Reconciling the FOPT and CIPT predictions for τ hadronic spectral function moments

[part II in preparation]
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[arXiv:2111.09614]

in collaboration with André H. Hoang, Diogo Boito and Matthias Jamin

Miguel Benitez-Rathgeb, Vienna, 22.03.2022

CONTENT

- Introduction
- Renormalons
- Theoretical description of hadronic τ decay rate
- FOPT v CIPT
- RF - GC scheme

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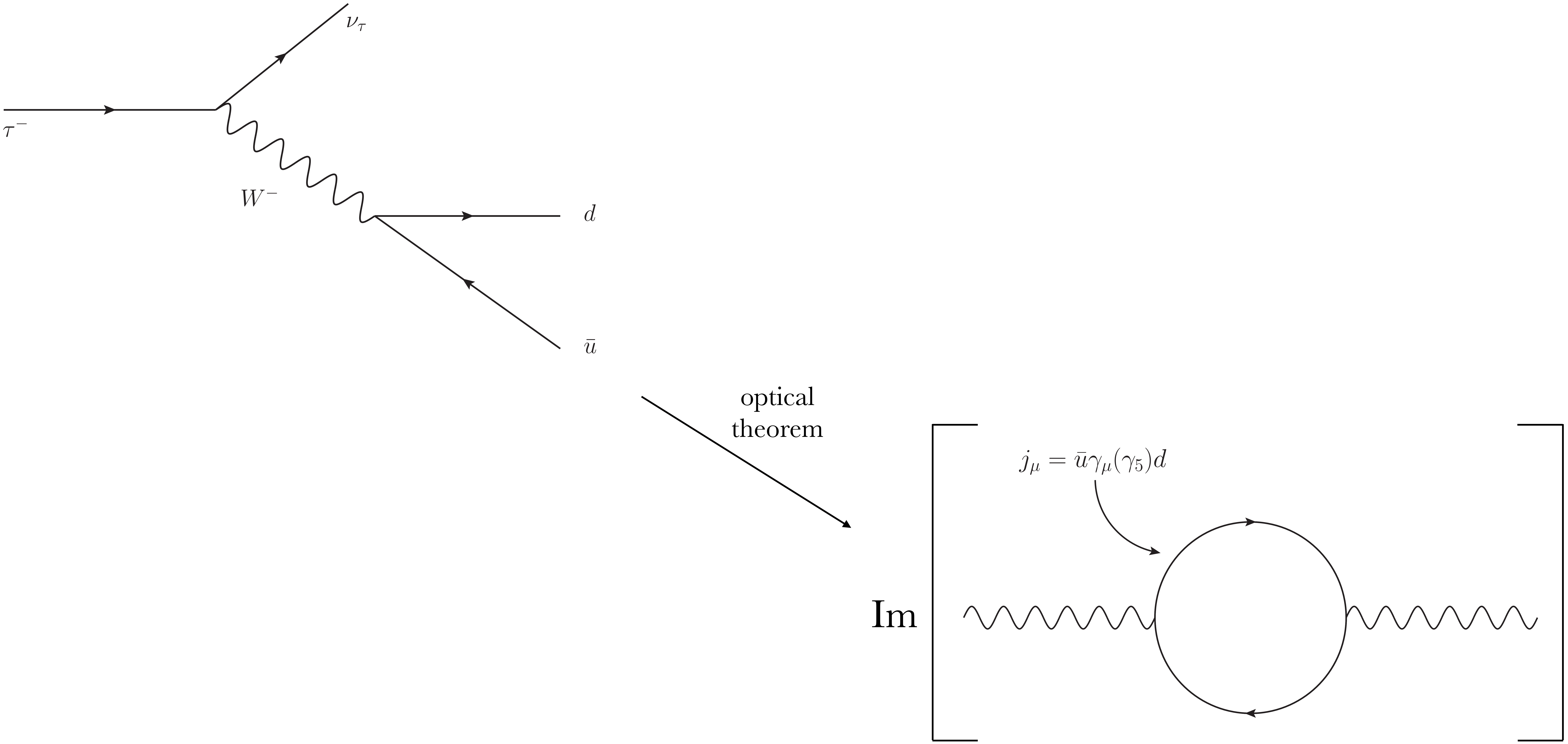

optical theorem

$j_\mu = \bar{u}\gamma_\mu(\gamma_5)d$

- In strong coupling determinations one considers

$$\underset{\substack{\uparrow \\ \text{spectral-function moment}}}{R^w(s_0)} = \int_0^{s_0} \frac{ds}{s_0}\, w(s) \frac{1}{\pi} \underset{\substack{\uparrow \\ \text{Experiment}}}{\mathrm{Im}\Pi(s)} = -\frac{1}{2\pi i} \oint_{|s|=s_0} \frac{ds}{s_0}\, w(s) \underset{\substack{\uparrow \\ \text{Theory}}}{\Pi(s)}$$

- Extraction of α_s from inclusive τ decay one of the most precise determinations of the QCD coupling
- Theoretical moments are of the form

$$-\frac{1}{2\pi i} \oint_{|s|=s_0} \frac{ds}{s_0}\, w(s)\Pi(s) \approx 1 + \underset{\substack{\uparrow \\ \text{depend on a prescription for setting renormalization scale } \mu}}{\delta^{(0)}} + \delta_{\mathrm{OPE}} + \delta_{\mathrm{DV}}$$

Two most widely employed prescriptions: Contour-Improved Perturbation Theory (**CIPT**) and Fixed-Order Perturbation Theory (**FOPT**)

Formally

FOPT:
$$\delta^{(0)}_{\mathrm{FO}} = \sum_{n=1}^{\infty} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n \sum_{k=1}^{n} k\ c_{n,k}\ J_{k-1}$$

CIPT:
$$\delta^{(0)}_{\mathrm{CI}} = \sum_{n=1}^{\infty} c_{n,1} \left[\frac{1}{2\pi i}\oint_{|s|=s_0} \frac{ds}{s_0} W(s) \left(\frac{\alpha_s(-s)}{\pi}\right)^n\right]$$

⟶ Lead to different predictions for the strong coupling

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QCD perturbation theory: $f(\alpha_s) = f(0) + \sum_{n=0}^{\infty} f_n \alpha_s^{n+1}$

Related series: $B[f](t) = f(0)\delta(t) + \sum_{n=0}^{\infty} \frac{f_n}{n!} t^n$ ≡ Borel transform (**BT**)

Due to $\int_0^{\infty} dz\ e^{-z/g} z^n = n! g^{n+1}$

recover original series: $f(\alpha_s) = \int_0^{\infty} dt\ e^{-t/\alpha_s} B[f](t)$ ≡ inverse BT (ambiguous)

Example:

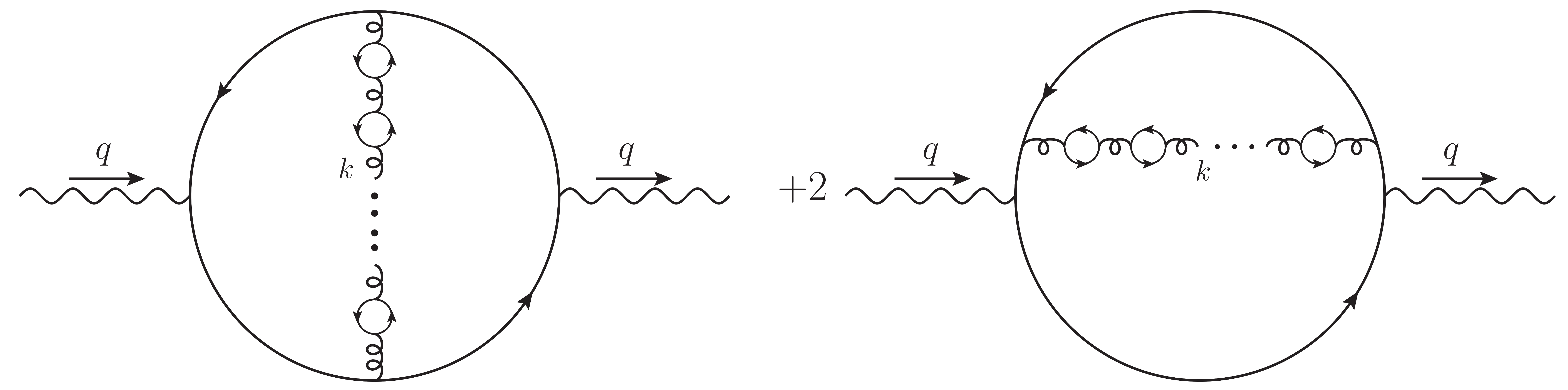
$$f_n = K a^n \Gamma(n+1+b) \xrightarrow{\text{BT}} B[f](t) = f(0)\delta(t) + \frac{K\Gamma(1+b)}{(1-at)^{1+b}}$$

