QCD Scattering Amplitudes at the Precision Frontier

Heribertus Bayu Hartanto

University of Vienna
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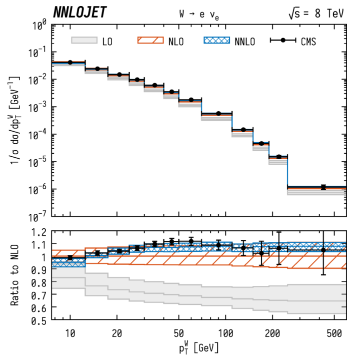
QCD corrections are important

$$d\sigma = d\sigma^{\mathrm{LO}} + \underbrace{d\sigma^{\mathrm{NLO}}}_{10-30\%} + \underbrace{d\sigma^{\mathrm{NNLO}}}_{1-10\%} + \ldots$$

- ✓ reduced scale dependence
- ✓ reliable normalization
- ✓ better agreement with data
- ✓ kinematic-dependent corrections



[Gehrmann-De Ridder,etal,2017]

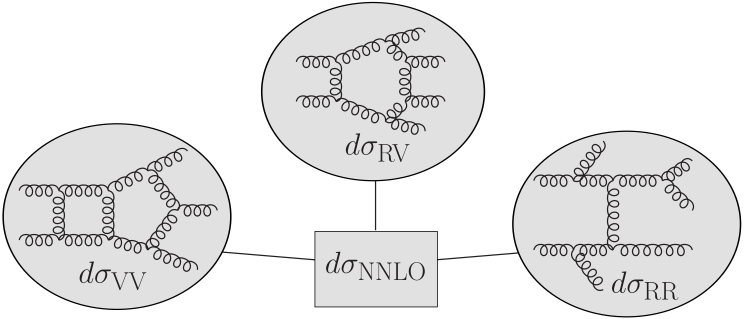
Precision frontier: NNLO for $2 \to 3$ ($pp \to \gamma\gamma\gamma$, $pp \to \gamma\gamma j$, $pp \to jjj$)

[Chawdry,Czakon,Mitov,Poncelet(2019,2021)][Kallweit,Sotnikov,Wiesemann(2020)][Czakon,Mitov,Poncelet(2021)]

- $pp \to jjj$: $R_{3/2}$, m_{jjj} $\Rightarrow$ α_s determination at multi-TeV range
- $pp \to \gamma\gamma j$: background to Higgs p_T, signal/background interference effects
- $pp \to Hjj$: Higgs p_T, background to VBF (probes Higgs coupling)
- $pp \to Vjj$: Vector boson p_T, W^+/W^- ratios, multiplicity scaling
- $pp \to VVj$: background for new physics

don't forget:

- EW corrections
- Resummation
- Showers



q_T subtraction

N-jettiness subtraction

projection to Born

antenna subtraction

CoLoRFulNNLO

STRIPPER

Nested Soft-Collinear Subtraction

geometric subtraction

$+ \ldots$

Two-Loop Calculation: General Strategy

$$A^{(2)} = \int [dk_1][dk_2] \sum_d \frac{N_d(k_i \cdot p_j, k_i \cdot \varepsilon_j, k_i \cdot k_j)}{(\text{propagators})_d}$$

generate Feynman diagrams + colour decomposition

$$= \sum_i c_i(\epsilon)\, G_i$$

Interference with tree-level, projectors, integrand reduction

$$= \sum_i d_i(\epsilon)\, \mathrm{MI}_i$$

IBP reduction to Master Integrals

$$= \frac{e_4}{\epsilon^4} + \frac{e_3}{\epsilon^3} + \frac{e_2}{\epsilon^2} + \frac{e_1}{\epsilon} + e_0$$

$$e_i = \sum r_i f_i, \quad r_i \to \{s_{ij}, \langle ij\rangle, [ij]\}, \quad f_i \to \{\pi, \ln, \mathrm{Li}_i, \ldots\}$$

$$= I^{(2)} A^{(0)} + I^{(1)} A^{(1)} + F^{(2)}$$

subtract universal pole structures

IBP identities: relations between integrals → reduce to independent set of integrals[Chetyrkin,Tkachov]

$$\int [dk] \frac{\partial}{\partial k_\mu} \frac{v_\mu(k,p)}{(\text{propagators})} = 0 \qquad \Rightarrow \qquad G_1 + G_2 + \cdots + G_n = 0$$

Public: AIR [Anastasiou,Lazopoulos], FIRE [Smirnov^2], Reduze [Studerus,Manteuffel], KIRA [Maierhoefer,Usovitsch,Uwer], LiteRed [Lee]

Master Integrals ⇒ Differential equation [Gehrmann,Remiddi,Henn]

⇒ Sector decomposition+numerical integration:SecDec[Borowka, etal],FIESTA[Smirnov,etal]

Two-Loop Calculation: General Strategy

$$
\begin{aligned}
A^{(2)} &= \int [dk_1][dk_2] \sum_d \frac{N_d(k_i \cdot p_j, k_i \cdot \varepsilon_j, k_i \cdot k_j)}{(\text{propagators})_d} \\
&= \sum_i c_i(\epsilon)\ G_i \\
&= \sum_i d_i(\epsilon)\ \mathrm{MI}_i \\
&= \frac{e_4}{\epsilon^4} + \frac{e_3}{\epsilon^3} + \frac{e_2}{\epsilon^2} + \frac{e_1}{\epsilon} + e_0 \\
&= I^{(2)} A^{(0)} + I^{(1)} A^{(1)} + F^{(2)}
\end{aligned}
$$

generate Feynman diagrams + colour decomposition

Interference with tree-level, projectors, integrand reduction

IBP reduction to Master Integrals

$e_i = \sum r_i f_i, \quad r_i \to \{s_{ij}, \langle ij \rangle, [ij]\}, \quad f_i \to \{\pi, \ln, \mathrm{Li}_i, ...\}$

