ASPECTS OF THRESHOLD RESUMMATION AT NEXT-TO-LEADING POWER

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OUTLINE

- Particle scattering near threshold
- Factorization and resummation at NLP
- New partonic channels in DIS and DY


*[arXiv: 2109.09752],*

*with M. van Beekveld and C. D. White.*
PARTICLE SCATTERING NEAR THRESHOLD
Hard scattering processes are calculated in perturbation theory.

Going beyond NNLO and N3LO is difficult, yet necessary to match the precision of current and forthcoming experiments!

Loop and phase space integrals:
- Analytic vs numerical evaluation
- Space of functions
- Infrared divergences
- Large logarithms

This talk
The presence of largely different scales gives rise to large logarithms:

\[ d\sigma \sim 1 + \alpha_s(L^2 + L + 1) + \alpha_s^2(L^4 + L^3 + L^2 + L + 1) + \ldots \]

\[ \sim \log^2(1 - z) \]

Large logarithms spoil the convergence of the perturbative series:

\[ \sim \log \frac{s}{-t} \]

\[ \sim \log \frac{m_H^2}{p_T^2} \]

Large logarithms spoil the convergence of the perturbative series:

\[ \rightarrow \text{ need resummation.} \]
Consider the DY invariant mass distribution:

\[
\frac{d\sigma}{dQ^2} = \tau \tilde{\sigma}_0(Q^2) \int_{\tau}^1 \frac{dz}{z} \mathcal{L}_{ab} \left( \frac{\tau}{z} \right) \Delta_{ab}(z),
\]

\[
\mathcal{L}_{ab}(y) = \int_{y}^{1} \frac{dx}{x} f_{a/A}(x) f_{b/B} \left( \frac{y}{x} \right).
\]

Near partonic threshold:

\[
\tau = \frac{Q^2}{s}, \quad z = \frac{Q^2}{\hat{s}}, \quad (z \geq \tau), \quad z \to 1,
\]

the partonic cross section has the singular expansion

\[
\Delta_{ab}(z) = \sum_{n=0}^{\infty} \alpha_s^n \left[ c_n \delta(1 - z) + \sum_{m=0}^{2n-1} \left( c_{nm} \frac{\ln^m(1 - z)}{1 - z} \right) + d_{nm} \ln^m(1 - z) \right] + \ldots.
\]
Large threshold logarithms spoil the reliability of the perturbative expansion and needs to be resummed.

Resummation of LP logarithms is well established: it relies on factorization and exponentiation properties of soft radiation.

The resummation of threshold logarithms leads to a more reliable perturbative expansion.

More relevant for the production of heavy final states ($HH, t\bar{t}, t\bar{t}W, t\bar{t}H, \ldots$).

\[
\frac{d\sigma}{dz} = \sum_{n=0}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^n \sum_{m=0}^{2n-1} c_{nm}^{(-1)} \log^m (1 - z) \right|_+ + \ldots
\]
PARTICLE SCATTERING NEAR THRESHOLD: NLP

What about NLP and higher power terms?

\[
\Delta_{ab}(z) = \sum_{n=0}^{\infty} a^n_s \left[ c_n \delta(1 - z) + \sum_{m=0}^{2n-1} \left( c_{nm} \left( \ln^m(1 - z) \right) \right) + d_{nm} \ln^m(1 - z) \right] + \ldots
\]

Can be relevant for precision physics!

Interesting problem: probes all-order structures beyond the semi-classical approximation.

Kramer, Laenen, Spira, 1998

Anastasiou, Duhr, Dulat, Herzog, Mistlberger, 2015
At NLP more production channels contribute. Convergence is faster for leading channels.
FACTORIZATION AND RESUMMATION AT NLP

\[
\frac{d\sigma}{d\xi} \sim \sum_{n=0}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^n \left[ c_n \delta(\xi) + \sum_{m=0}^{2n-1} \left( c_{nm} \frac{\ln^m(\xi)}{\xi} \right) + d_{nm} \ln^m(\xi) \right] + \ldots
\]

- Understanding the factorization and resummation of large logarithms at next-to-leading power (NLP) has been subject of intense work in the past few years!

- Drell-Yan, Higgs and DIS near threshold
  
  Bahjat-Abbas, Bonocore, Sinninghe Damsté, Laenen, Magnea, LV, White, 2019;
  van Beekveld, Beenakker, Laenen, White, 2019; van Beekveld, Laenen, Sinninghe Damsté, LV, 2021;
  Beneke, Broggio, Garny, Jaskiewicz, Szafron, LV, Wang, 2018;
  Beneke, Broggio, Jaskiewicz, LV, 2019;

- Operators and Anomalous dimensions
  
  Larkoski, Neill, Stewart 2014;
  Moult, Stewart, Vita 2017; Feige, Kolodrubetz, Moult, Stewart 2017;

- Thrust
  

- \( p_T \) and Rapidity logarithms
  
  Ebert, Moult, Stewart, Tackmann, Vita, 2018,
  Moult, Vita Yan 2019;
  Cieri, Oleari, Rocco, 2019; Oleari, Rocco 2020.

- Mass effects
  
  Liu, Mecaj, Neubert, Wang, 2020;
SCATTERING NEAR THRESHOLD:
LP VS NLP
FACTORISATION OF SOFT GLUONS AT LP

- Emission of soft gluons from an energetic parton (quark):

\[
\mathcal{M} \frac{\phi - k}{2p \cdot k} \gamma^\mu T^A u(p) \sim \mathcal{M} \frac{p^\mu}{p \cdot k} T^A u(p).
\]

- Emission of multiple soft gluons factorises:

\[
\sim \mathcal{M} S u(p), \quad S = \langle 0|\Phi_\beta(-\infty, 0)|0\rangle,
\]

\[
\Phi_\beta(\lambda_1, \lambda_2) = \mathcal{P} \exp \left\{ i g_s \int_{\lambda_1}^{\lambda_2} d\lambda \, \beta \cdot A(\lambda\beta) \right\}.
\]

- In general

\[
\sim \mathcal{M} S u(p_1) \bar{u}(p_2) \ldots \bar{u}(p_n), \quad S = \langle 0|\Phi_1 \ldots \Phi_n|0\rangle \sim e^{\mathcal{W}_E}.
\]

Collins, Soper, Sterman, 1989;
Gardi, Laenen, Stavenga, White, 2010;
Gardi, Smillie, White, 2013
EXPANSION BY REGIONS

- Beyond leading power one has non-trivial effects due to virtual gluons:

  \[ k = n_+ \cdot k \frac{n_-}{2} + n_- \cdot k \frac{n_+}{2} + k_, \]

  \[ n^2_{\pm} = 0, \quad n_- \cdot n_+ = 2, \quad n_- \sim \frac{p}{\hat{s}}, \]

  hard: \[ k \sim \sqrt{\hat{s}}(1,1,1) \]

  collinear: \[ k \sim \sqrt{\hat{s}}(1, \lambda, \lambda^2) \]

  anti-collinear: \[ k \sim \sqrt{\hat{s}}(\lambda^2, \lambda, 1) \]

  soft: \[ k \sim \sqrt{\hat{s}}(\lambda^2, \lambda^2, \lambda^2), \quad \lambda \ll 1. \]

Bonocore, Laenen, Magnea, LV, White, 2014
Virtual gluons gives non-analytical contributions $\propto$ to the scales of the problem: LP

$$|\mathcal{M}|^2 \propto \frac{\hat{s}^2}{tu} \left\{ C_F^2 \left( \frac{\mu^2}{-\hat{s}} \right)^\epsilon \left( -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} + \ldots \right) + C_A C_F \left( \frac{\hat{s} \mu^2}{tu} \right)^\epsilon \left( -\frac{1}{\epsilon^2} + \ldots \right) \right\} + \ldots$$

