Dispersive Approach(es) to Hadronic Light-by-Light Scattering for the Muon $g-2$

Jan Lüdtke

University of Vienna

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universität wien

Der Wissenschaftsfonds.

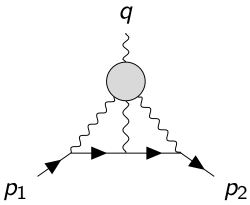
Doktoratskolleg Particles and Interactions

Outline

1. Experiment vs. Standard Model determination of the muon $g-2$: hadronic contributions

2. Dispersive approaches to hadronic light-by-light
   - Dispersion relations in four-point kinematics (present approach)
   - Dispersion relations in triangle kinematics (new approach)

     $\rightarrow$ The sub-process $\pi\pi \rightarrow \gamma\pi\pi$



3. Conclusions and outlook

in collaboration with Massimiliano Procura and Peter Stoffer

Introduction

The anomalous magnetic moment of the muon

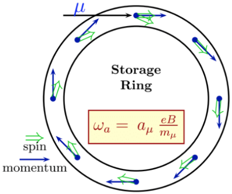
- **Dirac equation** gives $g_f = 2$ for fermions
- for leptons: permille-level deviations due to **radiative corrections**
  $\rightarrow$ define $a_l = \frac{g_l - 2}{2}$
- **high accuracy** in experiments and calculations (for electron and muon) allows for **strong tests** of the SM
- a_e prediction limited by knowledge of α_{QED} and no clear tension with experiment
- **different** for a_μ

Introduction

Measurement and Standard Model prediction for a_μ

Contribution	value $\times 10^{11}$	error $\times 10^{11}$
Experiment	116 592 061	41
QED	116 584 718.931	0.104
Electroweak	153.6	1.0
HVP LO	6931	40
HVP NLO	−98.3	0.7
HVP NNLO	12.4	0.1
HLbL LO	90	17
HLbL NLO	2	1
Sum SM	116 591 810	43

Aoyama et al., Phys. Rep. 2020 (WP 2020)



- combination of final BNL E821 result and run 1 of new FNAL E989
- experimental error expected to **reduce** to 16×10^{-11} in near future

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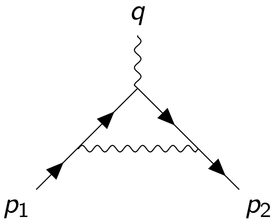
- dominated by



- known up to **5 loops**

  Aoyama et al. 2012, 2019

- **negligible uncertainty**

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- **much smaller** due to approximate scaling $\sim \frac{m_\mu^2}{\Lambda^2}$
- 2 loop calculation $+$ RGE estimate of 3 loop

  Gnendinger et al., Phys. Rev. D 2013
- model estimate for mixed EW/QCD
- **small** uncertainty

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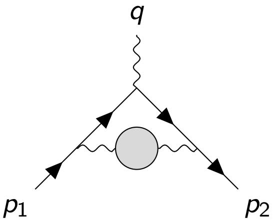

- related to $\sigma(e^+e^- \to \text{hadrons})$ by **unitarity**
- this data-driven evaluation is in tension with one lattice calculation: 7075(55)

  BMW collab., Science 2020
- **largest uncertainty**

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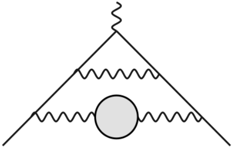
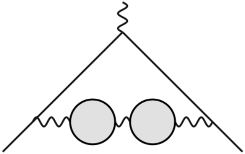
- diagrams like



and



- non-negligible at current precision, but **well known**

Introduction

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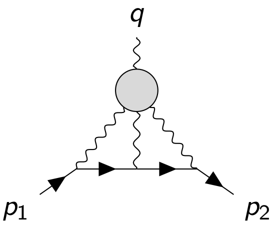

- more complicated than HVP due to **more legs** attached to blob
- **but:** 10 % precision sufficient
- number results from average between lattice and phenomenology (agree within uncertainties)

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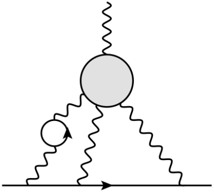

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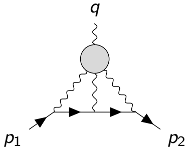
- sum **differs** by $251(59) \times 10^{-11}$ $(4.2\,\sigma)$ **from experiment**
- poorly understood effect on experimental or theory side or **new physics**?