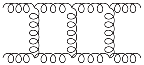
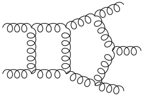
subtract universal pole structures

$$\text{loop amplitude} = \sum (\text{rational coefficients}) \times (\text{integral/special functions})$$

$$\text{finite remainder} = \text{loop amplitude} - \text{poles}$$

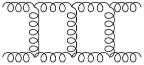
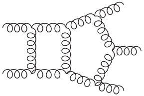
Compact analytic results ⇒ fast and stable for phenomenological applications

Algebraic complexity

	$gg \to gg$ 	$gg \to ggg$ 
# of Feynman diagrams	486	3540
# of Feynman integrals	$\mathcal{O}(1000)$	$\mathcal{O}(10000)$
integral reduction table	a few MB	$\sim$20 GB (compressed)
# of master integrals	7	61
finite remainder	a few KB	$\sim$10 MB

expression swell in the intermediate step $\Rightarrow$ evaluate numerically !!!

Algebraic complexity

	$gg \to gg$ 	$gg \to ggg$ 
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expression swell in the intermediate step ⇒ evaluate numerically !!!

Analytics from Numerics

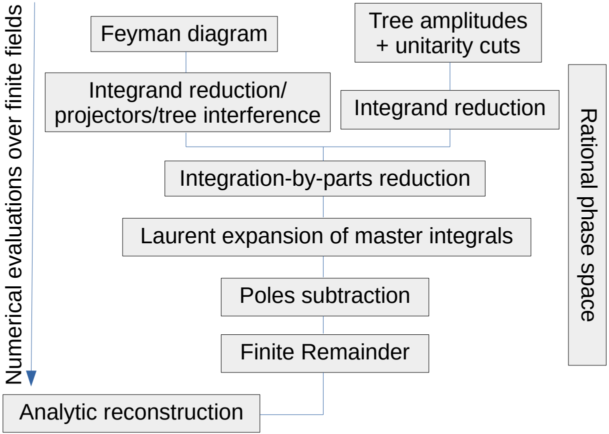
Multiscale process → potential large cancelation between terms

What kind of numerical evaluation?

- floating-point evaluation ($x = 4.744955523489933 \times 10^6$)
  ✓ fast ✗ limited precision
- evaluation over rational field $\mathcal{Q}$ ($x = 706998373/149$)
  ✓ exact ✗ can be slow and expensive
- evaluation over finite fields $\mathcal{Z}_p$ ($x \bmod_{11} = 8$)
  $\mathcal{Z}_p \Rightarrow$ the field of integer numbers modulo a prime p
  ✓ exact+fast ✗ some information lost
  → need to evaluate several finite fields to reconstruct rational number

Strategy ⇒ reconstruct analytic expressions from finite-field evaluations [Peraro(2016)]

Computational framework [Badger,Bronnum-Hansen,**HBH**,Peraro,Krys,Zoia]



QGRAF[Nogueira], FORM[Vermaseren,etal]
MATHEMATICA, SPINNEY[Cullen,etal]

finite field framework: FINITEFLOW[Peraro(2019)]

IBP identities generated using LITERED[Lee(2012)], solved numerically in FINITEFLOW using Laporta algorithm[Laporta(2000)]

Applications:

- $5g$, $q\bar{q}ggg$, $q\bar{q}Q\bar{Q}g$ (lc)
- $gg \to g\gamma\gamma$ (full)
- $u\bar{d} \to W^+ b\bar{b}$, $gg/q\bar{q} \to Hb\bar{b}$, $u\bar{d} \to W^+\gamma g$ (lc)
- $gg \to t\bar{t}$ (lc)

Momentum Twistor Variables

[Hodges(2009);Badger,Frellesvig,Zhang(2012)]

- helicity amplitudes: spinor components ($\langle ij\rangle$, $[ij]$) are not all independent

$$\langle ij\rangle = \bar{u}_-(p_i)v_-(p_j) \qquad [ij] = \bar{u}_+(p_i)v_+(p_j)$$

- rational parametrization of the n-point phase-space and the spinor components using $3n-10$ momentum-twistor variables
- 5-point parameterization:

$$|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad |2\rangle = \begin{pmatrix}0\\1\end{pmatrix}, \quad |3\rangle = \begin{pmatrix}\frac{1}{x_1}\\1\end{pmatrix}, \quad |4\rangle = \begin{pmatrix}\frac{1}{x_1}+\frac{1}{x_1 x_2}\\1\end{pmatrix}, \quad |5\rangle = \begin{pmatrix}\frac{1}{x_1}+\frac{1}{x_1 x_2}+\frac{1}{x_1 x_2 x_3}\\1\end{pmatrix},$$

$$|1] = \begin{pmatrix}1\\\frac{x_4-x_5}{x_4}\end{pmatrix}, \quad |2] = \begin{pmatrix}0\\x_1\end{pmatrix}, \quad |3] = \begin{pmatrix}x_1 x_4\\-x_1\end{pmatrix}, \quad |4] = \begin{pmatrix}x_1(x_2 x_3 - x_3 x_4 - x_4)\\-\frac{x_1 x_2 x_3 x_5}{x_4}\end{pmatrix}, \quad |5] = \begin{pmatrix}x_1 x_3(x_4-x_2)\\\frac{x_1 x_2 x_3 x_5}{x_4}\end{pmatrix}.$$

- phase information is lost: $|i\rangle \to t_i^{-1}|i\rangle$, $|i] \to t_i|i]$

$$A = A^{\text{phase}} \cdot \underbrace{\tilde{A}(x_1, x_2, x_3, x_4, x_5)}_{\text{phase}-\text{free}}$$