$$B[f](t) = f(0)\delta(t) + \frac{K\Gamma(1+b)}{(1-at)^{1+b}}$$

$\longrightarrow$ Poles in the BT of a pert. series ≡ Renormalon divergencies

Depending on momentum region from which singularity originates:
Infrared (**IR**) Renormalon or Ultra-Violett (**UV**) Renormalon

Intermezzo: Large-β_0 approximation



- Model used to investigate higher order behavior of QCD series
- Diagrams provide leading correction in large-n_f expansion
- Naive Non-Abelianization (**NNA**): $n_f \to -\frac{3}{2}\left(\frac{11}{3}C_A - \frac{2}{3}n_f\right) = -\frac{3}{2}\beta_0$
- NNA turns large-n_f expansion into large-β_0 expansion and asymptotic freedom is recovered

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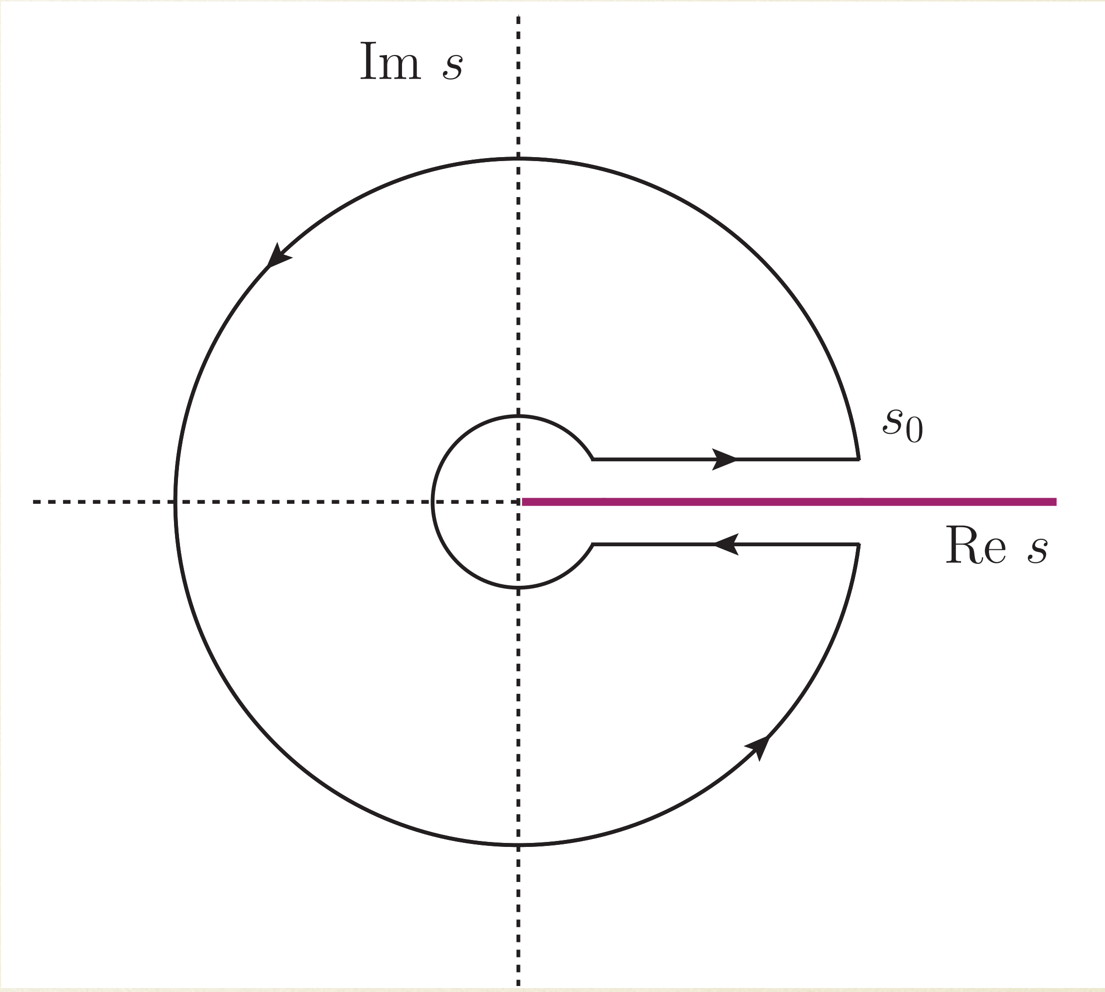
Starting point: Finite-Energy Sum Rule (**FESR**):

$$\int_0^{s_0} ds\ w(s)\rho(s) = -\frac{1}{2\pi i}\oint_{|s|=s_0} ds\ w(s)\Pi(s), \qquad \rho(s) = \frac{1}{\pi}\mathrm{Im}\Pi(s)$$



Vector correlation function (sufficient):

$$\Pi_{\mu\nu}(p) \equiv i\int \mathrm{d}x\, e^{ipx} \langle \Omega\,|\, T\{j_\mu(x) j_\nu(0)^\dagger\}\,|\,\Omega\rangle\,, \qquad j_\mu(x) = :\bar{u}(x)\gamma_\mu d(x):$$

$\Pi_{\mu\nu}(p)$ admits Lorentz decomposition

$$\Pi_{\mu\nu}(p) = (p_\mu p_\nu - g_{\mu\nu}p^2)\,\Pi^{(1+0)}(p^2) + g_{\mu\nu}\,p^2\,\Pi^{(0)}(p^2)$$

Correlator not a physical quantity since it contains a renorm. scale and scheme dependent subtraction constant

In the following consider (reduced) Adler function ($s \equiv p^2$, $\Pi(s) \equiv \Pi^{(1+0)}(s)$)

$$\frac{1}{4\pi^2}\left[1 + D(s)\right] \equiv - \; s \frac{\mathrm{d}}{\mathrm{d}s} \Pi(s)$$

Adler function subsequently expressed in terms of an Operator-Product Expansion (OPE):

$$D(s) = \hat{D}(s) + D^{\mathrm{OPE}}(s), \quad \text{with} \;\; \hat{D}(s) \; = \sum_{n=1}^{\infty} c_{n,1} \left(\frac{\alpha_s(-s)}{\pi} \right)^n$$

and

$$D^{\mathrm{OPE}}(s) \; = \frac{C_{4,0}(\alpha_s(-s))}{s^2} \langle \bar{\mathcal{O}}_{4,0} \rangle + \sum_{d=6}^{\infty} \frac{1}{(-s)^{d/2}} \sum_i C_{d,i}(\alpha_s(-s)) \langle \bar{\mathcal{O}}_{d,\gamma_i} \rangle$$

Correlator when computed with OPE in terms of quark and gluon fields does not equal hadronic counterpart

⟶ need to include Duality Violations (**DV**'s)

General form of theoretical moments entering the analysis

$$R_{V/A}^{(W)}(s_0) = \frac{N_c}{2} S_{\text{ew}} \, |V_{ud}|^2 \left[\delta_W^{\text{tree}} + \delta_W^{(0)}(s_0) + \sum_{d \geq 4} \delta_{W,V/A}^{(d)}(s_0) + \delta_{W,V/A}^{(\text{DV})}(s_0) \right]$$