Introduction

Current status in HLbL



- WP number is **combination** of

$$a_{\mu,\,\mathrm{phen}}^{\mathrm{HLbL}} = 92(19) \times 10^{-11}\,, \quad a_{\mu,\,\mathrm{lat}}^{\mathrm{HLbL}} = 79(35) \times 10^{-11} + c\text{-loop}$$

- both **compatible** with latest lattice result — Chao et al., EPJC 2021

$$a_{\mu,\,\mathrm{lat}}^{\mathrm{HLbL}} = 106.8(14.7) \times 10^{-11} + c\text{-loop}$$

- **data-driven approach** allowed for the first time to model-independently define individual contributions and assign numbers with small and reliable uncertainties — Colangelo, Hoferichter, Procura, Stoffer (CHPS) 2015, 2017

- sub-dominant contributions from **heavier resonances** can currently only be estimated and have **larger uncertainties** due to
  - lack of (precise) **data input**
  - **conceptual difficulties** in the present framework
    $\rightarrow$ will be addressed in this talk

Review of present approach

- 4-point function, tensor structures and master formula
- dispersion relations in **general kinematics**
- results and open questions

Tensor structures

Naive decomposition and Ward identities

- have to describe **hadronic correlator** of 4 photons/em-currents

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = -i \int \mathrm{d}^4x \; \mathrm{d}^4y \; \mathrm{d}^4z \, e^{-i(q_1\cdot x + q_2\cdot y + q_3\cdot z)} \times \langle 0| \, T\{j^\mu_{\mathrm{em}}(x) j^\nu_{\mathrm{em}}(y) j^\lambda_{\mathrm{em}}(z) j^\sigma_{\mathrm{em}}(0)\} \, |0\rangle$$

- **in general** there are 138 tensor structures consisting of q_i^α and $g^{\alpha\beta}$

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{138} L_i^{\mu\nu\lambda\sigma} \Xi_i$$

- **Ward identities** put 95 linear constraints on scalar functions Ξ_i

$$\{q_1^\mu, q_2^\nu, q_3^\lambda, q_4^\sigma\} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = 0$$

Tensor structures

BTT recipe and ambiguities

CHPS 2015, 2017

- **projection** gives basis for subspace fulfilling the Ward identities, but with **singularities** in the tensor structures
  $\to$ problematic for dispersion relations

- singularities can be **removed** Bardeen & Tung, Phys. Rev. 1968
  but: set becomes **incomplete** at specific kinematic points

- **add** non-singular tensor structures to obtain generating set at all kinematic points Tarrach, Nuovo Cim. A 1975

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

but: BTT set $T_i^{\mu\nu\lambda\sigma}$ is **overcomplete**, which implies **ambiguities** in the scalar coefficient functions Π_i

Tensor structures

Limit $q \to 0$ and Master formula

- for a_μ^{HLbL} we need **two-loop integral** over

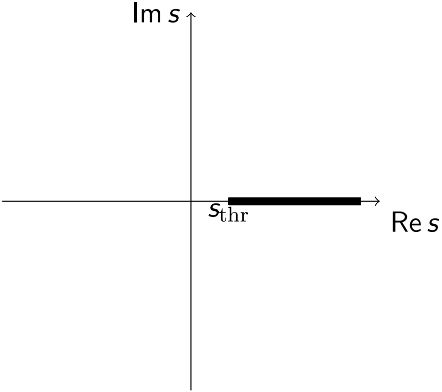
$$\lim_{q_4 \to 0} \frac{\partial}{\partial q_{4\rho}} \Pi^{\mu\nu\lambda\sigma}$$

- 35 linear combinations of the 54 structures **vanish** in this limit
- 5 of the 8 integrals can be performed in **full generality**
- due to **symmetry** only 12 linear combinations of scalar functions in the limit $q_4 \to 0$ enter the master formula

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2} \int_0^\infty \mathrm{d}Q_1 \int_0^\infty \mathrm{d}Q_2 \int_{-1}^1 \mathrm{d}\tau \sqrt{1-\tau^2} Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)$$