Factorisation

$$S\left( \frac{\hat{s} \mu^2}{tu}, \epsilon \right) \times H\left( \frac{\mu^2}{-\hat{s}}, \epsilon \right)$$

Factorisation: physics at different scales is uncorrelated.
Virtual gluons gives non-analytical contributions $\propto$ to the scales of the problem: NLP

$$|\mathcal{M}|^2 \propto C_F^2 \left\{ \begin{array}{l} \frac{N_{\text{LP}}}{tu} \frac{\hat{s}(t+u)}{t+u} \left( \frac{\mu^2}{-\hat{s}} \right)^\epsilon \left( -\frac{2}{\epsilon^2} - \frac{1}{\epsilon} + \ldots \right) + \left[ \frac{N_{\text{LP}}}{s} \left( \frac{\mu^2}{-t} \right)^\epsilon + \frac{N_{\text{LP anti-coll.}}}{u} \left( \frac{\mu^2}{-u} \right)^\epsilon \right] \left( -\frac{2}{\epsilon} + \ldots \right) \right] \\ + C_A C_F \frac{N_{\text{LP}}}{tu} \left( \frac{\hat{s} \mu^2}{tu} \right)^\epsilon \left( -\frac{1}{\epsilon^2} + \ldots \right) + \left[ \frac{N_{\text{LP}}}{s} \left( \frac{\mu^2}{-t} \right)^\epsilon + \frac{N_{\text{LP anti-coll.}}}{u} \left( \frac{\mu^2}{-u} \right)^\epsilon \right] \left( -\frac{5}{2} + \ldots \right) \right\} + \ldots$$

Factorization?

$$S \left[ \frac{\hat{s} \mu^2}{tu}, \epsilon \right] \times J \left[ \frac{\mu^2}{-t}, \epsilon \right] \times \bar{J} \left[ \frac{\mu^2}{-u}, \epsilon \right] \times H \left[ \frac{\mu^2}{-\hat{s}}, \epsilon \right]$$

Need an effective approach to take into account hard, collinear and soft modes.

Two approaches: $\sim$ Diagrammatic; $\sim$ Soft Collinear Effective Field Theory.
FACTORORIZATION AND RESUMMATION: DIAGRAMMATIC VS SCET
Describe momentum regions in terms of universal functions in QCD:

For instance, for Drell-Yan we have

- “Derivative” of the non-radiative amplitude,
- “Generalized” soft function,
- “Radiative” jet function,

\[ J_{\mu,a}(p, n, k) u(p) = \int d^D y \, e^{-i(p-k) \cdot y} \langle 0 | \Phi_n(\infty, y) \psi(y) j_{\mu,a}(0) | p \rangle. \]

One has

\[ F_p(-\infty, 0) = \mathcal{P} \exp \left[ g \int \frac{d^d k}{(2\pi)^d} A_\mu(k) \left( - \frac{p^\mu}{p \cdot k} + \frac{k^\mu}{2p \cdot k} - k^2 \frac{p^\mu}{2(p \cdot k)^2} \right) \right. \]

\[ \left. - \frac{i k^\nu \Sigma^\mu_\nu}{p \cdot k} \right] + \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d l}{(2\pi)^d} A_\mu(k) A_\nu(l) \left( \frac{\eta^\mu_\nu}{2p \cdot (k+l)} + \ldots \right). \]

\[ \mathcal{A}_{\mu,a}(p_j, k) = \sum_{i=1}^2 \left( \frac{1}{2} \widetilde{S}_{\mu,a}(p_j, k) + g T_{i,a} G_{i,\mu}^\nu \frac{\partial}{\partial p_i^\nu} + J_{\mu,a}(p_i, n_i, k) \right) \mathcal{A}(p_j) - \mathcal{A}_{\mu,a}(p_j, k), \]

for \( n_1 = p_2, n_2 = p_1 \).

(Removes soft-collinear overlap in the radiative jet)
• Exponentiation (resummation): investigate the combinatorial structure of all-order classes of diagrams, constructed with the functions defined before, contributing to a given logarithmic accuracy at NLP (more later).

  *Bahjat-Abbas, Bonocore, Sinninghe Damsté, Laenen, Magnea, LV, White, 2019; van Beekveld, LV, White 2021*

• The example discussed so far accurately describes $\text{DY}$ up to $\text{NNLO}$. In general, one needs to take into account
  • processes with more than two external directions;
  • factorization beyond one loop;
  • Multiple soft gluon emission.

• Task: obtain a classification of the jet-like structures, consisting of virtual radiation collinear to any of the $n$ external hard particles, contributing at subleading power in a parametrically small scale, corresponding to a fermion mass or a soft external momentum.
Work in progress: in QED one obtains the all-orders factorization formula

\[ M^{LP} = \left( \prod_{i=1}^{n} J^{i}_{(f)}(\hat{p}_i) \otimes H^{i}_{(f_{\gamma})} \right) S(n_i \cdot n_j), \]

\[ M^{NLP}_{\text{coll}} = \sum_{i=1}^{n} \left( \prod_{j \neq i} J^{j}_{(f)} \right) \left[ J^{i}_{(f_{\gamma})} \otimes H^{i}_{(f_{\gamma})} + J^{i}_{(f_{\partial \gamma})} \otimes H^{i}_{(f_{\partial \gamma})} \right] S + \sum_{i=1}^{n} \left( \prod_{j \neq i} J^{j}_{(f)} \right) J^{i}_{(f_{\gamma})} \otimes H^{i}_{(f_{\gamma})} S \]

+ \sum_{i=1}^{n} \left( \prod_{j \neq i} J^{j}_{(f)} \right) J^{i}_{(fff)} \otimes H^{i}_{(fff)} S + \sum_{1 \leq i \leq j \leq n} \left( \prod_{k \neq i,j} J^{k}_{(f)} \right) J^{i}_{(f_{\gamma})} J^{j}_{(f_{\gamma})} \otimes H^{i,j}_{(f_{\gamma})(f_{\gamma})} S.

The main message: in general, more types of jet functions are needed, involving two or more particles along a given collinear direction.

Possible drawbacks: soft-collinear overlap, power expansion of the factorized functions.
**FACTORIZATION AND RESUMMATION IN SCET AT LP**

- Effective Lagrangian and operators made of **collinear** and **soft** fields.

\[ \mathcal{L}_{\text{SCET}} = \sum_i \mathcal{L}_{ci} + \mathcal{L}_s, \]

\[ \mathcal{O}_n = \int dt_1 \ldots dt_n \mathcal{C}(t_1, \ldots, t_n) \phi_1(t_1 n_{1+}) \ldots \phi_n(t_n n_{n+}). \]

- Constructed to reproduce a scattering process as obtained with the **method of regions**.

- The cross section factorizes into a **hard scattering kernel**, and **matrix elements** of **soft** and **collinear** fields.

\[ \sigma \sim \mathcal{H} \otimes \mathcal{J}_1 \otimes \ldots \otimes \mathcal{J}_n \otimes S. \]

- Renormalize **UV** divergences of EFT operators and obtain **renormalization group equations**.

- Each function depends on a **single scale**: solving the RGE **resums** large logarithms.

See e.g. **Becher, Neubert 2006**

FACTORORIZATION IN SCET: LP VS NLP

- **Leading power (LP):**
  - \textit{N-jet} operators;
  - Soft-collinear decoupling.