Momentum Twistor Variables

- Example: MHV amplitudes

$$\mathcal{A}^{(0)}(1_g^-, 2_g^-, 3_g^+, 4_g^+, 5_g^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} = x_1^3 x_2^2 x_3$$

$$\mathcal{A}^{(0)}(1_g^-, 2_g^+, 3_g^-, 4_g^+, 5_g^+) = \frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} = x_1^3 x_2^2 x_3$$

- Example: Momentum conservation

$$\langle 1|2|5] + \langle 1|3|5] + \langle 1|4|5] = 0$$

$$-x_1^2 x_3(x_2 - x_4) - x_1^2 x_3(-x_2 + x_4 + x_2 x_5) + x_1^2 x_3 x_2 x_5 = 0$$

- Example: Schouten identity

$$\langle 12\rangle\langle 34\rangle + \langle 13\rangle\langle 42\rangle + \langle 14\rangle\langle 23\rangle = 0$$

$$-\frac{1}{x_1 x_2} + \frac{1 + x_2}{x_1 x_2} - \frac{1}{x_1} = 0$$

Reconstructing the finite remainders

$$F^{(2)}(\{p\}) = \sum_i r_i(\{p\})\, m_i(f) + \mathcal{O}(\epsilon),$$

- set one of the kinematic variables to one ($s_{12} = 1$ or $x_1 = 1$)
- Not all r_i coefficients independent
  $\Rightarrow$ find linear relations between coefficients and reconstruct the simpler ones

$$\sum_i y_i r_i = 0, \qquad y_i \in \mathbb{Q}$$

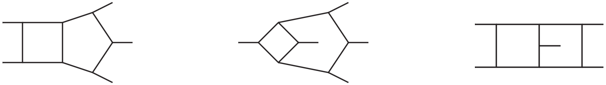
$\Rightarrow$ allow to supply known/candidate coefficients $\tilde{r}_j$

$$\sum_i y_i r_i + \sum_j \tilde{y}_j \tilde{r}_j = 0, \qquad y_i, \tilde{y}_j \in \mathbb{Q}$$

- guess the denominator $\rightarrow$ from letters [Abreu,etal(2019)][Abreu,etal(2020)] $dJ_i = \epsilon A_{ij}\, J_j$
- on-the-fly univariate partial fractioning $\rightarrow$ significant drop in complexity (degrees,sample pts,size)
- factor matching $\Rightarrow$ letters, spinor products/strings
- reconstructed expressions can be further simplified using MultivariateApart [Heller,von Manteuffel(2021)]

Massive progress in massless 2-loop 5-particle scattering

- All 2-loop 5-particle integrals are known



[Papadopoulos,Tommasini,Wever(2015)] [Gehrmann,Henn,Lo Presti(2015,2018)] [Abreu,Page,Zeng(2018)]
[Abreu,Dixon,Herrmann,Page,Zeng(2018,2019)] [Chicherin,Gehrmann,Henn,Wasser,Zhang,Zoia(2018,2019)][Chicherin,Sotnikov(2020)]

- Many 2-loop 5-particle QCD amplitudes known analytically

Leading colour $\Rightarrow$ $5g$, $2q3g$, $4q1g$, $2q3\gamma$, $2q1g2\gamma$

[Abreu,Agarwal,Badger,Brønnum-Hansen,Buccioni,Chawdhry,Czakon,Dormans,Febres Cordero, Gehrmann, **HBH**,Henn,Ita,Lo Presti,Mitov,Page,Peraro,Poncelet,Sotnikov,Tancredi,von Manteuffel,Zeng(2015-2021)]

Full colour $\Rightarrow$ $5g$ all-plus, $2q1g2\gamma$, $3g2\gamma$

[Badger,Chicherin,Gehrmann,Heinrich,Henn,Peraro,Wasser,Zhang,Zoia(2019)] [Agarwal,Buccioni,Tancredi,von Manteuffel(2021)]
[Badger,Bronnum-Hansen,Chicherin,Gehrmann,**HBH**,Henn,Marcoli,Moodie,Peraro,Zoia(2021)]

- NNLO QCD calculations for $2 \to 3$ processes

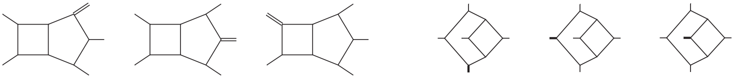
$pp \to \gamma\gamma\gamma$ [Chawdhry,Czakon,Mitov,Poncelet(2019)][Kallweit,Sotnikov,Wiesemann(2020)]

$pp \to \gamma\gamma j$ [Chawdhry,Czakon,Mitov,Poncelet(2021)] $pp \to jjj$ [Czakon,Mitov,Poncelet(2021)]

$pp(gg) \to \gamma\gamma j$ [Badger,Gehrmann,Marcoli,Moodie(2021)]

Scattering with an off-shell leg

- $pp \to Hjj$, $pp \to W/Z + jj$, $pp \to W/Z + \gamma j$, $pp \to W/Z + b\bar{b}$, $pp \to Hb\bar{b}$ (massless b)
- All planar integrals and non-planar hexabox family are known



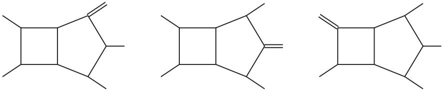
[Papadopoulos,Tommasini,Wever(2015)] [Papadopoulos,Wever(2019)] [Abreu,Ita,Moriello,Page,Tschernow,Zeng(2020)] [Canko,Papadopoulos,Syrrakos(2020)] [Syrrakos(2020)][Abreu,Ita,Page,Tschernow(2021)]

$\Rightarrow$ Planar one-mass pentagon functions, fast numerical evaluation [Chicherin,Sotnikov,Zoia(2021)]

- A number of QCD amplitudes known analytically at leading colour
  - $u\bar{d} \to Wb\bar{b}$ (on-shell W, massless b) [Badger,**HBH**,Zoia(2021)]
  - $gg/q\bar{q} \to Hb\bar{b}$ (massless b) [Badger,**HBH**,Krys,Zoia(2021)]
  - $q\bar{q}' \to \bar{\ell}\nu gg$ and $q\bar{q}' \to \bar{\ell}\nu Q\bar{Q}$ [Abreu,Febres Cordero,Ita,Klinkert,Page(2021)]
  - $u\bar{d} \to \bar{\ell}\nu\gamma g$ [Badger,**HBH**,Krys,Zoia(to appear)]

$\Rightarrow$ on-the-fly univariate partial fractioning has been crucial!!!

