Differential Cross Sections at Leading Order with Monte Carlo Methods

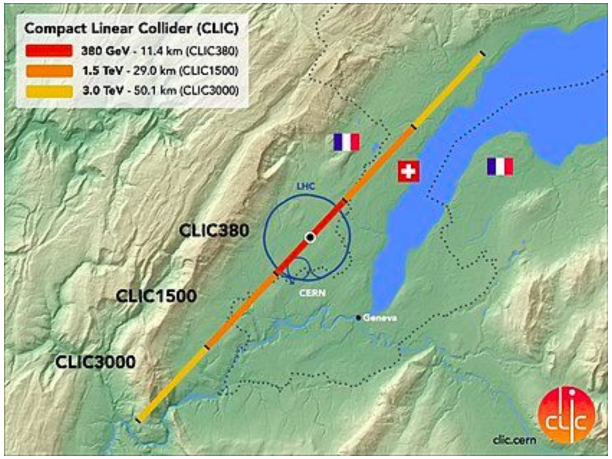
André H. Hoang, Simon Plätzer, Michael Seymour, Angelika Widl

Particle Physics Seminar
18/06/2021

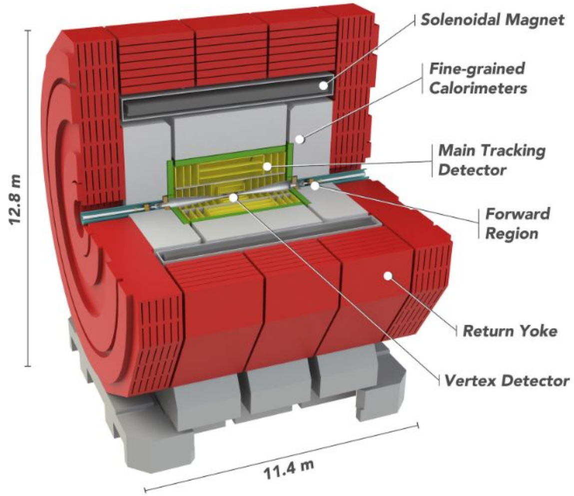
Outline

- Introduction
    - 2-particle phase space
- Matrix Element
    - helicity amplitudes - example
    - helicity amplitudes - formalism
- Phase Space Integration
    - Monte Carlo integration
    - energy-momentum conservation
        - Rambo
        - sequential phase space
    - improving precision
        - resonant propagators
        - Vegas
    - multichannel integration
- Our Project

Introduction

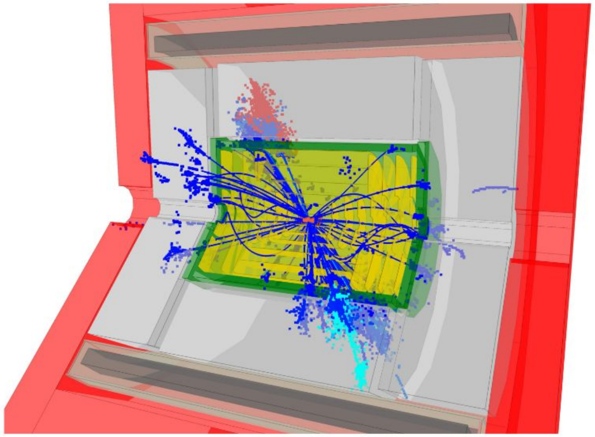


- to compare with experiment, we compare which particles arrive in the detectors
- for this we need to know in which direction the outgoing particles are going
- we need the differential cross section



(a)

1812.06018



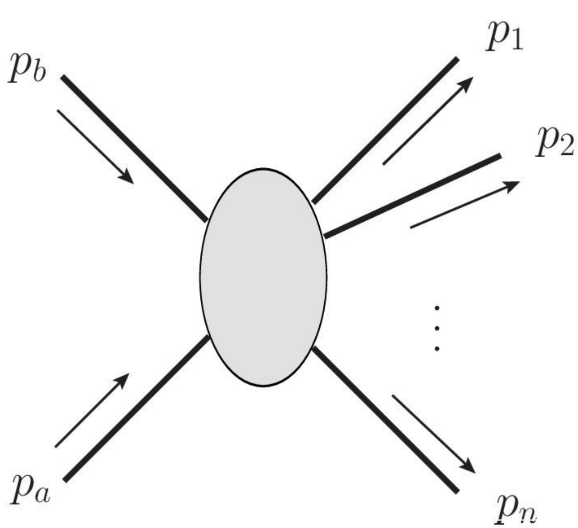
Differential Cross Section

$$d\sigma = d\Pi_n \, \frac{1}{2s} \boxed{\sum_{h_1 \ldots h_n} \left|\mathcal{M}_{a+b\to n}(p_1, \ \ldots, \ p_n)\right|^2}$$

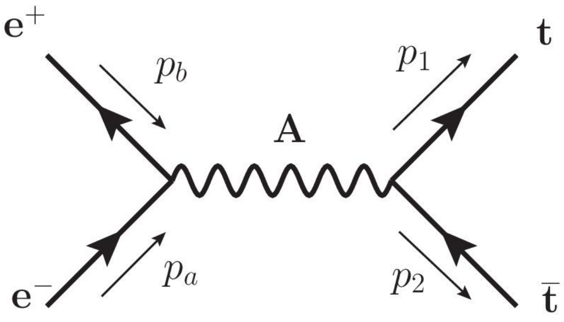
matrix element

$$\boxed{d\Pi_n = \left(\prod_{i=1}^{n} \frac{1}{(2\pi)^3} \frac{d^3 p_i}{2E_i} \right) \cdot (2\pi)^4 \, \delta^{(4)}(p_a + p_b - p_1 - \cdots - p_n)}$$

phase space integration



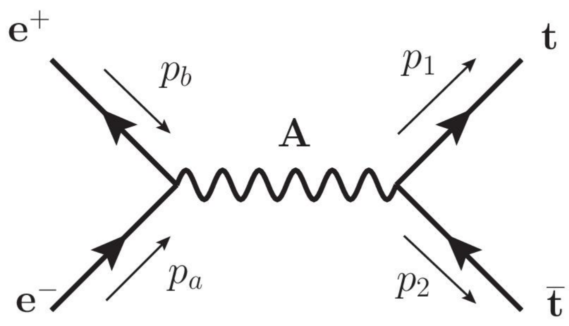
2-particle phase space



$$\mathrm{d}\sigma = \mathrm{d}\Pi_n \, \tfrac{1}{2s} \, \sum_{\mathrm{h}_1 \ldots \mathrm{h}_n} \left|\mathcal{M}_{a+b\,\to\,n}(p_1, \ \ldots, \, p_n)\right|^2$$

$$\mathrm{d}\Pi_n = \left(\prod_{i=1}^{n} \, \tfrac{1}{(2\pi)^3} \tfrac{\mathrm{d}^3 p_i}{2E_i} \right) \cdot (2\pi)^4 \, \delta^{(4)}(p_a + p_b - p_1 - \cdots - p_n)$$

2-particle phase space



$$\mathrm{d}\sigma = \mathrm{d}\Pi_n \, \frac{1}{2s} \boxed{\sum_{\mathrm{h}_1 \ldots \mathrm{h}_n} |\mathcal{M}_{a+b \to n}(p_1, \ \ldots, \ p_n)|^2}$$

$$\mathrm{d}\Pi_n = \left(\prod_{i=1}^{n} \frac{1}{(2\pi)^3} \frac{\mathrm{d}^3 p_i}{2E_i} \right) \cdot (2\pi)^4 \, \delta^{(4)}(p_a + p_b - p_1 - \cdots - p_n)$$

$$\frac{1}{4} \sum_{h_1, h_2} |\mathcal{M}|^2 = \frac{1}{4} \sum_{h_1, h_2} \frac{e^4}{s^2} \left[\overline{v}(p_b, h_b) \gamma_\mu u(p_a, h_a) \overline{u}(p_a, h_a) \gamma_\nu v(p_b, h_b) \right] \cdot$$

$$\cdot \left[\overline{u}(p_1, h_1) \gamma^\mu v(p_2, h_2) \overline{v}(p_2, h_2) \gamma^\nu v(p_1, h_1) \right]$$

take spin sums

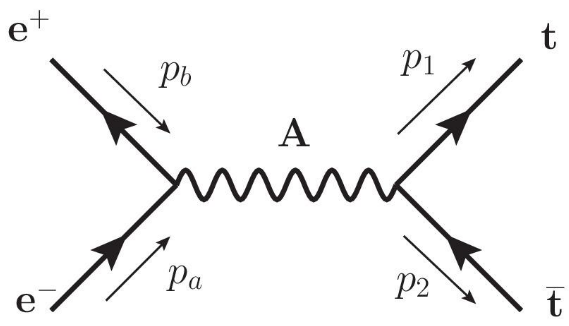
$$= \frac{e^4}{s^2} \mathrm{Tr}\left[\not{p}_b \gamma_\mu \not{p}_a \gamma_\nu \right] \mathrm{Tr}\left[\not{p}_1 \gamma^\mu \not{p}_2 \gamma^\nu \right]$$

calculate traces

$$= \frac{8e^4}{s^2} \Big((p_a \cdot p_1)(p_b \cdot p_2) + (p_a \cdot p_2)(p_b \cdot p_1) \Big)$$

sum of scalar products

2-particle phase space



$$\mathrm{d}\sigma = \mathrm{d}\Pi_n \; \tfrac{1}{2s} \; \sum_{\mathrm{h}_1\ldots\mathrm{h}_n} |\mathcal{M}_{a+b\to n}(p_1, \; \ldots, \; p_n)|^2$$

$$\boxed{\mathrm{d}\Pi_n = \left(\prod_{i=1}^n \tfrac{1}{(2\pi)^3} \tfrac{\mathrm{d}^3 p_i}{2E_i}\right) \cdot (2\pi)^4 \, \delta^{(4)}(p_a + p_b - p_1 - \cdots - p_n)}$$

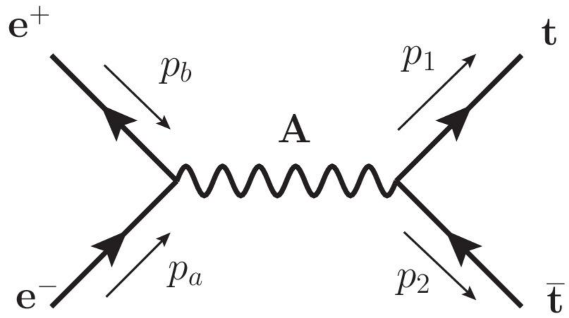
$$\int \mathrm{d}\Pi_2(P, p_1, p_2) = \int \frac{\mathrm{d}^3 p_1}{2E_1} \frac{1}{(2\pi)^3} \frac{\mathrm{d}^3 p_2}{2E_2} \frac{1}{(2\pi)^3} (2\pi)^4 \delta^{(4)}(P - p_1 - p_2)$$

$$= \frac{1}{(2\pi)^2} \frac{\sqrt{\lambda(P_0^2, m_1^2, m_2^2)}}{8P_0^2} \int \mathrm{d}\cos\theta \, \mathrm{d}\phi$$

- general 2-particle phase space in CM-frame of P
- only depends on angles

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$$

2-particle phase space

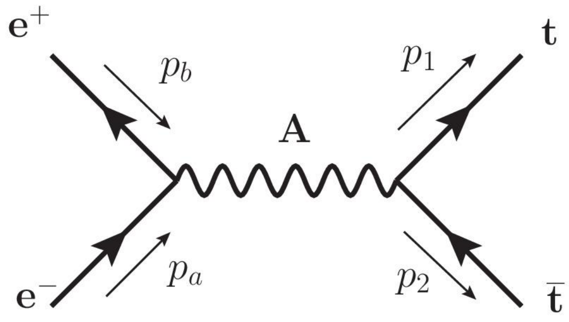


$$\frac{d\sigma}{d\cos\theta\, d\phi} = \frac{\alpha^4}{4s}\left(1 + \cos^2\theta\right)$$

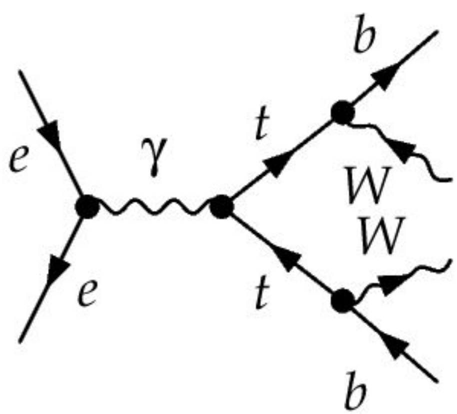
final state with 2 particles:

- traces from spin sums easily calculable
- analytic integration of phase space

2-particle phase space



$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta\,\mathrm{d}\phi} = \frac{\alpha^4}{4s}\left(1+\cos^2\theta\right)$$



final state with 2 particles:

- traces from spin sums easily calculable
- analytic integration of phase space

final state with >3 particles:

- matrix element with traces possible in principle, but computationally expensive in practice
- analytic integration of phase space not possible

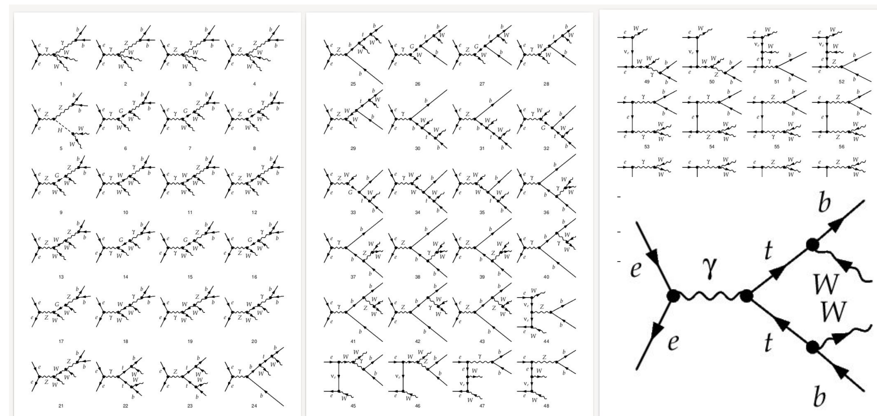
Matrix Element

Why are traces a problem for >3 particles in the final state?