Focus on $\left(x \equiv s/s_0\right)$

$$\delta_W^{(0)}(s_0) + \sum_{d \geq 4} \delta_W^{(d)}(s_0) = \frac{1}{2\pi i} \oint_{|s|=s_0} \frac{\mathrm{d}s}{s} W\!\left(\frac{s}{s_0}\right) D(s) = \frac{1}{2\pi i} \oint_{|x|=1} \frac{\mathrm{d}x}{x} W(x) \, D(x s_0)$$

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Take a step back and consider general structure of vector correlator

$$\Pi^V(s) \sim \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n \sum_{k=0}^{n+1} c_{n,k} \ln^k \left(\frac{-s}{\mu^2}\right)$$

For Adler function

$$D(s) \sim \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n \sum_{k=1}^{n+1} k\ c_{n,k} \ln^{k-1} \left(\frac{-s}{\mu^2}\right)$$

Adler function satisfies homogeneous RGE $\Rightarrow$ *Resumm logarithms*

FOPT $(\mu^2 = s_0)$

$$\delta_W^{(0),\mathrm{FO}}(s_0) = \sum_{n=1}^{\infty} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n \sum_{k=1}^{n} k\ c_{n,k}\ \frac{1}{2\pi i} \oint\limits_{|x|=1} \frac{\mathrm{d}x}{x} W(x)\ \ln^k(-x)$$

CIPT $(\mu^2 = -xs_0)$

$$\delta_W^{(0),\mathrm{CI}}(s_0) = \sum_{n=1}^{\infty} c_{n,1}\ \frac{1}{2\pi i} \oint\limits_{|x|=1} \frac{\mathrm{d}x}{x} W(x) \left(\frac{\alpha_s(-xs_0)}{\pi}\right)^n$$

FOPT $(\mu^2 = s_0)$

$$\delta_W^{(0),\mathrm{FO}}(s_0) = \sum_{n=1}^{\infty} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n \sum_{k=1}^{n} k\, c_{n,k}\, \frac{1}{2\pi i} \oint_{|x|=1} \frac{\mathrm{d}x}{x}\, W(x)\, \ln^k(-x)$$

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[Hoang&Regner]

$$= \sum_{n=1}^{\infty} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n c_{n,1}\, \frac{1}{2\pi i} \oint_{|x|=1} \frac{\mathrm{d}x}{x}\, W(x) \left(\frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\right)^n$$

For the Borel representations of the respective series one obtains

- FOPT Borel representation (FOPT Borel sum (**BS**))

$$\delta^{(0),\mathrm{FO}}_{W,\mathrm{Borel}}(s_0) = \mathrm{PV}\int_0^\infty \mathrm{du}\,\frac{1}{2\pi\mathrm{i}}\oint\limits_{|\mathrm{x}|=1}\frac{\mathrm{dx}}{\mathrm{x}}\,\mathrm{W(x)}\,\mathrm{B}[\hat{\mathrm{D}}](\mathrm{u})\ \mathrm{e}^{-\frac{4\pi\mathrm{u}}{\beta_0\alpha_\mathrm{s}(-\mathrm{x}\mathrm{s}_0)}}$$

- CIPT Borel representation

$$\delta^{(0),\mathrm{CI}}_{W,\mathrm{Borel}}(s_0) = \int_0^\infty d\bar{u}\,\frac{1}{2\pi i}\oint\limits_{C_x}\frac{dx}{x}\,W(x)\left(\frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\right)\,B[\hat{D}]\left(\frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\bar{u}\right)\ e^{-\frac{4\pi\bar{u}}{\beta_0\alpha_s(s_0)}}$$

Define Asymptotic separation (**AS**)

$$\Delta_W(s_0) \equiv \delta^{(0),\mathrm{CI}}_{W,\mathrm{Borel}}(s_0) - \delta^{(0),\mathrm{FO}}_{W,\mathrm{Borel}}(s_0)$$

In large-β_0 exact form of BT known

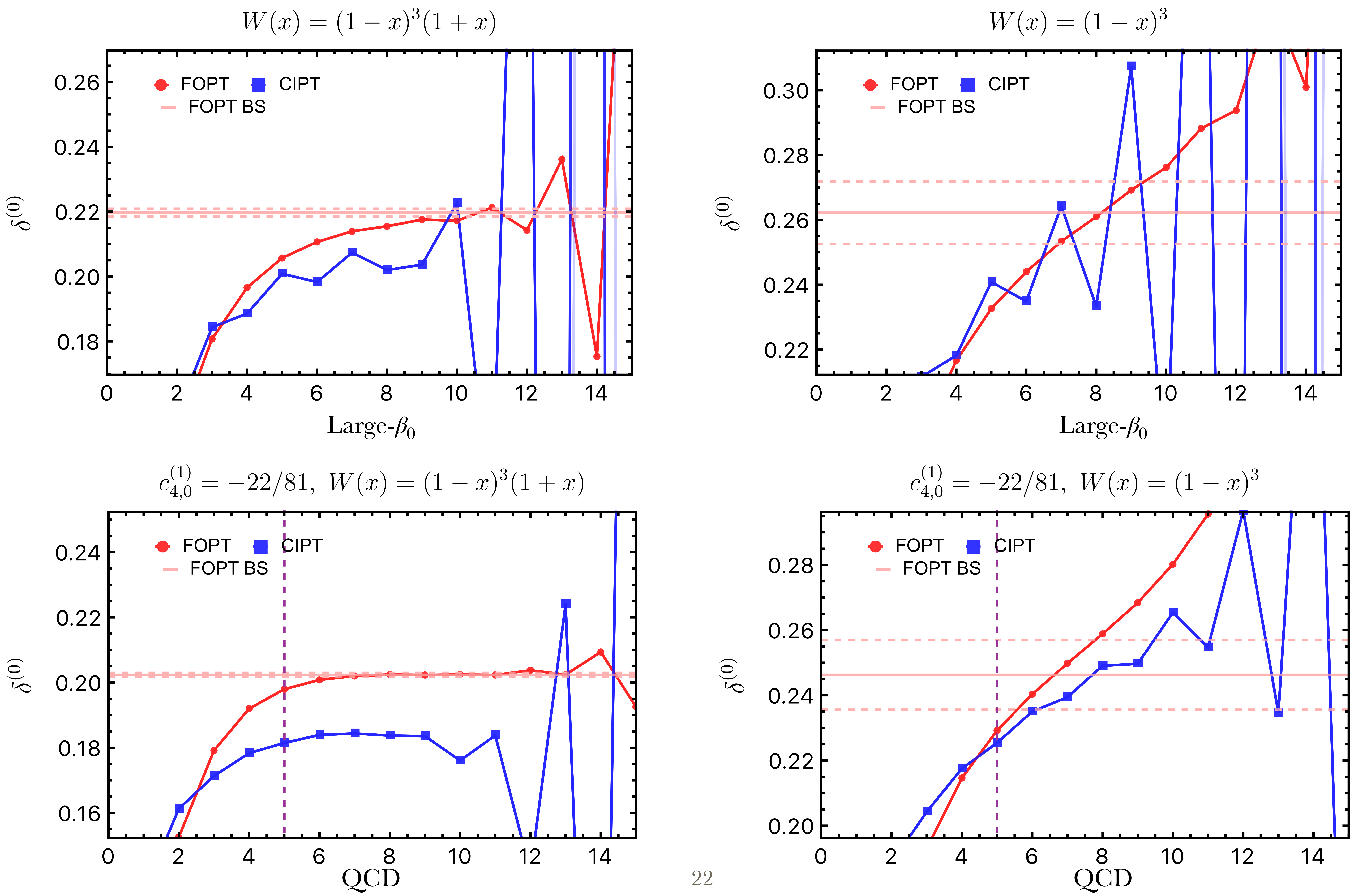
[Broadhurst]

$$B[\hat{D}](u) \;=\; \frac{128}{3\beta_0}\, e^{5u/3}\left\{\frac{3}{16(2-u)} + \sum_{p=3}^{\infty}\left[\frac{d_2(p)}{(p-u)^2} - \frac{d_1(p)}{p-u}\right] - \sum_{p=-1}^{-\infty}\left[\frac{d_2(p)}{(u-p)^2} + \frac{d_1(p)}{u-p}\right]\right\}$$

with $d_2(p) = \dfrac{(-1)^p}{4(p-1)(p-2)}$ and $d_1(p) = \dfrac{(-1)^p(3-2p)}{4(p-1)^2(p-2)^2}$

