronic processes

• Consider scalar function $F(s)$ that is analytic apart from branch cut on real axis

• Cauchy’s Theorem:
  
  $$F(s) = \frac{1}{2\pi i} \oint_{\Gamma} ds' \frac{F(s')}{s'-s}$$

Jan Lüdtke (Uni Wien)
Dispersion relations for low-energy hadronic processes

- Consider scalar function $F(s)$ that is analytic apart from branch cut on real axis
- Cauchy’s Theorem:
  \[ F(s) = \frac{1}{2\pi i} \oint \frac{F(s')}{s'-s} \, ds' \]
- deform integration contour

Jan Lüdtke (Uni Wien)
Dispersion relations for low-energy hadronic processes

- Consider scalar function $F(s)$ that is analytic apart from \textbf{branch cut} on real axis

- Cauchy’s Theorem:
  $$F(s) = \frac{1}{2\pi i} \oint_{\Gamma} ds' \frac{F(s')}{s' - s}$$

- deform integration contour

- If $F$ falls off \textbf{sufficiently fast}, only $C_+$ and $C_-$ contribute

  $$F(s) = \frac{1}{2\pi i} \int_{s_{\text{thr}}}^{\infty} ds' \text{disc} F(s'),$$

  $$\text{disc} F(s) = F(s + i\epsilon) - F(s - i\epsilon) = 2i\text{Im} F(s)$$
Dispersion relations for low-energy hadronic processes

Unitarity relations

- imaginary part determined from **unitarity** of the $S$-matrix ($SS^\dagger = 1$)
- plug in $S = 1 + iT$: $i(T^\dagger - T) = TT^\dagger$
- $T$-invariance implies $T^T = T$ and thus $i(T^\ast - T) = 2\text{Im}T = TT^\ast$
- sandwiching this between states gives
  \[
  \text{Im} \langle f | T | i \rangle = \sum_s \langle f | T | s \rangle \langle s | T | i \rangle^\ast
  \]
- can be visualized by **unitarity diagrams**

- at **low energies**, light states with low multiplicity **dominate** in sum
Dispersion relations in four-point kinematics
Application to HLbL

- write dispersion relations for $\Pi_i$ (overcomplete generating set free of kinematic singularities) in $s, t, u$ for fixed $q_i^2$

- single out lightest states in unitarity relations in each channel
  $\rightarrow$ imaginary parts given in terms of (simpler) sub-processes

- allows model-independent definition of individual contributions

- use experimental data on sub-processes to evaluate contributions to $a_{\mu}^{\text{HLbL}}$ due to light intermediate states
  $\rightarrow$ reliable uncertainty estimate for each contribution
Dispersion relations in four-point kinematics

Topologies

• focus on 1 and 2 particle intermediates states

\[ \begin{array}{c}
\begin{array}{c}
q_1 \quad \text{\emph{\parbox{1.2in}{\centering \includegraphics[width=1.2in]{diagram1}}}} \\
q_2 \quad q_3 \\
\end{array}
= \begin{array}{c}
\begin{array}{c}
q_1 \quad \text{\emph{\parbox{1.2in}{\centering \includegraphics[width=1.2in]{diagram2}}}} \\
q_2 \quad q_3 \\
\end{array}
+ \begin{array}{c}
\begin{array}{c}
q_1 \quad \text{\emph{\parbox{1.2in}{\centering \includegraphics[width=1.2in]{diagram3}}}} \\
q_2 \quad q_3 \\
\end{array}
+ \begin{array}{c}
\begin{array}{c}
q_1 \quad \text{\emph{\parbox{1.2in}{\centering \includegraphics[width=1.2in]{diagram4}}}} \\
q_2 \quad q_3 \\
\end{array}
+ \ldots
\end{array}
\end{array}
\end{array}
\right]

