Stefano Frixione

Collinear factorisation for e^+e^- collisions

Based on: 1909.03886 (SF), 1911.12040 (Bertone, Cacciari, SF, Stagnitto)
2105.06688 (SF), and work in progress within
MadGraph5_aMC@NLO (2108.10261, SF, Mattelaer, Zaro, Zhao)

Vienna, 9/11/2021

Assumption:

Somewhere, someone will build an e^+e^- collider

(linear or circular)

Consider the production of a system X at an e^+e^- collider:

$$e^+(P_{e^+}) + e^-(P_{e^-}) \longrightarrow X$$

Its cross section is written as follows:

$$d\Sigma_{e^+e^-}(P_{e^+}, P_{e^-}) = \sum_{kl=e^+e^-\gamma} \int dy_+ dy_- \, \mathcal{B}_{kl}(y_+, y_-) \, d\sigma_{kl}(y_+ P_{e^+}, y_- P_{e^-})$$

Here:

- $d\Sigma_{e^+e^-}$: the collider-level cross section
- $d\sigma_{kl}$: the particle-level cross section
- $\mathcal{B}_{kl}(y_+, y_-)$: describes beam dynamics (including beamstrahlung)
- e^+, e^- on the lhs: the beams
- e^+, e^-, γ on the rhs: the particles

I'll mostly be concerned with computing $d\sigma_{kl}$ in the rest of the talk

The particle-level cross section $d\sigma$ embeds all that is not beam dynamics

It is perturbatively computable, but plagued by $\log(m/E)$ terms to all orders. Fortunately, the dominant classes of these are factorisable:

$$d\sigma\left(\log(m/E), m/E\right) = \mathcal{K}\left(\log(m/E)\right) \otimes d\hat{\sigma}\left(m/E\right)$$

The idea is to compute $d\hat{\sigma}$ to some fixed order in perturbation theory, and $\mathcal{K}$ to all orders (so that logs are resummed)

The definitions of $\mathcal{K}$ and of the convolution $(\otimes)$ determine unambiguously how the logs are resummed. Historically (LEP), simulations have been predominantly done by adopting the YFS formalism

Therefore, two things to be done:

1. Compute $d\hat{\sigma}$

2. Compute $\mathcal{K}$ to all orders within a definite convolution scheme

Therefore, two things to be done:

1. Compute $d\hat{\sigma}$

   With the exception of dedicated, high-accuracy computations, the way to go is *automation*. With MadGraph5_aMC@NLO, both LO and NLO results can be obtained for arbitrary processes, for any combination $\alpha_S^k \alpha^p$

   (theoretical basis in 1405.0301, 1804.10017)

   $\longrightarrow$

Process		Syntax	Cross section (pb)			
Heavy quarks and jets			LO 1 TeV		NLO 1 TeV	
i.1	$e^+e^- \to jj$	`e+ e- > j j`	$6.223 \pm 0.005 \cdot 10^{-1}$	$^{+0.0\%}_{-0.0\%}$	$6.389 \pm 0.013 \cdot 10^{-1}$	$^{+0.2\%}_{-0.2\%}$
i.2	$e^+e^- \to jjj$	`e+ e- > j j j`	$3.401 \pm 0.002 \cdot 10^{-1}$	$^{+9.6\%}_{-8.0\%}$	$3.166 \pm 0.019 \cdot 10^{-1}$	$^{+0.2\%}_{-2.1\%}$
i.3	$e^+e^- \to jjjj$	`e+ e- > j j j j`	$1.047 \pm 0.001 \cdot 10^{-1}$	$^{+20.0\%}_{-15.3\%}$	$1.090 \pm 0.006 \cdot 10^{-1}$	$^{+0.0\%}_{-2.8\%}$
i.4	$e^+e^- \to jjjjj$	`e+ e- > j j j j j`	$2.211 \pm 0.006 \cdot 10^{-2}$	$^{+31.4\%}_{-22.0\%}$	$2.771 \pm 0.021 \cdot 10^{-2}$	$^{+4.4\%}_{-8.6\%}$
i.5	$e^+e^- \to t\bar{t}$	`e+ e- > t t~`	$1.662 \pm 0.002 \cdot 10^{-1}$	$^{+0.0\%}_{-0.0\%}$	$1.745 \pm 0.006 \cdot 10^{-1}$	$^{+0.4\%}_{-0.4\%}$
i.6	$e^+e^- \to t\bar{t}j$	`e+ e- > t t~ j`	$4.813 \pm 0.005 \cdot 10^{-2}$	$^{+9.3\%}_{-7.8\%}$	$5.276 \pm 0.022 \cdot 10^{-2}$	$^{+1.3\%}_{-2.1\%}$
i.7*	$e^+e^- \to t\bar{t}jj$	`e+ e- > t t~ j j`	$8.614 \pm 0.009 \cdot 10^{-3}$	$^{+19.4\%}_{-15.0\%}$	$1.094 \pm 0.005 \cdot 10^{-2}$	$^{+5.0\%}_{-6.3\%}$
i.8*	$e^+e^- \to t\bar{t}jjj$	`e+ e- > t t~ j j j`	$1.044 \pm 0.002 \cdot 10^{-3}$	$^{+30.5\%}_{-21.6\%}$	$1.546 \pm 0.010 \cdot 10^{-3}$	$^{+10.6\%}_{-11.6\%}$
i.9*	$e^+e^- \to t\bar{t}t\bar{t}$	`e+ e- > t t~ t t~`	$6.456 \pm 0.016 \cdot 10^{-7}$	$^{+19.1\%}_{-14.8\%}$	$1.221 \pm 0.005 \cdot 10^{-6}$	$^{+13.2\%}_{-11.2\%}$
i.10*	$e^+e^- \to t\bar{t}t\bar{t}j$	`e+ e- > t t~ t t~ j`	$2.719 \pm 0.005 \cdot 10^{-8}$	$^{+29.9\%}_{-21.3\%}$	$5.338 \pm 0.027 \cdot 10^{-8}$	$^{+18.3\%}_{-15.4\%}$
i.11	$e^+e^- \to b\bar{b}$ (4f)	`e+ e- > b b~`	$9.198 \pm 0.004 \cdot 10^{-2}$	$^{+0.0\%}_{-0.0\%}$	$9.282 \pm 0.031 \cdot 10^{-2}$	$^{+0.0\%}_{-0.0\%}$
i.12	$e^+e^- \to b\bar{b}j$ (4f)	`e+ e- > b b~ j`	$5.029 \pm 0.003 \cdot 10^{-2}$	$^{+9.5\%}_{-8.0\%}$	$4.826 \pm 0.026 \cdot 10^{-2}$	$^{+0.5\%}_{-2.5\%}$
i.13*	$e^+e^- \to b\bar{b}jj$ (4f)	`e+ e- > b b~ j j`	$1.621 \pm 0.001 \cdot 10^{-2}$	$^{+20.0\%}_{-15.3\%}$	$1.817 \pm 0.009 \cdot 10^{-2}$	$^{+0.0\%}_{-3.1\%}$
i.14*	$e^+e^- \to b\bar{b}jjj$ (4f)	`e+ e- > b b~ j j j`	$3.641 \pm 0.009 \cdot 10^{-3}$	$^{+31.4\%}_{-22.1\%}$	$4.936 \pm 0.038 \cdot 10^{-3}$	$^{+4.8\%}_{-8.9\%}$
i.15*	$e^+e^- \to b\bar{b}b\bar{b}$ (4f)	`e+ e- > b b~ b b~`	$1.644 \pm 0.003 \cdot 10^{-4}$	$^{+19.9\%}_{-15.3\%}$	$3.601 \pm 0.017 \cdot 10^{-4}$	$^{+15.2\%}_{-12.5\%}$
i.16*	$e^+e^- \to b\bar{b}b\bar{b}j$ (4f)	`e+ e- > b b~ b b~ j`	$7.660 \pm 0.022 \cdot 10^{-5}$	$^{+31.3\%}_{-22.0\%}$	$1.537 \pm 0.011 \cdot 10^{-4}$	$^{+17.9\%}_{-15.3\%}$
i.17*	$e^+e^- \to t\bar{t}b\bar{b}$ (4f)	`e+ e- > t t~ b b~`	$1.819 \pm 0.003 \cdot 10^{-4}$	$^{+19.5\%}_{-15.0\%}$	$2.923 \pm 0.011 \cdot 10^{-4}$	$^{+9.2\%}_{-8.9\%}$
i.18*	$e^+e^- \to t\bar{t}b\bar{b}j$ (4f)	`e+ e- > t t~ b b~ j`	$4.045 \pm 0.011 \cdot 10^{-5}$	$^{+30.5\%}_{-21.6\%}$	$7.049 \pm 0.052 \cdot 10^{-5}$	$^{+13.7\%}_{-13.1\%}$