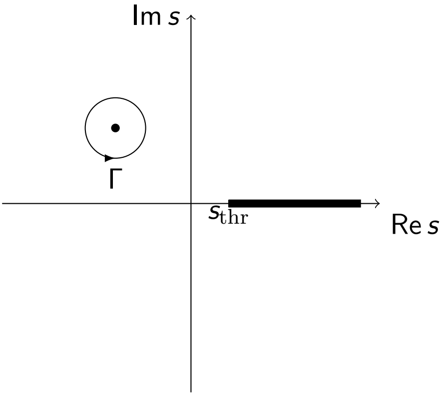
- $Q_i = \sqrt{-q_i^2}$, $\tau = \tau(Q_1, Q_2, Q_3)$
- kernel functions T_i **known** analytically
  $\to$ aim of dispersive approach(es) is to **reconstruct** $\bar{\Pi}_i$

Dispersion relations for low-energy hadronic processes



- Consider scalar function $F(s)$ that is analytic apart from **branch cut** on real axis

Dispersion relations for low-energy hadronic processes



- Consider scalar function $F(s)$ that is analytic apart from **branch cut** on real axis
- Cauchy's Theorem: $F(s) = \frac{1}{2\pi i} \oint_\Gamma \mathrm{d}s' \frac{F(s')}{s'-s}$

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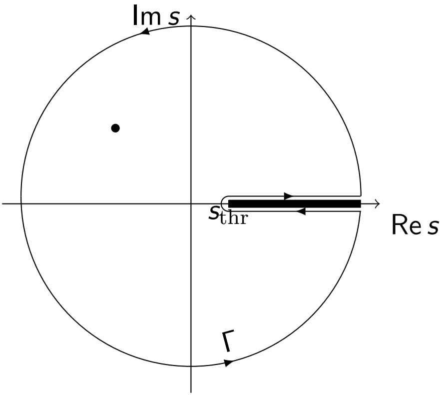

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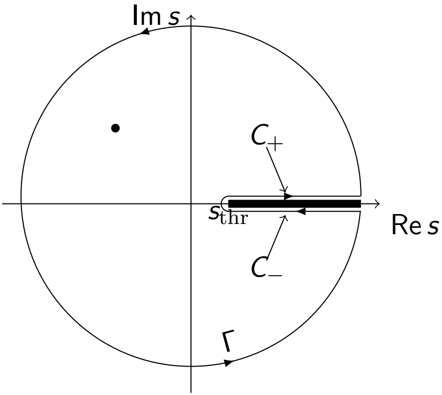
- deform integration contour

Dispersion relations for low-energy hadronic processes



- Consider scalar function $F(s)$ that is analytic apart from **branch cut** on real axis
- Cauchy's Theorem: $F(s) = \frac{1}{2\pi i} \oint_\Gamma \mathrm{d}s' \frac{F(s')}{s'-s}$
- deform integration contour
- If F falls off **sufficiently fast**, only C_+ and C_- contribute

$$F(s) = \frac{1}{2\pi i} \int_{s_{\text{thr}}}^{\infty} \mathrm{d}s' \frac{\text{disc}\, F(s')}{s'-s},$$

$$\text{disc}\, F(s) = F(s+i\epsilon) - F(s-i\epsilon) = 2i\,\mathrm{Im} F(s)$$

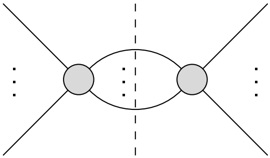
Dispersion relations for low-energy hadronic processes

Unitarity relations

- imaginary part determined from **unitarity** of the S-matrix ($SS^\dagger = 1$)
- plug in $S = 1 + iT$: $i(T^\dagger - T) = TT^\dagger$
- T-invariance implies $T^T = T$ and thus $i(T^* - T) = 2\mathrm{Im}\,T = TT^*$
- sandwiching this between states gives

$$\mathrm{Im}\,\langle f|\,T\,|i\rangle = \sum_s \langle f|\,T\,|s\rangle \langle s|\,T\,|i\rangle^*$$

