Differential Cross Sections at Leading Order with Monte Carlo Methods

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Outline

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● Matrix Element
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  ○ helicity amplitudes - formalism

● Phase Space Integration
  ○ Monte Carlo integration
  ○ energy-momentum conservation
    ■ Rambo
    ■ sequential phase space
  ○ improving precision
    ■ resonant propagators
    ■ Vegas
  ○ multichannel integration

● Our Project
Introduction
- to compare with experiment, we compare which particles arrive in the detectors
- for this we need to know in which direction the outgoing particles are going
- we need the differential cross section
Differential Cross Section

\[ d\sigma = d\Pi_n \frac{1}{2s} \sum_{h_1\ldots h_n} \left| M_{a+b \to n}(p_1, \ldots, p_n) \right|^2 \]

\[ d\Pi_n = \left( \prod_{i=1}^{n} \frac{1}{(2\pi)^3} \frac{d^3p_i}{2E_i} \right) \cdot (2\pi)^4 \delta^4(p_a + p_b - p_n - \cdots - p_n) \]

matrix element

phase space integration

\[ p_b \]
\[ p_1 \]
\[ p_2 \]
\[ p_n \]
\[ \vdots \]
2-particle phase space

\[ d\sigma = d\Pi_n \frac{1}{2s} \sum_{h_1 \ldots h_n} |\mathcal{M}_{a+b \rightarrow n}(p_1, \ldots, p_n)|^2 \]

\[ d\Pi_n = \left( \prod_{i=1}^{n} \frac{1}{(2\pi)^3} \frac{d^3 p_i}{2E_i} \right) \cdot (2\pi)^4 \delta^{(4)}(p_a + p_b - p_1 - \cdots - p_n) \]
2-particle phase space

\[
\frac{1}{4} \sum_{h_1, h_2} |\mathcal{M}|^2 = \frac{1}{4} \sum_{h_1, h_2} \frac{e^4}{s^2} \left[ \bar{u}(p_b, h_b) \gamma_\mu u(p_a, h_a) \bar{u}(p_a, h_a) \gamma_\nu v(p_b, h_b) \right] \cdot \left[ \bar{u}(p_1, h_1) \gamma_\mu v(p_2, h_2) \bar{u}(p_2, h_2) \gamma_\nu v(p_1, h_1) \right] \\
= \frac{e^4}{s^2} \text{Tr} \left[ \slashed{p}_b \gamma_\mu \slashed{p}_a \gamma_\nu \right] \text{Tr} \left[ \slashed{p}_1 \gamma_\mu \slashed{p}_2 \gamma_\nu \right] \\
= \frac{8e^4}{s^2} \left( (p_a \cdot p_1)(p_b \cdot p_2) + (p_a \cdot p_2)(p_b \cdot p_1) \right)
\]

d\sigma = d\Pi_n \frac{1}{2s} \sum_{h_1 \cdots h_n} |\mathcal{M}_{a+b \rightarrow n}(p_1, \ldots, p_n)|^2

d\Pi_n = \left( \prod_{i=1}^{n} \frac{1}{(2\pi)^3 \frac{d^3 p_i}{2E_i}} \right) \cdot (2\pi)^4 \delta^{(4)}(p_a + p_b - p_1 - \cdots - p_n)

take spin sums

calculate traces

sum of scalar products
2-particle phase space

\[ d\sigma = d\Pi_n \frac{1}{2s} \sum_{h_1 \ldots h_n} |M_{a+b \to n}(p_1, \ldots, p_n)|^2 \]

\[ d\Pi_n = \left( \prod_{i=1}^{n} \frac{1}{(2\pi)^3} \frac{d^3 p_i}{2E_i} \right) \cdot (2\pi)^4 \delta^{(4)}(p_a + p_b - p_1 - \cdots - p_n) \]

\[ \int d\Pi_2(P, p_1, p_2) = \int \frac{d^3 p_1}{2E_1} \frac{1}{(2\pi)^3} \frac{d^3 p_2}{2E_2} \frac{1}{(2\pi)^3} (2\pi)^4 \delta^{(4)}(P - p_1 - p_2) = \frac{1}{(2\pi)^2} \sqrt{\lambda(P_0^2, m_1^2, m_2^2)} \frac{1}{8P_0^2} \int d\cos \theta \, d\phi \]

- general 2-particle phase space in CM-frame of P
- only depends on angles

\[ \lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc \]
2-particle phase space

\[
\frac{d\sigma}{d\cos \theta \, d\phi} = \frac{\alpha^4}{4s} \left( 1 + \cos^2 \theta \right)
\]

final state with 2 particles:
- traces from spin sums easily calculable
- analytic integration of phase space
final state with 2 particles:
- traces from spin sums easily calculable
- analytic integration of phase space

final state with >3 particles:
- matrix element with traces possible in principle, but computationally expensive in practice
- analytic integration of phase space not possible
Matrix Element
Why are traces a problem for >3 particles in the final state?
Why are traces a problem for >3 particles in the final state?

example: \( e^+e^- \rightarrow b\bar{b}W^+W^- \)

- 62 diagrams
- \( 62 \times 62 / 2 = 1922 \) terms
- in general:
  - \( \frac{N^2}{2} \) terms for \( N \) diagrams
  - the number of diagrams grows with the number of external legs \( n \) faster than \( n! \)
  - grows faster than \( (n!)^2/2 \)

- traces get complicated
- each of the 1922 terms has a form similar to the image on the right
Is there a better method?
Is there a better method?