From 1405.0301; this is NLO in α_S

Process		Syntax	Cross section (pb)			
Top quarks +bosons			LO 1 TeV		NLO 1 TeV	
j.1	$e^+e^- \to t\bar{t}H$	`e+ e- > t t~ h`	$2.018 \pm 0.003 \cdot 10^{-3}$	$^{+0.0\%}_{-0.0\%}$	$1.911 \pm 0.006 \cdot 10^{-3}$	$^{+0.4\%}_{-0.5\%}$
j.2*	$e^+e^- \to t\bar{t}Hj$	`e+ e- > t t~ h j`	$2.533 \pm 0.003 \cdot 10^{-4}$	$^{+9.2\%}_{-7.8\%}$	$2.658 \pm 0.009 \cdot 10^{-4}$	$^{+0.5\%}_{-1.5\%}$
j.3*	$e^+e^- \to t\bar{t}Hjj$	`e+ e- > t t~ h j j`	$2.663 \pm 0.004 \cdot 10^{-5}$	$^{+19.3\%}_{-14.9\%}$	$3.278 \pm 0.017 \cdot 10^{-5}$	$^{+4.0\%}_{-5.7\%}$
j.4*	$e^+e^- \to t\bar{t}\gamma$	`e+ e- > t t~ a`	$1.270 \pm 0.002 \cdot 10^{-2}$	$^{+0.0\%}_{-0.0\%}$	$1.335 \pm 0.004 \cdot 10^{-2}$	$^{+0.5\%}_{-0.4\%}$
j.5*	$e^+e^- \to t\bar{t}\gamma j$	`e+ e- > t t~ a j`	$2.355 \pm 0.002 \cdot 10^{-3}$	$^{+9.3\%}_{-7.9\%}$	$2.617 \pm 0.010 \cdot 10^{-3}$	$^{+1.6\%}_{-2.4\%}$
j.6*	$e^+e^- \to t\bar{t}\gamma jj$	`e+ e- > t t~ a j j`	$3.103 \pm 0.005 \cdot 10^{-4}$	$^{+19.5\%}_{-15.0\%}$	$4.002 \pm 0.021 \cdot 10^{-4}$	$^{+5.4\%}_{-6.6\%}$
j.7*	$e^+e^- \to t\bar{t}Z$	`e+ e- > t t~ z`	$4.642 \pm 0.006 \cdot 10^{-3}$	$^{+0.0\%}_{-0.0\%}$	$4.949 \pm 0.014 \cdot 10^{-3}$	$^{+0.6\%}_{-0.5\%}$
j.8*	$e^+e^- \to t\bar{t}Zj$	`e+ e- > t t~ z j`	$6.059 \pm 0.006 \cdot 10^{-4}$	$^{+9.3\%}_{-7.8\%}$	$6.940 \pm 0.028 \cdot 10^{-4}$	$^{+2.0\%}_{-2.6\%}$
j.9*	$e^+e^- \to t\bar{t}Zjj$	`e+ e- > t t~ z j j`	$6.351 \pm 0.028 \cdot 10^{-5}$	$^{+19.4\%}_{-15.0\%}$	$8.439 \pm 0.051 \cdot 10^{-5}$	$^{+5.8\%}_{-6.8\%}$
j.10*	$e^+e^- \to t\bar{t}W^\pm jj$	`e+ e- > t t~ wpm j j`	$2.400 \pm 0.004 \cdot 10^{-7}$	$^{+19.3\%}_{-14.9\%}$	$3.723 \pm 0.012 \cdot 10^{-7}$	$^{+9.6\%}_{-9.1\%}$
j.11*	$e^+e^- \to t\bar{t}HZ$	`e+ e- > t t~ h z`	$3.600 \pm 0.006 \cdot 10^{-5}$	$^{+0.0\%}_{-0.0\%}$	$3.579 \pm 0.013 \cdot 10^{-5}$	$^{+0.1\%}_{-0.0\%}$
j.12*	$e^+e^- \to t\bar{t}\gamma Z$	`e+ e- > t t~ a z`	$2.212 \pm 0.003 \cdot 10^{-4}$	$^{+0.0\%}_{-0.0\%}$	$2.364 \pm 0.006 \cdot 10^{-4}$	$^{+0.6\%}_{-0.5\%}$
j.13*	$e^+e^- \to t\bar{t}\gamma H$	`e+ e- > t t~ a h`	$9.756 \pm 0.016 \cdot 10^{-5}$	$^{+0.0\%}_{-0.0\%}$	$9.423 \pm 0.032 \cdot 10^{-5}$	$^{+0.3\%}_{-0.4\%}$
j.14*	$e^+e^- \to t\bar{t}\gamma\gamma$	`e+ e- > t t~ a a`	$3.650 \pm 0.008 \cdot 10^{-4}$	$^{+0.0\%}_{-0.0\%}$	$3.833 \pm 0.013 \cdot 10^{-4}$	$^{+0.4\%}_{-0.4\%}$
j.15*	$e^+e^- \to t\bar{t}ZZ$	`e+ e- > t t~ z z`	$3.788 \pm 0.004 \cdot 10^{-5}$	$^{+0.0\%}_{-0.0\%}$	$4.007 \pm 0.013 \cdot 10^{-5}$	$^{+0.5\%}_{-0.5\%}$
j.16*	$e^+e^- \to t\bar{t}HH$	`e+ e- > t t~ h h`	$1.358 \pm 0.001 \cdot 10^{-5}$	$^{+0.0\%}_{-0.0\%}$	$1.206 \pm 0.003 \cdot 10^{-5}$	$^{+0.9\%}_{-1.1\%}$
j.17*	$e^+e^- \to t\bar{t}W^+W^-$	`e+ e- > t t~ w+ w-`	$1.372 \pm 0.003 \cdot 10^{-4}$	$^{+0.0\%}_{-0.0\%}$	$1.540 \pm 0.006 \cdot 10^{-4}$	$^{+1.0\%}_{-0.9\%}$