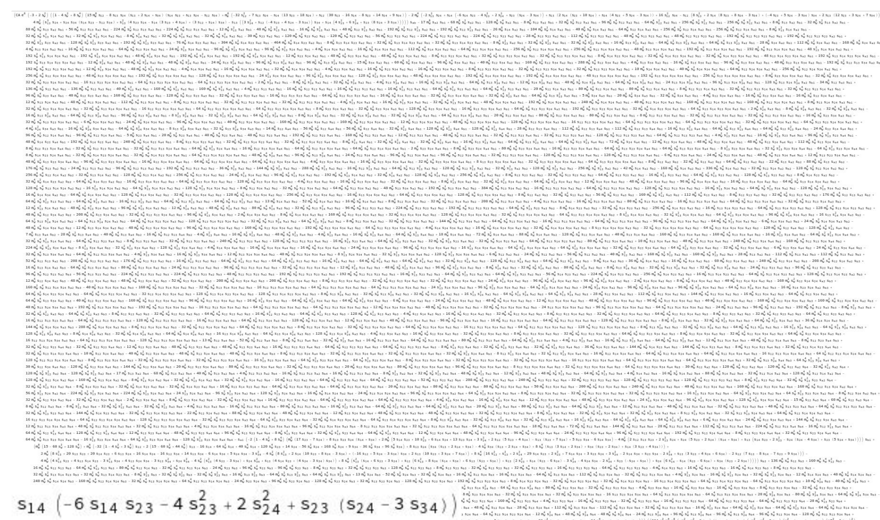
Why are traces a problem for >3 particles in the final state?

example: $e^+e^- \longrightarrow b\,\bar{b}\,W^+W^-$

- 62 diagrams
- 62 x 62 / 2 = 1922 terms
- in general:
  - $N^2/2$ terms for N diagrams
  - the number of diagrams grows with the number of external legs n faster than $n!$
  - grows faster than $(n!)^2/2$



- traces get complicated
- each of the 1922 terms has a form similar to the image on the right



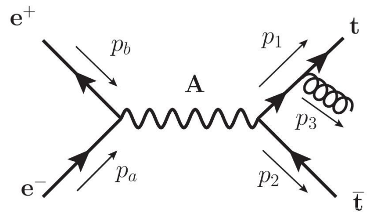
Is there a better method?

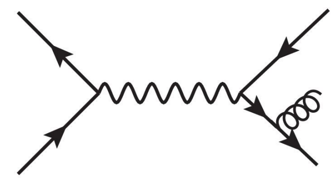
Is there a better method?

Yes, calculate helicity amplitudes

- number of helicity amplitudes grows approximately as 2^n with the number of external particles n
- no trace evaluation necessary
- additional simplifications

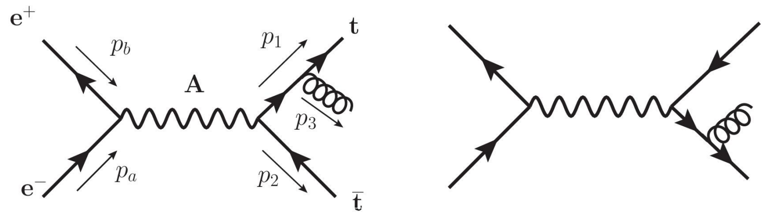
Example





$$
\begin{aligned}
\sum_{h_a,\, h_b,\, h_1,\, h_2,\, h_3} |\mathcal{M}_1 + \mathcal{M}_2|^2 &= \sum_{h_a,\, h_b,\, h_1,\, h_2,\, h_3} |\mathcal{M}_{h_a\, h_b\, h_1\, h_2\, h_3}|^2 \\
&= \sum_{\text{colors}} |\mathcal{M}_{+++++}|^2 + |\mathcal{M}_{++++-}|^2 + |\mathcal{M}_{+++-+}|^2 + |\mathcal{M}_{++-++}|^2 + |\mathcal{M}_{+++--}|^2 + |\mathcal{M}_{++-+-}|^2 + |\mathcal{M}_{++--+}|^2 + |\mathcal{M}_{++---}|^2 \\
&\quad + |\mathcal{M}_{+-+++}|^2 + |\mathcal{M}_{+-++-}|^2 + |\mathcal{M}_{+---+}|^2 + |\mathcal{M}_{+----}|^2 + |\mathcal{M}_{+-+-+}|^2 + |\mathcal{M}_{+--++}|^2 + |\mathcal{M}_{+-+--}|^2 + |\mathcal{M}_{+--+-}|^2 \\
&\quad + |\mathcal{M}_{-++++}|^2 + |\mathcal{M}_{-+++-}|^2 + |\mathcal{M}_{-+--+}|^2 + |\mathcal{M}_{-+---}|^2 + |\mathcal{M}_{-++-+}|^2 + |\mathcal{M}_{-+-++}|^2 + |\mathcal{M}_{-++--}|^2 + |\mathcal{M}_{-+-+-}|^2 \\
&\quad + |\mathcal{M}_{--+++}|^2 + |\mathcal{M}_{--++-}|^2 + |\mathcal{M}_{--+-+}|^2 + |\mathcal{M}_{---++}|^2 + |\mathcal{M}_{--+--}|^2 + |\mathcal{M}_{---+-}|^2 + |\mathcal{M}_{----+}|^2 + |\mathcal{M}_{-----}|^2
\end{aligned}
$$

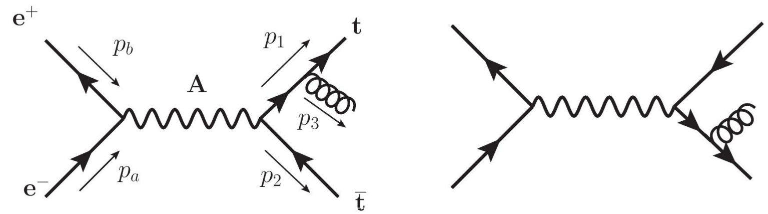
Example



$$
\begin{aligned}
\sum_{h_a,\,h_b,\,h_1,\,h_2,\,h_3} |\mathcal{M}_1 + \mathcal{M}_2|^2 &= \sum_{h_a,\,h_b,\,h_1,\,h_2,\,h_3} |\mathcal{M}_{h_a\,h_b\,h_1\,h_2\,h_3}|^2 \\
&= \sum_{\text{colors}} |\mathcal{M}_{+++++}|^2 + |\mathcal{M}_{++++-}|^2 + |\mathcal{M}_{+++-+}|^2 + |\mathcal{M}_{++-++}|^2 + |\mathcal{M}_{+++--}|^2 + |\mathcal{M}_{++-+-}|^2 + |\mathcal{M}_{++--+}|^2 + |\mathcal{M}_{++---}|^2 \\
&\quad + |\mathcal{M}_{+-+++}|^2 + |\mathcal{M}_{+-++-}|^2 + |\mathcal{M}_{+---+}|^2 + |\mathcal{M}_{+----}|^2 + |\mathcal{M}_{+-+-+}|^2 + |\mathcal{M}_{+--++}|^2 + |\mathcal{M}_{+-+--}|^2 + |\mathcal{M}_{+--+-}|^2 \\
&\quad + |\mathcal{M}_{-++++}|^2 + |\mathcal{M}_{-+++-}|^2 + |\mathcal{M}_{-+--+}|^2 + |\mathcal{M}_{-+---}|^2 + |\mathcal{M}_{-++-+}|^2 + |\mathcal{M}_{-+-++}|^2 + |\mathcal{M}_{-++--}|^2 + |\mathcal{M}_{-+-+-}|^2 \\
&\quad + |\mathcal{M}_{--+++}|^2 + |\mathcal{M}_{--++-}|^2 + |\mathcal{M}_{--+-+}|^2 + |\mathcal{M}_{---++}|^2 + |\mathcal{M}_{--+--}|^2 + |\mathcal{M}_{---+-}|^2 + |\mathcal{M}_{----+}|^2 + |\mathcal{M}_{-----}|^2
\end{aligned}
$$

1) terms violating helicity conservation drop out

Example

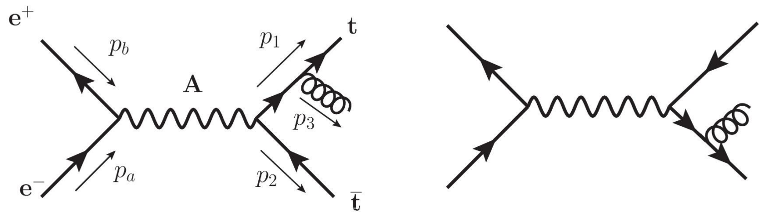


$$\sum_{h_a,\,h_b,\,h_1,\,h_2,\,h_3} |\mathcal{M}_1 + \mathcal{M}_2|^2 = \sum_{h_a,\,h_b,\,h_1,\,h_2,\,h_3} |\mathcal{M}_{h_a\,h_b\,h_1\,h_2\,h_3}|^2$$

$$\begin{aligned} = \sum_{\text{colors}} & |\mathcal{M}_{+++++}|^2 + |\mathcal{M}_{++++-}|^2 + |\mathcal{M}_{+++-+}|^2 + |\mathcal{M}_{++-++}|^2 + |\mathcal{M}_{+++--}|^2 + |\mathcal{M}_{++-+-}|^2 + |\mathcal{M}_{++--+}|^2 + |\mathcal{M}_{++---}|^2 \\ & + |\mathcal{M}_{+-+++}|^2 + |\mathcal{M}_{+-++-}|^2 + |\mathcal{M}_{+---+}|^2 + |\mathcal{M}_{+----}|^2 + |\mathcal{M}_{1,+-+-+}|^2 + |\mathcal{M}_{1,+--++}|^2 + |\mathcal{M}_{1,+-+--}|^2 + |\mathcal{M}_{1,+--+-}|^2 \\ & + |\mathcal{M}_{-++++}|^2 + |\mathcal{M}_{-+++-}|^2 + |\mathcal{M}_{-+--+}|^2 + |\mathcal{M}_{-+---}|^2 + |\mathcal{M}_{1,-++-+}|^2 + |\mathcal{M}_{1,-+-++}|^2 + |\mathcal{M}_{1,-++--}|^2 + |\mathcal{M}_{1,-+-+-}|^2 \\ & + |\mathcal{M}_{--+++}|^2 + |\mathcal{M}_{--++-}|^2 + |\mathcal{M}_{--+-+}|^2 + |\mathcal{M}_{---++}|^2 + |\mathcal{M}_{--+--}|^2 + |\mathcal{M}_{---+-}|^2 + |\mathcal{M}_{----+}|^2 + |\mathcal{M}_{-----}|^2 \end{aligned}$$

1) terms violating helicity conservation drop out
2) diagrams can be dropped by clever choice of gluon polarization vectors