Yes, calculate helicity amplitudes

- number of helicity amplitudes grows approximately as $2^n$ with the number of external particles $n$
- no trace evaluation necessary
- additional simplifications
\[
\sum_{h_a, h_b, h_1, h_2, h_3} |M_1 + M_2|^2 = \sum_{h_a, h_b, h_1, h_2, h_3} |M_{h_a, h_b, h_1, h_2, h_3}|^2
\]

\[
= \sum_{\text{colors}} |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2
\]

\[
+ |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2
\]

\[
+ |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2
\]

\[
+ |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2 + |M_{+-+-+}|^2
\]
\[
\sum_{a, b, c, d} |\mathcal{M}_1 + \mathcal{M}_2|^2 = \sum_{a_1, b_1, c_1, d_1} |\mathcal{M}_{a_1 b_1 c_1 d_1}|^2
\]

\[
= \sum_\text{colors} |\mathcal{M}_{++--}|^2 + |\mathcal{M}_{+-++}|^2 + |\mathcal{M}_{+--+}|^2 + |\mathcal{M}_{+-+-}|^2 + |\mathcal{M}_{+---}|^2 + |\mathcal{M}_{+-++}|^2 + |\mathcal{M}_{+-++}|^2 + |\mathcal{M}_{-++-}|^2 + |\mathcal{M}_{-++-}|^2 + |\mathcal{M}_{+-++}|^2 + |\mathcal{M}_{-++-}|^2 + |\mathcal{M}_{-++-}|^2 + |\mathcal{M}_{+-++}|^2 + |\mathcal{M}_{-++-}|^2 + |\mathcal{M}_{-++-}|^2 + |\mathcal{M}_{-++-}|^2 + |\mathcal{M}_{-++-}|^2 + |\mathcal{M}_{-++-}|^2 + |\mathcal{M}_{-++-}|^2 + |\mathcal{M}_{-++-}|^2
\]

1) terms violating helicity conservation drop out
\[
\sum_{h_a, h_b, h_1, h_2, h_3} |M_1 + M_2|^2 = \sum_{h_a, h_b, h_1, h_2, h_3} |M_{h_a h_b h_1 h_2 h_3}|^2
\]

\[
= \sum_{\text{colors}} |M_{+++++}|^2 + |M_{++++-}|^2 + |M_{++++-}|^2 + |M_{++++-}|^2 + |M_{++++-}|^2 + |M_{++++-}|^2 + |M_{++++-}|^2
\]

1) terms violating helicity conservation drop out

2) diagrams can be dropped by clever choice of gluon polarization vectors
Example

\[ \sum_{h_a, h_b, h_1, h_2, h_3} |M_{1, h_a, h_b, h_1, h_2, h_3}|^2 = \sum_{\text{colors}} |M_{1, ++-+}|^2 + |M_{1, +--+}|^2 + |M_{1, --++}|^2 + |M_{1, -++-}|^2 \\
+ |M_{1, +-++}|^2 + |M_{1, -++-}|^2 + |M_{1, +-+-}|^2 + |M_{1, --+-}|^2 \\
= \sum_{\text{colors}} |(\bar{v}_+ (p_b) \gamma_\mu u_-(p_a)) M_{1, ++-+}^\mu|^2 + |(\bar{v}_- (p_b) \gamma_\mu u_+(p_a)) M_{1, ++-+}^\mu|^2 \\
+ |(\bar{v}_+ (p_b) \gamma_\mu u_-(p_a)) M_{1, +-++}^\mu|^2 + |(\bar{v}_- (p_b) \gamma_\mu u_+(p_a)) M_{1, +-++}^\mu|^2 \\
+ |(\bar{v}_+ (p_b) \gamma_\mu u_-(p_a)) M_{1, -++-}^\mu|^2 + |(\bar{v}_- (p_b) \gamma_\mu u_+(p_a)) M_{1, -++-}^\mu|^2 \\
+ |(\bar{v}_+ (p_b) \gamma_\mu u_-(p_a)) M_{1, +-+-}^\mu|^2 + |(\bar{v}_- (p_b) \gamma_\mu u_+(p_a)) M_{1, +-+-}^\mu|^2 \]