- **Next-to-leading power (NLP):**
  - Kinematic suppression;
  - \textit{Multi-particle emission} along the same collinear direction;
  - No soft-collinear decoupling.

DRELL-YAN AT LP IN SCET

- Drell-Yan momentum modes:

\[ p_c = (n_+ p_c, n_- p_c, p_c \perp) \sim Q(1, \lambda^2, \lambda), \quad p_c-PDF \sim (Q, \Lambda^2_{QCD}/Q, \Lambda_{QCD}), \]
\[ p_\bar{c} = (n_+ p_\bar{c}, n_- p_\bar{c}, p_\bar{c} \perp) \sim Q(\lambda^2, 1, \lambda), \quad p_\bar{c}-PDF \sim (\Lambda^2_{QCD}/Q, Q, \Lambda_{QCD}), \]
\[ p_s = (n_+ p_s, n_- p_s, p_s \perp) \sim Q(\lambda^2, \lambda^2, \lambda^2), \quad Q^2 \lambda^2 = Q(1 - z) \gg \Lambda_{QCD}. \]

- At LP the QCD DY current matches to

\[ \bar{\psi} \gamma_\mu \psi = \int dt \, d\bar{t} \, \tilde{C}^{A0}(t, \bar{t}) \, J^{A0, A0}_\mu(t, \bar{t}), \]

where

\[ J^{A0A0}_\rho = \bar{\chi}_c(\bar{t}n_-) \gamma_\perp \rho \chi_c(tn_+) \]
\[ = \bar{\chi}^{(0)}_c(\bar{t}n_-) Y^{(0)}_- \gamma_\perp \rho Y^{(0)}_+ \chi^{(0)}_c(tn_+), \]

- After decoupling, soft interaction factorise into soft Wilson lines.
- Initial and final state factorise into soft and PDF-collinear sector:

\[ |X\rangle = |X^{PDF}_c\rangle |X^{PDF}_c\rangle |X_s\rangle. \]
**DRELL-YAN AT LP IN SCET**

- The invariant mass distribution at LP reads

\[
\frac{d\sigma_{DY}}{dQ^2} = \frac{4\pi\alpha^2_{EM}}{3N_c Q^4} \sum_{a,b} \int_0^1 dx_a dx_b f_{a/A}(x_a)f_{b/B}(x_b) \hat{\sigma}^{LP}_{q\bar{q}}(z);
\]

where

\[
\hat{\sigma}^{LP}_{q\bar{q}}(z) = |C(Q^2)|^2 Q S_{DY}(Q(1 - z));
\]

- The soft function is a vacuum matrix element of Wilson lines:

\[
S_{DY}(\Omega) = \int \frac{dx^0}{4\pi} e^{ix^0/\Omega} \frac{1}{N_c} \text{Tr}(0| \bar{T}[(Y^+_Y_-)(x^0)]T[(Y^+_Y_+)(x^0)]|0).
\]
DRELL-YAN AT NLP IN SCET

- Schematic factorization formula at NLP: we expect

\[
\frac{d\sigma_{\text{DY}}}{dQ^2} = \frac{4\pi\alpha_{\text{EM}}^2}{3N_c Q^4} \sum_{a,b} \int_0^1 dx_a dx_b f_{a/A}(x_a) f_{b/B}(x_b) \hat{\sigma}_{q\bar{q}}^{\text{NLP}}(z),
\]

where

\[
\hat{\sigma}_{\text{NLP}} = \sum_{\text{terms}} \left[ C \otimes J \otimes \bar{J} \right] \otimes S,
\]

- \(C\) is the hard Wilson matching coefficient,

- \(S\) is a generalized soft function,

- \(J\) is a new collinear function.

- The collinear function is trivial at LP, because all threshold collinear modes are scaleless.

- The collinear scale is induced by the injection of a soft momentum.
This is provided at subleading power by soft-collinear interactions. For instance:

\[ J_{c}^{T2}(t) = i \int d^4 z \, T \left[ \chi_c(t n_+) \mathcal{L}_{2z}^{(2)}(z) \right], \]

where

\[ \mathcal{L}_{2z}^{(2)}(z) = \frac{1}{2} \tilde{\chi}_c(z) z_\perp z_\perp \left[ i \partial_\nu \, i n_\perp \partial \mathcal{B}_\mu^+(z_-) \right] \frac{\gamma_+}{2} \chi_c(z), \quad \mathcal{B}_\mu^\perp = Y_\perp^\dagger [i D_s^\mu Y_\perp], \]

The matrix element reads

\[ \langle X | \overline{\psi} \gamma^\rho \psi(0) | A(p_A) B(p_B) \rangle = \int dt \, d\bar{t} \, C_{A0, A0}(t, \bar{t}) \langle X_{c}^{PDF} | \tilde{\chi}_{c, \alpha a}(\bar{t} n_-) | B(p_B) \rangle \gamma_\perp, \alpha \gamma \\
\times i \int d^4 z \langle X_{c}^{PDF} | \frac{1}{2} z_\perp z_\perp (i n_\perp \partial_z)^2 T \left[ \chi_{c, \gamma f } (t n_+) \tilde{\chi}_c(z) \mathcal{T}^A \frac{\gamma_+}{2} \chi_c(z) \right] | A(p_A) \rangle \\
\times \langle X_s | T \left( \left[ Y_\perp^\dagger(0) Y_\perp(0) \right]_{af} \frac{i \partial_\mu}{2} \mathcal{B}_{\perp \nu}^A (z_-) \right) | 0 \rangle. \]

Non-scaleless collinear matrix element

(Recall)

\[ J_{\mu, a} (p, n, k) u(p) = \int d^d y \, e^{-i(p-k)\cdot y} \left\langle 0 \mid \Phi_n(\infty, y) \psi(y) \right\rangle \langle \mu, a \rangle \mid p \rangle. \]

\[-g \overline{\psi}(x) \gamma^\mu T_a \psi(x) +...\]
DRELL-YAN AT NLP IN SCET

- This is easily generalized at any subleading power: there can be many Lagrangian insertions, each with its own $\omega_i$ conjugate to the large component of the collinear momentum.

$$i^m \int \{d^4 z_j \} \ T \left[ \{ \psi_c(t_k n_+) \} \times \{ \mathcal{L}^{(l)}(z_j) \} \right]$$

$$= 2\pi \sum_i \int du \int \{ dz_{j-} \} \ \tilde{J}_i(\{ t_k \}, u; \{ z_{j-} \}) \ \chi_{c \text{PDF}}(u n_+) \ s_i( \{ z_{j-} \}) ,$$

- After taking the matrix element squared, this gives a generalized soft functions:

$$S(\Omega, \omega) = \int \frac{dx^0}{4\pi} e^{ix^0 \Omega/2} \left( \prod_{j=1}^{n} \int \frac{d(z_{-j})}{4\pi} e^{-i\omega_j z_{-j}} \right)$$

$$\times \ \text{Tr} \langle 0 | \mathbf{T} \left[ (Y_+^\dagger Y_-)(x^0) \right] \mathbf{T} \left[ (Y_-^\dagger Y_+)(x^0) \times \mathcal{L}_s^n(z_{1-}) \times \ldots \times \mathcal{L}_s^n(z_{n-}) \right] | 0 \rangle .$$

which are equivalent to the generalized Wilson lines built in terms of NLP webs in the diagrammatic approach.