In full QCD one relies on models (C-scheme used for our studies, denoted by barred quantities in the following) [Boito,Jamin,Miravitllas]

$$B[\hat{D}(s)]_{\mathrm{mr}}(u) \sim b^{(0)} + b^{(1)}u + \frac{N_{4,0}}{(2-u)^{\gamma_4}} + \frac{N_{6,0}}{(3-u)^{\gamma_6}} + \frac{N_{-2}}{(1+u)^{\gamma_{(-2)}}}$$

$W(x) = (1-x)^3(1+x)$
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FOPT
CIPT
FOPT BS
$\delta^{(0)}$
0.26
0.24
0.22
0.20
0.18
0.30
0.28
0.26
0.24
0.22
0.16
0.20
0
2
4
6
8
10
12
14
Large-β_0
QCD

Conclusion from Analysis/Motivation of our work:

- CIPT/FOPT discrepancy is systematic, not accidental and not an effect due to missing higher orders
- CIPT not compatible with standard OPE
- Asymptotic separation vanishes if IR Renormalons are absent
- CIPT and FOPT should become consistent for IR-subtracted PT

Intermezzo II: The C-scheme

- We use the *C*-scheme for the QCD coupling

$$\frac{\pi}{\bar{\alpha}_s(Q^2)} + \frac{\beta_1}{4\beta_0}\ln(\bar{\alpha}_s(Q^2)) = \frac{\pi}{\alpha_s(Q^2)} + \frac{\beta_1}{4\beta_0}\ln(\alpha_s(Q^2)) + \frac{\beta_0}{2}\int_0^{\alpha_s(Q^2)} d\tilde{\alpha}\left[\frac{1}{\beta(\tilde{\alpha})} + \frac{2\pi}{\beta_0\tilde{\alpha}^2} - \frac{\beta_1}{2\beta_0^2\tilde{\alpha}}\right]$$

[Boito,Jamin,Miravitllas]

- The β-function is exact in this scheme

$$\frac{d\bar{\alpha}_s(Q^2)}{d\ln Q} = \bar{\beta}(\bar{\alpha}_s(Q^2)) \equiv -2\,\bar{\alpha}_s(Q^2)\frac{\beta_0\,\bar{\alpha}_s(Q^2)}{4\pi - \frac{\beta_1}{\beta_0}\bar{\alpha}_s(Q^2)}$$

- Final numerical results all transformed back to usual $\overline{\text{MS}}$-scheme

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$$D(s) = \hat{D}(s) + D^{\mathrm{OPE}}(s)$$

- IR Renormalons induce ambiguity in the Borel integral
- Physical observables are ambiguity free
- Higher dimensional OPE contributions cancel ambiguities caused by IR Renormalons in $\hat{D}(s)$ (cancel order-by-order in pert. th. for asymptotically increasing behavior of QCD corrections)
- Idea is to define a scale-independent renormalon-free Gluon Condensate (**GC**) where order-dependent compensating contribution is made explicit

Leading $d = 4$ OPE correction to Adler function (for massless quarks)

$$\delta D_{4,0}^{\mathrm{OPE}}(-Q^2) = \frac{1}{Q^4}\frac{2\pi^2}{3}\left[1 + \bar{c}_{4,0}^{(1)}\,\bar{a}_Q\right]\langle \bar{G}^2\rangle\,, \quad \text{with} \quad \bar{a}_Q \equiv \frac{\beta_0\,\bar{\alpha}_s(Q^2)}{4\pi}$$

Term in the Euclidean Adler function's Borel function that corresponds to the GC OPE correction

$$B_{4,0}(u) = \left[1 + \bar{c}_{4,0}^{(1)}\,\bar{a}_Q\right]\frac{N_{4,0}}{(2-u)^{1+4\hat{b}_1}}\,, \quad \text{with} \quad \hat{b}_1 \equiv \frac{\beta_1}{2\beta_0^2}$$

Borel function term $B_{4,0}(u)$ fully quantifies $u = 2$ IR Renormalon contribution in coefficients of pert. series associated to OPE correction term $\delta D_{4,0}^{\mathrm{OPE}}(-Q^2)$

$$\delta\hat{D}_{4,0}(-Q^2) \;=\; N_{4,0}\left[1 + \bar{c}_{4,0}^{(1)}\,\bar{a}_Q\right]\sum_{\ell=1}^{\infty} r_\ell^{(4,0)}\,\bar{a}_Q^{\ell} \quad \text{with} \quad r_\ell^{(4,0)} = \left(\frac{1}{2}\right)^{\ell+4\hat{b}_1}\frac{\Gamma(\ell+4\hat{b}_1)}{\Gamma(1+4\hat{b}_1)}$$

Relation between original order-dependent GC $\langle \bar{G}^2 \rangle^{(n)}$ in the $\overline{\text{MS}}$-scheme and new Renormalon-Free (**RF**) order-independent GC $\langle G^2 \rangle(R)$

$$\langle \bar{G}^2 \rangle^{(n)} \equiv \langle G^2 \rangle(R^2) - R^4 \sum_{\ell=1}^{n} N_g \, r_\ell^{(4,0)} \, \bar{a}_R^\ell , \qquad N_g = \frac{3}{2\pi^2} N_{4,0}$$

Purpose of RF GC is to reshuffle series on RHS back into pert. series for Euclidean Adler function $\hat{D}(-Q^2)$ s.t. effects of GC Renormalon from original series are eliminated.

Resulting subtraction series generated by inverse BT

$$\delta \hat{D}_{4,0}(-Q^2, R^2) = - \left[1 + \bar{c}_{4,0}^{(1)} \, \bar{a}_Q \right] \int_0^\infty \mathrm{d}u \left[\frac{R^4}{Q^4} \frac{N_{4,0}}{(2-u)^{1+4\hat{b}_1}} \right]_{\text{Taylor}} e^{-\frac{u}{\bar{a}_R}}$$

Need to expand resulting series in $\bar{a}_R$ in $\bar{a}_Q$ and truncate at same order as original series to cancel GC Renormalon

At the moment we have

$$\Delta\hat{D}_{4,0}(-Q^2, R^2) = \left[1 + \bar{c}^{(1)}_{4,0}\,\bar{a}_Q\right] N_{4,0} \int_0^\infty \mathrm{d}u \left[\frac{e^{-\frac{u}{\bar{a}_Q}}}{(2-u)^{1+4\hat{b}_1}} - \frac{R^4}{Q^4}\frac{e^{-\frac{u}{\bar{a}_R}}}{(2-u)^{1+4\hat{b}_1}} \right]$$

$\longrightarrow$ Ambiguity due to cuts cancels in the difference

Next step: Get rid of R-dependence of $\langle G^2\rangle(R^2)$

Take additional step to define scale-invariant RF GC matrix element

Need a function that obeys the same *R*-evolution equation as subtraction series

Obtained from BS of subtraction series

$$\bar{c}_0(R^2) \equiv R^4 \;\; \mathrm{PV} \int_0^\infty \frac{\mathrm{d}u \;\, e^{-\frac{u}{\bar{a}_R}}}{(2-u)^{1+4\hat{b}_1}}$$

$$= \begin{cases} -\dfrac{R^4\, e^{-\frac{2}{\bar{a}_R}}}{(-\bar{a}_R)^{4\hat{b}_1}}\,\Gamma\Big(-4\hat{b}_1, -\dfrac{2}{\bar{a}_R}\Big) - \mathrm{sig}[\mathrm{Im}[a_R]]\,\dfrac{i\pi\, R^4\, e^{-\frac{2}{\bar{a}_R}}}{\Gamma(1+4\hat{b}_1)\,\bar{a}_R^{4\hat{b}_1}} & \text{for } \; \mathrm{Im}[R^2] \neq 0 \\[2ex] -\dfrac{R^4\, e^{-\frac{2}{\bar{a}_R}}}{(\bar{a}_R)^{4\hat{b}_1}}\,\mathrm{Re}\left[e^{4\pi\hat{b}_1 i}\,\Gamma\Big(-4\hat{b}_1, -\dfrac{2}{\bar{a}_R}\Big)\right] & \text{for } \; \mathrm{Im}[R^2] = 0 \end{cases}$$