- can be visualized by **unitarity diagrams**



- at **low energies**, light states with low multiplicity **dominate** in sum

Dispersion relations in four-point kinematics

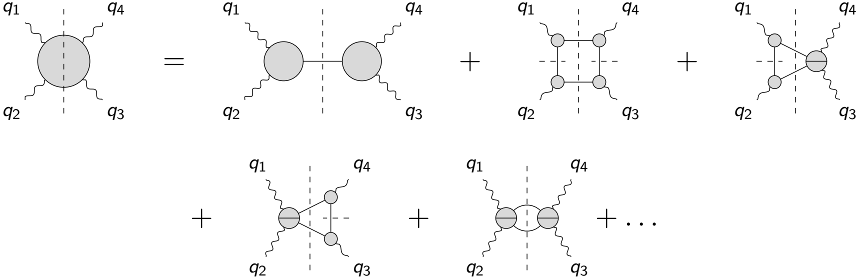
Application to HLbL

- write dispersion relations for $\boldsymbol{\Pi_i}$ (overcomplete generating set free of kinematic singularities) in s, t, u for **fixed** q_i^2
- single out **lightest** states in unitarity relations in **each channel**
  $\rightarrow$ imaginary parts given in terms of (simpler) sub-processes
- allows **model-independent definition** of individual contributions
- use **experimental data** on sub-processes to evaluate contributions to a_μ^{HLbL} due to light intermediate states
  $\rightarrow$ **reliable** uncertainty estimate for each contribution

Dispersion relations in four-point kinematics

Topologies

- focus on **1 and 2 particle** intermediates states



- crossed diagrams not shown
- s-channel π^0 pole contributes to **only** 1 scalar function

$$\Pi_1^{\pi^0-\text{pole}} = \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) F_{\pi^0\gamma^*\gamma^*}(q_3^2, q_4^2)}{s - m_\pi^2}$$

$$\xrightarrow{q_4 \to 0} \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) F_{\pi^0\gamma^*\gamma^*}(q_3^2, 0)}{q_3^2 - m_\pi^2}$$

Dispersion relations in four-point kinematics

Results and current status

- contributions of 1 and 2 pseudoscalar states under **good control**
  - pseudoscalar poles: $93.8(4.0) \times 10^{-11}$

    Hoferichter et al., JHEP 2018, Masjuan et al., Phys. Rev. D 2017
  - π- and K-loops: $-16.4(2) \times 10^{-11}$ CHPS, JHEP 2017, WP 2020
  - S-wave $\pi\pi$ rescattering: $-8(1) \times 10^{-11}$ CHPS, JHEP 2017
  - huge improvement in **precision** and **reliability** compared to earlier (model) estimates

- full description of 3 particle and higher intermediate states very challenging

  $\rightarrow$ describe through resonances in **narrow width approximation**

- but: data very **scarce** $\rightarrow$ only rough estimates possible, mostly based on models

- at high energies pQCD and OPE used to constrain the Π_i

Melnikov & Vainshtein, Phys. Rev. D 2004, Colangelo et al. JHEP 2020, JL & Procura, EPJC 2020

Dispersion relations in four-point kinematics

Singly on-shell basis and sum rules

CHPS, JHEP 2017

- sufficient to consider $q_4^2 = 0$
- in this limit a Lorentz **basis free of kinematic singularities** in s, t, u exists ($\check{\Pi}_i$)
- $\check{\Pi}_i$ have **different** mass dimensions
  $\rightarrow$ $\check{\Pi}_i$ with lower mass dimension fall off faster at high energies
  $\rightarrow$ implies **sum rules** of form $\int \mathrm{d}s' \, \mathrm{Im}\check{\Pi}_i(s') = 0$
- sum rules guarantee **basis independence** of a_μ^{HLbL}
- but: sum rules **only** fulfilled for (infinite) **sum** over intermediate states
  $\rightarrow$ individual contributions **basis dependent**
- **exception:** pseudoscalar poles and loops fulfill sum rules individually

Dispersion relations in four-point kinematics

Current limitations due to singularities in photon virtualities

- in addition: $\check{\Pi}_i$ have **singularities in q_i^2**
  $\rightarrow$ residues vanish due to **sum rules** for (infinite) **sum** over intermediate states

- poles lead to **non-convergent** master formula integrals for **individual** contributions
  $\rightarrow$ must **subtract** poles using same prescription for all contributions
  $\rightarrow$ additional **ambiguity**

- in original basis this affects contributions with spin ≥ 1

- by basis change it can be **avoided** for axial-vector mesons

  Colangelo et al., EPJC 2021

- **without additional** sum rules singularities are **unavoidable** for intermediate states with spin ≥ 2

How can we overcome this limitation?