*Beneke, Broggio, Jaskiewicz, LV, 2019*
DRELL-YAN AT NLP IN SCET

- Up to NLP one has:

\[
\Delta_{\text{NLP}}^\text{dyn}(z) = -\frac{2}{(1 - \epsilon)} Q \left[ \left( \frac{\eta_-}{4} \right) \gamma_{\perp \rho} \left( \frac{\eta_+}{4} \right) \gamma^\rho \right]_{\beta \gamma} \\
\times \int d(n+p) \ C^{A_0, A_0} (n+p, x_b n-p_B) C^{*A_0A_0} (x_a n+p_A, x_b n-p_B) \\
\times \sum_{i=1}^{5} \int \{d\omega_j\} J_i \ (n+p, x_a n+p_A; \{\omega_j\}) \ S_i(\Omega; \{\omega_j\}) + \text{h.c.}
\]

- Compare fixed order expansion against results in literature.
  - At NLO this gives the contribution:

\[
\Delta_{\text{NLP}}^{\text{dyn}}^{(1)}(z) = 4 Q H^{(0)} (Q^2) \int d\omega J_{1,2}^{(0)} (x_a n+p_A; \omega) \ S_{1}^{(1)}(\Omega; \omega)
\]
DRELL-YAN AT NLP IN SCET

• At NNLO:

\[
\Delta^{\text{dy}}_{\text{NLP-hard}}(z) = 2 Q \int d\omega \, S^{(1)}_1(\Omega; \omega) \left( (H^{(1)}(Q^2) \, J^{(0)}_{1,2}(x_a n_p A; \omega) \
- C^{A0}(x_a n_p A, x_b n_p B) \, J^{(0)}_{1,1}(x_a n_p A; \omega) \right) \\
\times \frac{\partial}{\partial x_a(n+p_A)} C^{A0}(x_a n_p A, x_b n_p B)) + \text{h.c.,}
\]

\[
\Delta^{\text{dy}}_{\text{NLP-coll}}(z) = 4 Q \, H^{(0)}(Q^2) \int d\omega \, J^{(1)}_{1,2}(x_a n_p A; \omega) \, S^{(1)}_1(\Omega; \omega),
\]

\[
\Delta^{\text{dy}}_{\text{NLP-soft}}(z) = -\frac{4}{(1 - \epsilon)} Q \left[ \left( \frac{\eta -}{4} \right) \gamma_{1}^{\rho} \left( \frac{\eta +}{4} \right) \gamma_{1}^{\rho} \right]_{\beta\gamma} H^{(0)}(Q^2) \\
\times \sum_{i=1}^{5} \int \{d\omega_j\} \, J^{(0)}_i(x_a n_p A; \{\omega_j\}) \, S^{(2)}_i(\Omega; \{\omega_j\}).
\]

Beneke, Broggio, Jaskiewicz, LV, 2019

Broggio, Jaskiewicz, LV, 2021
DRELL-YAN AT NLP IN SCET

- We reproduce the 1real 1-virtual and 2-real contribution to DY.
- The ingredients to reproduce 1r1v correction in DY are the 1-loop collinear and soft functions:

\[
J_{1,2;\gamma\beta,fq}^{K(1)}(n+p_A;\omega) = \frac{\alpha_s e^{\epsilon\gamma_E}}{4\pi} \frac{1}{(n+p_A)} \delta_{\gamma\beta} T_{fq}^K \frac{\omega_{n+p_A}}{\mu^2} \left(\frac{\omega_{n+p_A}}{\mu^2}\right)^{-\epsilon} \frac{\Gamma[1+\epsilon]\Gamma[1-\epsilon]^2}{(\epsilon-1)(\epsilon+1)\Gamma[2-2\epsilon]} \\
\times \left(C_F \left(-\frac{4}{\epsilon} + 3 + 8\epsilon + \epsilon^2\right) - C_A \left(-5 + 8\epsilon + \epsilon^2\right)\right),
\]

\[
S_1(\Omega,\omega) = \frac{\alpha_s C_F}{2\pi} \frac{\mu^{2\epsilon} e^{\epsilon\gamma_E}}{\Gamma[1-\epsilon]} \frac{1}{\omega^{1+\epsilon}} \frac{1}{(\Omega - \omega)^\epsilon} \theta(\omega)\theta(\Omega - \omega) + O(\alpha_s^2).
\]

- However: the convolution is regularized by dimensional regularization. For resummation, we treat the two object independently, and expand in $\epsilon$ prior to performing the convolution.

\[
\int d\omega \left(\frac{n+p}{\omega}\right)^{-\epsilon} \frac{1}{\omega^{1+\epsilon}} \frac{1}{(\Omega - \omega)^\epsilon}.
\]

- Technical problem that needs to be solved to achieve resummation beyond LL.

LL RESUMMATION AT NLP

• For leading channels like $q\bar{q}$ in Drell-Yan or $gg$ in Higgs production, it turns out that the collinear function contributes only starting at NLL accuracy.

• This means that at LL accuracy only the hard and soft functions contribute. The divergent contribution problem can be easily overcome, and LLs can be resummed.

SCET: Beneke, Broggio, Garny, Jaskiewicz, Szafron, LV, Wang, 2018;
Diagrammatic: Bahjat-Abbas, Bonocore, Sinninghe Damsté, Laenen, Magnea, LV, White, 2019

van Beekveld, Laenen, Sinninghe Damsté, LV, 2021.
ENDPOINT DIVERGENCES AND RESUMMATION

Addressing the problem of endpoint divergences in the leading channels is not an easy task: one has to deal with several operators/contributions.

- Things are different for subleading channels (e.g. $qg$ in Drell-Yan): they start at NLP, and LLs are generated by all momentum regions. Better place to start investigating endpoint divergences.
THRESHOLD RESUMMATION AT NLP IN OFF-DIAGONAL CHANNELS: DEEP INELASTIC SCATTERING
Deep inelastic scattering (DIS) near threshold develops a hierarchy of scales:

\[ Q^2 \gg P_X^2 \sim Q^2(1 - x), \quad \text{with} \quad x \equiv \frac{Q^2}{2p \cdot q} \to 1. \]

Factorization and resummation well understood at LP:

\[ W_\phi = \frac{1}{8\pi Q^2} \int d^4x \ e^{i q \cdot x} \left< N(P) \left[ G^{A}_{\mu \nu} G^{\mu \nu A} \right] (x) \left[ G^{B}_{\rho \sigma} G^{\rho \sigma B} \right] (0) \left| N(P) \right> \]

\[ = |C(Q^2, \mu)|^2 \int_x^1 \frac{d \xi}{\xi} J\left( Q^2 \frac{1 - \xi}{\xi}, \mu \right) x \frac{f_g}{\xi} \left( \frac{x}{\xi}, \mu \right). \]

Short-distance coefficient and jet function are single scale object – resummation obtained by solving the corresponding RGE.
**DIS: OFF-DIAGONAL CHANNEL**

- The off-diagonal channel \( q(p) + \phi^*(q) \to X(p_X) \) contributes to DIS at NLP. Consider the partonic structure function

\[
W_{\phi,q}|_{q^* \to qg} = \int_0^1 dz \left( \frac{\mu^2}{s_{qg}z\bar{z}} \right) \mathcal{P}_{qg}(s_{qg}, z) \bigg|_{s_{qg} = Q^2 \frac{1-x}{x}} , \quad \mathcal{P}_{qg}(s_{qg}, z) \equiv \frac{e^{\gamma E} Q^2}{16\pi^2 \Gamma(1 - \epsilon)} \frac{|\mathcal{M}_{q\phi^* \to qg}|^2}{|\mathcal{M}_0|^2} .
\]

with momentum fraction \( z \equiv \frac{n-p_1}{n-p_1 + n-p_2} \), and \( \bar{z} = 1 - z \).