From 1405.0301; this is NLO in α_S

Process	Syntax	Cross section (in pb)		Correction (in %)
		LO	NLO	
$pp \to e^+\nu_e$	`p p > e+ ve QCD=0 QED=2 [QED]`	$5.2498 \pm 0.0005 \cdot 10^3$	$5.2113 \pm 0.0006 \cdot 10^3$	-0.73 ± 0.01
$pp \to e^+\nu_e j$	`p p > e+ ve j QCD=1 QED=2 [QED]`	$9.1468 \pm 0.0012 \cdot 10^2$	$9.0449 \pm 0.0014 \cdot 10^2$	-1.11 ± 0.02
$pp \to e^+\nu_e jj$	`p p > e+ ve j j QCD=2 QED=2 [QED]`	$3.1562 \pm 0.0003 \cdot 10^2$	$3.0985 \pm 0.0005 \cdot 10^2$	-1.83 ± 0.02
$pp \to e^+e^-$	`p p > e+ e- QCD=0 QED=2 [QED]`	$7.5367 \pm 0.0008 \cdot 10^2$	$7.4997 \pm 0.0010 \cdot 10^2$	-0.49 ± 0.02
$pp \to e^+e^- j$	`p p > e+ e- j QCD=1 QED=2 [QED]`	$1.5059 \pm 0.0001 \cdot 10^2$	$1.4909 \pm 0.0002 \cdot 10^2$	-1.00 ± 0.02
$pp \to e^+e^- jj$	`p p > e+ e- j j QCD=2 QED=2 [QED]`	$5.1424 \pm 0.0004 \cdot 10^1$	$5.0410 \pm 0.0007 \cdot 10^1$	-1.97 ± 0.02
$pp \to e^+e^-\mu^+\mu^-$	`p p > e+ e- mu+ mu- QCD=0 QED=4 [QED]`	$1.2750 \pm 0.0000 \cdot 10^{-2}$	$1.2083 \pm 0.0001 \cdot 10^{-2}$	-5.23 ± 0.01
$pp \to e^+\nu_e\mu^-\bar{\nu}_\mu$	`p p > e+ ve mu- vm~ QCD=0 QED=4 [QED]`	$5.1144 \pm 0.0007 \cdot 10^{-1}$	$5.3019 \pm 0.0009 \cdot 10^{-1}$	$+3.67 \pm 0.02$
$pp \to He^+\nu_e$	`p p > h e+ ve QCD=0 QED=3 [QED]`	$6.7643 \pm 0.0001 \cdot 10^{-2}$	$6.4914 \pm 0.0012 \cdot 10^{-2}$	-4.03 ± 0.02
$pp \to He^+e^-$	`p p > h e+ e- QCD=0 QED=3 [QED]`	$1.4554 \pm 0.0001 \cdot 10^{-2}$	$1.3700 \pm 0.0002 \cdot 10^{-2}$	-5.87 ± 0.02
$pp \to Hjj$	`p p > h j j QCD=0 QED=3 [QED]`	$2.8268 \pm 0.0002 \cdot 10^0$	$2.7075 \pm 0.0003 \cdot 10^0$	-4.22 ± 0.01
$pp \to W^+W^-W^+$	`p p > w+ w- w+ QCD=0 QED=3 [QED]`	$8.2874 \pm 0.0004 \cdot 10^{-2}$	$8.8017 \pm 0.0012 \cdot 10^{-2}$	$+6.21 \pm 0.02$
$pp \to ZZW^+$	`p p > z z w+ QCD=0 QED=3 [QED]`	$1.9874 \pm 0.0001 \cdot 10^{-2}$	$2.0189 \pm 0.0003 \cdot 10^{-2}$	$+1.58 \pm 0.02$
$pp \to ZZZ$	`p p > z z z QCD=0 QED=3 [QED]`	$1.0761 \pm 0.0001 \cdot 10^{-2}$	$0.9741 \pm 0.0001 \cdot 10^{-2}$	-9.47 ± 0.02
$pp \to HZZ$	`p p > h z z QCD=0 QED=3 [QED]`	$2.1005 \pm 0.0003 \cdot 10^{-3}$	$1.9155 \pm 0.0003 \cdot 10^{-3}$	-8.81 ± 0.02
$pp \to HZW^+$	`p p > h z w+ QCD=0 QED=3 [QED]`	$2.4408 \pm 0.0000 \cdot 10^{-3}$	$2.4809 \pm 0.0005 \cdot 10^{-3}$	$+1.64 \pm 0.02$
$pp \to HHW^+$	`p p > h h w+ QCD=0 QED=3 [QED]`	$2.7827 \pm 0.0001 \cdot 10^{-4}$	$2.4259 \pm 0.0027 \cdot 10^{-4}$	-12.82 ± 0.10
$pp \to HHZ$	`p p > h h z QCD=0 QED=3 [QED]`	$2.6914 \pm 0.0003 \cdot 10^{-4}$	$2.3926 \pm 0.0003 \cdot 10^{-4}$	-11.10 ± 0.02
$pp \to t\bar{t}W^+$	`p p > t t~ w+ QCD=2 QED=1 [QED]`	$2.4119 \pm 0.0003 \cdot 10^{-1}$	$2.3025 \pm 0.0003 \cdot 10^{-1}$	-4.54 ± 0.02
$pp \to t\bar{t}Z$	`p p > t t~ z QCD=2 QED=1 [QED]`	$5.0456 \pm 0.0006 \cdot 10^{-1}$	$5.0033 \pm 0.0007 \cdot 10^{-1}$	-0.84 ± 0.02
$pp \to t\bar{t}H$	`p p > t t~ h QCD=2 QED=1 [QED]`	$3.4480 \pm 0.0004 \cdot 10^{-1}$	$3.5102 \pm 0.0005 \cdot 10^{-1}$	$+1.81 \pm 0.02$
$pp \to t\bar{t}j$	`p p > t t j QCD=3 QED=0 [QED]`	$3.0277 \pm 0.0003 \cdot 10^2$	$2.9683 \pm 0.0004 \cdot 10^2$	-1.96 ± 0.02
$pp \to jjj$	`p p > j j j QCD=3 QED=0 [QED]`	$7.9639 \pm 0.0010 \cdot 10^6$	$7.9472 \pm 0.0011 \cdot 10^6$	-0.21 ± 0.02
$pp \to tj$	`p p > t j QCD=0 QED=2 [QED]`	$1.0613 \pm 0.0001 \cdot 10^2$	$1.0539 \pm 0.0001 \cdot 10^2$	-0.70 ± 0.02

From 1804.10017; this is NLO in α; e^+e^- results can be obtained as easily as these ones, provided a definite scheme for item 2. above has been chosen (as is now the case)