$\rightarrow$ Dispersion relations at $q_4 = 0$

Dispersion relations in triangle kinematics

General idea and advantages

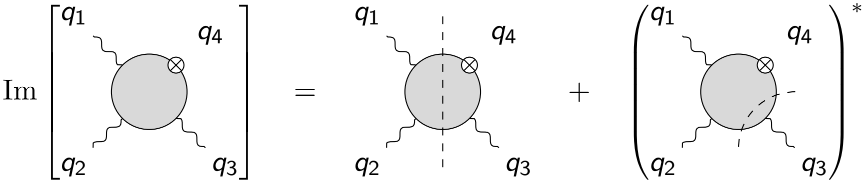
- has been realized that dispersion relations can also be written at $q_4 \to 0$ Colangelo et al., JHEP 2020
- at $q_4 \to 0$, all **ambiguities disappear** and a Lorentz basis free of kinematic singularities ($\hat{\Pi}_i$) exists
- dispersion relations for them **avoid ambiguities** coming from subtraction of spurious **poles**
- will allow to include D-wave $\pi\pi$ rescattering, tensor-meson poles, ...

Dispersion relations in triangle kinematics

Addition of cuts

- suspend additional arguments and use simplified notation
  $\hat{\Pi}_i(q_3^2) = \lim_{s \to q_3^2} \check{\Pi}_i(s, q_3^2)$ with $s = (q_3 + q_4)^2$

$$\mathrm{Im}\,\hat{\Pi}_i(q_3^2) = \lim_{s \to q_3^2} \left[\mathrm{Im}_s \check{\Pi}_i(s, q_3^2 + i\epsilon) + \mathrm{Im}_3 \check{\Pi}_i(s + i\epsilon, q_3^2)^* \right]$$

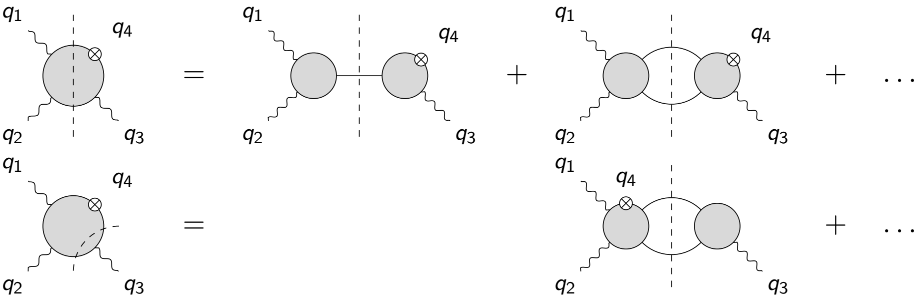


$\to$ s- and q_3^2-channel cuts have to be **added**

Dispersion relations in triangle kinematics

Topologies and sub-processes

- s- and q_3^2-channel cuts with 1 and 2 pion intermediate states



$\rightarrow$ all sub-processes **except** for $\gamma^*\gamma^* \rightarrow \pi\pi\gamma$ well-known
$\rightarrow$ **cancellation** of **infrared divergences** in $\pi^+\pi^-$ intermediate states between s- and q_3^2-cuts **demonstrated**

- s-channel **resonance contributions** given in terms of transition form factors (including axials and tensor mesons ...)