• crossed diagrams not shown

• s-channel \( \pi^0 \) pole contributes to only 1 scalar function

\[
\Pi_1^{\pi^0 - \text{pole}} = \frac{F_{\pi^0 \gamma^* \gamma^*}(q_1^2, q_2^2)F_{\pi^0 \gamma^* \gamma^*}(q_3^2, q_4^2)}{s - m_{\pi}^2}
\]

\[
\begin{align*}
q_4 \to 0 \quad & F_{\pi^0 \gamma^* \gamma^*}(q_1^2, q_2^2)F_{\pi^0 \gamma^* \gamma^*}(q_3^2, 0) \\
& \to \frac{F_{\pi^0 \gamma^* \gamma^*}(q_1^2, q_2^2)}{q_3^2 - m_{\pi}^2}
\end{align*}
\]
Dispersion relations in four-point kinematics

Results and current status

• contributions of 1 and 2 pseudoscalar states under **good control**
  ▶ pseudoscalar poles: $93.8(4.0) \times 10^{-11}$
  ▶ $\pi$- and $K$-loops: $-16.4(2) \times 10^{-11}$
    - CHPS, JHEP 2017, WP 2020
  ▶ $S$-wave $\pi\pi$ rescattering: $-8(1) \times 10^{-11}$
    - CHPS, JHEP 2017
  ▶ huge improvement in **precision** and **reliability** compared to earlier (model) estimates

• full description of 3 particle and higher intermediate states very challenging
  → describe through resonances in **narrow width approximation**

• but: data very **scarce** → only rough estimates possible, mostly based on models

• at high energies pQCD and OPE used to constrain the $\Pi_i$
Dispersion relations in four-point kinematics
Singly on-shell basis and sum rules

• sufficient to consider $q_4^2 = 0$

• in this limit a Lorentz basis free of kinematic singularities in $s, t, u$ exists ($\check{\Pi}_i$)

• $\check{\Pi}_i$ have different mass dimensions
  → $\check{\Pi}_i$ with lower mass dimension fall off faster at high energies
  → implies sum rules of form $\int ds' \text{Im}\check{\Pi}_i(s') = 0$

• sum rules guarantee basis independence of $a^\text{HLbL}_\mu$

• but: sum rules only fulfilled for (infinite) sum over intermediate states
  → individual contributions basis dependent

• exception: pseudoscalar poles and loops fulfill sum rules individually
Dispersion relations in four-point kinematics

Current limitations due to singularities in photon virtualities

• in addition: $\tilde{\Pi}_i$ have singularities in $q_i^2$
  $\rightarrow$ residues vanish due to sum rules for (infinite) sum over intermediate states

• poles lead to non-convergent master formula integrals for individual contributions
  $\rightarrow$ must subtract poles using same prescription for all contributions
  $\rightarrow$ additional ambiguity

• in original basis this affects contributions with spin $\geq 1$

• by basis change it can be avoided for axial-vector mesons

• without additional sum rules singularities are unavoidable for intermediate states with spin $\geq 2$

Colangelo et al., EPJC 2021
How can we overcome this limitation?

→ Dispersion relations at $q_4 = 0$
Dispersion relations in triangle kinematics

General idea and advantages

- has been realized that dispersion relations can also be written at $q_4 \to 0$

- at $q_4 \to 0$, all **ambiguities disappear** and a Lorentz basis free of kinematic singularities ($\hat{\Pi}_i$) exists

- dispersion relations for them **avoid ambiguities** coming from subtraction of spurious poles

- will allow to include $D$-wave $\pi\pi$ rescattering, tensor-meson poles, ...
Dispersion relations in triangle kinematics

Addition of cuts

- suppress additional arguments and use simplified notation

\[ \hat{\Pi}_i(q_3^2) = \lim_{s \to q_3^2} \check{\Pi}_i(s, q_3^2) \text{ with } s = (q_3 + q_4)^2 \]

\[
\text{Im} \hat{\Pi}_i(q_3^2) = \lim_{s \to q_3^2} \left[ \text{Im}_s \check{\Pi}_i(s, q_3^2 + i\epsilon) + \text{Im}_3 \check{\Pi}_i(s + i\epsilon, q_3^2)^\ast \right]
\]

\[
\text{Im}
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{bmatrix}
= \begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{bmatrix}
+ \begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{bmatrix}^\ast
\]

→ s- and $q_3^2$-channel cuts have to be added
Dispersion relations in triangle kinematics

Topologies and sub-processes

• $s$- and $q^2_3$-channel cuts with 1 and 2 pion intermediate states

\[ q_1 \quad = \quad q_1 + q_2 + q_3 + q_4 \]