Therefore, two things to be done:

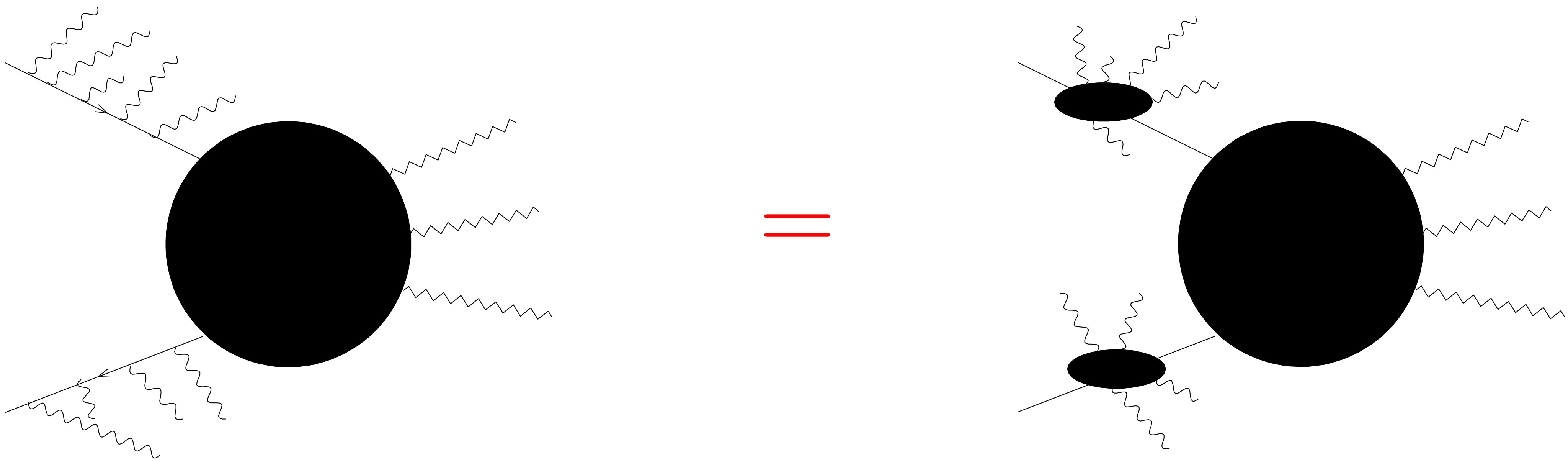
1. Compute $d\hat{\sigma}$

2. Compute $\mathcal{K}$ to all orders within a definite convolution scheme

   We adopt a collinear-factorisation approach. Comparisons with YFS-based predictions will help assess theoretical systematics in a comprehensive way

   (I'll concentrate here on ISR. Analogous formulae hold for FSR)

Collinear factorisation



$$d\sigma = \text{PDF} \star \text{PDF} \star d\hat{\sigma}$$

PDFs collect (universal) small-angle dynamics

$$d\sigma_{kl}(p_k, p_l) = \sum_{ij=e^+,e^-,\gamma} \int dz_+ dz_- \, \Gamma_{i/k}(z_+, \mu^2, m^2)\, \Gamma_{j/l}(z_-, \mu^2, m^2) \times d\hat{\sigma}_{ij}(z_+ p_k, z_- p_l, \mu^2) + \mathcal{O}\left(\left(\frac{m^2}{s}\right)^p\right)$$

where one calculates Γ and $d\hat{\sigma}$ to predict $d\sigma$

- $k\,, l = e^+\,, e^-\,, \gamma$ on the lhs: the particles that emerge from beamstrahlung
- $i\,, j = e^+\,, e^-\,, \gamma$ on the rhs: the partons
- $d\sigma_{kl}$: the particle-level (ie observable) cross section
- $d\hat{\sigma}_{ij}$: the subtracted parton-level cross section.
  Generally with $m = 0$ $\Longrightarrow$ power-suppressed terms in $d\sigma$ discarded
- $\Gamma_{i/k}$: the PDF of parton i inside particle k
- μ: the hard scale, $m^2 \ll \mu^2 \sim s$

Why this approach?

Because it allows one to exploit a significant amount of the technical knowledge we have acquired in two decades of LHC physics

[And: to cross-check YFS-based predictions, and to provide meaningful systematics]

Indeed, *very* similar to QCD, with some notable differences:

- PDFs and power-suppressed terms can be computed perturbatively
- An object (e.g. e^-) may play the role of both particle and parton

As in QCD, a particle is a physical object, a parton is not

As I have said, parton-level cross section computations are highly automated, and can now be carried out at the NLO in both α and α_S with MadGraph5_aMC@NLO

Conversely, until recently PDFs were only available at the LO+LL, which is insufficient in the context of NLO simulations

$\Longrightarrow$

z-space LO+LL PDFs $(\alpha \log(E/m))^k$:

$\sim$ 1992

- $0 \leq k \leq \infty$ for $z \simeq 1$ (Gribov, Lipatov)
- $0 \leq k \leq 3$ for $z < 1$ (Skrzypek, Jadach; Cacciari, Deandrea, Montagna, Nicrosini; Skrzypek)
- matching between these two regimes

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- matching between these two regimes

z-space NLO+NLL PDFs $(\alpha \log(E/m))^k + \alpha\,(\alpha \log(E/m))^{k-1}$:

$\longrightarrow$ 1909.03886, 1911.12040, 2105.06688

- $0 \leq k \leq \infty$ for $z \simeq 1$
- $0 \leq k \leq 3$ for $z < 1 \quad \Longleftrightarrow \quad \mathcal{O}(\alpha^3)$
- matching between these two regimes
- for e^+, e^-, and γ
- both numerical and analytical

Main tool: the solution of PDFs evolution equations

Henceforth, I consider the dominant production mechanism at an e^+e^- collider, namely that associated with partons inside an electron*

Simplified notation:

$$\Gamma_i(z,\mu^2) \equiv \Gamma_{i/e^-}(z,\mu^2)$$

*The case of the positron is identical, at least in QED, and will be understood

NLO initial conditions (1909.03886)

Conventions for the perturbative coefficients:

$$\Gamma_i = \Gamma_i^{[0]} + \frac{\alpha}{2\pi}\Gamma_i^{[1]} + \mathcal{O}(\alpha^2)$$

Results:

$$\begin{aligned}
\Gamma_i^{[0]}(z,\mu_0^2) &= \delta_{ie^-}\delta(1-z) \\
\Gamma_{e^-}^{[1]}(z,\mu_0^2) &= \left[\frac{1+z^2}{1-z}\left(\log\frac{\mu_0^2}{m^2} - 2\log(1-z) - 1\right)\right]_+ + K_{ee}(z) \\
\Gamma_{\gamma}^{[1]}(z,\mu_0^2) &= \frac{1+(1-z)^2}{z}\left(\log\frac{\mu_0^2}{m^2} - 2\log z - 1\right) + K_{\gamma e}(z) \\
\Gamma_{e^+}^{[1]}(z,\mu_0^2) &= 0
\end{aligned}$$

Note:

- Meaningful only if $\mu_0 \sim m$
- In $\overline{\rm MS}$, $K_{ij}(z) = 0$; in general, these functions *define* a factorisation scheme

NLL evolution (1911.12040, 2105.06688)

General idea: solve the evolution equations starting from the initial conditions computed previously

$$\frac{\partial \Gamma_i(z,\mu^2)}{\partial \log \mu^2} = \frac{\alpha(\mu)}{2\pi} \left[P_{ij} \otimes \Gamma_j\right](z,\mu^2) \iff \frac{\partial \Gamma(z,\mu^2)}{\partial \log \mu^2} = \frac{\alpha(\mu)}{2\pi} \left[\mathbb{P} \otimes \Gamma\right](z,\mu^2)\,,$$

Done conveniently in terms of non-singlet, singlet, and photon

Two ways:

- Mellin space: suited to both numerical solution and all-order, large-z analytical solution (called *asymptotic solution*). Dominant
- Directly in z space in an integrated form: suited to fixed-order, all-z analytical solution (called *recursive solution*). Subleading

Bear in mind that PDFs are fully defined only after adopting a definite *factorisation scheme*, which is the choice of the finite terms associated with the subtraction of the collinear poles

(done by means of the $K_{ij}(z)$ functions)

- 1911.12040 $\longrightarrow$ $\overline{\mathrm{MS}}$
- 2105.06688 $\longrightarrow$ a DIS-like scheme (called Δ)

A technicality: owing to the running of α, it is best to evolve in t rather than in μ, with: (∼ Furmanski, Petronzio)

$$
\begin{aligned}
t &= \frac{1}{2\pi b_0} \log \frac{\alpha(\mu)}{\alpha(\mu_0)} \\
&= \frac{\alpha(\mu)}{2\pi} L - \frac{\alpha^2(\mu)}{4\pi} \left(b_0 L^2 - \frac{2b_1}{b_0} L \right) + \mathcal{O}(\alpha^3)\,, \qquad L = \log \frac{\mu^2}{\mu_0^2}\,.
\end{aligned}
$$

Note:

- $t \longleftrightarrow \mu$; notation-wise, the dependence on t is equivalent to the dependence on μ
- $t = 0 \iff \mu = \mu_0$
- L is my "large log"
- Tricky: fixed-α expressions are obtained with $t = \alpha L/(2\pi)$ (and not $t = 0$)

Mellin space

Introduce the evolution operator $\mathbb{E}_N$

$$\Gamma_N(\mu^2) = \mathbb{E}_N(t)\,\Gamma_{0,N}\,, \qquad \mathbb{E}_N(0) = I\,, \qquad \Gamma_{0,N} \equiv \Gamma_N(\mu_0^2)$$

The PDFs evolution equations are then re-expressed by means of an evolution equation for the evolution operator:

$$\begin{aligned}\frac{\partial \mathbb{E}_N^{(K)}(t)}{\partial t} &= b_0 \alpha(\mu) \mathbb{K}_N \left(I + \frac{\alpha(\mu)}{2\pi}\,\mathbb{K}_N \right)^{-1} \mathbb{E}_N^{(K)}(t) \\ &+ \frac{b_0 \alpha^2(\mu)}{\beta(\alpha(\mu))} \sum_{k=0}^{\infty} \left(\frac{\alpha(\mu)}{2\pi} \right)^k \\ &\quad \times \left(I + \frac{\alpha(\mu)}{2\pi}\,\mathbb{K}_N \right) \mathbb{P}_N^{[k]} \left(I + \frac{\alpha(\mu)}{2\pi}\,\mathbb{K}_N \right)^{-1} \mathbb{E}_N^{(K)}(t)\end{aligned}$$

- Can be solved numerically
- Can be solved analytically in a closed form under simplifying assumptions. Chiefly: large-z is equivalent to large-N

Asymptotic $\overline{\rm MS}$ solution

Non-singlet $\equiv$ singlet; photon is more complicated

$$\Gamma_{\rm NLL}(z,\mu^2) \stackrel{z\to 1}{\longrightarrow} \frac{e^{-\gamma_{\rm E}\xi_1} e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)} \xi_1 (1-z)^{-1+\xi_1}$$
$$\times \left\{ 1 + \frac{\alpha(\mu_0)}{\pi} \left[\left(L_0 - 1\right) \left(A(\xi_1) + \frac{3}{4}\right) - 2B(\xi_1) + \frac{7}{4} \right.\right.$$
$$\left.\left. + \left(L_0 - 1 - 2A(\xi_1)\right) \log(1-z) - \log^2(1-z) \right] \right\}$$

where $L_0 = \log \mu_0^2/m^2$, and:

$$A(\kappa) = -\gamma_{\rm E} - \psi_0(\kappa)$$
$$B(\kappa) = \frac{1}{2}\gamma_{\rm E}^2 + \frac{\pi^2}{12} + \gamma_{\rm E}\,\psi_0(\kappa) + \frac{1}{2}\psi_0(\kappa)^2 - \frac{1}{2}\psi_1(\kappa)$$

with:

$$\begin{aligned}
\xi_1 &= 2t - \frac{\alpha(\mu)}{4\pi^2 b_0}\left(1 - e^{-2\pi b_0 t}\right)\left(\frac{20}{9}n_F + \frac{4\pi b_1}{b_0}\right) \\
&= 2t + \mathcal{O}(\alpha t) = \eta_0 + \ldots \\
\hat{\xi}_1 &= \frac{3}{2}\,t + \frac{\alpha(\mu)}{4\pi^2 b_0}\left(1 - e^{-2\pi b_0 t}\right)\left(\lambda_1 - \frac{3\pi b_1}{b_0}\right) \\
&= \frac{3}{2}\,t + \mathcal{O}(\alpha t) = \lambda_0\eta_0 + \ldots \\
\lambda_1 &= \frac{3}{8} - \frac{\pi^2}{2} + 6\zeta_3 - \frac{n_F}{18}(3 + 4\pi^2)
\end{aligned}$$

Remember that:

$$\begin{aligned}
t &= \frac{1}{2\pi b_0}\log\frac{\alpha(\mu)}{\alpha(\mu_0)} \\
&= \frac{\alpha(\mu)}{2\pi}\,L - \frac{\alpha^2(\mu)}{4\pi}\left(b_0 L^2 - \frac{2b_1}{b_0}L\right) + \mathcal{O}(\alpha^3)\,, \qquad L = \log\frac{\mu^2}{\mu_0^2}\,.
\end{aligned}$$

Asymptotic Δ solution

Non-singlet $\equiv$ singlet; photon is trivial

$$\Gamma_{\rm NLL}(z,\mu^2) \xrightarrow{z\to 1} \frac{e^{-\gamma_{\rm E}\xi_1} e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)} \xi_1 (1-z)^{-1+\xi_1} \times \left[\left(1 + \frac{3\alpha(\mu_0)}{4\pi} L_0\right) \sum_{p=0}^{\infty} \mathcal{S}_{1,p}(z) - \frac{\alpha(\mu_0)}{\pi} L_0 \sum_{p=0}^{\infty} \mathcal{S}_{2,p}(z) \right]$$