- At LO one has

\[
\mathcal{P}_{qg}(s_{qg})_{\text{tree}} = \frac{\alpha_s C_F}{2\pi} \frac{z^2}{z} , \quad \Rightarrow \quad W_{\phi,q}|_{\mathcal{O}(\alpha_s), \text{leading pole}} = -\frac{1}{\epsilon} \frac{\alpha_s C_F}{2\pi} \left( \frac{\mu^2}{Q^2(1-x)} \right)^\epsilon .
\]

The single pole originate from \( z \to 0 \), due to the \( 1/z \) of the momentum distribution function.

- At NLO:

\[
\mathcal{P}_{qg}(s_{qg}, z)|_{\text{1-loop}} = \mathcal{P}_{qg}(s_{qg}, z)|_{\text{tree}} \frac{\alpha_s 1}{\pi} \frac{1}{\epsilon^2}
\]

\[
\cdot \left( T_1 \cdot T_0 \left( \frac{\mu^2}{zQ^2} \right)^\epsilon + T_2 \cdot T_0 \left( \frac{\mu^2}{\bar{z}Q^2} \right)^\epsilon \right)
\]

\[
+ T_1 \cdot T_2 \left[ \left( \frac{\mu^2}{Q^2} \right)^\epsilon - \left( \frac{\mu^2}{zQ^2} \right)^\epsilon + \left( \frac{\mu^2}{z s_{qg}} \right)^\epsilon \right] + \mathcal{O}(\epsilon^{-1}).
\]
ON THE ENDPOINT DIVERGENCES

\[ P_{qg}(s_{qq}, z) |_{1 \text{-loop}} = P_{qg}(s_{qq}, z) \big|_{\text{tree}} \left( \frac{\alpha_s}{\pi} \frac{1}{\epsilon^2} \left( T_1 \cdot T_0 \left( \frac{\mu^2}{z Q^2} \right)^\epsilon + T_2 \cdot T_0 \left( \frac{\mu^2}{z^2 Q^2} \right)^\epsilon \right) + T_1 \cdot T_2 \left[ \left( \frac{\mu^2}{Q^2} \right)^\epsilon - \left( \frac{\mu^2}{z Q^2} \right)^\epsilon + \left( \frac{\mu^2}{z s_{qq}} \right)^\epsilon \right] \right) + \mathcal{O}(\epsilon^{-1}). \]

- The **T1.T2** term contains a single pole, but: promoted to **leading pole** after integration!
- Compare **exact** integration:
  \[
  \frac{1}{\epsilon^2} \int_0^1 dz \frac{1}{z^{1+\epsilon}} (1 - z^{-\epsilon}) = -\frac{1}{2\epsilon^3},
  \]
  vs integration **after expansion**:
  \[
  \frac{1}{\epsilon^2} \int_0^1 dz \frac{1}{z^{1+\epsilon}} \left( \epsilon \ln z - \frac{\epsilon^2}{2!} \ln^2 z + \frac{\epsilon^2}{3!} \ln^3 z + \cdots \right) = -\frac{1}{\epsilon^3} + \frac{1}{\epsilon^3} - \frac{1}{\epsilon^3} + \cdots.
  \]
- Expansion in **\( \epsilon \)** not possible before integration!
- The pole associated to **T1.T2** does not originate from the standard cups anomalous dimension.
BREACKDOWN OF FACTORIZATION NEAR THE ENDPOINT

- What happens for $z \to 0$?

- Dynamic scale: $zQ^2$.
- In the endpoint region new counting parameter, $\lambda^2 \ll z \ll 1$.
- New modes contribute: need “z-SCET”.
- z-modes are non-physical! Not related to external scales of the problem.

<table>
<thead>
<tr>
<th>Name</th>
<th>$(n_+, l_\perp, n_-)$</th>
<th>virtuality $l^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>hard $[h]$</td>
<td>$Q(1, 1, 1)$</td>
<td>$Q^2$</td>
</tr>
<tr>
<td>$z$-hardcollinear $[z - hc]$</td>
<td>$Q(1, \sqrt{z}, z)$</td>
<td>$zQ^2$</td>
</tr>
<tr>
<td>$z$-anti-hardcollinear $[z - \bar{hc}]$</td>
<td>$Q(z, \sqrt{z}, 1)$</td>
<td>$zQ^2$</td>
</tr>
<tr>
<td>$z$-soft $[z - s]$</td>
<td>$Q(z, z, z)$</td>
<td>$z^2Q^2$</td>
</tr>
<tr>
<td>$z$-anti-softcollinear $[z - \bar{s}c]$</td>
<td>$Q(\lambda^2, \sqrt{z}\lambda, z)$</td>
<td>$z\lambda^2Q^2$</td>
</tr>
</tbody>
</table>

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Beneke, Garny, Jaskiewicz, Szafron, LV, Wang, 2020
**BREAKDOWN OF FACTORIZATION NEAR THE ENDPOINT**

- What happens for $z \to 0$?

  - Dynamic scale: $zQ^2$.
  - In the endpoint region new counting parameter, $\lambda^2 \ll z \ll 1$.
  - New modes contribute: need “$z$-SCET”.
  - $z$-modes are non-physical! Not related to external scales of the problem.

- Need re-factorization:

  $$ C^{B_1}(Q, z) \underset{\text{multi-scale function}}{\longrightarrow} J^{B_1}(z) \overset{z \to 0}{\longrightarrow} C^{A_0}(Q^2) \int d^4x \, T \left[ J^{A_0}, L_{q_{z-s\bar{s}}}(x) \right] = C^{A_0}(Q^2) D^{B_1}(zQ^2, \mu^2) J^{B_1}_{z-s\bar{s}}. $$

DIS FACTORIZATION

- Re-factorization is nontrivial: needs to be embedded in a complete EFT description of DIS:

- Physical modes:
  
  **Perturbative modes**
  
  - Hard: \( p^2 = Q^2 \),
  - Hard-collinear: \( p^2 = Q^2 \lambda^2 = Q^2/N \),
  - Collinear: \( p^2 = \Lambda^2 \),
  - Soft-collinear: \( p^2 = \Lambda^2 \lambda^2 = \Lambda^2/N \).

  **Non-perturbative modes**

  "z-SCET" is here

  \[ g(N) \equiv \int_0^1 dx \: x^{N-1} \: g(x), \quad x \to 1 \iff N \to \infty. \]

- Both terms contain endpoint divergences in the convolution integral.

- We could reshuffle factorization theorem;
  → however, use d-dimensional consistency conditions to start with.

\[ \text{Beneke, Garny, Jaskiewicz, Szafron, LV, Wang, 2020} \]
**D-DIMENSIONAL CONSISTENCY CONDITIONS**

- **Hadronic structure function** is finite:

\[
W = \sum_i W_{\phi,i} f_i = \sum_i \tilde{C}_{\phi,k} \tilde{f}_k, \quad \text{with} \quad \tilde{f}_k = Z_{ki} f_i, \quad W_{\phi,i} = \tilde{C}_{\phi,k} Z_{ki}.
\]

- Focus on the **bare** functions: at LP a single channel contribute:

\[
\sum_i (W_{\phi,i} f_i)^{LP} = W_{\phi,g}^{LP} f_g^{LP}.
\]

- Work in **d-dimensions**: \( \epsilon \) regularizes **endpoint divergences** in convolution integrals. The general expansion of the cross section reads

\[
W_{\phi,g} f_g = f_g(\Lambda) \times \sum_n \left(\frac{\alpha_s}{4\pi}\right)^n \frac{1}{\epsilon^{2n}} \sum_{k=0}^{n} \sum_{j=0}^{n} b_{kj}^{(n)}(\epsilon) \left( \frac{\mu^{2n} N^j}{Q^{2k} \Lambda^{2(n-k)}} \right)^\epsilon + O\left(\frac{1}{N}\right).
\]