Example



$$
\begin{aligned}
\sum_{h_a,h_b,h_1,h_2,h_3} |\mathcal{M}_{1,h_a,h_b,h_1,h_2,h_3}|^2 &= \sum_{\text{colors}} |\mathcal{M}_{1,+-+-+}|^2 + |\mathcal{M}_{1,+--++}|^2 + |\mathcal{M}_{1,+-+--}|^2 + |\mathcal{M}_{1,+--+-}|^2 \\
&\quad + |\mathcal{M}_{1,-++-+}|^2 + |\mathcal{M}_{1,-+-++}|^2 + |\mathcal{M}_{1,-++--}|^2 + |\mathcal{M}_{1,-+-+-}|^2 \\
&= \sum_{\text{colors}} \left|(\overline{v}_+(p_b)\gamma_\mu u_-(p_a))\,\mathcal{M}^\mu_{1,+-+}\right|^2 + \left|(\overline{v}_-(p_b)\gamma_\mu u_+(p_a))\,\mathcal{M}^\mu_{1,+-+}\right|^2 \\
&\quad + \left|(\overline{v}_+(p_b)\gamma_\mu u_-(p_a))\,\mathcal{M}^\mu_{1,-++}\right|^2 + \left|(\overline{v}_-(p_b)\gamma_\mu u_+(p_a))\,\mathcal{M}^\mu_{1,-++}\right|^2 \\
&\quad + \left|(\overline{v}_+(p_b)\gamma_\mu u_-(p_a))\,\mathcal{M}^\mu_{1,+--}\right|^2 + \left|(\overline{v}_-(p_b)\gamma_\mu u_+(p_a))\,\mathcal{M}^\mu_{1,+--}\right|^2 \\
&\quad + \left|(\overline{v}_+(p_b)\gamma_\mu u_-(p_a))\,\mathcal{M}^\mu_{1,-+-}\right|^2 + \left|(\overline{v}_-(p_b)\gamma_\mu u_+(p_a))\,\mathcal{M}^\mu_{1,-+-}\right|^2
\end{aligned}
$$

1) terms violating helicity conservation drop out
2) diagrams can be dropped by clever choice of gluon polarization vectors
3) subparts of amplitudes can be reused easily

Formalism

massless Dirac equation $$p_\mu \gamma^\mu \; u_s(p) = \; 0$$

chiral basis $$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \qquad \begin{array}{l} \sigma^\mu = (\,\mathbb{1}, \quad \vec{\sigma}\,) \\ \overline{\sigma}^\mu = (\,\mathbb{1}, \; -\vec{\sigma}) \end{array}$$

solution in chiral basis $$u_\pm(p) = \frac{1}{\sqrt{2\,p_0}} \begin{pmatrix} (\sigma \cdot p)\, \chi_{\vec{n},\,\pm} \\ (\overline{\sigma} \cdot p)\, \chi_{\vec{n},\,\pm} \end{pmatrix}$$

2-spinors with spin along $\vec{n}$ axis $$(\vec{\sigma} \cdot \vec{n})\, \chi_{\vec{n},\,\pm} \; = \; \pm\, \chi_{\vec{n},\,\pm}$$

$$\chi_{\vec{n},\,+} = \frac{1}{\sqrt{2(1-n_3)}} \begin{pmatrix} n_1 - i\, n_2 \\ 1 - n_3 \end{pmatrix} \mathrm{e}^{i\,\phi_1}$$
$$\chi_{\vec{n},\,-} = \frac{1}{\sqrt{2(1-n_3)}} \begin{pmatrix} 1 - n_3 \\ -n_1 - i\, n_2 \end{pmatrix} \mathrm{e}^{i\,\phi_2}$$

choose $\vec{n} = \vec{p}/|\vec{p}|$ (the spin of a massless particle is aligned (anti-)parallel to the momentum):

$$u_+(p) = \frac{1}{\sqrt{p_0 - p_3}} \begin{pmatrix} 0 \\ 0 \\ p_1 - i\, p_2 \\ p_0 - p_3 \end{pmatrix} \mathrm{e}^{i\phi_1}, \quad u_-(p) = \frac{1}{\sqrt{p_0 - p_3}} \begin{pmatrix} p_0 - p_3 \\ -p_1 - i\, p_2 \\ 0 \\ 0 \end{pmatrix} \mathrm{e}^{i\phi_2}$$

Formalism

gamma matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix},\ \gamma_5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

chiral projectors

$$P_R = \frac{\mathbb{1} + \gamma_5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$
$$P_L = \frac{\mathbb{1} - \gamma_5}{2} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\gamma_\mu P_R = P_L \gamma_\mu$$

$$\begin{aligned} P_R\, u_+ &= u_+ & \overline{u}_+\, P_L &= \overline{u}_+ \\ P_L\, u_- &= u_- & \overline{u}_-\, P_R &= \overline{u}_- \\ P_R\, v_- &= v_- & \overline{v}_-\, P_L &= \overline{v}_- \\ P_L\, v_+ &= v_+ & \overline{v}_+\, P_R &= \overline{v}_+ \end{aligned}$$

Formalism

gamma matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix},\ \gamma_5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

chiral projectors

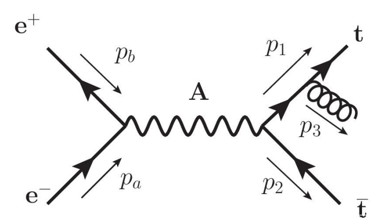
$$P_R = \frac{\mathbb{1}+\gamma_5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

$$P_L = \frac{\mathbb{1}-\gamma_5}{2} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\gamma_\mu P_R = P_L \gamma_\mu$$

$$\begin{aligned} P_R\, u_+ &= u_+ & \overline{u}_+\, P_L &= \overline{u}_+ \\ P_L\, u_- &= u_- & \overline{u}_-\, P_R &= \overline{u}_- \\ P_R\, v_- &= v_- & \overline{v}_-\, P_L &= \overline{v}_- \\ P_L\, v_+ &= v_+ & \overline{v}_+\, P_R &= \overline{v}_+ \end{aligned}$$

terms violating helicity conservation drop out:



$$\begin{aligned} \mathcal{M}_1 &= \overline{u}_+(p_1)\, \gamma_\mu \gamma_\nu \gamma_\rho\, v_+(p_2) \cdot \mathcal{M}_1^{\mu\nu\rho} \\ &= \overline{u}_+(p_1)\, P_L\ \gamma_\mu \gamma_\nu \gamma_\rho\ P_L\, v_+(p_2) \cdot \mathcal{M}_1^{\mu\nu\rho} \\ &= \overline{u}_+(p_1)\, \gamma_\mu \gamma_\nu \gamma_\rho\ P_R\, P_L\ v_+(p_2) \cdot \mathcal{M}_1^{\mu\nu\rho} \\ &= 0 \end{aligned}$$

Formalism

complex polarization vectors are defined by:

$$\epsilon_\pm(p)\cdot p = 0$$
$$\epsilon_\pm \cdot \epsilon_\pm^* = -1$$
$$\epsilon_\pm \cdot \epsilon_\pm = 0$$
$$\epsilon_\pm \cdot \epsilon_\mp = -1$$

fix polarization with:

$$n\cdot\epsilon_\pm(p,n) = 0\,, \quad n^2 = 0$$

polarization sum:

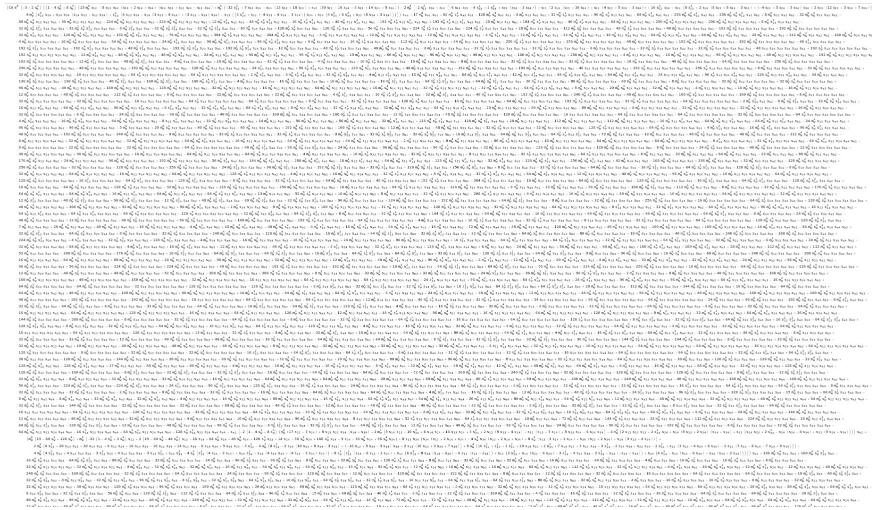
$$\sum_{\lambda=\pm} \epsilon_\lambda^\mu(p)\,\epsilon_\lambda^\nu(p)^* = -g^{\mu\nu} + \tfrac{p_\mu n_\nu + n_\mu\, p_\nu}{p\cdot n}$$

write polarization vectors with spinors:

$$\epsilon_+^\mu(p,\,n) \;=\; \frac{\overline{u}_+(p)\,\gamma^\mu\,v_-(n)}{\sqrt{2}\,\overline{v}_+(n)\,u_+(p)}\;, \quad \epsilon_-^\mu(p,\,n) \;=\; \frac{\overline{u}_-(p)\,\gamma^\mu\,v_+(n)}{\sqrt{2}\,\overline{v}_-(n)\,u_-(p)}$$

$e^+e^- \longrightarrow b\,\bar{b}\,W^+W^-$:

1922 terms of the form:

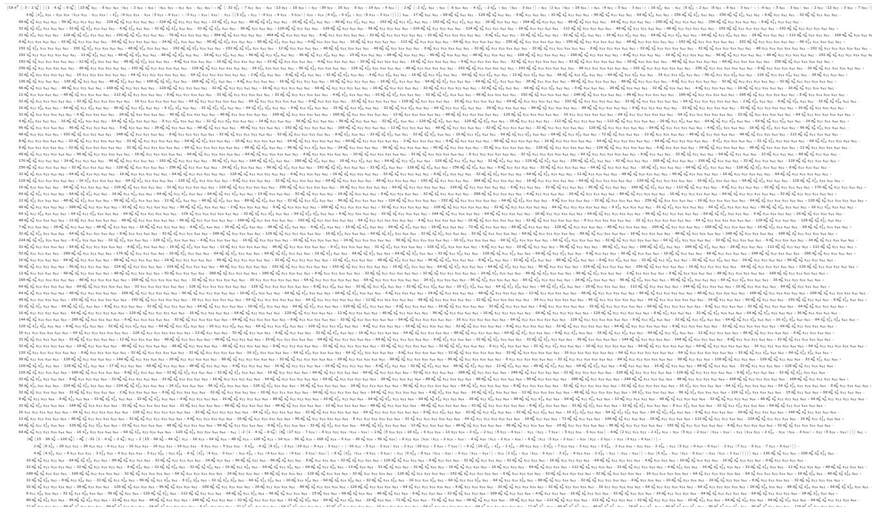


number of additions/subtractions/divisions/multiplications: 30000

trace method

$e^+e^- \longrightarrow b\,\bar{b}\,W^+W^-$:

1922 terms of the form:



number of additions/subtractions/divisions/multiplications: 30000

trace method

1333 terms of the form:

```
amplitude[33][18] = A46*gZbL*(SC5*SC64 - 4.*L2a*(c33*R1b - R13*R3b)*s17*sqrt2);
```

number of additions/subtractions/divisions/multiplications: 11

helicity amplitudes

400 subexpressions of the form:

```
SC64 = (-2.*(c39*L13 + c19*L1b)*R3b)/L13;
```