A basis of special functions (1)



① [Abreu,Ita,Moriello,Page,Tschernow,Zeng(2020)]

- ✓ planar alphabet identified (58 letters, 3 square-roots), canonical DEs derived
- ✓ Integrate DEs numerically using generalised series expansions [Moriello(2019)]
- ✗ analytically reconstructing MI coefficients is still too complicated

② [Canko,Papadopoulos,Syrrakos(2020)][Syrrakos(2020)]

- ✓ Construct Simplified Differential Equations (SDEs) using known canonical basis
- ✓ Analytic solutions in term of Goncharov PolyLogarithms (GPLs) $G(1,x) = \ln(1-x)$
- ✗ GPLs not linearly independent: no analytic pole cancellations

③ [Chicherin,Sotnikov,Zoia(2021)]

- ✓ Pentagon function basis allows for analytic pole cancellation, fast numerical evaluation
- ✗ basis given for $ij \to klM$, need to rederive analytic expressions for IS-FS crossings
- ✗ can't be used for $M \to ijkl$ decay

A basis of special functions (2)

- use the components of the ϵ-expansion of the MIs as special functions

$$\mathrm{MI}_i(s) = \sum_{w\geq 0} \epsilon^w \mathrm{MI}_i^{(w)}(s)$$

- starting from canonical DEs[Abreu,etal(2020)] write MIs in terms of Chen's iterated integrals[Chen(1977)] for example:

$$\mathrm{MI}_i^{(2)}(s) = I_\gamma(w_1, w_2; s_0, s) + I_\gamma(w_1, w_3; s_0, s) + \cdots + \mathrm{tc}_j^{(2)}(s_0)$$

where

$$I_\gamma(w_{i_1}, \ldots, w_{i_n}; s_0, s) = \int_\gamma d\log w_{i_n}(s') I_\gamma(w_{i_1}, \ldots, w_{i_{n-1}}; s_0, s'), \qquad I_\gamma(\ldots; s_0, s_0) = 0$$

weight $\rightarrow$ # of integrations

- Use GPL expressions[Canko,etal(2020)][Syrrakos(2020)] + PSLQ algorithm to prepare the boundary values
- Shuffle algebra to remove products of lower-weight functions
  + linear algebra to extract linearly independent functions

$$\left\{\mathrm{MI}_i^{(w)}(s)\right\} \Longrightarrow \left\{f_i^{(w)}(s)\right\}$$

- $f_i^{(w)}$ can be evaluated using GPLs or generalised series expansions or pentagon functions

$u\bar{d} \to Wb\bar{b}$ amplitude [Badger,**HBH**,Zoia arXiv:2102.02516]

$$\bar{d}(p_1) + u(p_2) \to b(p_3) + \bar{b}(p_4) + W^+(p_5)$$

- colour decomposition at leading colour $\to$ only planar contribution

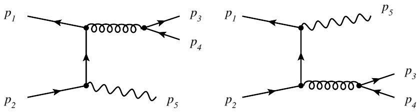
$$\mathcal{A}^{(2)}(1_{\bar{d}}, 2_u, 3_b, 4_{\bar{b}}, 5_W) \sim g_s^6 g_W \, N_c^2 \, \delta_{i_1}^{\,\bar{i}_4} \delta_{i_3}^{\,\bar{i}_2} \, A^{(2)}(1_{\bar{d}}, 2_u, 3_b, 4_{\bar{b}}, 5_W)$$

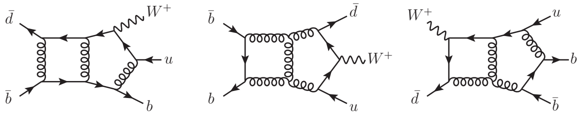
- massless b quarks, $p_3^2 = p_4^2 = 0$
- onshell W boson

$$p_5^2 = m_W^2, \qquad \sum_\lambda \varepsilon_W^{\mu *}(p_5, \lambda) \varepsilon_W^{\nu}(p_5, \lambda) = -g^{\mu\nu} + \frac{p_5^\mu p_5^\nu}{m_W^2}$$

Invariants:

$$s_{12} = (p_1 + p_2)^2\,, \quad s_{23} = (p_2 - p_3)^2\,, \quad s_{34} = (p_3 + p_4)^2\,,$$
$$s_{45} = (p_4 + p_5)^2\,, \quad s_{15} = (p_1 - p_5)^2\,, \quad s_5 = p_5^2\,,$$
$$\mathrm{tr}_5 = 4 i \epsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu p_3^\rho p_4^\sigma\,.$$





- Two-loop amplitude interfered with tree level

$$M^{(2)} = \sum_{\text{spin}} A^{(0)*} A^{(2)} = M^{(2)}_{\text{even}} + \text{tr}_5 \, M^{(2)}_{\text{odd}}$$

- Numerators containing: $\text{tr}(\cdots)$ and $\text{tr}(\cdots\gamma_5\cdots\gamma_5\cdots) \Rightarrow$ anti-commuting γ_5 prescription
  $\text{tr}(\cdots\gamma_5\cdots) \Rightarrow$ Larin's prescription [Larin(1993)]
- Cross-check with helicity amplitude computations (numerically) in tHV scheme
  $\Rightarrow$ results agree at the level of finite remainder!!
- Univariate partial fraction (UPF) in s_{23} and reconstruct in the remaining variables $(s_{34}, s_{45}, s_{15}, s_5)$
- Reconstruction data (even): all coeffs (63/62), indep coeffs (54/54), UPF in s_{23} (31/4)
  evaluation time: 40s $\to$ 1000s with UPF, 38663 points, 2 prime fields