→ all sub-processes except for $\gamma^* \gamma^* \rightarrow \pi\pi\gamma$ well-known
→ cancellation of infrared divergences in $\pi^+\pi^-$ intermediate states between $s$- and $q^2_3$-cuts demonstrated

• $s$-channel resonance contributions given in terms of transition form factors (including axials and tensor mesons . . . )
Dispersion relations in triangle kinematics

Topologies and sub-processes

• $s$- and $q_3^2$-channel cuts with 1 and 2 pion intermediate states

→ all sub-processes except for $\gamma^*\gamma^* \rightarrow \pi\pi\gamma$ well-known

• again 2 different ($s$-channel) cuts

→ all sub-processes except for $\pi\pi \rightarrow \pi\pi\gamma$ well-known
The sub-process $\pi\pi \rightarrow \gamma\pi\pi$

- derivation of dispersive description to be inserted into HLbL in triangle kinematics

- shares many features with $\gamma^*\gamma^* \rightarrow \pi\pi\gamma$ and HLbL

- Lorentz structure much simpler

→ focus on this for the rest of the talk
The sub-process $\pi \pi \rightarrow \gamma \pi \pi$

Kinematics and Lorentz decomposition

- amplitude $M(\pi^0 \pi^0 \rightarrow \pi^+ \pi^- \gamma) = \epsilon^{*}_\mu M^\mu$

- charged channel $(\pi^+ \pi^- \rightarrow \pi^+ \pi^- \gamma)$ also needed, but related to mixed channel through isospin symmetry

  Kuhn 1999, Ecker & Unterdorfer 2002

- **BTT decomposition** $M^\mu = \sum_{i=1}^{6} \hat{T}^\mu_i M_i$ leads 3 Tarrach redundancies

  \[
  \begin{align*}
  \hat{T}^\mu_1 &= p^\mu_2 (p_3 \cdot q) - p^\mu_3 (p_2 \cdot q), & \hat{T}^\mu_2 &= p^\mu_3 (p_1 \cdot q) - p^\mu_1 (p_3 \cdot q), \\
  \hat{T}^\mu_3 &= p^\mu_1 (p_2 \cdot q) - p^\mu_2 (p_1 \cdot q), & \hat{T}^\mu_4 &= q^\mu (p_1 \cdot q) - p^\mu_1 q^2, \\
  \hat{T}^\mu_5 &= q^\mu (p_2 \cdot q) - p^\mu_2 q^2, & \hat{T}^\mu_6 &= q^\mu (p_3 \cdot q) - p^\mu_3 q^2
  \end{align*}
  \]

- 5-particle process has 10 kinematic invariants, 5 fixed by on-shell conditions
The sub-process $\pi\pi \rightarrow \gamma\pi\pi$

Soft-photon limit and singularities

- in principle only $\lim_{q \rightarrow 0} \frac{\partial}{\partial q_{\nu}} M^\mu$ needed
- Tarrach redundancies drop out in this limit and a 2D basis exists
- **but:** limit *does not exist* due to
  \[ p_1 \quad q \quad p_3 \]
  \[ p_2 \quad p_4 \]

  \[ \rightarrow \textbf{split} \quad M^\mu = M^\mu_{\text{sing}} + M^\mu_{\text{reg}} \]

- ambiguity to shift finite terms between $M^\mu_{\text{sing}}$ and $M^\mu_{\text{reg}}$
- for $M^\mu_{\text{reg}}$ the limit can be performed and the problem reduces to
  **4-point kinematics** \[ \rightarrow \text{Mandelstam variables} \]
- singularities cancel when plugged into HLbL
- need gauge-invariant non-perturbative definition of $M^\mu_{\text{sing}}$
The sub-process $\pi\pi \rightarrow \gamma\pi\pi$

Definition of $\mathcal{M}_{\text{sing}}^\mu$

- **Low's theorem**: terms of order $q^{-1}$ obtainable from
  
  with scalar QED vertex for soft photon,
  
  terms of order $q^0$ fixed by imposing Ward identity
  
  $\rightarrow$ terms up to order $q^0$ given in terms of $\pi\pi \rightarrow \pi\pi$