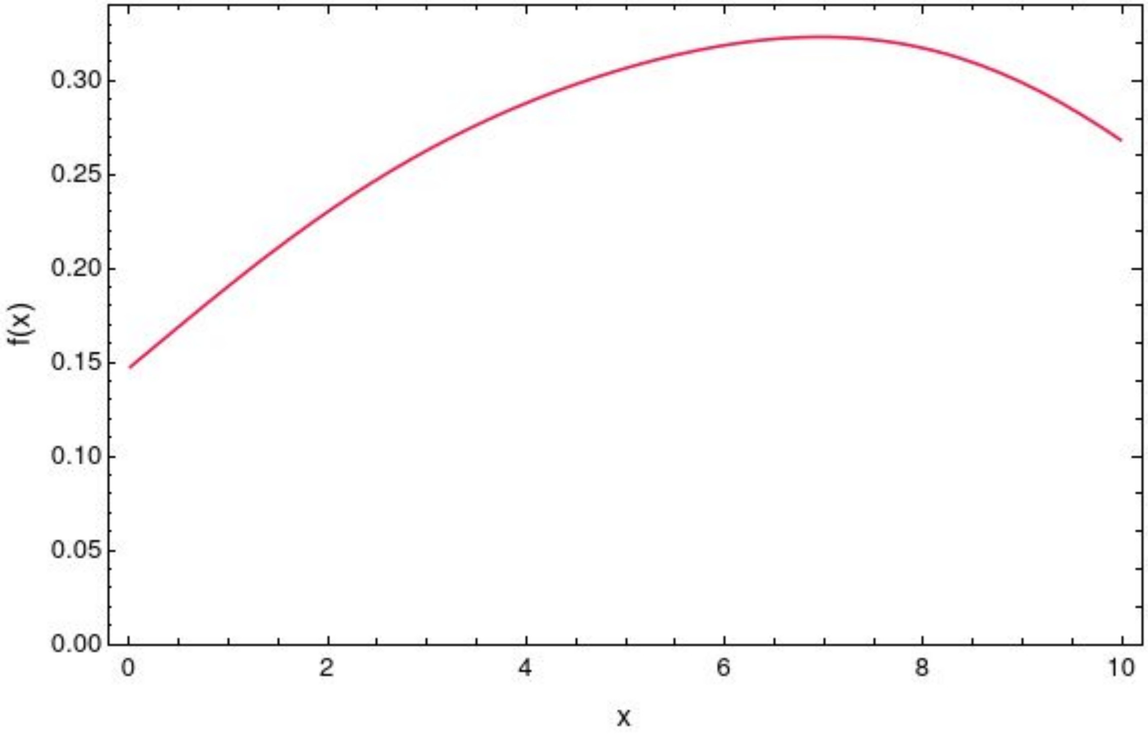
Phase Space Integration

$$\boxed{\mathrm{d}\sigma = \left(\prod_{i=1}^{n} \frac{1}{(2\pi)^3} \frac{\mathrm{d}^3 p_i}{2E_i} \right) \cdot (2\pi)^4 \, \delta^{(4)}(p_a + p_b - p_1 - \cdots - p_n) \cdot |\mathcal{M}|^2}$$

- for *n > 3* not integrable analytically
- integrate numerically instead
- only option because of high dimensions: Monte Carlo integration

Monte Carlo Integration

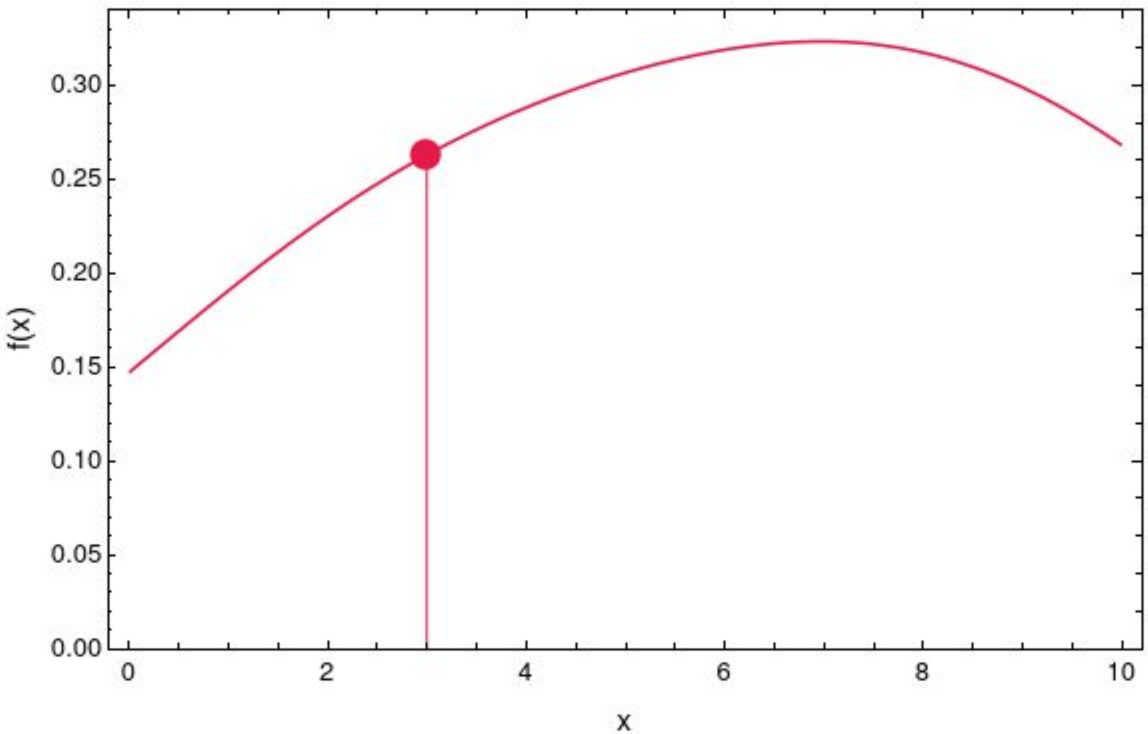
Want to integrate *f(x)* between *a = 0* and *b = 10*:



$$\int_a^b dx\, f(x) = ?$$

Monte Carlo Integration

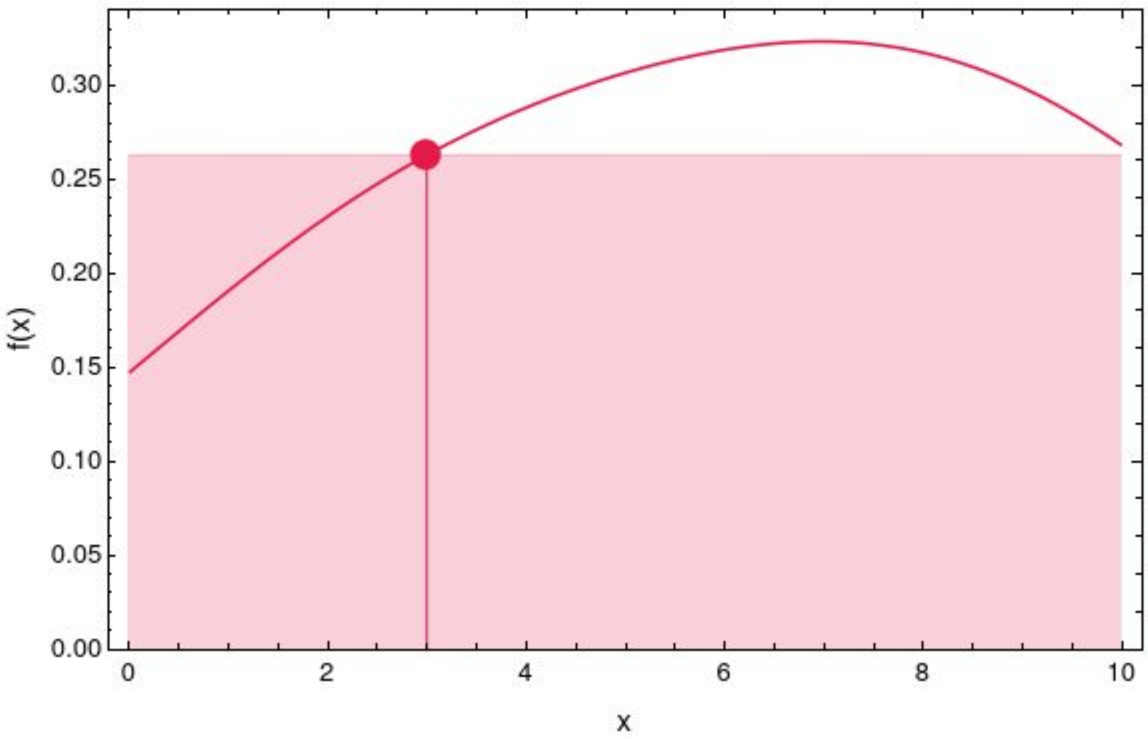
Want to integrate *f(x)* between *a = 0* and *b = 10*:



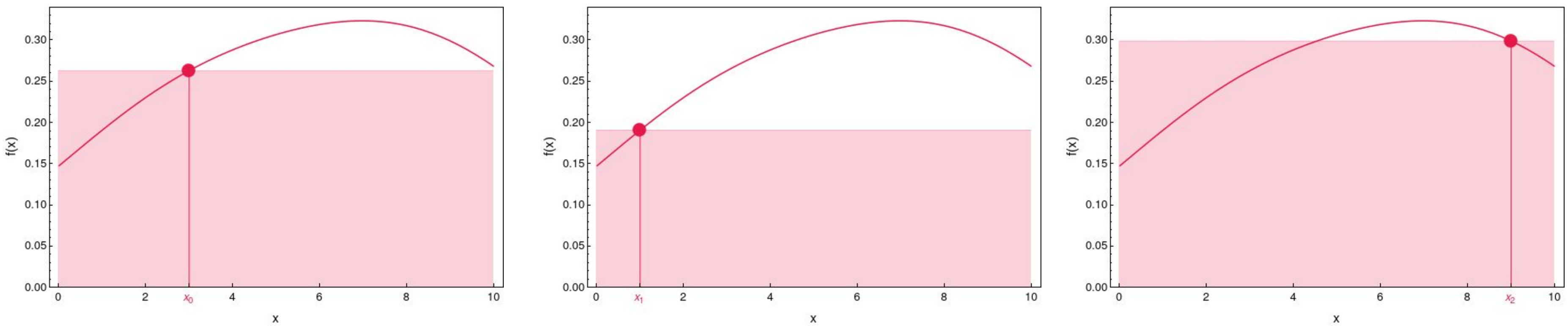
$$\int_a^b dx\, f(x) = ?$$

Monte Carlo Integration

First estimate:



$$\int_a^b dx\, f(x) \approx (b-a) \cdot f(x_0)$$



Better estimate: take average $$\bar{f} = \frac{1}{N}\sum_{i=1}^{N}(b-a)\cdot f(x_i)$$

Take average:

$$\bar{f}_N = \frac{1}{N}\sum_{i=1}^{N}(b-a)\cdot f(x_i)$$

$$I = \int_a^b dx\, f(x)$$

$$I \approx \bar{f}_N$$

Variance of *f(x)*:

$$S_f^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(\bar{f}_N - f(x_i)\right)^2$$

$$\sigma_f^2 = \int_a^b dx\,(I - f(x))^2$$

$$\sigma_f^2 \approx S_f^2$$

Take average:

$$\overline{f}_N = \frac{1}{N}\sum_{i=1}^{N}(b-a)\cdot f(x_i)$$

$$I = \int_a^b dx\, f(x)$$

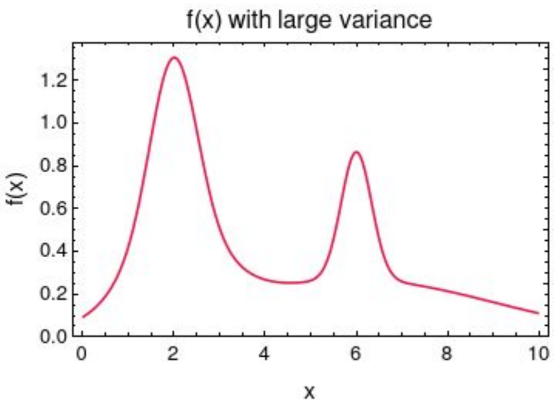
$$I \approx \overline{f}_N$$

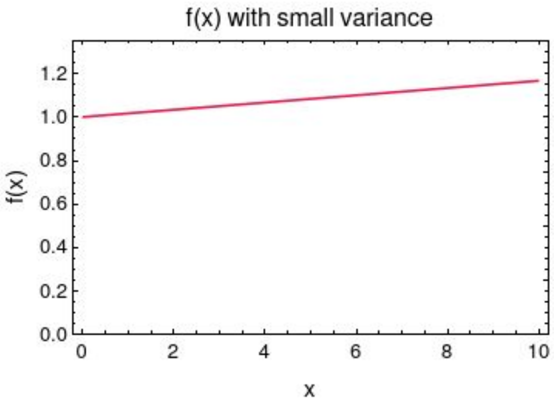
Variance of *f(x)*:

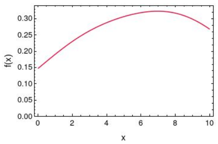
$$S_f^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(\overline{f}_N - f(x_i)\right)^2$$

$$\sigma_f^2 = \int_a^b dx\,(I - f(x))^2$$

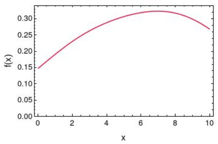
$$\sigma_f^2 \approx S_f^2$$





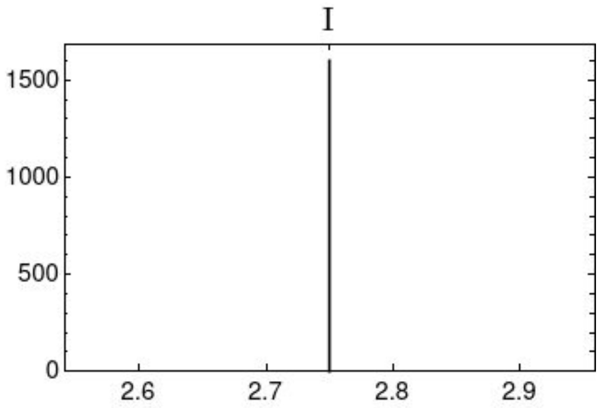
back to the example from before: 

What is the error of a Monte Carlo evaluation with 100 points?