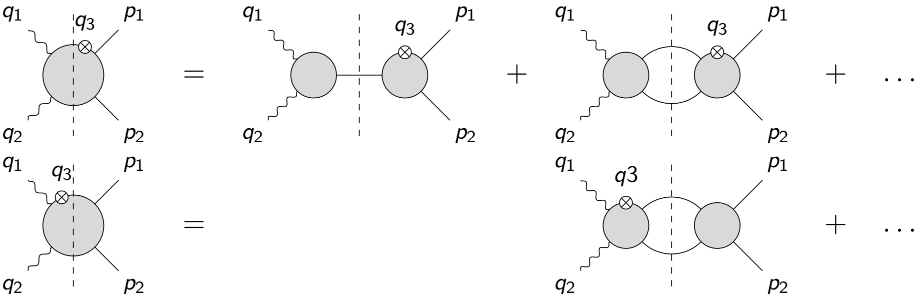
Dispersion relations in triangle kinematics

Topologies and sub-processes

- s- and q_3^2-channel cuts with 1 and 2 pion intermediate states

  $\rightarrow$ all sub-processes **except** for $\gamma^*\gamma^* \rightarrow \pi\pi\gamma$ well-known

- again 2 different (s-channel) cuts



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The sub-process $\pi\pi \to \gamma\pi\pi$

- derivation of dispersive description to be inserted into HLbL in triangle kinematics
- shares **many** features with $\gamma^*\gamma^* \to \pi\pi\gamma$ and HLbL
- Lorentz structure **much** simpler

$\to$ focus on this for the rest of the talk

The sub-process $\pi\pi \to \gamma\pi\pi$

Kinematics and Lorentz decomposition

- amplitude $\mathcal{M}(\pi^0\pi^0 \to \pi^+\pi^-\gamma) = \epsilon^*_\mu \mathcal{M}^\mu$
- charged channel $(\pi^+\pi^- \to \pi^+\pi^-\gamma)$ also needed, but related to mixed channel through **isospin symmetry** Kuhn 1999, Ecker & Unterdorfer 2002
- **BTT decomposition** $\mathcal{M}^\mu = \sum_{i=1}^{6} \hat{T}_i^\mu \mathcal{M}_i$ leads 3 Tarrach redundancies

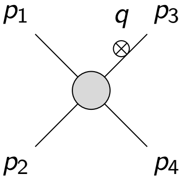
$$
\begin{aligned}
\hat{T}_1^\mu &= p_2^\mu (p_3 \cdot q) - p_3^\mu (p_2 \cdot q)\,, & \hat{T}_2^\mu &= p_3^\mu (p_1 \cdot q) - p_1^\mu (p_3 \cdot q)\,,\\
\hat{T}_3^\mu &= p_1^\mu (p_2 \cdot q) - p_2^\mu (p_1 \cdot q)\,, & \hat{T}_4^\mu &= q^\mu (p_1 \cdot q) - p_1^\mu q^2\,,\\
\hat{T}_5^\mu &= q^\mu (p_2 \cdot q) - p_2^\mu q^2\,, & \hat{T}_6^\mu &= q^\mu (p_3 \cdot q) - p_3^\mu q^2
\end{aligned}
$$

- 5-particle process has 10 kinematic invariants, 5 fixed by on-shell conditions

The sub-process $\pi\pi \to \gamma\pi\pi$

Soft-photon limit and singularities

- in principle only $\lim_{q\to 0} \frac{\partial}{\partial q_\nu}\mathcal{M}^\mu$ needed
- Tarrach redundancies drop out in this limit and a 2D basis exists
- **but**: limit **does not exist** due to

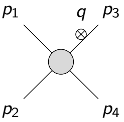


$\to$ **split** $\mathcal{M}^\mu = \mathcal{M}^\mu_{\text{sing}} + \mathcal{M}^\mu_{\text{reg}}$

- ambiguity to shift finite terms between $\mathcal{M}^\mu_{\text{sing}}$ and $\mathcal{M}^\mu_{\text{reg}}$
- for $\mathcal{M}^\mu_{\text{reg}}$ the limit can be performed and the problem reduces to **4-point kinematics** $\to$ Mandelstam variables
- singularities cancel when plugged into HLbL
- need gauge-invariant non-perturbative definition of $\mathcal{M}^\mu_{\text{sing}}$

The sub-process $\pi\pi \to \gamma\pi\pi$

Definition of $\mathcal{M}^\mu_{\text{sing}}$



Low, Phys. Rev. 1958

- **Low's theorem**: terms of order q^{-1} obtainable from with scalar QED vertex for soft photon,
  terms of order q^0 fixed by imposing Ward identity
  $\to$ terms up to order q^0 given in terms of $\pi\pi \to \pi\pi$