- but: $\lim_{q \rightarrow 0} \frac{\partial}{\partial q^\nu} (\mathcal{M}^\mu - \mathcal{M}_{\text{sing}}^\mu)$ still does **not exist** due to terms like
  $q^\mu \frac{p_i \cdot q}{p_j \cdot q}$ (of order $q^1$, but limit depends on **direction** of $q$)
  
  $\rightarrow$ need definition of $\mathcal{M}_{\text{sing}}^\mu$ that includes **all** singular terms

- achieved from **unitarity** with a single-pion intermediate state
  
  $\rightarrow$ also only depends on $\pi\pi \rightarrow \pi\pi$ amplitude $\mathcal{T}$

\[
\mathcal{M}_{\text{sing}}^\mu = F_V^\pi(q^2) \left( \frac{(2p_3 + q)^\mu}{(p_3 + q)^2 - m_\pi^2} \mathcal{T}(s, \tilde{t} - u) - \frac{(2p_4 + q)^\mu}{(p_4 + q)^2 - m_\pi^2} \mathcal{T}(s, t - \tilde{u}) - 2(p_1 - p_2)^\mu \Delta \mathcal{T} \right)
\]

\[
\Delta \mathcal{T} = \frac{\mathcal{T}(s, \tilde{t} - u) - \mathcal{T}(s, t - \tilde{u})}{\tilde{t} - u - t + \tilde{u}}
\]
The sub-process $\pi\pi \to \gamma\pi\pi$

Unitarity relation and Cancellation of singularities

- two-pion intermediate states in unitarity relations involve $\pi\pi \to \gamma\pi\pi$ as a \textbf{sub-process} (similarly in $t$- and $u$-channels)

\[ p_1 \quad q \quad p_3 \]
\[ p_2 \quad p_4 \]
\[ p_1 \quad q \quad p_3 \]
\[ p_2 \quad p_4 \]

\[ = \]

\[ p_1 \quad q \quad p_3 \]
\[ p_2 \quad p_4 \]
\[ p_1 \quad q \quad p_3 \]
\[ p_2 \quad p_4 \]

\[ = \quad + \quad \ldots \]

\[ \to \] contains the \textbf{soft-singular} piece $\mathcal{M}^\mu_{\text{sing}}$

- checked that \textbf{sum of cuts} reproduces the singularities of $\text{Im}\pi\pi\mathcal{M}^\mu_{\text{sing}}$ \to \textbf{finite difference} is $\text{Im}\pi\pi\mathcal{M}^\mu_{\text{reg}}$ and can be projected onto basis Lorentz structures in limit $q \to 0$
The sub-process $\pi\pi \rightarrow \gamma\pi\pi$

Imaginary parts

- $t$-channel imaginary part of 1 scalar function

\[
\text{Im}_{t}^{\pi-\pi^0} \tilde{M}_1 = \frac{1}{8} \sqrt{1 - \frac{4m_{\pi}^2}{t}} \Re \sum_l (2l + 1) \int d\Omega_l(t_1^1(t) + t_2^2(t)) P_l(z'_t) \\
\times \left[ \left( 2 \frac{z'_t - z''_t}{1 - z_t} + \frac{z'_t + z''_t}{1 + z_t} - 5 \right) \tilde{M}_1(t, z''_t) + \left( \frac{z'_t + z''_t}{1 + z_t} - 1 \right) \tilde{M}_2(t, z''_t) \right] \\
+ \Delta_1
\]

- $\Delta_1$ finite remainder of terms involving $M_{\text{sing}}^\mu$ \\
  \rightarrow only depends on $\pi\pi \rightarrow \pi\pi$ partial waves $t_i^j$

- $s$-channel imaginary parts depend on both $s$- and $t$-channel amplitudes

- have checked imaginary parts at one loop in $\chi^\text{PT}$
The sub-process $\pi\pi \rightarrow \gamma\pi\pi$

Dispersion relations

- from **fixed-s dispersion relations** for $\bar{M}_i$, make ansatz for **partial-wave expansion** in $t$-channel: $\bar{M}_i(t, z_t) = \sum_{l=0}^{\infty} (2l + 1)g^i_l(t)P_l(z_t)$
  