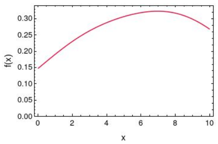
back to the example from before: 

What is the error of a Monte Carlo evaluation with 100 points?

1) perform integration many times to get error from Gauss distribution

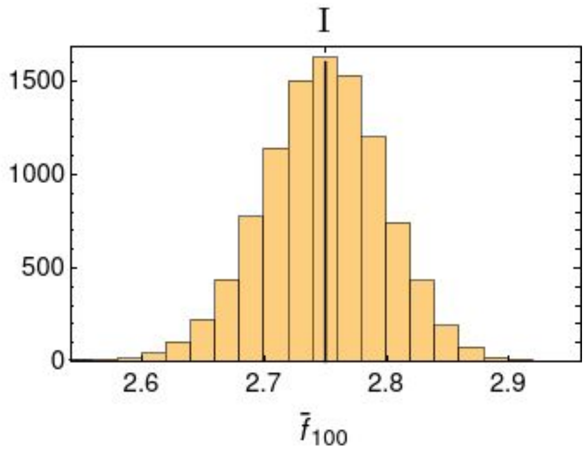


exact (analytical) integration result

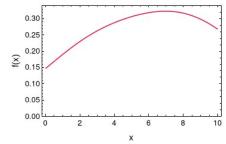
back to the example from before: 

What is the error of a Monte Carlo evaluation with 100 points?

1) perform integration many times to get error from Gauss distribution

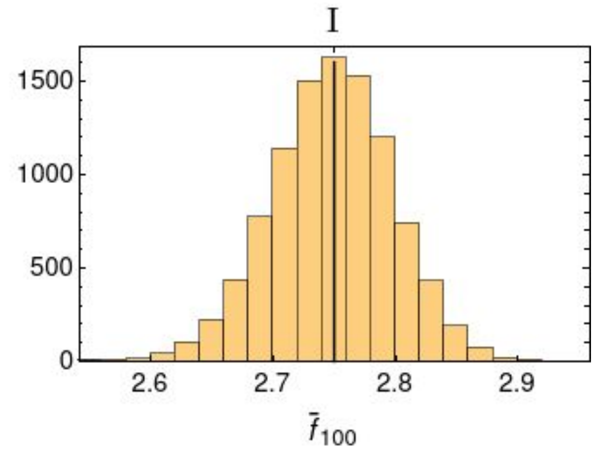


histogram of 10 000 evaluations of $\bar{f}_{100}$

back to the example from before: 

What is the error of a Monte Carlo evaluation with 100 points?

1) perform integration many times to get error from Gauss distribution



histogram of 10 000 evaluations of $\overline{f}_{100}$

or

2) calculate variance directly

$$\sigma^2\left[\overline{f}_N\right] = \sigma^2\left[\frac{(b-a)}{N}\sum_{i=1}^{N} f(x_i)\right] = \frac{(b-a)^2}{N^2} \cdot N\,\sigma^2[f(x)]$$

$$\boxed{\sigma\left[\overline{f}_N\right] = \frac{\sigma_f}{\sqrt{N}}(b-a)}$$

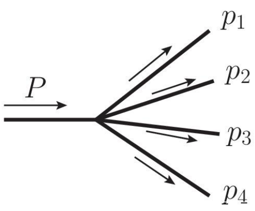
Energy-Momentum Conservation

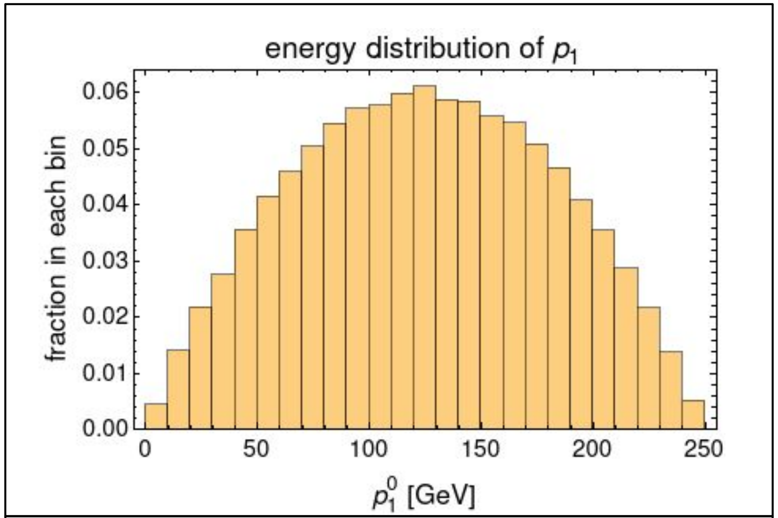
$$\mathrm{d}\sigma = \left(\prod_{i=1}^{n} \frac{1}{(2\pi)^3} \frac{\mathrm{d}^3 p_i}{2E_i} \right) \cdot (2\pi)^4 \, \delta^{(4)}(p_a + p_b - p_1 - \cdots - p_n) \cdot |\mathcal{M}|^2$$

- how to satisfy energy-momentum conservation?

option 1: Rambo [Kleiss, Stirling 1986]

- produce *n* massless 4-momenta
  (energy from finite distribution and momentum in random direction)
- Lorentz transform all momenta to CM frame
- rescale momenta to obtain correct CM energy
- transform to massive momenta

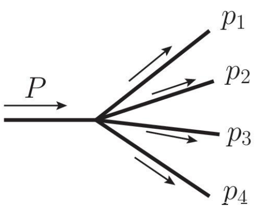


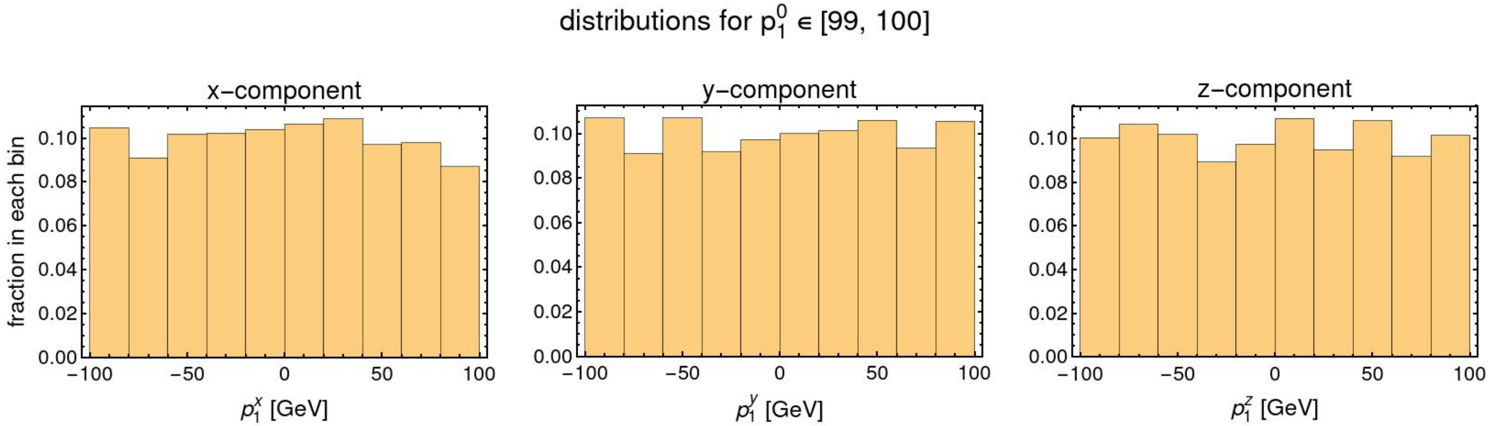


finite energy distribution

option 1: Rambo [Kleiss, Stirling 1986]

- produce *n* massless 4-momenta
  (energy from finite distribution and momentum in random direction)
- Lorentz transform all momenta to CM frame
- rescale momenta to obtain correct CM energy
- transform to massive momenta





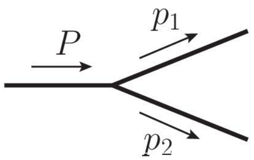
momenta are evenly distributed for fixed energy

option 2: sequential phase space

- rewrite *n* particle phase space into *2*-particle phase spaces

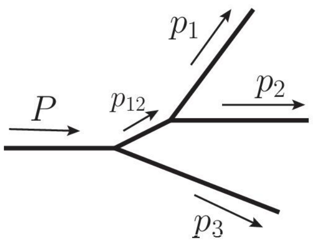
2-particle phase space known

$$\int \mathrm{d}\Pi_2(P, p_1, p_2) = \frac{1}{(2\pi)^2} \frac{\sqrt{\lambda(P_0^2, m_1^2, m_2^2)}}{8P_0^2} \int \mathrm{d}\cos\theta \, d\phi$$



rewrite *n*-particle phase space
into *2*-particle phase spaces

$$\int \mathrm{d}\Pi_3(P, p_1, p_2, p_3) = \int \mathrm{d}\Pi_2(P, p_{12}, p_3) \, \mathrm{d}\Pi_2(p_{12}, p_1, p_2) \, dp_{12}^2$$

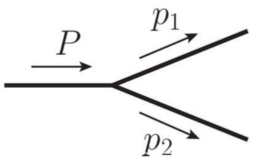


option 2: sequential phase space

- rewrite *n* particle phase space into *2*-particle phase spaces

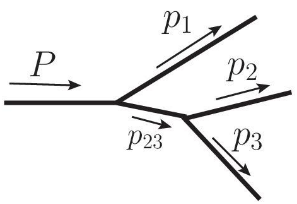
2-particle phase space known

$$\int \mathrm{d}\Pi_2(P, p_1, p_2) = \frac{1}{(2\pi)^2} \frac{\sqrt{\lambda(P_0^2, m_1^2, m_2^2)}}{8P_0^2} \int \mathrm{d}\cos\theta \, d\phi$$



rewrite *n*-particle phase space into *2*-particle phase spaces

$$\begin{aligned}\int \mathrm{d}\Pi_3(P, p_1, p_2, p_3) &= \int \mathrm{d}\Pi_2(P, p_{12}, p_3)\, \mathrm{d}\Pi_2(p_{12}, p_1, p_2)\, dp_{12}^2 \\ &= \int \mathrm{d}\Pi_2(P, p_1, p_{23})\, \mathrm{d}\Pi_2(p_{23}, p_2, p_3)\, dp_{23}^2\end{aligned}$$

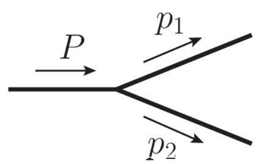


option 2: sequential phase space

- rewrite *n* particle phase space into *2*-particle phase spaces

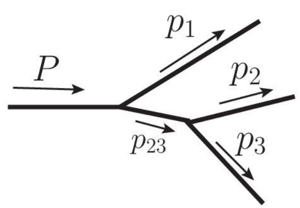
2-particle phase space known

$$\int \mathrm{d}\Pi_2(P,p_1,p_2) = \frac{1}{(2\pi)^2}\frac{\sqrt{\lambda(P_0^2,m_1^2,m_2^2)}}{8P_0^2}\int \mathrm{d}\cos\theta\, d\phi$$



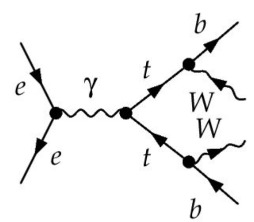
rewrite *n*-particle phase space into *2*-particle phase spaces

$$\int \mathrm{d}\Pi_3(P,p_1,p_2,p_3) = \int \mathrm{d}\Pi_2(P,p_{12},p_3)\,\mathrm{d}\Pi_2(p_{12},p_1,p_2)\,dp_{12}^2$$
$$= \int \mathrm{d}\Pi_2(P,p_1,p_{23})\,\mathrm{d}\Pi_2(p_{23},p_2,p_3)\,dp_{23}^2$$