Adding $\bar{c}_0(R^2)$ to $\Delta\hat{D}_{4,0}(-Q^2, R^2)$ shifts the resulting series back to original FOPT BS („Minimal scheme“ in this sense)

For the pert. series for the Adler function we obtain (in the RF GC scheme)

$$\hat{D}^{\mathrm{RF}}(s,R^2) = \frac{1}{s^2}\left[1+\bar{c}^{(1)}_{4,0}\bar{a}(-s)\right]N_{4,0}\,\bar{c}_0(R^2) \,+\, \int_0^\infty \mathrm{d}u\,\left[B[\hat{D}(s)](u)\right]_{\mathrm{Taylor}} e^{-\frac{u}{\bar{a}(-s)}}$$

$$-\,\left[1+\bar{c}^{(1)}_{4,0}\,\bar{a}(-s)\right]N_{4,0}\,\frac{R^4}{s^2}\int_0^\infty \mathrm{d}u\,\left[\frac{1}{(2-u)^{1+4\hat{b}_1}}\right]_{\mathrm{Taylor}} e^{-\frac{u}{\bar{a}_R}}$$

$$= \frac{1}{s^2}\left[1+\bar{c}^{(1)}_{4,0}\bar{a}(-s)\right]N_{4,0}\,\bar{c}_0(R^2) \,+\, \sum_{\ell=1}^{\infty}\bar{c}_\ell\,\bar{a}^\ell(-s)$$

($\bar{c}_\ell$ C-scheme analog to $c_{n,1}$)

$$-\,\left[1+\bar{c}^{(1)}_{4,0}\,\bar{a}(-s)\right]N_{4,0}\,\frac{R^4}{s^2}\sum_{\ell=1}^{\infty}\left(\frac{1}{2}\right)^{\ell+4\hat{b}_1}\frac{\Gamma(\ell+4\hat{b}_1)}{\Gamma(1+4\hat{b}_1)}\,\bar{a}_R^\ell$$

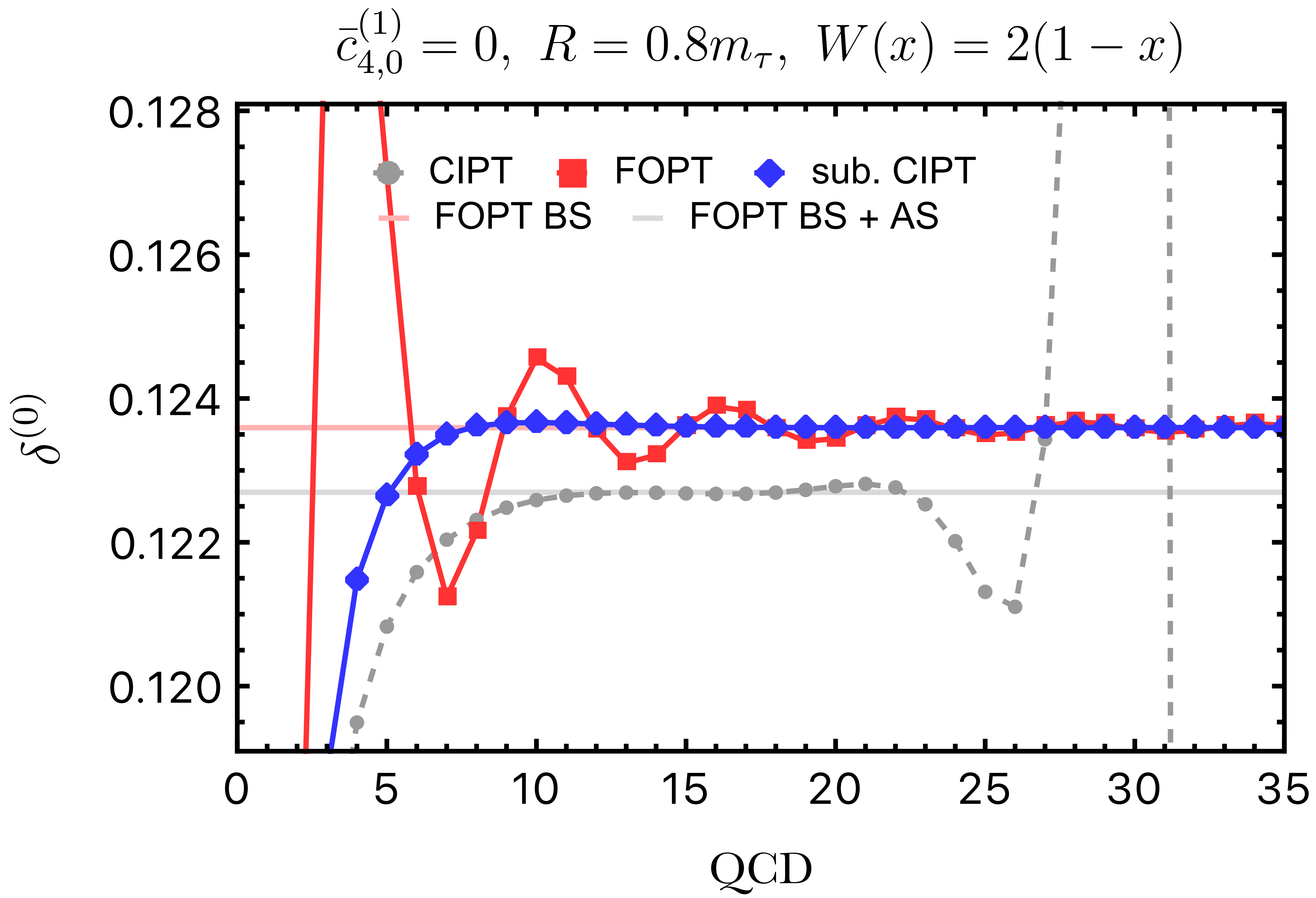
Consistently expand and truncate using α_s at a common renormalization scale

Toy Model analysis:

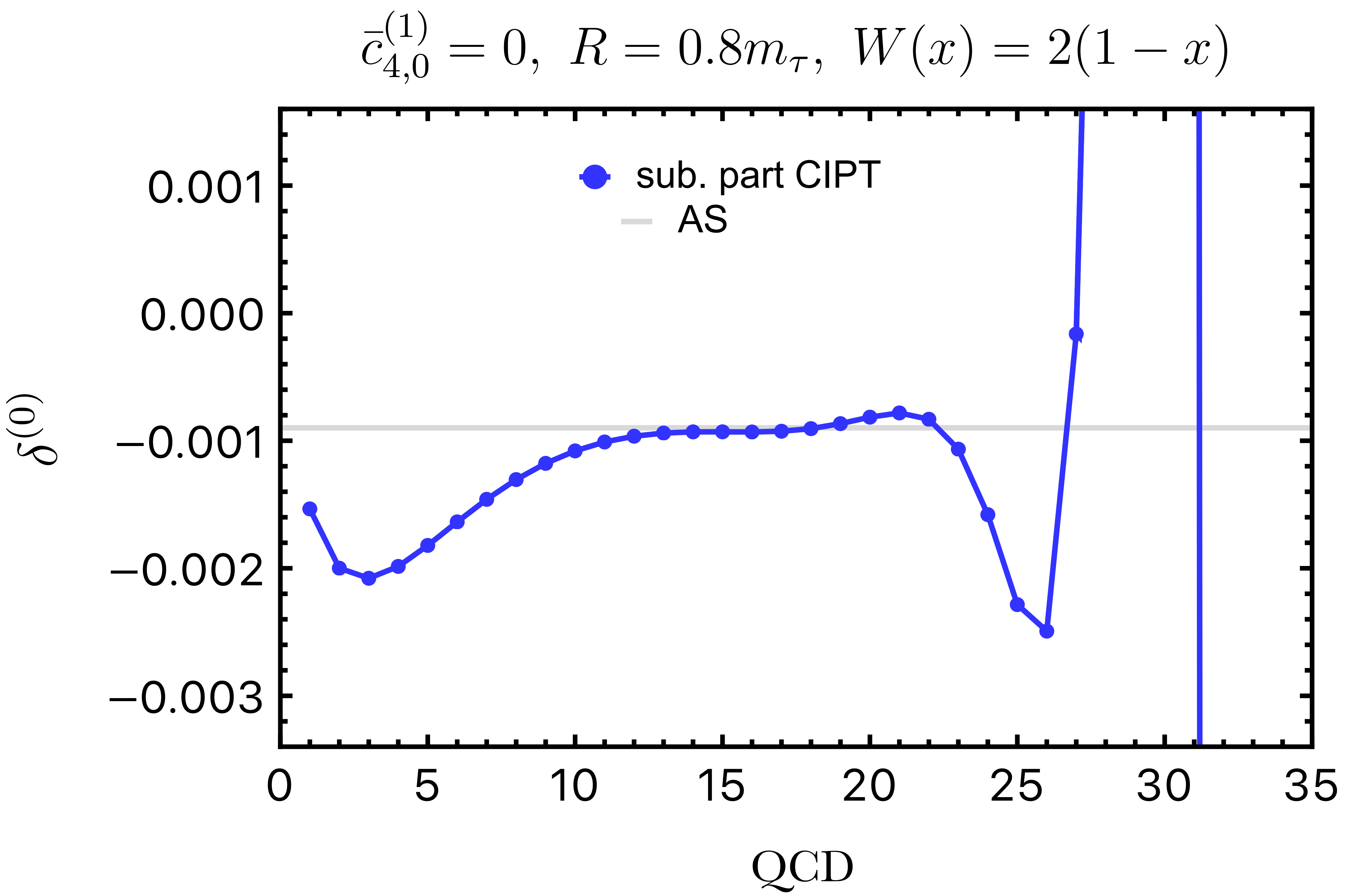
- Take as Borel function

$$B[\hat{D}_{\text{toy}}(s)](u) = \frac{1}{(2-u)^{1+4\hat{b}_1}}$$

- Consider vanishing $\mathcal{O}(\alpha_s)$ correction in GC Wilson coefficient
- Use $W(x) = 2(1-x)$
- GC suppressed for this moment
- Here and in the following: $s_0 = m_\tau^2$ and $\alpha_s(m_\tau^2) = 0.315$
- IR factorization scale R taken as $0.8 m_\tau$