$\Rightarrow \sim 4$ times speed up with UPF

Numerical evaluation

Only 19 linear combinations of $f_i^{(4)}$ appear in the two-loop finite remainder ⇒ define a new basis $g_i^{(w)}$

$$\left\{f_i^{(w)}(s)\right\} \Longrightarrow \left\{g_i^{(w)}(s)\right\}$$

Evaluate numerically the $g_i^{(w)}$ basis directly

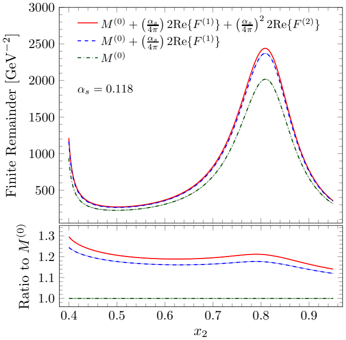
$$\vec{g} = \begin{pmatrix} \epsilon^4 g_i^{(4)} \\ \epsilon^3 g_i^{(3)} \\ \epsilon^2 g_i^{(2)} \\ \epsilon g_i^{(1)} \\ 1 \end{pmatrix}$$

$$d\vec{g} = \epsilon d\tilde{B} \cdot \vec{g}$$

Much simpler than the DEs for the master integrals

Use generalised series expansion approach [Moriello(2019)] as implemented in DiffExp [Hidding(2020)]

Evaluation time ~260s/pt using basic DiffExp setup

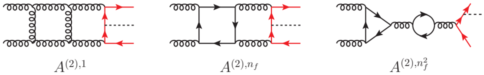


Update: map $f_i^{(w)}$ to pentagon functions of [Chicherin,Sotnikov,Zoia(2021)], evaluation time now < 1s (dp)

$$f_i^{(w)} = \sum_j m_{ij}\, p_j^{(w)}$$

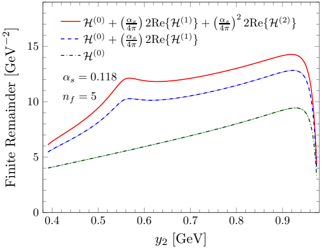
Update: amplitude for $u\bar{d} \to (W \to \nu\bar{\ell})b\bar{b}$ has also been computed ⇒ ready for pheno!!!

$pp \to Hb\bar{b}$ [Badger,**HBH**,Krys,Zoia arXiv:2107.14733]



$$0 \to \bar{b}(p_1) + b(p_2) + g(p_3) + g(p_4) + H(p_5)$$
$$0 \to \bar{b}(p_1) + b(p_2) + \bar{q}(p_3) + q(p_4) + H(p_5)$$
$$0 \to \bar{b}(p_1) + b(p_2) + \bar{b}(p_3) + b(p_4) + H(p_5)$$

$\bar{q}q \to b\bar{b}H$ finite remainder



Compute helicity amplitudes:
$\Rightarrow$ need momentum twistor parameterization

- generate mom twistor for $(q_1, \cdots, q_6)$, $q_i^2 = 0$
$$p_1 = q_1, p_2 = q_2, p_3 = q_3, p_4 = q_4, p_5 = q_5 + q_6.$$
- Fix direction of q_6: $\langle q_2 q_6 \rangle = 0$, $[q_2 q_6] = 0$
$\Rightarrow$ (8-2) = 6 independent variables $(x_1, \cdots, x_6)$

Same function basis as the $u\bar{d} \to Wb\bar{b}$ computation

Construct new function basis for the finite remainders

Numerical evaluation with DiffExp [Hidding(2020)]
$\Rightarrow$ will be updated to use pentagon functions [Chicherin,Sotnikov,Zoia(2021)]

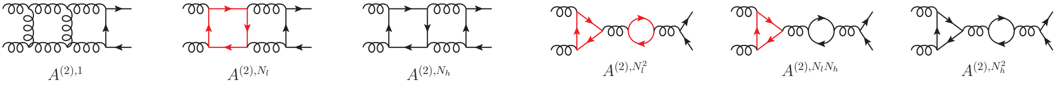
$\bar{b}bggH$	hel	$r_i(x)$	indep $r_i(x)$	UPF in x_5	points
$F^{(2),1}$	$++++$	63/57	52/46	20/6	3361
	$+++-$	135/134	119/120	28/22	24901
	$++--$	105/111	105/111	22/12	4797

So far ...

- finite-field method is applied for massless fermions
- analytic approach is employed for massless internal particles

- how to take into account massive fermions using our finite-field framework?
- analytic computation for processes with massive internal particles?
- i.e. can we do $t\bar{t}$, $t\bar{t}j$, $t\bar{t}V$, ... ?

$gg \to t\bar{t}$ revisited [Badger,Chaubey,**HBH**,Marzucca arXiv:2021.13450]



- Derive analytic form of helicity amplitudes for the leading colour contribution

  $\Rightarrow$ focus on amplitude with a top-quark loop ($A^{(2),N_h}$)

- Massive spinor formalism allows to include top decay efficiently in NWA $\left(p^\mu = p^{\flat,\mu} + \frac{m^2}{2p^\flat \cdot n} n^\mu\right)$

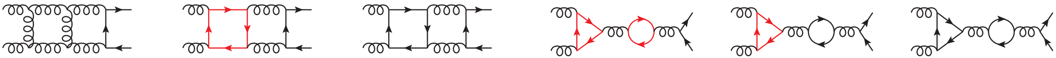
$$\bar{u}_+(p,m) = \frac{\langle n|(\not{p}+m)}{\langle np^\flat\rangle}, \quad \bar{u}_-(p,m) = \frac{[n|(\not{p}+m)}{[np^\flat]}, \quad v_+(p,m) = \frac{(\not{p}-m)|n\rangle}{\langle p^\flat n\rangle}, \quad v_-(p,m) = \frac{(\not{p}-m)|n]}{[p^\flat n]}.$$