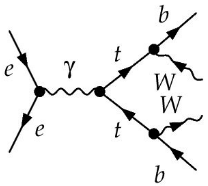


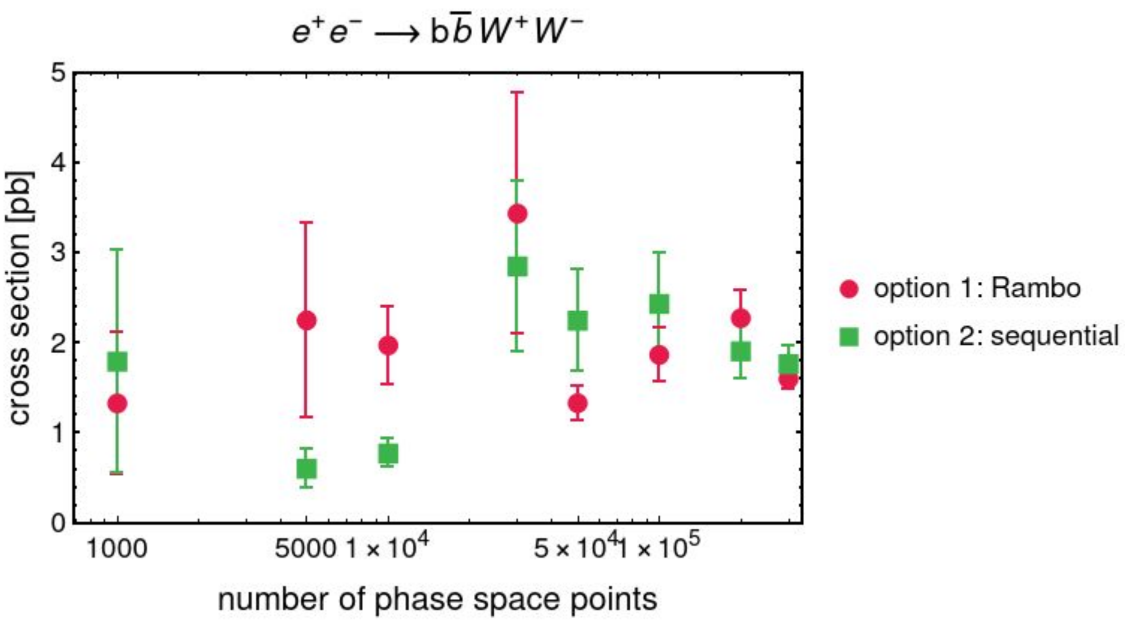
example:

$$\int \mathrm{d}\Pi_4(P,p_b,p_{\bar b},p_{W^+},p_{W^-}) = \int \mathrm{d}\Pi_2(P,p_t,p_{\bar t})\,\mathrm{d}\Pi_2(p_t,p_b,p_{W^+})\,\mathrm{d}\Pi_2(p_{\bar t},p_{\bar b},p_{W^-})\,dp_t^2\,dp_{\bar t}^2$$
$$= \int (d\cos\theta_t\ d\phi_t)\ \ (d\cos\theta_b\ d\phi_b)\,(d\cos\theta_{\bar b}\ d\phi_{\bar b})\,dp_t^2\,dp_{\bar t}^2\ \cdot(\dots)$$



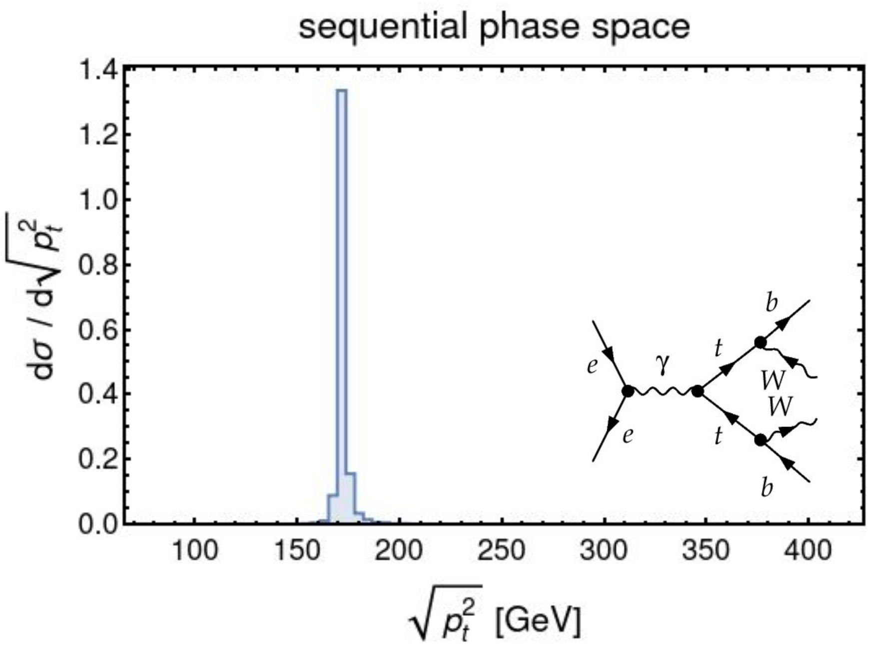
Monte Carlo integration error for $e^+e^- \longrightarrow b\,\bar{b}\,W^+W^-$



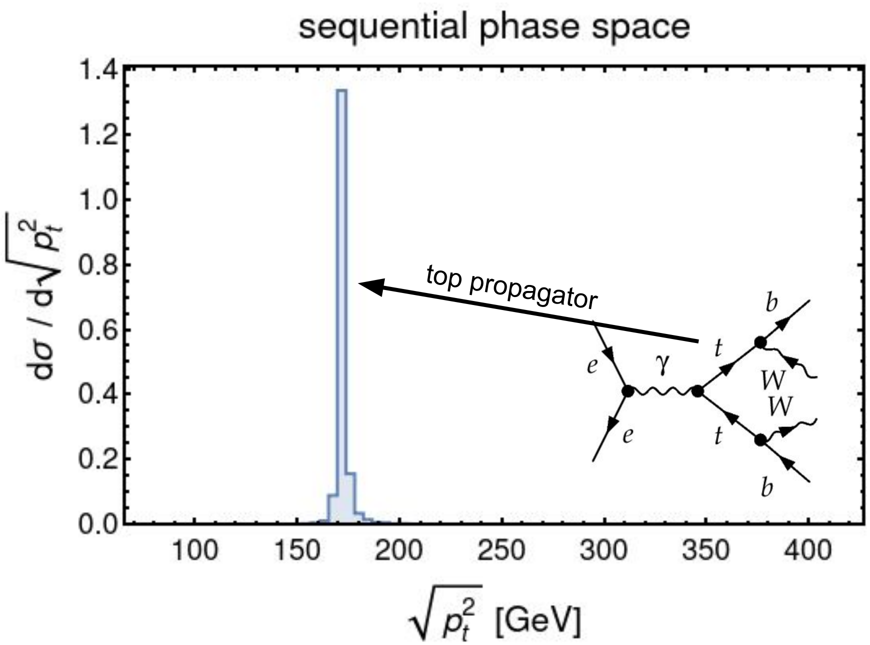


can we do better?

Where does the large and unstable integration error come from?



- the large function variance leads to a large Monte Carlo integration error example:



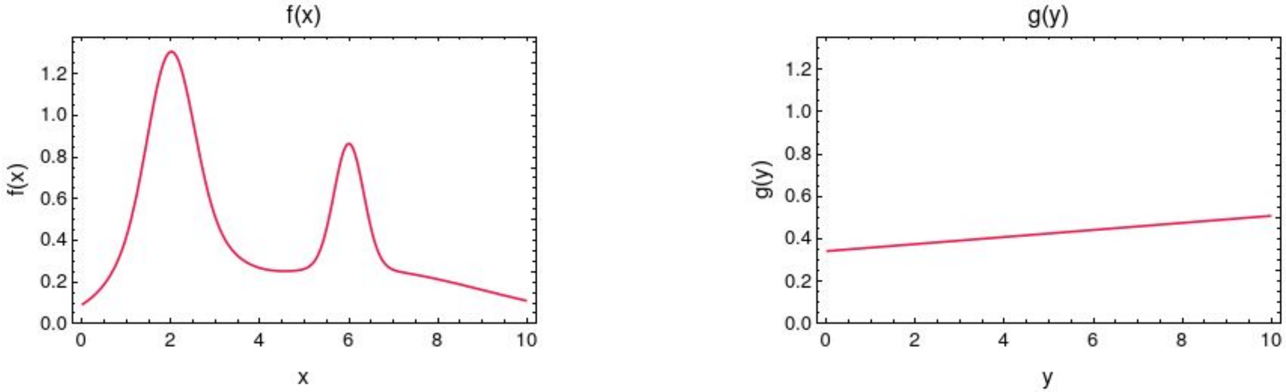
- the large function variance leads to a large Monte Carlo integration error
- it comes from the top quark propagator, which is large for $p_t^2 \approx m_t^2$:

$$\left| \frac{1}{p_t^2 - m_t^2 + i\,\Gamma_t\, m_t} \right|^2 = \frac{1}{\left(p_t^2 - m_t^2\right)^2 + m_t^2\,\Gamma_t^2}$$

Improving Precision: Variable Transformations

Basic Principle: Do a variable transformation to get a function with smaller variance.

$$\int dx\, f(x) = \int dy\, f(y) \cdot (dx(y)/dy)$$
$$= \int dy\, g(y)$$



large function variance
⇨ large Monte Carlo integration error

small function variance
⇨ small Monte Carlo integration error

Improving Precision: Variable Transformations

Basic Principle: Do a variable transformation to get a function with smaller variance.

$$\begin{aligned}\int dx\, f(x) &= \int dy\ f(y) \cdot (dx(y)/dy) \\ &= \int dy\ g(y)\end{aligned}$$

method 1: function known (example resonant propagators)

method 2: function unknown (Vegas)

method 1: function known

example: resonant propagator $f(p^2) = \frac{1}{(p^2-m^2)^2+m^2\Gamma^2}$

variable transformation:

$$\int dp^2\ f(p^2)\ =\ \int d\theta\ f\big(p^2(\theta)\big)\ \cdot\ \big(dp^2(\theta)/d\theta\big)$$
$$= \int d\theta\ \frac{\big(p^2(\theta)-m^2\big)^2+\Gamma^2 m^2}{\Gamma\, m}\ f\big(p^2(\theta)\big)$$
$$= \int d\theta\ \frac{1}{\Gamma\, m}$$

$$\theta(p^2) = \arctan\left(\frac{p^2-m^2}{\Gamma\, m}\right)$$
$$p^2(\theta) = m^2\ +\ \Gamma\, m \cdot \tan(\theta)$$

method 1: function known

example: resonant propagator $f(p^2) = \frac{1}{(p^2-m^2)^2+m^2\Gamma^2}$

variable transformation:

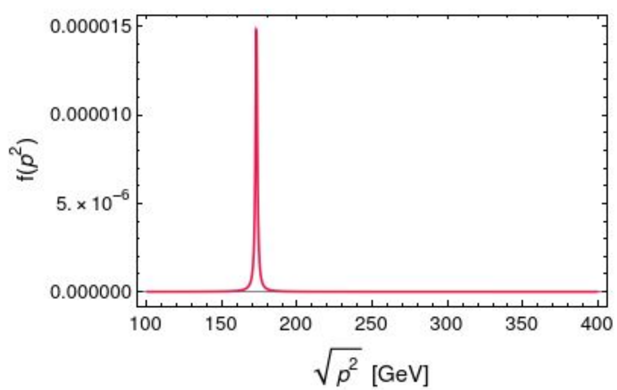
$$\int dp^2 \ f(p^2) \ = \ \int d\theta \ f(p^2(\theta)) \ \cdot \ (dp^2(\theta)/d\theta)$$

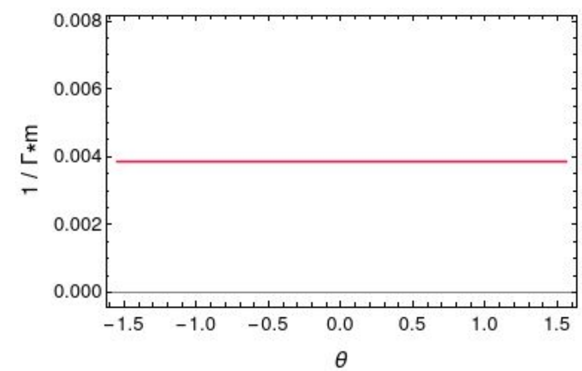
$$= \int d\theta \ \frac{(p^2(\theta)-m^2)^2+\Gamma^2 m^2}{\Gamma\, m} \ f(p^2(\theta))$$

$$= \int d\theta \ \frac{1}{\Gamma\, m}$$

$$\theta(p^2) = \arctan\left(\frac{p^2-m^2}{\Gamma\, m}\right)$$

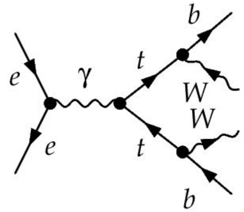
$$p^2(\theta) = m^2 \ + \ \Gamma\, m \cdot \tan(\theta)$$