- In this equation:
  - Each **hard** loop gives \( \left( \frac{\mu^2}{Q^2} \right) \epsilon \),
  - Each **hard-collinear** loop gives \( \left( \frac{\mu^2}{Q^2 N} \right) \epsilon \),
  - Each **collinear** loop gives \( \left( \frac{\mu^2}{\Lambda^2} \right) \epsilon \),
  - Each **soft-collinear** loop gives \( \left( \frac{\mu^2}{\Lambda^2 N} \right) \epsilon \).
D-DIMENSIONAL CONSISTENCY CONDITIONS

- Invoking pole cancellations one has the obvious condition:
  \[
  \sum_{k=0}^{n} \sum_{j=0}^{n} b_{k,j}^{(n)} = 0 ,
  \]

  however, also all poles of the type
  \[
  (\ln N)^r \left( \ln \frac{\Lambda}{Q} \right)^s \times \frac{1}{\epsilon^{2n-r-s}} .
  \]
  need to cancel. In the end one has the conditions
  \[
  \sum_{k=0}^{n} \sum_{j=0}^{n} j^r k^s b_{k,j}^{(n)} = 0 \quad \text{for } s + r < 2n, \ r, s \geq 0 .
  \]

- At order \( n \) this gives \( n^2 + 2n \) equations, for \( (n+1)^2 \) coefficients \( b_{k,j}^{(n)} \): the system can be solved in terms of \( n \) unknown coefficient, one per order \( n \).

- Let us consider the \( n \)-loop hard coefficients \( b_{n0}^{(n)} \): assuming exponentiation of the one-loop hard coefficient \( b_{10}^{(1)} \) one has
  \[
  W_{\phi,g}^{LP,LL}_{\text{hard loops}} = \exp \left[ -\frac{\alpha_s C_A}{\pi} \frac{1}{\epsilon^2} \left( \frac{\mu^2}{Q^2} \right)^\epsilon \right] , \quad \Rightarrow \quad b_{n0}^{(n)} = (-4C_A)^n .
  \]

- This provides the single condition required at LP to fix all of the \( b_{k,j}^{(n)} \):
  \[
  (W_{\phi,g} f_g)^{LP,LL} = \exp \left[ \frac{\alpha_s C_A}{\pi} \frac{1}{\epsilon^2} \left\{ \left( \frac{\mu^2}{Q^2} \right)^\epsilon - \left( \frac{\mu^2}{\Lambda^2} \right)^\epsilon \right\} (N^\epsilon - 1) \right] f_g(\Lambda) .
  \]
**RESUMMATION AT LP**

- The exponentiation of the one loop correction is actually well known. In SCET hard modes are given as a short-distance coefficient of the LP operator

\[ J^{A0} = 2g_{\mu\nu} n_- \partial A^\mu_{\perp hc}(sn_-) n_+ \partial A^\nu_{\perp c}(tn_+) , \]

whose anomalous dimension reads

\[ \Gamma(\alpha_s, \mu) \equiv -\frac{dZ}{d \ln \mu} Z^{-1} = \Gamma_{\text{cusp}}(\alpha_s) \ln \frac{Q^2}{\mu^2} + \gamma(\alpha_s), \quad \text{with} \quad \frac{d\alpha_s}{d \ln \mu} \equiv -2\epsilon\alpha_s + \beta(\alpha_s). \]

- The bare coefficient function is given by

\[ C^{A0}_{\text{bare}} = Z(\mu) C^{A0}(\mu), \quad \text{with} \quad \ln Z(\mu) = \int_0^{\alpha_s(\mu)} \frac{d\alpha}{\alpha} \frac{1}{2\epsilon - \beta(\alpha)/\alpha} \left( \Gamma(\alpha, \mu) - \int_0^\alpha \frac{d\alpha'}{\alpha'} \frac{2\Gamma_{\text{cusp}}(\alpha')}{2\epsilon - \beta(\alpha')/\alpha'} \right). \]

- At LL accuracy

\[ \Gamma_{\text{cusp}}(\alpha_s) = \frac{\alpha_s C_A}{\pi}, \quad \gamma(\alpha_s) = 0, \quad \beta(\alpha_s) = 0, \]

- Which gives

\[ W^{LP,LL}_{\phi,g} \bigg|_{\text{hard loops}} = |C^{A0}_{\text{bare}}|^2_{LL} = \exp \left( 2 \ln Z^{LL}(Q) \right) = \exp \left[ -\frac{\alpha_s C_A}{\pi} \frac{1}{\epsilon^2} \left( \frac{\mu^2}{Q^2} \right)^\epsilon \right]. \]
Consider now the expansion to NLP:

\[
\sum_i (W_{\phi,i} f_i)^{NLP} = W_{\phi,q} f_q^{LP} + W_{\phi,q} f_q^{LP} + W_{\phi,g} f_g^{LP} + W_{\phi,g} f_g^{NLP}.
\]

The general expansion of the cross section reads.

\[
\sum_i (W_{\phi,i} f_i)^{NLP} = f_q(\Lambda) \times \frac{1}{N} \sum_{n=1} \left( \frac{\alpha_s}{4\pi} \right)^n \frac{1}{\epsilon^{2n-1}} \sum_{k=0}^{n} \sum_{j=0}^{n} c_{k,j}^{(n)} (\epsilon) \left( \frac{\mu^{2n} N^j}{Q 2k \Lambda^{2(n-k)}} \right) + f_q(\Lambda), f_g(\Lambda) \text{ terms}.
\]

Invoking pole cancellations:

\[
\sum_{k=0}^{n} \sum_{j=0}^{n} j^r k^s c_{k,j}^{(n)} = 0 \quad \text{for } s + r < 2n - 1, \ r, s \geq 0,
\]

allows \((n+1)^2\) coefficients \(c_{k,j}^{(n)}\) to be determined from \(2n^2-n\) equations.

Three unknowns: these can be reduced using information from the region expansion:

- \(c_{n0}^{(n)} = 0\) (final state cannot be purely hard)
- \(c_{00}^{(n)} = 0\) (without any hard or anti-hardcollinear loops there must be at least one soft-collinear loop).
- Non-trivial condition is given by \(c_{n1}^{(n)}\) : this is the n-loop hard region!
D-DIMENSIONAL CONSISTENCY CONDITIONS

- Assume exponentiation of 1-loop result:

\[ P_{gg}(s_{gg}, z)|_{1\text{-loop}} = P_{gg}(s_{gg}, z)|_{\text{tree}} \frac{\alpha_s}{\pi} \frac{1}{\epsilon^2} \left( T_1 \cdot T_0 \left( \frac{\mu^2}{zQ^2} \right)^{\epsilon} + T_2 \cdot T_0 \left( \frac{\mu^2}{zQ^2} \right)^{\epsilon} + T_1 \cdot T_2 \left[ \left( \frac{\mu^2}{Q^2} \right)^{\epsilon} - \left( \frac{\mu^2}{zQ^2} \right)^{\epsilon} + \left( \frac{\mu^2}{z s_{gg}} \right)^{\epsilon} \right] \right) + \mathcal{O}(\epsilon^{-1}). \]

- Restricting to the hard region and substituting color operators one has

\[ P_{gg, \text{hard}}(s_{gg}, z) = \frac{\alpha_s C_F}{2\pi} \frac{1}{z} \exp \left[ \frac{\alpha_s}{\pi} \frac{1}{\epsilon^2} \left( -C_A \left( \frac{\mu^2}{Q^2} \right)^{\epsilon} + (C_A - C_F) \left( \frac{\mu^2}{zQ^2} \right)^{\epsilon} \right) \right]. \]