The $\mathcal{S}_{i,p}(z)$ functions are increasingly suppressed at $z \to 1$ with growing p. The dominant behaviour is:

$$\Gamma_{\rm NLL}(z,\mu^2) \xrightarrow{z\to 1} \frac{e^{-\gamma_{\rm E}\xi_1} e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)} \xi_1 (1-z)^{-1+\xi_1} \times \left[\frac{\alpha(\mu)}{\alpha(\mu_0)} + \frac{\alpha(\mu)}{\pi} L_0 \left(A(\xi_1) + \log(1-z) + \frac{3}{4} \right) \right]$$

- A vastly different logarithmic behaviour w.r.t. the $\overline{\rm MS}$ case

  However, $\Gamma_{\rm NLL}^{(\overline{\rm MS})} - \Gamma_{\rm NLL}^{(\Delta)} = \mathcal{O}(\alpha^2)$

Key facts

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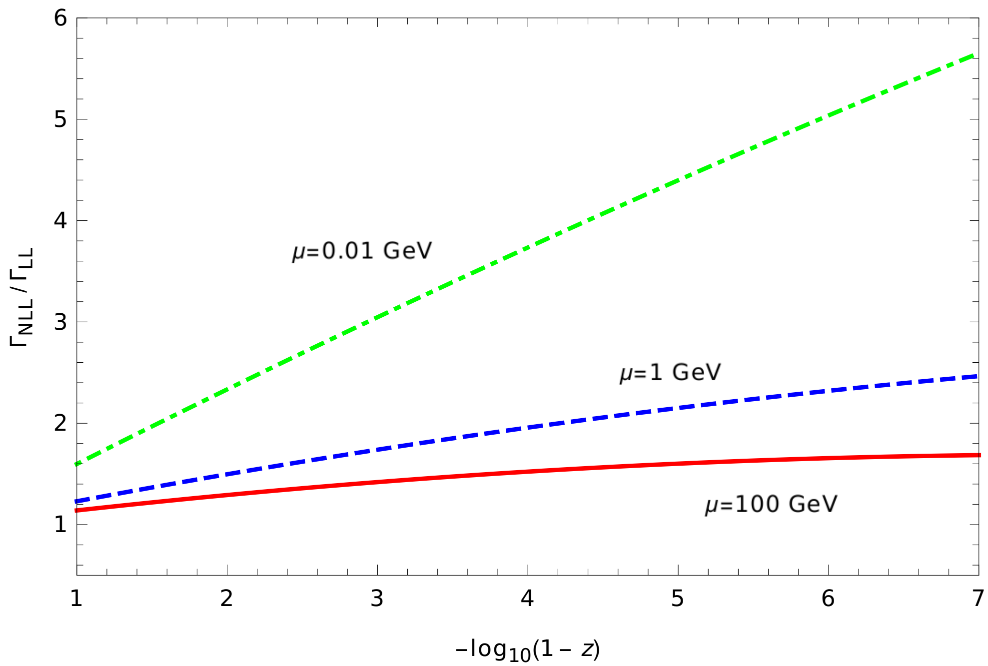
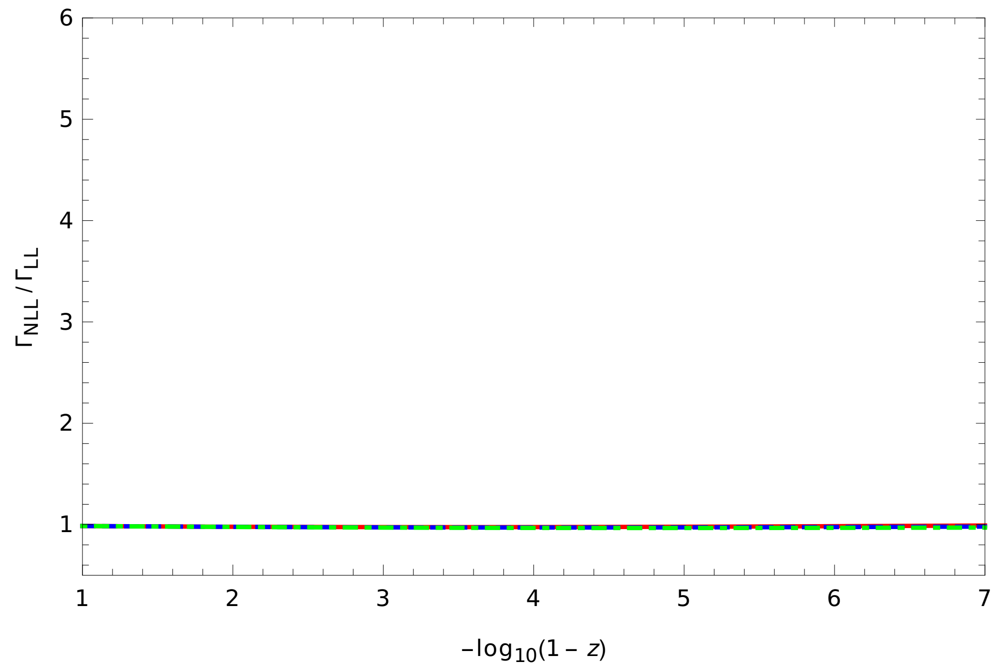
- In addition to that, in $\overline{\mathrm{MS}}$ there are single and double logarithmic terms $\longrightarrow$

$\Gamma_{\rm NLL}/\Gamma_{\rm LL}$ at large z $(\mu_0 = m)$



$\overline{\rm MS}$ scheme



Δ scheme

In $\overline{\rm MS}$, significant scale dependence, and significant differences w.r.t. LL results. This doesn't happen in Δ (note the y ranges in the plots)

This *does not mean* NLO and LO cross sections will differ by large factors:
PDFs are unphysical, and there are huge cancellations with partonic cross sections.
Also, bear in mind that $\Gamma_{\rm NLL}^{(\overline{\rm MS})} - \Gamma_{\rm NLL}^{(\Delta)} = \mathcal{O}(\alpha^2)$

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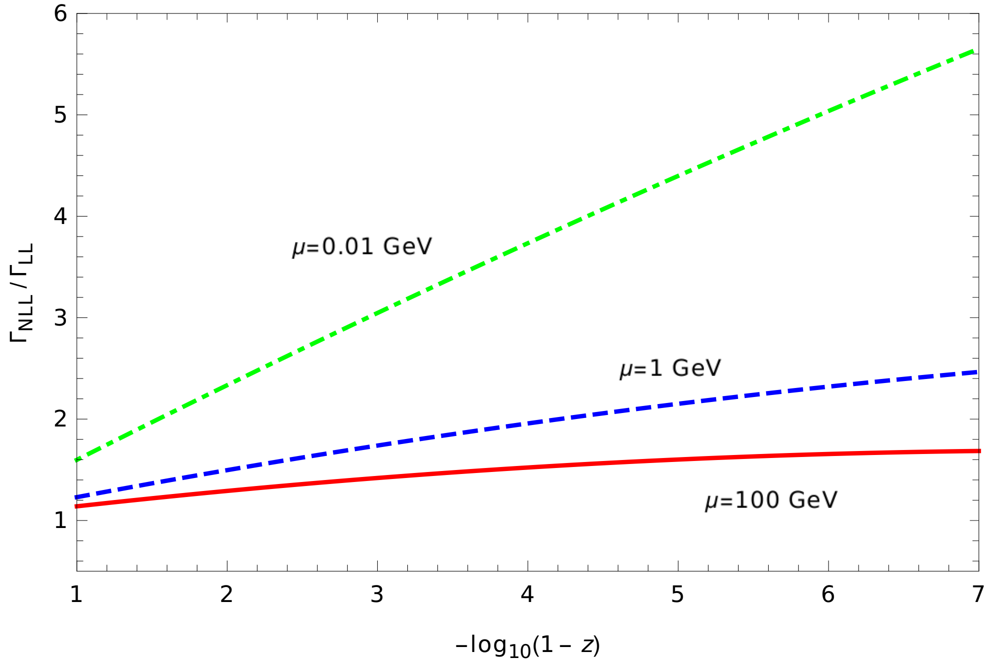