- Introduce a method to deal with massive fermions within the finite-field framework

$$A(1_{\bar{t}}^+, 2_t^+, 3_g, 4_g; n_1, n_2) = \langle n_1 n_2\rangle A_1 + \langle n_1 3\rangle\langle n_2 4\rangle A_2 + \langle n_1 3\rangle\langle n_2 3\rangle A_3 + \langle n_1 4\rangle\langle n_2 4\rangle A_4$$

$$(n_1, n_2) = \{(p_3,p_3),(p_4,p_4),(p_3,p_4),(p_4,p_3)\}$$

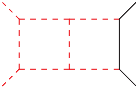
$A(1_{\bar{t}}^+, 2_t^+, 3_g, 4_g; p_3, p_3)$ depends on 3 variables



$A^{(2),1}$	$A^{(2),N_l}$	$A^{(2),N_h}$	$A^{(2),N_l^2}$	$A^{(2),N_l N_h}$	$A^{(2),N_h^2}$
GPLs	GPLs	Elliptics+GPLs	GPLs	GPLs	GPLs

Analytic solutions of master integrals:

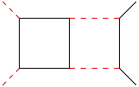
- PolyLogarithms
  [Bonciani,Ferroglia,Gehrmann,von Manteuffel,Studerus 2011,2013][Mastrolia,Passera,Primo,Schubert 2017]



$$= \ldots + \epsilon^4 G(0, x, 0, 1; y) + \ldots$$

$G(i_1, \ldots, i_n; x) \to$ Goncharov Polylogarithms of weight n, $\quad G(1, x) = \ln(1 - x)$

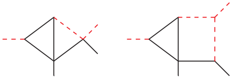
- Elliptics [Adams,Chaubey,Weinzierl 2018]



$$= \ldots + \epsilon^4 I_\gamma(\eta^{(b)}_{1,1}, \omega_{0,4}, \omega_4, \omega_{0,4}; \lambda) + \ldots$$

$$\eta^{(b)}_{1,1} = \frac{x - 1}{(3x^2 - 2xy - 4x + 3)(x + 1)} \frac{\pi}{\Psi^{(b)}_1} dx \qquad \Psi^{(b)}_1 \sim \text{EllipticK}(\ldots)$$

$\Rightarrow$ not all integrals known analytically



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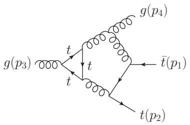
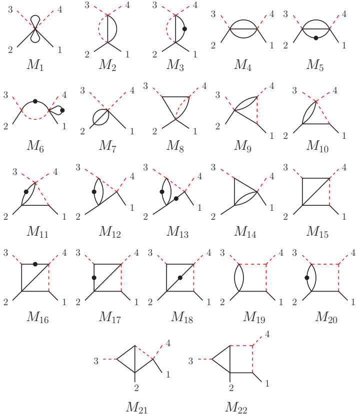




Kinematic variables

$$-\frac{s}{m_t^2} = \frac{(1-x)^2}{x}, \quad \frac{t}{m_t^2} = y$$

Differential equation for $\vec{M}$

$$d\vec{M} = A(\epsilon, x, y)\vec{M}$$

Changing basis $\vec{J} = U(\epsilon, x, y)\vec{M}$ (Polylogarithms)

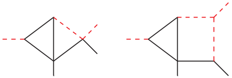
$$d\vec{J} = \epsilon \tilde{A}_1 \vec{J}$$

$$\tilde{A}_i = \tilde{A}_i(x, y)$$

Solve $\vec{J}$ in terms of iterated integrals $I_\gamma(\cdots, \lambda)$

$$J_i = \sum_{k=0}^{4} \epsilon^k J_i^k + \mathcal{O}(\epsilon^5)$$

$\Rightarrow$ not all integrals known analytically



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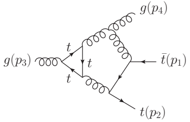

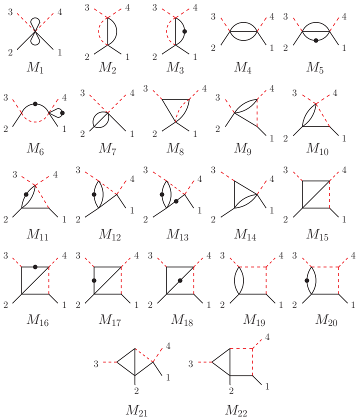

Kinematic variables

$$-\frac{s}{m_t^2} = \frac{(1-x)^2}{x}, \quad \frac{t}{m_t^2} = y$$

Differential equation for $\vec{M}$

$$d\vec{M} = A(\epsilon, x, y)\vec{M}$$

Changing basis $\vec{J} = U(\epsilon, x, y, \Psi, d\Psi)\vec{M}$ (Elliptics)

$$d\vec{J} = (\tilde{A}_0 + \epsilon\tilde{A}_1)\vec{J}$$

$$\tilde{A}_i = \tilde{A}_i(x, y, \Psi_i^{(a,b,c)}, d\Psi_i^{(a,b,c)})$$

Solve $\vec{J}$ in terms of iterated integrals $I_\gamma(\cdots, \lambda)$

$$J_i = \sum_{k=0}^{4} \epsilon^k J_i^k + \mathcal{O}(\epsilon^5)$$

Assembling the $A^{(2),N_h}$ amplitude

- Finite remainder

$$F^{(2),Nh} = \sum_i c_i\left(x, y, \Psi_k^{(a,b,c)}, d\Psi_k^{(a,b,c)}\right) m_i(G, I_\gamma) + \mathcal{O}(\epsilon)$$