- With \( f_i(\mu) = U_{ij}(\mu)f_j(\Lambda) \) one has

\[ \sum_i (W_{\phi, i} f_i)^{NLP} \bigg|_{\alpha f_q(\Lambda)} = (W_{\phi, q}^{NLP} U_{qq}^{LP} + W_{\phi, q}^{LP} U_{\phi q}^{NLP}) f_q(\Lambda). \]

- Inserting the result above in the end one has

\[ W_{\phi, q}^{NLP, LP} = -\frac{1}{2N} \frac{C_F}{C_F - C_A} \frac{\epsilon N^\epsilon}{N^\epsilon - 1} \left( \exp \left[ \frac{\alpha_s C_F}{\pi} \frac{1}{\epsilon^2} \left( \frac{\mu^2}{Q^2} \right)^{\epsilon} (N^\epsilon - 1) \right] - \exp \left[ \frac{\alpha_s C_A}{\pi} \frac{1}{\epsilon^2} \left( \frac{\mu^2}{Q^2} \right)^{\epsilon} (N^\epsilon - 1) \right] \right), \]

\[ U_{\phi q}^{NLP, LP} = -\frac{1}{2N} \frac{C_F}{C_F - C_A} \frac{\epsilon N^\epsilon}{N^\epsilon - 1} \left( \exp \left[ -\frac{\alpha_s C_F}{\pi} \frac{1}{\epsilon^2} \left( \frac{\mu^2}{\Lambda^2} \right)^{\epsilon} (N^\epsilon - 1) \right] - \exp \left[ -\frac{\alpha_s C_A}{\pi} \frac{1}{\epsilon^2} \left( \frac{\mu^2}{\Lambda^2} \right)^{\epsilon} (N^\epsilon - 1) \right] \right). \]

**Similar conjecture “soft quark Sudakov” in Moult, Stewart, Vita, Zhu, 2019.**

**Beneke, Garny, Jaskiewicz, Szafron, LV, Wang, 2020**
RESUMMATION FROM RE-FACTORIZATION: A GLIMPSE

- Is it possible to achieve this in SCET? Another look at re-factorization:

\[ C^{B_1}(Q, z) J^{B_1}(z) \xrightarrow{z \to 0} C^{A_0}(Q^2) \int d^4 x \mathcal{T} \left[ J^{A_0}, \mathcal{L}_{Q_x \to z_{\overline{8}}} (x) \right] = C^{A_0}(Q^2) D^{B_1}(zQ^2, \mu^2) J^{B_1}_{z \to 8c}. \]

- Integrate out hard modes (solve RGEs in d-dimensions)

\[ \frac{d}{d \ln \mu} C^{A_0}(Q^2, \mu^2) = \frac{\alpha_s C_A}{\pi} \ln \frac{Q^2}{\mu^2} C^{A_0}(Q^2, \mu^2). \]

\[ \Rightarrow \quad [C^{A_0}(Q^2, \mu^2)]_{\text{bare}} = C^{A_0}(Q^2, Q^2) \exp \left[ -\frac{\alpha_s C_A}{2\pi} \frac{1}{\epsilon^2} \left( \frac{Q^2}{\mu^2} \right)^{-\epsilon} \right]. \]

- Integrate out z-hardcollinear modes

\[ \frac{d}{d \ln \mu} D^{B_1}(zQ^2, \mu^2) = \frac{\alpha_s}{\pi} (C_F - C_A) \ln \frac{zQ^2}{\mu^2} D^{B_1}(zQ^2, \mu^2). \]

\[ \Rightarrow \quad [D^{B_1}(zQ^2, \mu^2)]_{\text{bare}} = D^{B_1}(zQ^2, zQ^2) \exp \left[ -\frac{\alpha_s}{2\pi} (C_F - C_A) \frac{1}{\epsilon^2} \left( \frac{zQ^2}{\mu^2} \right)^{-\epsilon} \right]. \]

- This reproduces

\[ \mathcal{P}_{qq,\text{hard}}(s_{qq}, z) = \frac{\alpha_s C_F}{2\pi} \frac{1}{z} \exp \left[ \frac{\alpha_s}{\pi} \frac{1}{\epsilon^2} \left( -C_A \left( \frac{\mu^2}{Q^2} \right)^\epsilon + (C_A - C_F) \left( \frac{\mu^2}{zQ^2} \right)^\epsilon \right) \right] \]
Integrating $P_{qg}$ we get $W$:

$$W_{\phi,q}|_{q\phi^*\rightarrow qg} = \int_0^1 dz \left( \frac{\mu^2}{s_{qg}z\bar{z}} \right)^\epsilon P_{qg}(s_{qg}, z) \bigg|_{s_{qg}=Q^2\frac{1-x}{x}},$$

Furthermore, recall

$$W = \sum_i W_{\phi,i} f_i = \sum_k \tilde{C}_{\phi,k} f_k, \quad \Rightarrow \quad W^{\text{NLP}} = \tilde{C}^{\text{NLP}} Z_{qq} + \tilde{C}^{\text{LP}} Z_{gq}.$$  

We obtain a solution for $\tilde{C}$:

$$\tilde{C}_{\phi,q}^{\text{NLP,LL}}|_{\epsilon\rightarrow 0} = \frac{1}{2N \ln N} \frac{C_F}{C_F - C_A} \left( B_0(a) \exp \left[ C_A \frac{\alpha_s}{\pi} \left( \frac{1}{2} \ln^2 N + \ln N \ln \frac{\mu^2}{Q^2} \right) \right] \right), \quad \text{with} \quad a = \frac{\alpha_s}{\pi} (C_F - C_A) \ln^2 N,$$

and

$$P_{ij} = -\gamma_{ij} = \frac{dZ_{ik}}{d \ln \mu} (Z^{-1})_{kj}, \quad \gamma_{qg}^{\text{NLP,LL}}(N) = -\frac{1}{N} \frac{\alpha_s C_F}{\pi} B_0(a), \quad B_0(x) = \sum_{n=0}^{\infty} \frac{B_n}{(n!)^2} x^n,$$

with Bernoulli numbers $B_0 = 1, B_1 = -1/2, ...$

Resummation has been obtained assuming the exponentiation of the 1-real 1 virtual-hard contribution to the partonic structure function. The exponentiation can be phrased in the context of refactorization in SCET.

Resummation of the whole structure function can be achieved starting from the exponentiation of a single region, because of consistency conditions for pole cancellation.

Exponentiation of the hard region fixes the tower of coefficients $c_{n1}^{(n)}$.

Determining another tower of coefficients should allow one to achieve resummation as well. The tower of all-real soft emissions $c_{nn}^{(n)}$ is particularly suitable for a diagrammatic approach.

van Beekveld, LV, White 2021
The tower of soft real emissions can be determined based on the following considerations:

In a physical polarization gauge in which only ladder diagrams contribute to the LLs.