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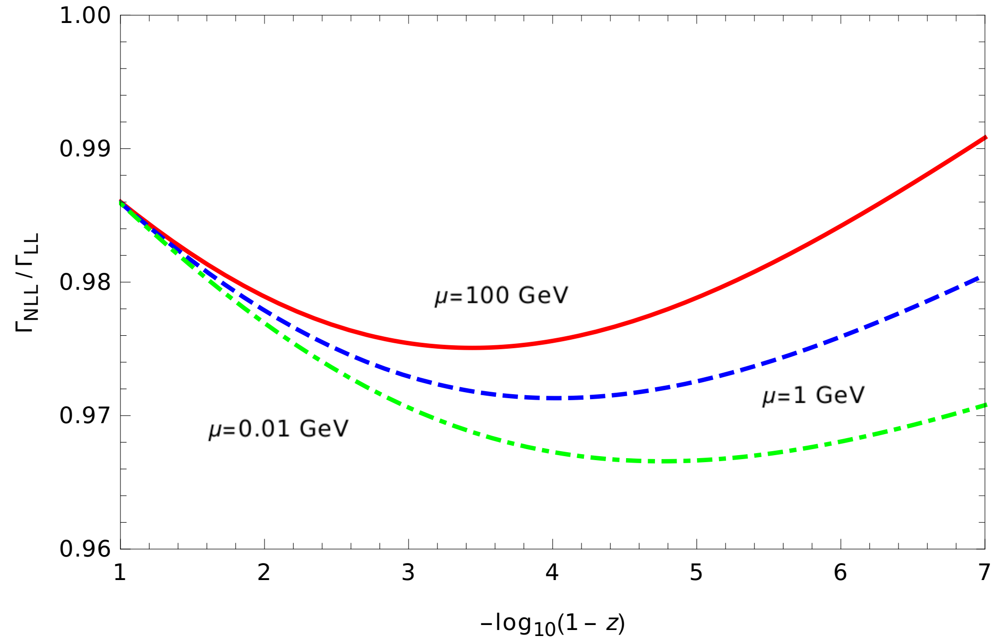

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Key facts

- Both $\overline{\mathrm{MS}}$ and Δ results feature an integrable singularity at $z \to 1$, essentially identical to the LL one

- In addition to that, in $\overline{\mathrm{MS}}$ there are single and double logarithmic terms

- Owing to the integrable singularity, it is essential to have large-z analytical results: the PDFs convoluted with cross sections are obtained by matching the small- and intermediate-z numerical solution with the large-z analytical one

  Analytical recursive solutions are used as cross-checks

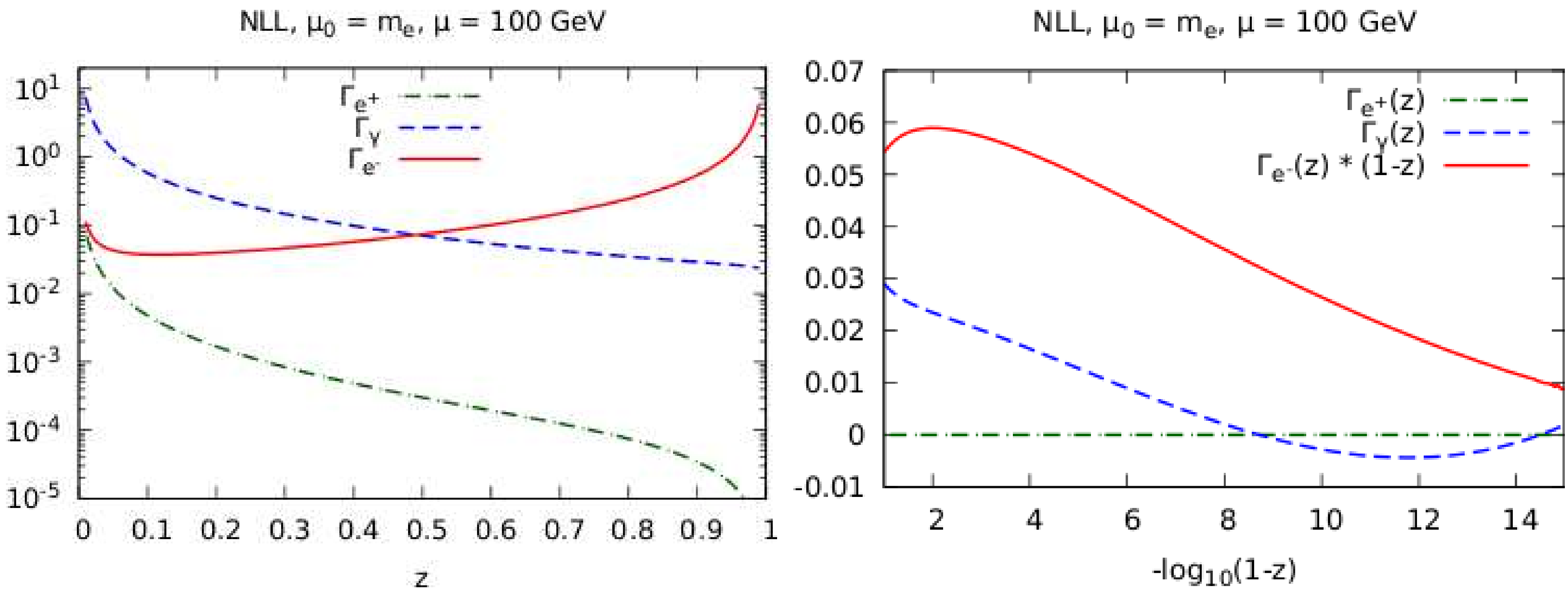
A look at the photon:

$$\Gamma_\gamma^{(\overline{\mathrm{MS}})}(z,\mu^2) \quad \stackrel{z\to 1}{\longrightarrow} \quad \frac{t\alpha(\mu_0)^2}{\alpha(\mu)}\,\frac{3}{2\pi\xi_1}\,\log(1-z) - \frac{t\alpha(\mu_0)^3}{\alpha(\mu)}\,\frac{1}{2\pi^2\xi_1}\,\log^3(1-z)$$

$$\Gamma_\gamma^{(\Delta)}(z,\mu^2) \quad \stackrel{z\to 1}{\longrightarrow} \quad \frac{1}{2\pi}\,\frac{\alpha^2(\mu_0)}{\alpha(\mu)}\,\frac{1+(1-z)^2}{z}\,L_0 + \frac{1}{2\pi\xi_1}\,\frac{t\,\alpha^2(\mu_0)}{\alpha(\mu)}\,L_0 - \frac{t\,\alpha(\mu)}{2\pi\xi_1}\,\frac{e^{-\gamma_{\mathrm{E}}\xi_1}e^{\hat{\xi}_1}}{\Gamma\left(1+\xi_1\right)}\,(1-z)^{\xi_1}\,L_0\,.$$