- relations beyond shuffle algebra needed for analytic pole cancellation

$$I(a_{3,3}^{(b)}, f, \ldots) = \int a_{3,3}^{(b)} I(f, \ldots) = \int d\left(\psi_1^{(b)} \frac{x(-1+y)}{\pi(-1+x)^2}\right) \cdot I(f, \ldots) = \left[\psi_1^{(b)} \frac{x(-1+y)}{\pi(-1+x)^2} I(f, \ldots)\right]_{(0,1)}^{(x,y)} - I\left(\psi_1^{(b)} \frac{x(-1+y)}{\pi(-1+x)^2} f, \ldots\right)$$

  $\Rightarrow$ more relations expected in the finite part

- analytic continuation needs further investigation

  $\Rightarrow$ a few integrals (involving 3 elliptic curves) are still computed using pySecDec

- test evaluations reproduced known numerical results [Baernreuther,Czakon,Fiedler(2014)]

  $\Rightarrow$ first cross check on $A^{(2),N_h}$ from (semi) analytic calculation !!!

lots more to understand!!!

Back-up Slides

Interlude: 4D Spinor Helicity Formalism

For massless four-vector p_i, define spinor products:

$$\langle ij\rangle = \bar{u}_-(p_i)u_+(p_j), \quad [ij] = \bar{u}_+(p_i)u_-(p_j), \quad \langle ij\rangle[ji] = 2p_i \cdot p_j.$$

where

$$u_+(p) = P_R u(p) \qquad u_-(p) = P_L u(p)$$

Spinor sandwiches

$$\langle i|m \cdots n|j] = \bar{u}_-(p_i)\, \not{p}_m \cdots \not{p}_n\, u_-(p_j) \qquad (\text{odd } \# \text{ of } p)$$
$$\langle i|m \cdots n|j\rangle = \bar{u}_-(p_i)\, \not{p}_m \cdots \not{p}_n\, u_+(p_j) \qquad (\text{even } \# \text{ of } p)$$
$$[i|m \cdots n|j] = \bar{u}_+(p_i)\, \not{p}_m \cdots \not{p}_n\, u_-(p_j) \qquad (\text{even } \# \text{ of } p)$$

Polarization vectors

$$\epsilon_+^\mu(k, q) = \frac{\langle q|\gamma^\mu|k]}{\sqrt{2}\langle qk\rangle}, \quad \epsilon_-^\mu(k, q) = \frac{[q|\gamma^\mu|k\rangle}{\sqrt{2}[qk]}.$$

Momentum Twistor Variables

[Hodges]

$$p_i \cdot \sigma_{a\dot{a}} = \lambda_{ia}\tilde{\lambda}_{i\dot{a}} \qquad p_i^\mu = x_i^\mu - x_{i-1}^\mu \qquad \mu_i^{\dot{a}} = x_i \cdot \tilde{\sigma}^{\dot{a}a}\lambda_{ia}$$

Momentum twistor variables $Z_i(\lambda_i, \mu_i)$ for each momentum

$\tilde{\lambda}_i$ are obtained via

$$\tilde{\lambda}_i = \frac{\langle i, i+1\rangle \mu_{i-1} + \langle i+1, i-1\rangle \mu_i + \langle i-1, i\rangle \mu_{i+1}}{\langle i, i+1\rangle\langle i-1, i\rangle}$$

5-point parameterization:

$$Z = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{1}{x_1} & \frac{1}{x_1} + \frac{1}{x_1 x_2} & \frac{1}{x_1} + \frac{1}{x_1 x_2} + \frac{1}{x_1 x_2 x_3} \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{x_4}{x_2} & 1 \\ 0 & 0 & 1 & 1 & 1 - \frac{x_5}{x_4} \end{pmatrix}$$

$$s_{12} = x_1, \; s_{23} = x_1 x_4, \; s_{45} = x_1 x_5$$

$$\langle 12\rangle = 1, \; [12] = -x_1, \; \langle 23\rangle = -\frac{1}{x_1}, \; [23] = x_1^2 x_4, \; \langle 45\rangle = -\frac{1}{x_1 x_2 x_3}, \; [45] = x_1^2 x_2 x_3 x_5$$

Analytic Reconstruction from Numerical Evaluations

Functional reconstruction techniques:

- Univariate polynomials: Newton's interpolation formula

$$f(z) = \sum_{r=0}^{R} a_r \prod_{i=0}^{r-1} (z - y_i)$$
$$= a_0 + (z - y_0)\Big(a_1 + (z - y_1)\Big(a_2 + (z - y_2)(\cdots + (z - y_{r-1})\, a_r)\Big)\Big)$$

- Univariate rational function: Thiele's interpolation formula

$$f(z) = a_0 + \cfrac{z - y_0}{a_1 + \cfrac{z - y_1}{a_2 + \cfrac{z - y_3}{\cdots + \cfrac{z - y_{r-1}}{a_N}}}}$$
$$= a_0 + (z - y_0)\left(a_1 + (z - y_1)\left(a_2 + (z - y_2)\left(\cdots + \frac{z - y_{N-1}}{a_N}\right)^{-1}\right)^{-1}\right)^{-1}$$

Analytic Reconstruction from Numerical Evaluations

- Multivariate polynomials: recursive application of Newton's interpolation

$$f(z_1, \dots, z_n) = \sum_{r=0}^{R} a_r(z_2, \dots, z_n) \prod_{i=0}^{r-1} (z_1 - y_i)$$

- Multivariate rational function: combination of Newton's and Thiele's interpolation. Example: $f(z_1, z_2)$
  - For $(z_1, z_2) \to (tz_1, tz_2)$, find $f(t)$ using Thiele's formula

$$f(tz_1, tz_2) = \frac{f_0(z_1, z_2) + f_1(z_1, z_2)t + f_2(z_1, z_2)t^2}{1 + g_1(z_1, z_2)t + g_2(z_1, z_2)t^2}$$