The power suppression is given by the soft quark polarization sum; gluon emissions are eikonal (LP):

\[
|\mathcal{M}_{q_h \to q_{g_1} \ldots g_n}|^2 = \frac{|\lambda|^{2n+1} C_F^{m-1} C_A^{m-m} g_s^{2(n+1)}}{8\mu^{(d-4)(n+1)}} \left( \prod_{i=1}^{n} \frac{2q \cdot p \cdot k_i}{q' \cdot k_i} \right) \text{Tr}[\gamma^\beta k_q \gamma^\alpha] \left( -\eta_{\alpha\beta} + \frac{q' \cdot p_{H,\beta} + q' \cdot p_{H,\alpha}}{q' \cdot p_H} \right) \\
\times \frac{1}{(p \cdot k_1)^2[p \cdot (k_1 + k_2)]^2 \ldots [p \cdot (k_1 + \ldots + k_m + k_q)]^2 \ldots [p \cdot (k_1 + \ldots + k_n + k_q)]^2}.
\]

Phase space can be also approximated to LP, and factorizes in Laplace space:

\[
\int d\Phi^{(n+2)} = \frac{2\pi}{(4\pi)^{(n+1)d/2}} (Q^2)^{n+(n+1)d/2} x^{-(n+1)(d-4)/2} (1 - x)^{-(n+1)d/2} \cdot \int_{-\infty}^{\infty} \frac{dT}{2\pi i} e^{T(1-x)} \\
\times \int d\alpha_q d\beta_q (\alpha_q \beta_q)^{d-4/2} e^{-T(\alpha_q + \beta_q)} \left[ \prod_{i=1}^{n} \int d\bar{\alpha}_i d\bar{\beta}_i (\bar{\alpha}_i \bar{\beta}_i)^{d-4/2} e^{-T(\bar{\alpha}_i + \bar{\beta}_i)} \right], \quad \bar{\alpha}_i \simeq \frac{q' \cdot k}{p \cdot q}, \quad \bar{\beta}_i \simeq \frac{p \cdot k}{p \cdot q}.
\]
In Mellin space one obtains

\[ W^{(n+1)}_{\phi,q} = - \left( \sum_{m=0}^{n} C_F^{m+1} C_A^{n-m} \right) \frac{2 N^\epsilon}{\epsilon N} \left( \frac{4 N^\epsilon}{\epsilon^2} \right)^n \frac{1}{(n+1)!}, \]

which sums to

\[ W_{\phi,q} \bigg|_{LL} = \sum_{n=1}^{\infty} a_s^n W^{(n)}_{\phi,q} = - \frac{2 a_s C_F N^\epsilon}{\epsilon N C_F - C_A} \frac{1}{\left( \frac{4 a_s N^\epsilon}{\epsilon^2} \right)} \left\{ \exp \left[ \frac{4 a_s C_F N^\epsilon}{\epsilon^2} \right] - \exp \left[ \frac{4 a_s C_A N^\epsilon}{\epsilon^2} \right] \right\}. \]

The full result is found requiring that virtual corrections modify the real emission contributions at each order, removing singularities which are simultaneously soft and collinear: this leads to the parametrization

\[ W_{\phi,q} \bigg|_{LL} = - \frac{2 a_s C_F N^\epsilon}{\epsilon} \frac{1}{N C_F - C_A} \left( \frac{4 a_s (N^\epsilon + \lambda_1)}{\epsilon^2} \right)^{-1} \exp \left[ \frac{4 a_s (\lambda_2 C_F + \lambda_3 C_A)}{\epsilon^2} \right] \times \left\{ \exp \left[ \frac{4 a_s C_F (N^\epsilon + \lambda_4)}{\epsilon^2} \right] - \exp \left[ \frac{4 a_s C_A (N^\epsilon + \lambda_5)}{\epsilon^2} \right] \right\}. \]

which can be easily fixed, given that the system is overconstrained. We find

\[ W_{\phi,q} \bigg|_{LL} = - \frac{2 a_s C_F N^\epsilon}{\epsilon} \frac{1}{N C_F - C_A} \left( \frac{4 a_s (N^\epsilon - 1)}{\epsilon^2} \right)^{-1} \left\{ \exp \left[ \frac{4 a_s C_F (N^\epsilon - 1)}{\epsilon^2} \right] - \exp \left[ \frac{4 a_s C_A (N^\epsilon - 1)}{\epsilon^2} \right] \right\}, \]

which reproduces the previous result.

van Beekveld, LV, White 2021
OFF-DIAGONAL DIS: THE DIAGRAMMATIC WAY

- The same procedures can be easily adapted to the subleading $qg$ channel in Drell-Yan (and Higgs production).

- **Consistency conditions** can be studied to determine the smallest set of parameters necessary to determine the whole partonic cross section;

- The set of parameters can be determined
  - by assuming exponentiation of a given region, justified within a refactorization approach.
  - by direct calculation of the ladder diagrams contributing to the real emission.

- Either way, one in the end reproduces an earlier conjecture in Lo Presti, Almasy, Vogt 2014:

\[
W_{DY,gq}^{\text{LL}} = -\frac{T_R}{2(C_F - C_A)} \frac{1}{N} \frac{\epsilon(N^\epsilon - 1)}{N^\epsilon - 1} \exp \left( \frac{4a_s C_F (N^\epsilon - 1)}{\epsilon^2} \right) \\
\quad \times \left\{ \exp \left[ \frac{4a_s C_F N^\epsilon (N^\epsilon - 1)}{\epsilon^2} \right] - \exp \left[ \frac{4a_s C_A N^\epsilon (N^\epsilon - 1)}{\epsilon^2} \right] \right\},
\]

\[
\tilde{C}_{DY,gq}^{\text{LL}} = \frac{T_R}{C_A - C_F} \frac{1}{2N \ln N} \left[ \epsilon^{8C_F a_s \ln^2 N} B_0 [4a_s (C_A - C_F) \ln^2 N] - \epsilon^{(2C_F + 6C_A) a_s \ln^2 N} \right].
\]
CONCLUSION

- The general structure of factorization theorems describing soft radiation beyond LP is now understood.

- However, it involves **divergences** in convolution integrals: new “internal” modes appear due to endpoint divergences. Rigorous factorization and resummation **still possible**, but **highly non-trivial**.

- Relationships between **bare** and **renormalized** objects need to be **better understood**.

- We provided an explicit example (**off-diagonal DIS qg channel**) where these problems can be studied, obtaining **new insight**.

- To do:
  - derive a **refactorization** framework for resummation near threshold in SCET;
  - extend the **diagrammatic approach** beyond LLs.

- More in general:
  - **phenomenological analysis** of relevant processes for the LHC;
  - extend resummation at NLP to **other kinematic limits** (small $p_T$, small $\beta$, etc).
EXTRA
Hard scattering processes are calculated in perturbation theory.

Going beyond NNLO and N3LO is difficult, yet necessary to match the precision of current and forthcoming experiments!

Loop and phase space integrals:
- Analytic vs numerical evaluation
- Space of functions
- Infrared divergences
- Large logarithms

High-energy limit
The presence of largely different scales gives rise to large logarithms:

\[ d\sigma \sim 1 + \alpha_s(L + 1) + \alpha_s^2(L^2 + L + 1) + \ldots \]

\[ \sim \log^2(1 - z) \]

Studies of scattering amplitudes to high-perturbative order, bootstrap approach for infrared divergences, phenomenology.
Developed a framework for the calculation of amplitudes in the **high-energy limit**, by solving iteratively rapidity evolution equation (Baliktsky-JIMWLK).

<table>
<thead>
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<th>Order</th>
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</tr>
</tbody>
</table>

**Analysed to 2 loops in** Del Duca, Falcioni, Magnea, Vernazza 2014;

**Calculated to 3 loops in** Caron-Huot, Gardi, LV, 2017;

**Calculated to 4 loops in** Falcioni, Gardi, Milloy, LV, 2020; Falcioni, Gardi, Maher, Milloy, LV, 2021.

**IR divergences calculated to all orders in** Caron-Huot, Gardi, Reichel, LV, 2017;

**Finite terms calculated to 13 loops in** Caron-Huot, Gardi, Reichel, LV, 2020.