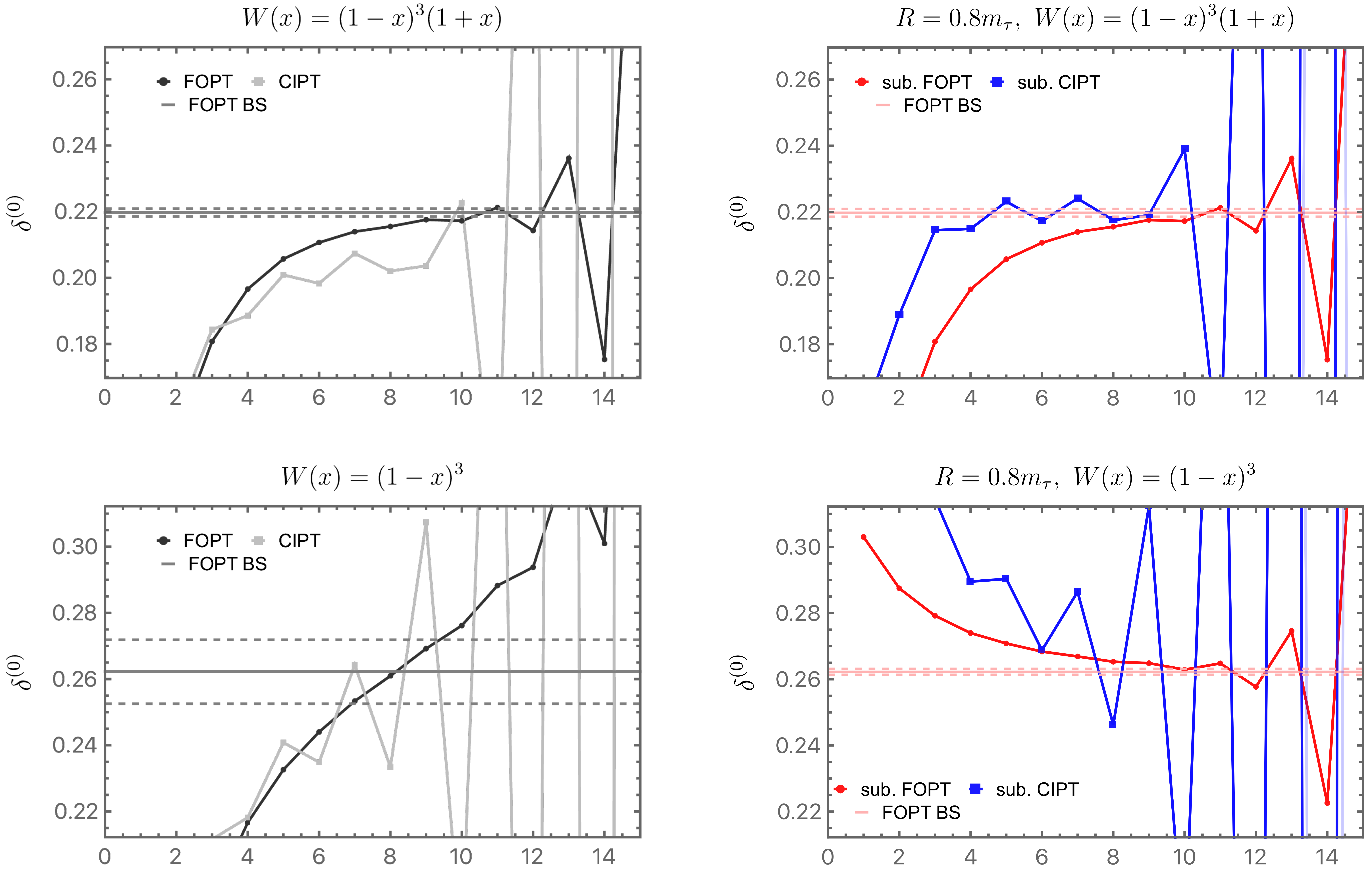


$\bar{c}_{4,0}^{(1)} = 0,\ R = 0.8 m_\tau,\ W(x) = 2(1-x)$



$\bar{c}_{4,0}^{(1)} = 0,\ R = 0.8 m_\tau,\ W(x) = 2(1-x)$

Numerical Results: Large-β_0 approximation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$R = 0.8m_\tau,\ W(x) = (1-x)^3(1+x)$
$W(x) = (1-x)^3$
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$\delta^{(0)}$
FOPT
CIPT
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sub. FOPT
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0.18
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0.26
0.28
0.30
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2
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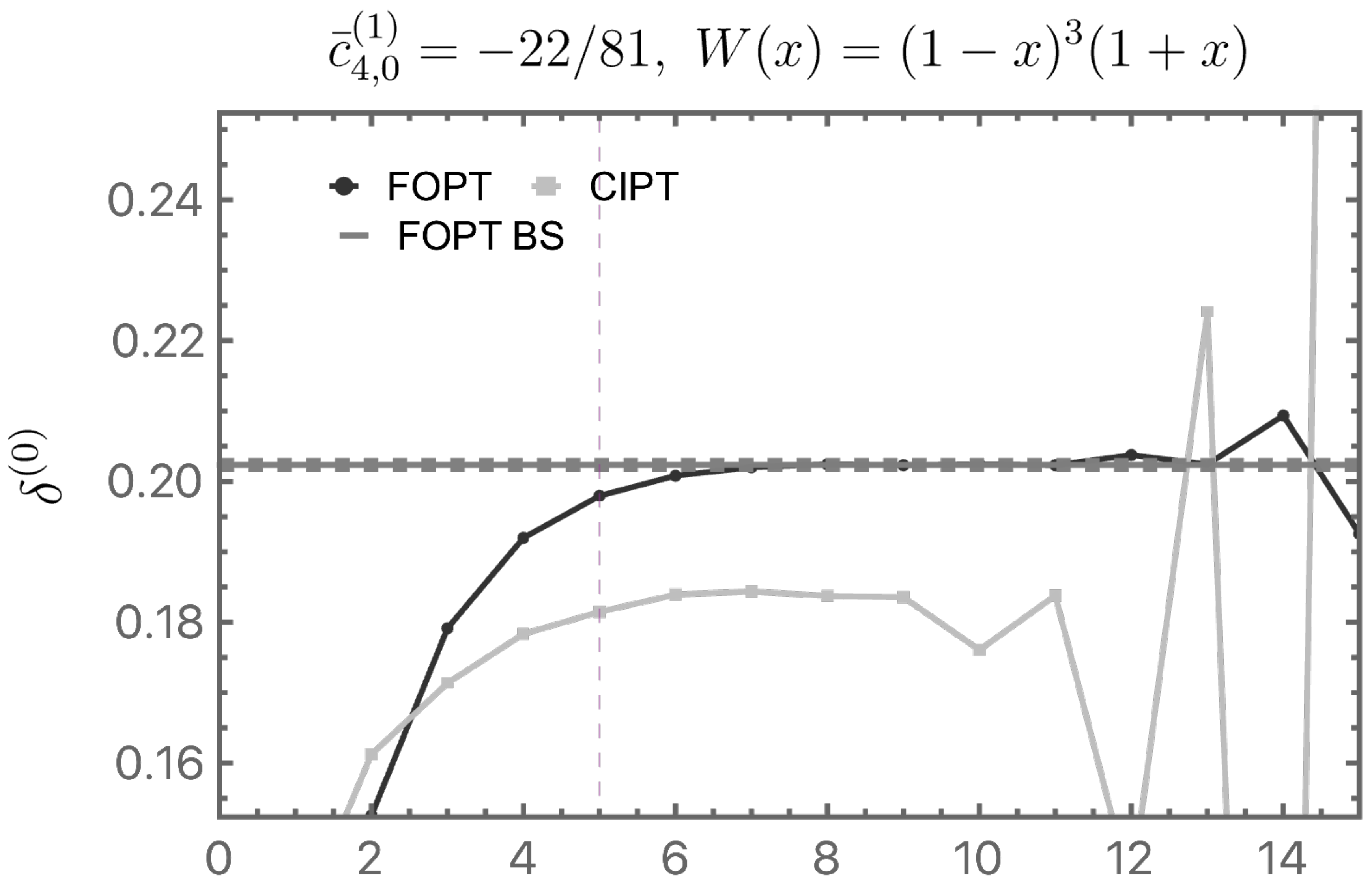
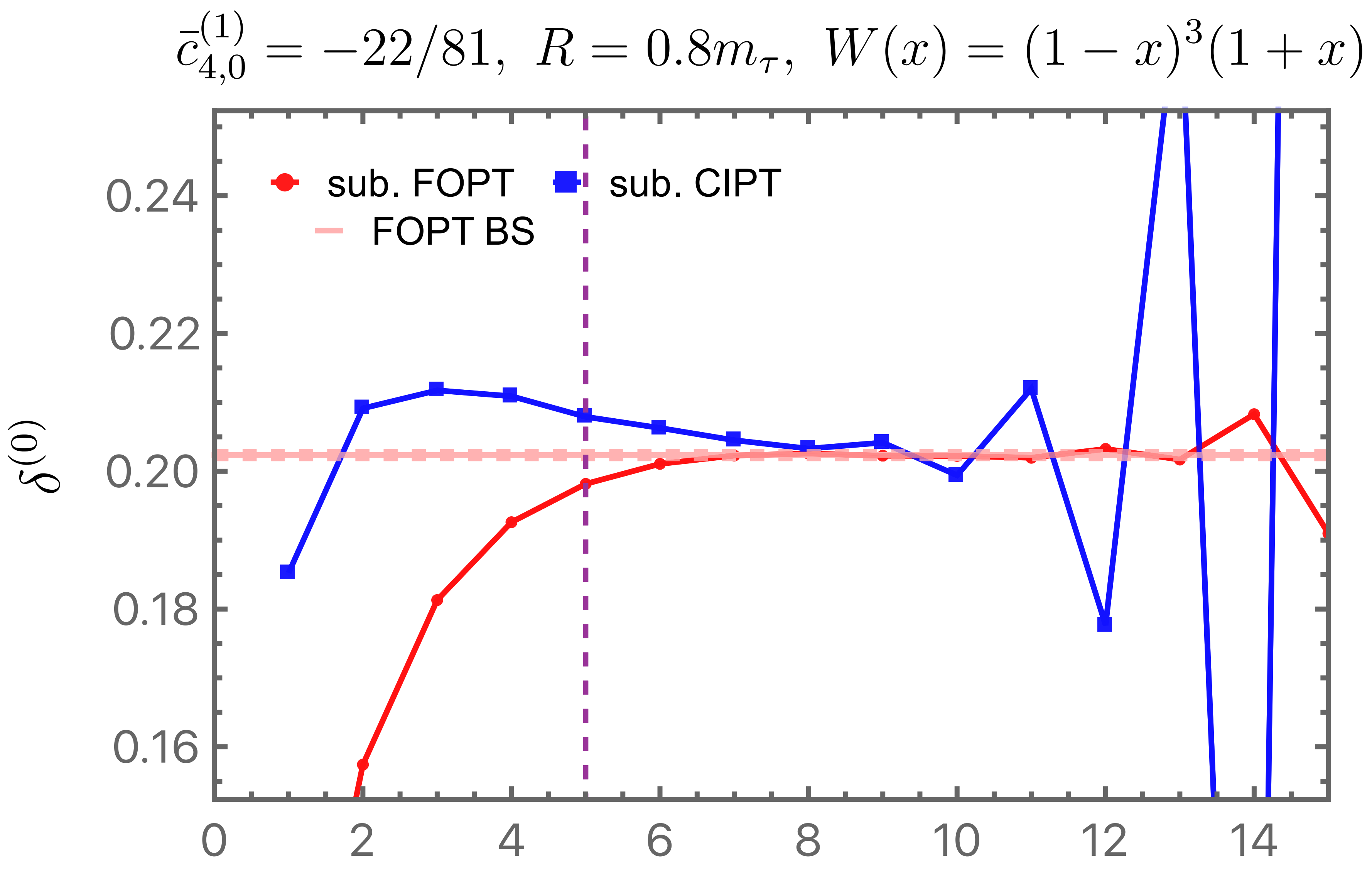
Numerical Results: Full QCD