- $\overline{\mathrm{MS}}$ vs Δ exhibits the same pattern as for (non-)singlet: logarithmic terms dominate at $z \to 1$ in $\overline{\mathrm{MS}}$, but are absent in Δ

$\overline{\mathrm{MS}}$ results



e^- vs γ vs e^+. Note that e^- in the right-hand panel is strongly damped

As expected, electron dominance, but photons may play a role in the production of very massive objects

Cross sections

The results for these are not yet public; we are double-checking them.
Some *preliminary* findings are the following:

- The inclusion of NLL contributions into the electron PDF has an impact between 0.1% and 0.5% (on average: results are expected to be observable dependent)
- This estimate does not include the effects of the photon PDF
- The comparison between $\overline{\mathrm{MS}}$- and Δ-based results shows differences compatible with non-zero $\mathcal{O}(\alpha^2)$ effects, as expected

Conclusions

- We have computed all NLO initial conditions for PDFs and FFs (1909.03886), unpolarised

- We have NLL-evolved those relevant to the electron PDFs (1911.12040, 2105.06688), both analytically and numerically

- We have released the first version of MadGraph5_aMC@NLO (2108.10261) that includes both $e^{\pm}$ PDFs and beamstrahlung effects

Many results are based on establishing a "dictionary" QCD $\longrightarrow$ QED, which works at any order in α_S and α

Being done/to be done

- Present results for physical cross sections
- Add the resummation of soft non-collinear logarithms
- Fragmentation functions (also relevant to hadron colliders)
- Polarisations?
- Higher logarithmic accuracy?

EXTRA SLIDES

z space

Use integrated PDFs (so as to simplify the treatment of endpoints)

$$\mathcal{F}(z,t) = \int_0^1 dy\, \Theta(y-z)\, \Gamma(y,\mu^2) \quad \Longrightarrow \quad \Gamma(z,\mu^2) = -\frac{\partial}{\partial z}\mathcal{F}(z,t)$$

in terms of which the formal solution of the evolution equation is:

$$\mathcal{F}(z,t) = \mathcal{F}(z,0) + \int_0^t du\, \frac{b_0 \alpha^2(u)}{\beta(\alpha(u))} \left[\mathbb{P}\,\overline{\otimes}\,\mathcal{F}\right](z,u)$$

By inserting the representation:

$$\mathcal{F}(z,t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\mathcal{J}_k^{\mathrm{LL}}(z) + \frac{\alpha(t)}{2\pi}\, \mathcal{J}_k^{\mathrm{NLL}}(z) \right)$$

on both sides of the solution, one obtains recursive equations, whereby a $\mathcal{J}_k$ is determined by all $\mathcal{J}_p$ with $p < k$. The recursion starts from $\mathcal{J}_0$, which are the integrated initial conditions

For the record, the recursive equations are:

$$\begin{aligned}
\mathcal{J}_k^{\mathrm{LL}} &= \mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1}^{\mathrm{LL}} \\
\mathcal{J}_k^{\mathrm{NLL}} &= (-)^k (2\pi b_0)^k \mathcal{F}^{[1]}(\mu_0^2) \\
&\quad + \sum_{p=0}^{k-1} (-)^p (2\pi b_0)^p \Bigg(\mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\mathrm{NLL}} + \mathbb{P}^{[1]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\mathrm{LL}} \\
&\qquad\qquad - \frac{2\pi b_1}{b_0} \mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\mathrm{LL}} \Bigg)
\end{aligned}$$

We have computed these for $k \leq 3$ ($\mathcal{J}^{\mathrm{LL}}$) and $k \leq 2$ ($\mathcal{J}^{\mathrm{NLL}}$), ie to $\mathcal{O}(\alpha^3)$

Results in 1911.12040 and its ancillary files

Large-z singlet and photon

As for the non-singlet, start from the asymptotic AP kernel expressions:

$$\mathbb{P}_{\mathrm{S},N} \quad \xrightarrow{N\to\infty} \quad \begin{pmatrix} -2\log\bar{N}+2\lambda_0 & 0 \\ 0 & -\frac{2}{3}\,n_F \end{pmatrix} + \frac{\alpha}{2\pi}\begin{pmatrix} \frac{20}{9}n_F\log\bar{N}+\lambda_1 & 0 \\ 0 & -n_F \end{pmatrix} + \mathcal{O}(1/N) + \mathcal{O}(\alpha^2)$$

This implies

$$\begin{aligned} (\mathbb{E}_N)_{SS} &= E_N \\ M^{-1}\left[\,(\mathbb{E}_N)_{\gamma\gamma}\,\right] &= \frac{\alpha(\mu_0)}{\alpha(\mu)}\,\delta(1-z) \end{aligned}$$

$\Rightarrow$ Singlet $\equiv$ non-singlet

Photon $\equiv$ initial condition + $\alpha(0)$ scheme

Photon $\equiv$ initial condition + $\alpha(0)$ scheme $\Longrightarrow$

$$\Gamma_\gamma(z,\mu^2) = \frac{1}{2\pi}\,\frac{\alpha(\mu_0)^2}{\alpha(\mu)}\,\frac{1+(1-z)^2}{z}\left(\log\frac{\mu_0^2}{m^2} - 2\log z - 1\right)\,.$$

Or: $\sim$ Weizsaecker-Williams function, plus the natural emergence of a small scale in the argument of α

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But: vastly different from the numerical (exact) solution

$\longrightarrow$ $1/N$ suppression of off-diagonal terms in the evolution operator is over-compensated by the δ-like peak of the electron initial-condition

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But: vastly different from the numerical (exact) solution

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By solving the 2×2 system e.g. in $\overline{\rm MS}$:

$$\Gamma_\gamma^{(\overline{\rm MS})}(z,\mu^2) \quad \stackrel{z\to 1}{\longrightarrow} \quad \frac{t\alpha(\mu_0)^2}{\alpha(\mu)}\,\frac{3}{2\pi\xi_1}\,\log(1-z) - \frac{t\alpha(\mu_0)^3}{\alpha(\mu)}\,\frac{1}{2\pi^2\xi_1}\,\log^3(1-z)$$

A remarkable fact

Our asymptotic solutions, expanded in α, feature *all* of the terms:

$$\frac{\log^q(1-z)}{1-z} \qquad \text{singlet, non-singlet}$$

$$\log^q(1-z) \qquad \text{photon}$$

of our recursive solutions

Non-trivial; stems from keeping subleading terms (at $z \to 1$) in the AP kernels