- but: $\lim_{q\to 0} \frac{\partial}{\partial q_\nu}(\mathcal{M}^\mu - \mathcal{M}^\mu_{\text{sing}})$ still does **not exist** due to terms like $q^\mu \frac{p_i \cdot q}{p_j \cdot q}$ (of order q^1, but limit depends on **direction** of q)
  $\to$ need definition of $\mathcal{M}^\mu_{\text{sing}}$ that includes **all** singular terms

- achieved from **unitarity** with a single-pion intermediate state
  $\to$ also only depends on $\pi\pi \to \pi\pi$ amplitude $\mathcal{T}$

$$\mathcal{M}^\mu_{\text{sing}} = F^V_\pi(q^2)\left(\frac{(2p_3+q)^\mu}{(p_3+q)^2 - m_\pi^2}\mathcal{T}(s, \tilde{t} - u) - \frac{(2p_4+q)^\mu}{(p_4+q)^2 - m_\pi^2}\mathcal{T}(s, t - \tilde{u}) - 2(p_1 - p_2)^\mu \Delta\mathcal{T}\right)$$

$$\Delta\mathcal{T} = \frac{\mathcal{T}(s, \tilde{t} - u) - \mathcal{T}(s, t - \tilde{u})}{\tilde{t} - u - t + \tilde{u}}$$

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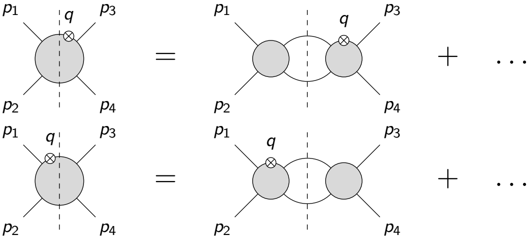
Unitarity relation and Cancellation of singularities

- two-pion intermediate states in unitarity relations involve $\pi\pi \to \gamma\pi\pi$ as a **sub-process** (similarly in t- and u-channels)

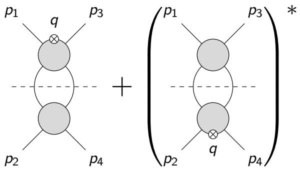


$\to$ contains the **soft-singular** piece $\mathcal{M}^{\mu}_{\text{sing}}$

- checked that **sum of cuts** reproduces the singularities of $\text{Im}^{\pi\pi}\mathcal{M}^{\mu}_{\text{sing}}$ $\to$ **finite difference** is $\text{Im}^{\pi\pi}\mathcal{M}^{\mu}_{\text{reg}}$ and can be projected onto basis Lorentz structures in limit $q \to 0$

The sub-process $\pi\pi \to \gamma\pi\pi$

Imaginary parts



- t-channel imaginary part of 1 scalar function

$$\mathrm{Im}_t^{\pi^-\pi^0}\bar{\mathcal{M}}_1 = \frac{1}{8}\frac{\sqrt{1-\frac{4m_\pi^2}{t}}}{16\pi^2}\mathrm{Re}\sum_l (2l+1)\int \mathrm{d}\Omega_t'(t_l^{1*}(t)+t_l^{2*}(t))P_l(z_t')$$
$$\times\left[\left(2\frac{z_t'-z_t''}{1-z_t}+\frac{z_t'+z_t''}{1+z_t}-5\right)\bar{\mathcal{M}}_1(t,z_t'')+\left(\frac{z_t'+z_t''}{1+z_t}-1\right)\bar{\mathcal{M}}_2(t,z_t'')\right]$$
$$+\Delta_1$$

- Δ_1 **finite remainder** of terms involving $\mathcal{M}^\mu_{\mathrm{sing}}$
  $\to$ only depends on $\pi\pi \to \pi\pi$ partial waves t_l^i
- s-channel imaginary parts depend on both s- and t-channel amplitudes
- have **checked** imaginary parts at one loop in χPT