  - allows to perform angular integral
  - considering $\pi\pi$ scattering only up to $D$-waves truncates tower of $g^i_l$

- **project** dispersion relation onto partial waves to obtain **coupled integral equations** for $g^i_l(t) \rightarrow$ Roy–Steiner equations

\[
g^i_l(t) = \sum_{j', l} \sum_{i'=1}^{2} \frac{1}{\pi} \int_{4m^2_\pi}^{\infty} dt' K_{jj'}^{ii'}(t, t') \text{Re}[ (t^1_{i'}(t') + t^2_{i'}(t')) g^j_{i'}(t') ] + \Delta^i_l(t)
\]

- **similar** equations exist for $s$-channel partial waves (include integrals over $t$-channel partial waves $g^i_l$)

- solving this will complete the dispersive reconstruction of the sub-process $\pi\pi \rightarrow \gamma\pi\pi$
Conclusions

• **discrepancy** between measurement and Standard Model prediction of \(a_\mu\) could be due to **New Physics**: higher precision needed

• established dispersive formalism for HLbL **very successful** for most important contributions

• complementary dispersive approach **promises** to overcome roadblocks in inclusion of higher-spin intermediate states

• important steps towards complete calculation in new approach **already achieved**:
  - unitarity relations
  - cancellation of infrared divergences
  - new dispersion relations for \(\pi \pi \rightarrow \gamma \pi \pi\)
Outlook

• solution of Roy–Steiner equations will **complete** study of $\pi\pi \rightarrow \gamma\pi\pi$

• with $\pi\pi \rightarrow \gamma\pi\pi$ as input, **similar study** possible for $\gamma^*\gamma^* \rightarrow \pi\pi\gamma$
  ▶ more complicated Lorentz structure
  ▶ but: **many similarities** concerning kinematics, cancellation of IR singularities, phase-space integrals, ... expected

• this will allow for a complete treatment of all 1- and 2-particle intermediate states in HLbL with **arbitrary angular momenta**

• study in detail **reshuffling** of contributions between the 2 dispersive approaches to HLbL
  ▶ learn how to combine them to include as many contributions as possible

• study dispersive representation of pQCD quark loop to **incorporate short-distance constraints**
  → Michael’s work
Thank you for your attention!
Backup
Addition of cuts

- suppress additional arguments and use simplified notation

\[ \text{Im} \hat{\Pi}_i(q_3^2) = \lim_{s \to q_3^2} \tilde{\Pi}_i(s, q_3^2) \]

\[ \text{Im} \hat{\Pi}_i(s) = \lim_{q_3^2 \to s} \frac{\tilde{\Pi}_i(s + i \epsilon, q_3^2 + i \epsilon) - \tilde{\Pi}_i(s - i \epsilon, q_3^2 - i \epsilon)}{2i} \]

\[ = \lim_{q_3^2 \to s} \left[ \frac{\tilde{\Pi}_i(s + i \epsilon, q_3^2 + i \epsilon) - \tilde{\Pi}_i(s - i \epsilon, q_3^2 + i \epsilon)}{2i} \right. \]

\[ + \left. \frac{\tilde{\Pi}_i(s - i \epsilon, q_3^2 + i \epsilon) - \tilde{\Pi}_i(s - i \epsilon, q_3^2 - i \epsilon)}{2i} \right] \]

\[ = \lim_{q_3^2 \to s} \left[ \frac{\tilde{\Pi}_i(s + i \epsilon, q_3^2 + i \epsilon) - \tilde{\Pi}_i(s - i \epsilon, q_3^2 + i \epsilon)}{2i} \right. \]

\[ + \left. \left( \frac{\tilde{\Pi}_i(s + i \epsilon, q_3^2 + i \epsilon) - \tilde{\Pi}_i(s + i \epsilon, q_3^2 - i \epsilon)}{2i} \right)^* \right] \]

\[ =: \lim_{q_3^2 \to s} \left[ \text{Im}_s \tilde{\Pi}_i(s, q_3^2 + i \epsilon) + \text{Im}_3 \tilde{\Pi}_i(s + i \epsilon, q_3^2)^* \right] \]