  - Reconstruct f_i and $g_i \Rightarrow$ multivariate polynomials
  - Shift denominator if needed
  - Reconstruct coefficients $(a_{\vec{ij}}, b_{\vec{ij}})$ in $\mathcal{Z}_p$, promote to $\mathcal{Q}$

More details on FF and functional reconstruction: [Peraro,arXiv:1608.01902]

Finite Fields

- used in computer algebra systems (polynomial factorization/GCD, linear solver)

Finite field: a field that contains a finite number of elements

- Consider finite fields $\mathcal{Z}_p$, with p prime $\Rightarrow \mathcal{Z}_p = \{0, \ldots, p-1\}$
- addition, subtraction, and multiplication via modular arithmetic

$$(5+7) \bmod 11 = 1 \qquad (5\times 7) \bmod 11 = 2 \qquad (5-7) \bmod 11 = 9$$

- Every $a \in \mathcal{Z}_p$ has a multiplicative inverse, a^{-1}

$$5^{-1} \bmod 11 = 9$$

- From $\mathcal{Q}$ to $\mathcal{Z}_p$

$$q = a/b \in \mathcal{Q} \quad \rightarrow \quad q \bmod p \equiv a \times (b^{-1} \bmod p) \bmod p$$

- From $\mathcal{Z}_p$ to $\mathcal{Q}$
  find a, b with rational reconstruction algorithm, correct when $a, b \leq \sqrt{p}$ [Wang,1981]
  make p large enough by using Chinese Reminder Theorem
  $\rightarrow$ solution in $\mathcal{Z}_{p_1}, \mathcal{Z}_{p_2}, \cdots \Rightarrow$ solution in $\mathcal{Z}_{p_1 p_2 \cdots}$
- Rational operation is well defined in $\mathcal{Z}_p$, but no square roots

$u\bar{d} \to (W \to \nu\bar{\ell})b\bar{b}$ amplitude

$$0 \to d(p_1) + \bar{u}(p_2) + b(p_3) + \bar{b}(p_4) + \nu(p_5) + \ell^+(p_6)$$

- Detach $W \to \ell\nu$ decay, decompose 5-pt amplitude into form factors

$$A_6^{(L)} = A_5^{(L)\mu}\, D_\mu\, P(s_{56}), \qquad M_6^{(L)} = \sum_{\text{spin}} A_6^{(0)\dagger} A_6^{(L)} = M_5^{(L)\mu\nu}\, \mathcal{D}_{\mu\nu}\, |P(s_{56})|^2$$

$$M_5^{(L)\mu\nu} = \sum_{\text{spin}} A_5^{(0)\mu\dagger} A_5^{(L)\nu}, \quad \mathcal{D}_{\mu\nu} = \sum_{\text{spin}} D_\mu^\dagger D_\nu, \quad P(s) = \frac{1}{s - M_W^2 + i\Gamma_W}$$

$$M_5^{(L)\mu\nu} = \sum_{i=1}^{16} a_i^{(L)} v_i^{\mu\nu}, \qquad a_i^{(L)} = \sum_j \Delta_{ij}^{-1}\, \tilde{M}_{5,j}^{(L)}, \qquad \Delta_{ij} = v_{i\mu\nu} v_j^{\mu\nu}, \qquad v_i^{\mu\nu} \in \{p_1^\mu, p_2^\mu, p_3^\mu, p_W^\mu\}$$

  9 non-vanishing contracted amplitude $\tilde{M}_{5,j}^{(L)} = v_{i\mu\nu} M_5^{(L)\mu\nu}$ $\Rightarrow$ reconstruct in one go
- Use the machinery from on-shell calculation (integrand processing, reconstruction, function basis)
- Cross checked against 6-pt helicity amplitude computation and $q\bar{q}' \to \bar{\ell}\nu Q\bar{Q}$ result from [Abreu,Febres Cordero,Ita,Klinkert,Page(2021)]
- Use pentagon functions for numerical evaluation $\Rightarrow$ ready for pheno!!!

Univariate partial fraction decomposition over finite fields

Perform on-the-fly univariate partial fraction w.r.t to y

$$f(x,y) = \frac{N(x,y)}{\prod_{i=1}^{s} \mathcal{D}_i^{e_i}(x,y)}, \qquad \begin{aligned} d_N &= \text{max degree of } N(x,y) \text{ in } y \\ d_i &= \text{max degree of } \mathcal{D}_i(x,y) \text{ in } y \end{aligned}$$

$f(x,y) \Rightarrow$ *black-box* evaluation, d_N, d_i, $\mathcal{D}_i$ are known from univariate slice

Ansatz (w.r.t. y):

$$f(x,y) = \sum_{i=1}^{s} \sum_{j=1}^{e_i} \sum_{t=0}^{d_i-1} \frac{u_{ijt}(x) y^t}{\mathcal{D}_i^j(x,y)} + r(x) + \sum_{h=1}^{d_N - \sum_{i=1}^{s} e_i d_i} v_h(x) y^h$$

$\Rightarrow$ Linear fit to reconstruct the unknown functions: $u_{ijt}(x)$, $r(x)$, $v_h(x)$

Example:

$$f(x,y) = \frac{y^4 + 13xy^2 + x^2}{(y-x)(y+x)^2}, \quad \text{ansatz} \Rightarrow f(x,y) = \frac{u_{110}(x)}{y-x} + \frac{u_{210}(x)}{y+x} + \frac{u_{220}(x)}{(y+x)^2} + r(x) + v_1(x) y$$

for each numerical values of x, y is sampled several time to reconstruct its analytic dependence
higher computational cost but one fewer variables and (usually) lower degrees in x
$\Rightarrow$ fewer sample points needed for analytic reconstruction