The sub-process $\pi\pi \to \gamma\pi\pi$

Dispersion relations

- from **fixed-s dispersion relations** for $\bar{\mathcal{M}}_i$, make ansatz for **partial-wave expansion** in t-channel: $\bar{\mathcal{M}}_i(t, z_t) = \sum_{l=0}^{\infty}(2l+1)g_l^i(t)P_l(z_t)$
  - **allows** to perform angular integral
  - considering $\pi\pi$ scattering only up to D-waves truncates tower of g_l^i
- **project** dispersion relation onto partial waves to obtain **coupled integral equations** for $g_l^i(t) \to$ **Roy–Steiner equations**

$$g_j^i(t) = \sum_{j',l}\sum_{i'=1}^{2}\frac{1}{\pi}\int_{4m_\pi^2}^{\infty} dt' K_{jj'l}^{ii'}(t,t')\mathrm{Re}[(t_l^{1*}(t') + t_l^{2*}(t'))g_{j'}^{i'}(t')] + \Delta_j^i(t)$$

- **similar** equations exist for s-channel partial waves (include integrals over t-channel partial waves g_l^i)
- solving this will complete the dispersive reconstruction of the sub-process $\pi\pi \to \gamma\pi\pi$

Conclusions

- **discrepancy** between measurement and Standard Model prediction of a_μ could be due to **New Physics**: higher precision needed
- established dispersive formalism for HLbL **very successful** for most important contributions
- complementary dispersive approach **promises** to overcome roadblocks in inclusion of higher-spin intermediate states
- important steps towards complete calculation in new approach **already achieved**:
  - unitarity relations
  - cancellation of infrared divergences
  - new dispersion relations for $\pi\pi \to \gamma\pi\pi$

Outlook

- solution of Roy–Steiner equations will **complete** study of $\pi\pi \to \gamma\pi\pi$
- with $\pi\pi \to \gamma\pi\pi$ as input, **similar study** possible for $\gamma^*\gamma^* \to \pi\pi\gamma$
  - more complicated Lorentz structure
  - but: **many similarities** concerning kinematics, cancellation of IR singularities, phase-space integrals, ... expected
- this will allow for a complete treatment of all 1- and 2-particle intermediate states in HLbL with **arbitrary angular momenta**
- study in detail **reshuffling** of contributions between the 2 dispersive approaches to HLbL
  - learn how to combine them to include as many contributions as possible
- study dispersive representation of pQCD quark loop to **incorporate short-distance constraints**

$\to$ Michael's work

Thank you for your attention!

Backup

Addition of cuts

- suppress additional arguments and use simplified notation $\mathrm{Im}\hat{\Pi}_i(q_3^2) = \lim_{s\to q_3^2} \check{\Pi}_i(s, q_3^2)$ with $s = (q_3 + q_4)^2$

$$
\begin{aligned}
\mathrm{Im}\,\hat{\Pi}_i(s) &= \lim_{q_3^2\to s} \frac{\check{\Pi}_i(s+i\epsilon, q_3^2+i\epsilon) - \check{\Pi}_i(s-i\epsilon, q_3^2-i\epsilon)}{2i} \\
&= \lim_{q_3^2\to s} \Bigg[\frac{\check{\Pi}_i(s+i\epsilon, q_3^2+i\epsilon) - \check{\Pi}_i(s-i\epsilon, q_3^2+i\epsilon)}{2i} \\
&\qquad + \frac{\check{\Pi}_i(s-i\epsilon, q_3^2+i\epsilon) - \check{\Pi}_i(s-i\epsilon, q_3^2-i\epsilon)}{2i}\Bigg] \\
&= \lim_{q_3^2\to s} \Bigg[\frac{\check{\Pi}_i(s+i\epsilon, q_3^2+i\epsilon) - \check{\Pi}_i(s-i\epsilon, q_3^2+i\epsilon)}{2i} \\
&\qquad + \left(\frac{\check{\Pi}_i(s+i\epsilon, q_3^2+i\epsilon) - \check{\Pi}_i(s+i\epsilon, q_3^2-i\epsilon)}{2i}\right)^*\Bigg] \\
&=: \lim_{q_3^2\to s} \left[\mathrm{Im}_s\check{\Pi}_i(s, q_3^2+i\epsilon) + \mathrm{Im}_3\check{\Pi}_i(s+i\epsilon, q_3^2)^*\right]
\end{aligned}
$$