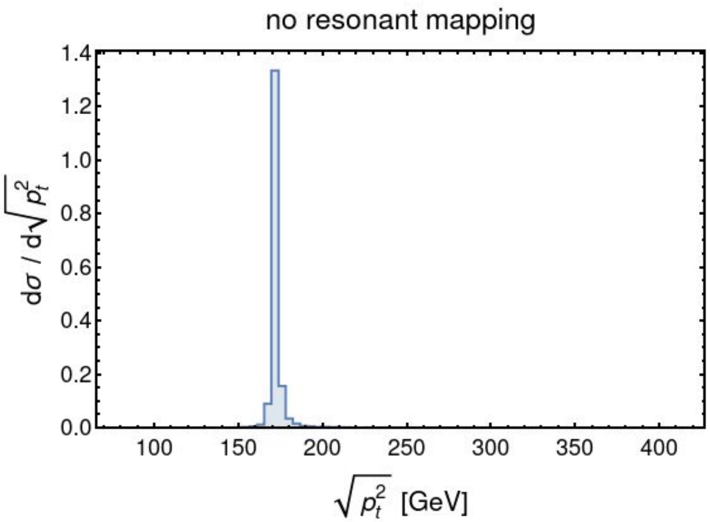


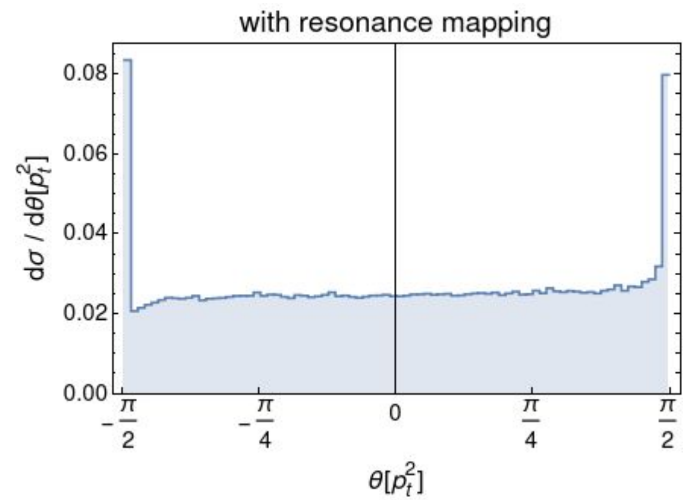


method 1: function known

example: $e^+e^- \longrightarrow b\,\bar{b}\,W^+W^-$

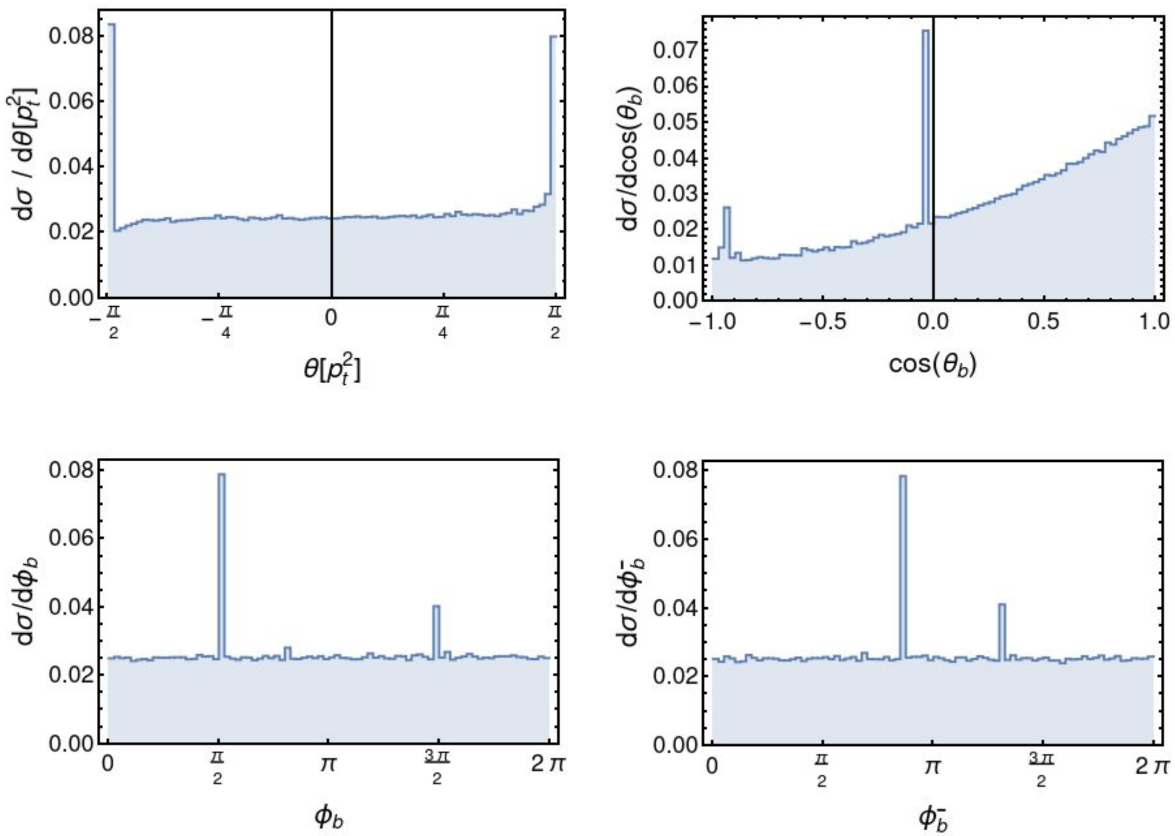






method 1: function known

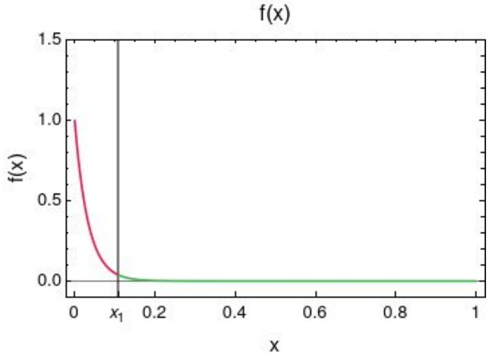
example: $e^+e^- \longrightarrow b\,\bar{b}\,W^+W^-$



method 2: function unknown (Vegas)

example: 1) separate integration region into 2 intervals of different size

$$\int_0^1 dx\, f(x) = \int_0^{x_1} dx\, f(x) + \int_{x_1}^1 dx\, f(x)$$

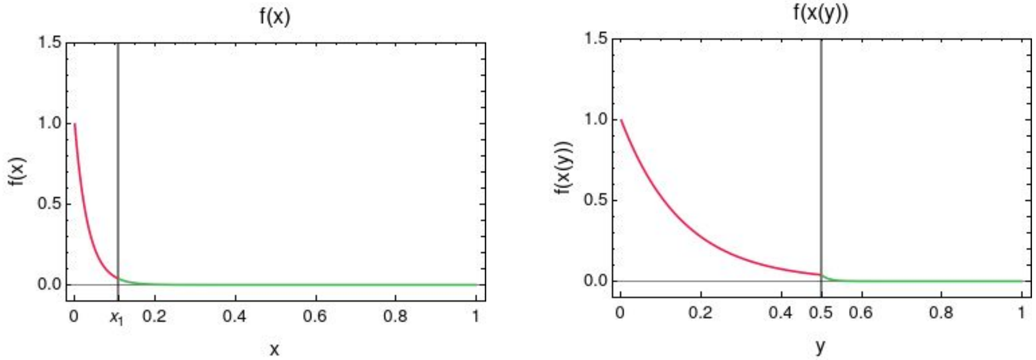


method 2: function unknown (Vegas)

example: 1) separate integration region into 2 intervals of different size

2) make intervals of same size by stretching one interval and compressing the other

$$\int_0^1 dx\, f(x) = \int_0^{x_1} dx\, f(x) + \int_{x_1}^1 dx\, f(x)$$

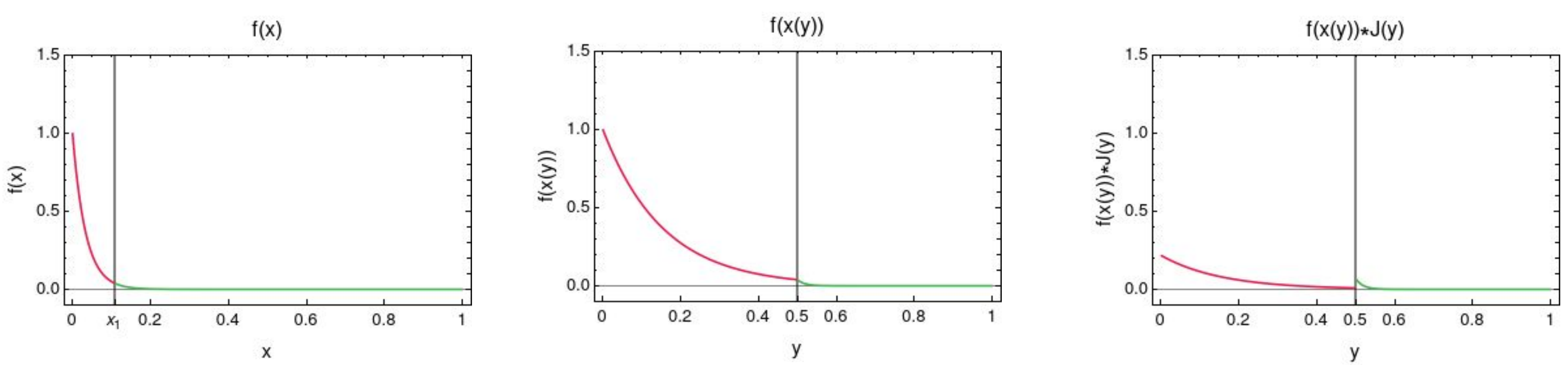


method 2: function unknown (Vegas)

example:
1) separate integration region into 2 intervals of different size
2) make intervals of same size by stretching one interval and compressing the other
3) add Jacobian factor

variable transformation: $x(y) = \begin{cases} 0 \;\; + \Delta x_0 \cdot 2y & 0 < y < 1/2 \\ x_1 + \Delta x_1 \cdot (2y-1) & 1/2 < y < 1 \end{cases} , \qquad \begin{array}{l} \Delta x_0 = x_1 - 0 \\ \Delta x_1 = 1 - x_1 \end{array}$

$$\int_0^1 dx\, f(x) = \int_0^{x_1} dx\, f(x) + \int_{x_1}^1 dx\, f(x) = \int_0^{1/2} dy\, f(x(y)) \cdot J(y) + \int_{1/2}^1 dy\, f(x(y)) \cdot J(y)$$



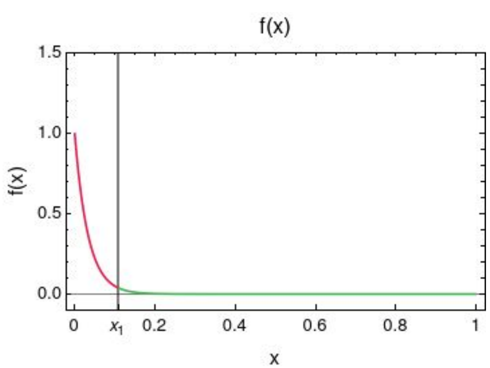
method 2: function unknown (Vegas)

example:

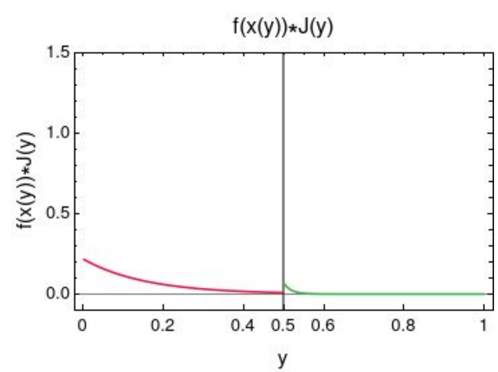
1) separate integration region into 2 intervals of different size
2) make intervals of same size by stretching one interval and compressing the other
3) add Jacobian factor

variable transformation: $x(y) = \begin{cases} 0 + \Delta x_0 \cdot 2y & 0 < y < 1/2 \\ x_1 + \Delta x_1 \cdot (2y-1) & 1/2 < y < 1 \end{cases}, \qquad \begin{aligned} \Delta x_0 &= x_1 - 0 \\ \Delta x_1 &= 1 - x_1 \end{aligned}$

$$\int_0^1 dx\, f(x) = \int_0^{x_1} dx\, f(x) + \int_{x_1}^1 dx\, f(x) = \int_0^{1/2} dy\, f(x(y)) \cdot J(y) + \int_{1/2}^1 dy\, f(x(y)) \cdot J(y)$$