$\bar{c}_{4,0}^{(1)} = -22/81,\ W(x) = (1-x)^3(1+x)$



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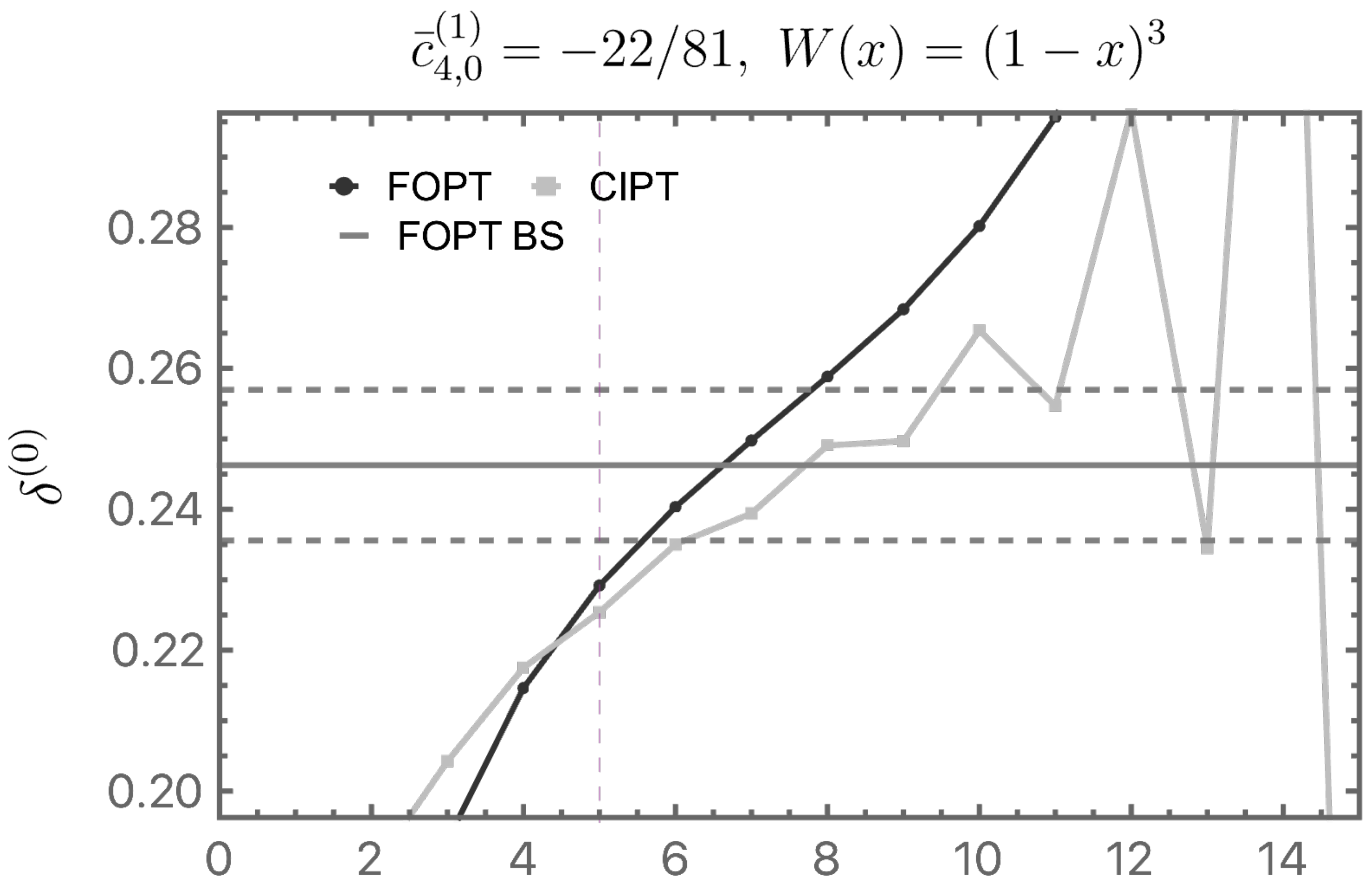
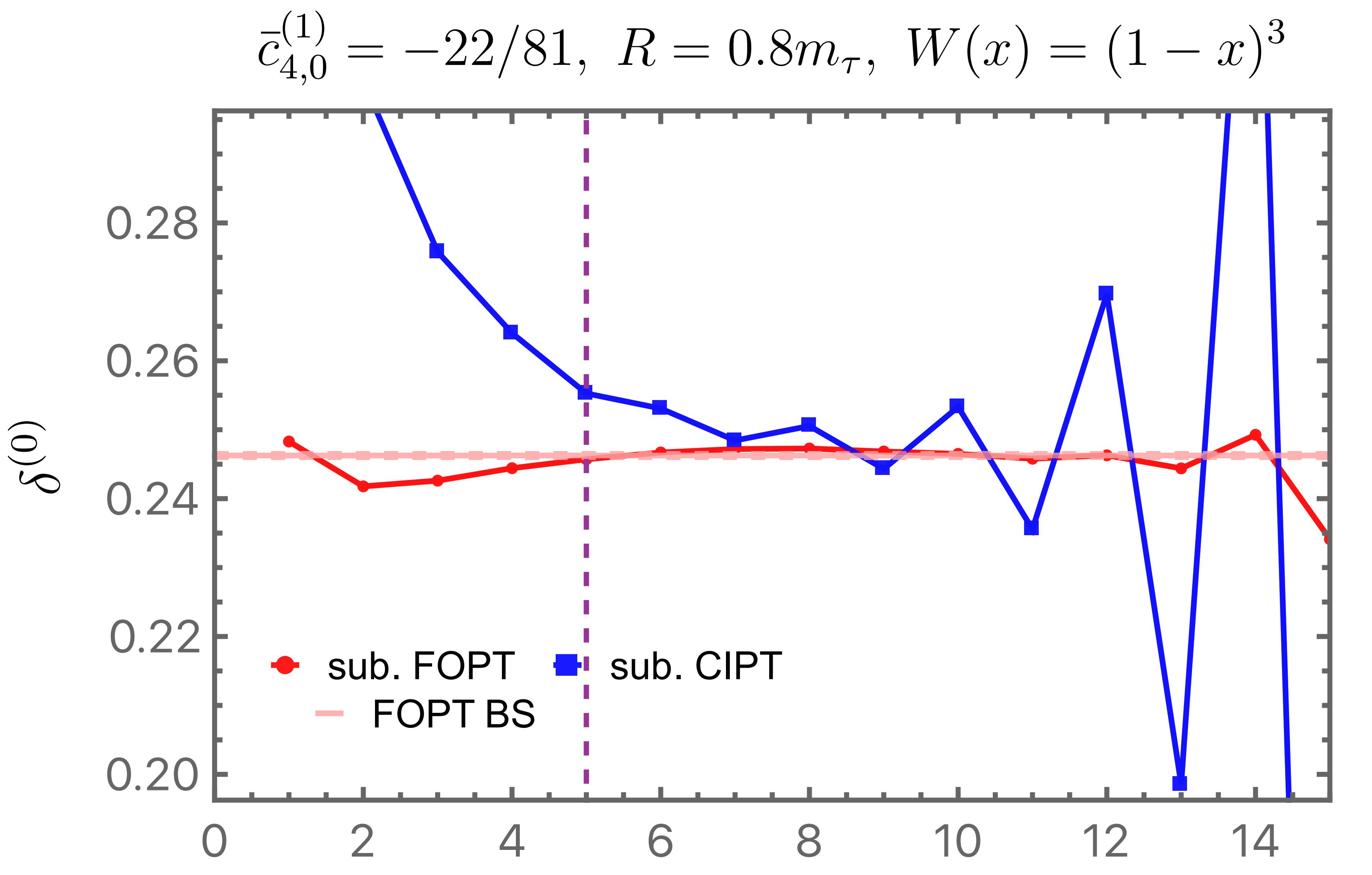


$\bar{c}_{4,0}^{(1)} = -22/81,\ W(x) = (1-x)^3$



$\bar{c}_{4,0}^{(1)} = -22/81,\ R = 0.8m_\tau,\ W(x) = (1-x)^3$

Conclusions:

- In RF scheme, size of discrepancy between FOPT and CIPT for moments which suppress GC is strongly diminished
- Subtracted FOPT and CIPT expansions approach FOPT BS
- Results suggest that α_s extractions using CIPT/FOPT in RF scheme should lead to much better agreement in comparison to original FOPT/CIPT expansions
- In RF scheme, moments which enhance the GC display much better pert. behavior
- GC enhancing moments which are nowadays excluded in most phenomenological analyses might be employed in high-precision determinations of α_s in the future