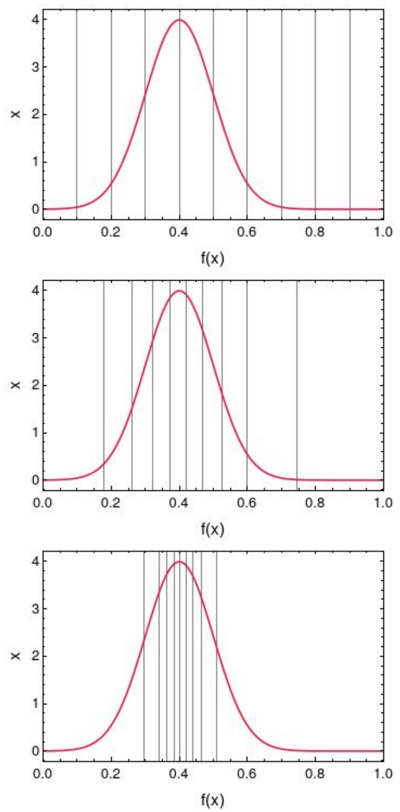


large variance



small variance

method 2: function unknown (Vegas)



in practice:

- take N intervals

- calculate contribution to variance in each interval
- change size of interval accordingly
- repeat

method 2: function unknown (Vegas)

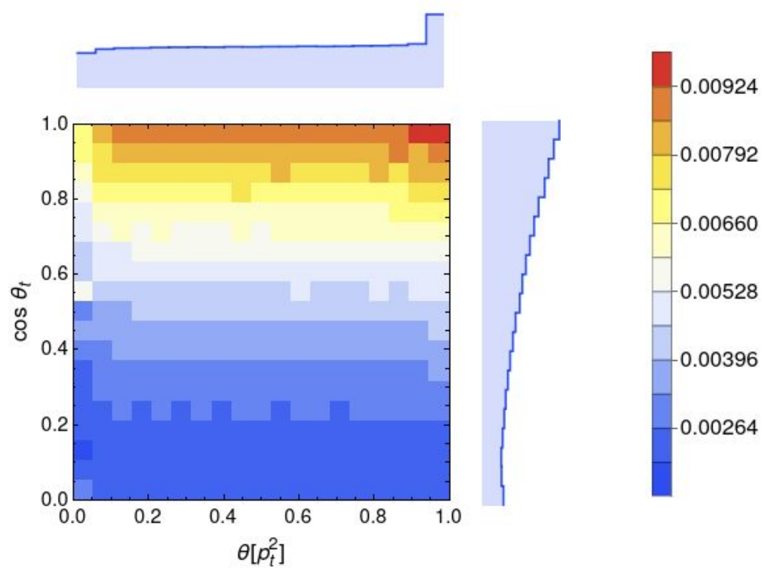
in practice:

- take N intervals

- calculate contribution to variance in each interval
- change size of interval accordingly
- repeat

- for higher dimensions: project on each dimension and divide each axis separately into intervals

$$e^+e^- \longrightarrow b\,\bar{b}\,W^+W^-$$



method 2: function unknown (Vegas)

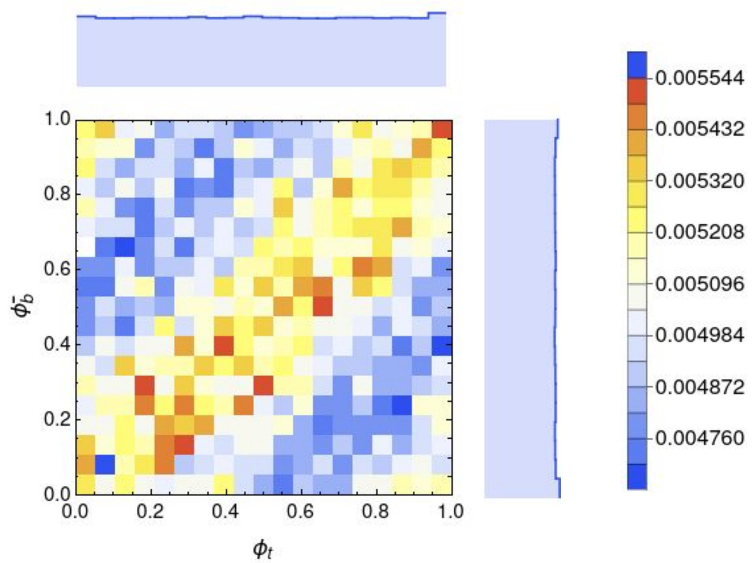
in practice:

- take *N* intervals

- calculate contribution to variance in each interval
- change size of interval accordingly
- repeat

- for higher dimensions: project on each dimension and divide each axis separately into intervals

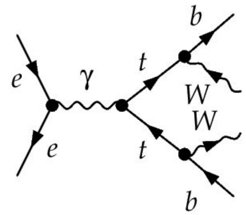
$$e^+e^- \longrightarrow b\,\bar{b}\,W^+W^-$$



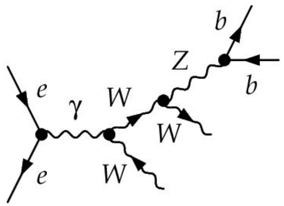
caveat: cannot adapt to diagonal structures

Multichannel Integration

- discussed how to use the structure of one diagram for integration



- what about all other 61 diagram structures?



separate into parts, where the resonant propagators are coming mainly from one diagram:

$$
\begin{aligned}
\int d\Pi \, |\mathcal{M}_1 + \cdots + \mathcal{M}_n|^2 &= \int d\Pi \, |\mathcal{M}|^2 \\
&= \int d\Pi \; |\mathcal{M}|^2 \, \frac{|\mathcal{M}_1|^2 + \cdots + |\mathcal{M}_n|^2}{|\mathcal{M}_1|^2 + \cdots + |\mathcal{M}_n|^2} \\
&= \int d\Pi^{(1)} \; |\mathcal{M}_1|^2 \, \frac{|\mathcal{M}|^2}{\sum_{i=1}^n |\mathcal{M}_i|^2} \; + \; \ldots \; + \; \int d\Pi^{(n)} \, |\mathcal{M}_n|^2 \, \frac{|\mathcal{M}|^2}{\sum_{i=1}^n |\mathcal{M}_i|^2}
\end{aligned}
$$

separate into parts, where the resonant propagators are coming mainly from one diagram:

$$\int d\Pi \, |\mathcal{M}_1 + \cdots + \mathcal{M}_n|^2 = \int d\Pi \, |\mathcal{M}|^2$$

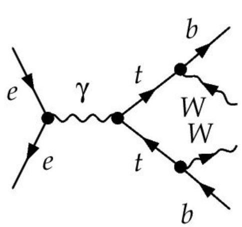
$$= \int d\Pi \; |\mathcal{M}|^2 \, \frac{|\mathcal{M}_1|^2 + \cdots + |\mathcal{M}_n|^2}{|\mathcal{M}_1|^2 + \cdots + |\mathcal{M}_n|^2}$$

$$= \int d\Pi^{(1)} \; |\mathcal{M}_1|^2 \, \frac{|\mathcal{M}|^2}{\sum_{i=1}^n |\mathcal{M}_i|^2} \; + \; \ldots \; + \; \int d\Pi^{(n)} \, |\mathcal{M}_n|^2 \, \frac{|\mathcal{M}|^2}{\sum_{i=1}^n |\mathcal{M}_i|^2}$$

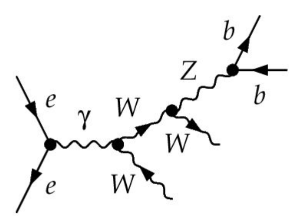
adapt phase space structures to diagrams (“integration channels”):

$$\int \mathrm{d}\Pi^{(1)} = \int \mathrm{d}\Pi_2(P, p_t, p_{\bar{t}}) \, \mathrm{d}\Pi_2(p_t, p_b, p_{W^+}) \, \mathrm{d}\Pi_2(p_{\bar{t}}, p_{\bar{b}}, p_{W^-}) \, dp_t^2 \, dp_{\bar{t}}^2$$

$$\int \mathrm{d}\Pi^{(n)} = \int \mathrm{d}\Pi_2(P, p_{W^-}, p_{W_{\mathrm{intermediate}}}) \, \mathrm{d}\Pi_2(p_{W_{\mathrm{intermediate}}}, p_{W^+}, \; p_Z) \, \mathrm{d}\Pi_2(p_Z, p_b, p_{\bar{b}}) \, dp^2_{W_{\mathrm{intermediate}}} \, dp_Z^2$$



1



n

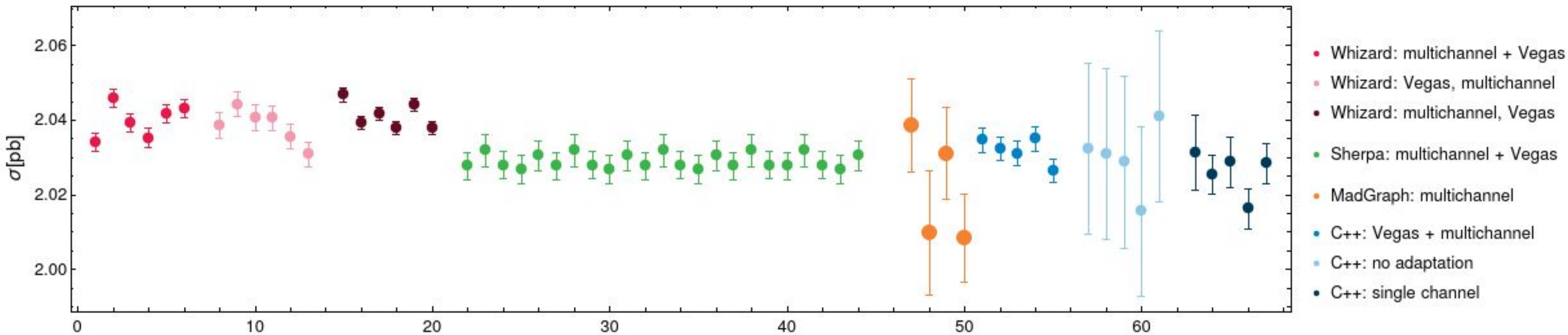
multiply with factors $\alpha_1, \ \ldots, \ \alpha_n$ (= “channel weights”)

$$\begin{aligned}
\int d\Pi \, |\mathcal{M}_1 + \cdots + \mathcal{M}_n|^2 &= \int d\Pi \, |\mathcal{M}|^2 \\
&= \int d\Pi \ |\mathcal{M}|^2 \, \frac{\alpha_1 |\mathcal{M}_1|^2 + \cdots + \alpha_n |\mathcal{M}_n|^2}{\alpha_1 |\mathcal{M}_1|^2 + \cdots + \alpha_n |\mathcal{M}_n|^2} \\
&= \alpha_1 \int d\Pi^{(1)} \ |\mathcal{M}_1|^2 \, \frac{|\mathcal{M}|^2}{\sum_{i=1}^n \alpha_i |\mathcal{M}_i|^2} \ + \ \ldots \ + \ \alpha_n \int d\Pi^{(n)} \, |\mathcal{M}_n|^2 \, \frac{|\mathcal{M}|^2}{\sum_{i=1}^n \alpha_i |\mathcal{M}_i|^2}
\end{aligned}$$

→ channel weights are adapted to minimize the integration error

Comparison of methods and generators:
(each point evaluated with 300 000 phase space points)

$$e^+e^- \longrightarrow b\,\bar{b}\,W^+W^-$$



Project

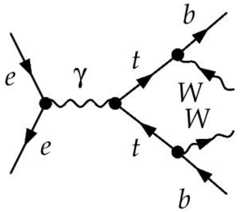
- want to improve Monte Carlo event generation for top quark production:
  - NLL threshold corrections for the differential cross section for top quark pair production at the threshold
  - parton showers off intermediate top quarks
  - ....
- we are building an NLO Monte Carlo for $e^+e^- \longrightarrow b\,\bar{b}\,W^+W^-$ with resonant subtraction

Conclusion

- matrix elements can be efficiently calculated with helicity amplitudes
- phase space integration usually uses multiple integration channels
- each channel uses variable transformations to reduce the integration error
- we have finished a LO C++ Monte Carlo for $e^+e^- \longrightarrow b\,\bar{b}\,W^+W^-$ and are now building an NLO code with resonant subtraction

Conclusion

- matrix elements can be efficiently calculated with helicity amplitudes
- phase space integration usually uses multiple integration channels
- each channel uses variable transformations to reduce the integration error
- we have finished a LO C++ Monte Carlo for $e^+e^- \longrightarrow b\,\bar{b}\,W^+W^-$ and are now building an NLO code with resonant subtraction



Thank you!