1) terms violating helicity conservation drop out
2) diagrams can be dropped by clever choice of gluon polarization vectors
3) subparts of amplitudes can be reused easily
Formalism

massless Dirac equation

\[ p_\mu \gamma^\mu \ u_s(p) = 0 \]

chiral basis

\[ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \quad \sigma^\mu = (1, -\vec{\sigma}) \quad \overline{\sigma}^\mu = (1, -\vec{\sigma}) \]

solution in chiral basis

\[ u_{\pm}(p) = \frac{1}{\sqrt{2 p_0}} \begin{pmatrix} \sigma \cdot p \chi_{\vec{n}, \pm} \\ (\overline{\sigma} \cdot p) \chi_{\vec{n}, \pm} \end{pmatrix} \]

\[ (\overline{\sigma} \cdot \vec{n}) \chi_{\vec{n}, \pm} = \pm \chi_{\vec{n}, \pm} \]

2-spinors with spin along \( \vec{n} \) axis

\[ \chi_{\vec{n}, +} = \frac{1}{\sqrt{2(1-n_3)}} \begin{pmatrix} n_1 - i n_2 \\ 1 - n_3 \end{pmatrix} e^{i \phi_1} \]
\[ \chi_{\vec{n}, -} = \frac{1}{\sqrt{2(1-n_3)}} \begin{pmatrix} 1 - n_3 \\ -n_1 - i n_2 \end{pmatrix} e^{i \phi_2} \]

choose \( \vec{n} = \vec{p} / |\vec{p}| \) (the spin of a massless particle is aligned (anti-)parallel to the momentum):

\[ u_{\pm}(p) = \frac{1}{\sqrt{p_0 - p_3}} \begin{pmatrix} 0 \\ 0 \\ p_1 - i p_2 \\ p_0 - p_3 \end{pmatrix} e^{i \phi_1}, \quad u_{\pm}(p) = \frac{1}{\sqrt{p_0 - p_3}} \begin{pmatrix} p_0 - p_3 \\ -p_1 - i p_2 \\ 0 \\ 0 \end{pmatrix} e^{i \phi_2} \]
Formalism

Gamma matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Chiral projectors

$$P_R = \frac{1 + \gamma_5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_\mu P_R = P_L \gamma_\mu$$

$$P_L = \frac{1 - \gamma_5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_R u_+ = u_+ \quad \bar{u}_+ P_L = \bar{u}_+$$
$$P_L u_- = u_- \quad \bar{u}_- P_R = \bar{u}_-$$
$$P_R v_- = v_- \quad \bar{v}_- P_L = \bar{v}_-$$
$$P_L v_+ = v_+ \quad \bar{v}_+ P_R = \bar{v}_+$$
Formalism

**gamma matrices**

\[
\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}
\]

**chiral projectors**

\[
P_R = \frac{1 + \gamma_5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \gamma_\mu P_R = P_L \gamma_\mu \quad P_R u_+ = u_+ \quad \bar{u}_+ P_L = \bar{u}_+
\]

\[
P_L = \frac{1 - \gamma_5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad P_L u_- = u_- \quad \bar{u}_- P_R = \bar{u}_-
\]

\[
P_R v_- = v_- \quad \bar{v}_- P_L = \bar{v}_- \quad P_L v_+ = v_+ \quad \bar{v}_+ P_R = \bar{v}_+
\]

**terms violating helicity conservation drop out:**

\[
\mathcal{M}_1 = \bar{u}_+ (p_1) \gamma_\mu \gamma_\nu \gamma_\rho v_+ (p_2) \cdot \mathcal{M}_1^{\mu \nu \rho}
\]

\[
= \bar{u}_+ (p_1) P_L \gamma_\mu \gamma_\nu \gamma_\rho P_L v_+ (p_2) \cdot \mathcal{M}_1^{\mu \nu \rho}
\]

\[
= \bar{u}_+ (p_1) \gamma_\mu \gamma_\nu \gamma_\rho P_R P_L v_+ (p_2) \cdot \mathcal{M}_1^{\mu \nu \rho}
\]

\[
= 0
\]
Formalism

complex polarization vectors are defined by:

\[ \epsilon_{\pm}(p) \cdot p = 0 \]
\[ \epsilon_{\pm} \cdot \epsilon_{\pm}^* = -1 \]
\[ \epsilon_{\pm} \cdot \epsilon_{\mp} = -1 \]

fix polarization with:

\[ n \cdot \epsilon_{\pm}(p, n) = 0, \quad n^2 = 0 \]

polarization sum:

\[ \sum_{\lambda=\pm} \epsilon_{\lambda}^\mu(p) \epsilon_{\lambda}^\nu(p)^* = -g^{\mu\nu} + \frac{p_\mu n_\nu + n_\mu p_\nu}{p \cdot n} \]

write polarization vectors with spinors:

\[ \epsilon_{\pm}^\mu(p, n) = \frac{\overline{u}_{\pm}(p) \gamma^\mu v_-(n)}{\sqrt{2} \overline{v}_+(n) u_+(p)}, \quad \epsilon_{\mp}^\mu(p, n) = \frac{\overline{u}_-(p) \gamma^\mu v_+(n)}{\sqrt{2} \overline{v}_-(n) u_-(p)} \]
$e^+ e^- \rightarrow b \bar{b} W^+ W^-$:

1922 terms of the form:

Trace method

Number of additions/subtractions/divisions/multiplications: 30000
\( e^+ e^- \rightarrow b \bar{b} W^+ W^- : \)

1922 terms of the form:

number of additions/subtractions/divisions/multiplications: 30000

1333 terms of the form:

\[
\text{amplitude}[33][18] = A46*gZbL*(SC5*SC64 - 4.*L2a*(c33*Rlb - R13*R3b)*s17*sqrt2);
\]

number of additions/subtractions/divisions/multiplications: 11

400 subexpressions of the form:

\[ SC64 = (-2.)*(c39*L13 + c19*L1b*R3b)/L13; \]
Phase Space Integration
\[
d\sigma = \left( \prod_{i=1}^{n} \frac{1}{(2\pi)^3} \frac{d^3p_i}{2E_i} \right) \cdot (2\pi)^4 \delta^{(4)}(p_a + p_b - p_1 - \cdots - p_n) \cdot |\mathcal{M}|^2
\]

- for \( n > 3 \) not integrable analytically
- integrate numerically instead
- only option because of high dimensions: Monte Carlo integration
Want to integrate $f(x)$ between $a = 0$ and $b = 10$:

$$\int_a^b dx \, f(x) = ?$$
Monte Carlo Integration

Want to integrate $f(x)$ between $a = 0$ and $b = 10$:

$$\int_{a}^{b} dx \ f(x) = ?$$
First estimate:

\[ \int_a^b dx \, f(x) \approx (b - a) \cdot f(x_0) \]
Better estimate: take average \( \bar{f} = \frac{1}{N} \sum_{i=1}^{N} (b - a) \cdot f(x_i) \)
Take average:

\[
\bar{f}_N = \frac{1}{N} \sum_{i=1}^{N} (b - a) \cdot f(x_i)
\]

\[
I = \int_{a}^{b} dx \ f(x)
\]

\[
I \approx \bar{f}_N
\]

Variance of \( f(x) \):

\[
S_f^2 = \frac{1}{N - 1} \sum_{i=1}^{N} \left( \bar{f}_N - f(x_i) \right)^2
\]

\[
\sigma_f^2 = \int_{a}^{b} dx \ (I - f(x))^2
\]

\[
\sigma_f^2 \approx S_f^2
\]
Take average:

$$\bar{f}_N = \frac{1}{N} \sum_{i=1}^{N} (b - a) \cdot f(x_i)$$

$$I = \int_{a}^{b} dx \, f(x)$$

$$I \approx \bar{f}_N$$

Variance of $f(x)$:

$$S_f^2 = \frac{1}{N - 1} \sum_{i=1}^{N} \left( \bar{f}_N - f(x_i) \right)^2$$

$$\sigma_f^2 = \int_{a}^{b} dx \, (I - f(x))^2$$

$$\sigma_f^2 \approx S_f^2$$
back to the example from before:

What is the error of a Monte Carlo evaluation with 100 points?
back to the example from before:

What is the error of a Monte Carlo evaluation with 100 points?

1) perform integration many times to get error from Gauss distribution

exact (analytical) integration result
back to the example from before:

What is the error of a Monte Carlo evaluation with 100 points?

1) perform integration many times to get error from Gauss distribution

histogram of 10 000 evaluations of $\bar{f}_{100}$
back to the example from before:

What is the error of a Monte Carlo evaluation with 100 points?

1) perform integration many times to get error from Gauss distribution

or

2) calculate variance directly

\[ \sigma^2[\bar{f}_N] = \sigma^2 \left[ \frac{(b-a)}{N} \sum_{i=1}^{N} f(x_i) \right] = \frac{(b-a)^2}{N^2} \cdot N \sigma^2[f(x)] \]

\[ \sigma[\bar{f}_N] = \frac{\sigma_f}{\sqrt{N}} (b-a) \]
\[ d\sigma = \left( \prod_{i=1}^{n} \frac{1}{(2\pi)^3} \frac{d^3 p_i}{2E_i} \right) \cdot (2\pi)^4 \delta^{(4)}(p_a + p_b - p_1 - \cdots - p_n) \cdot |\mathcal{M}|^2 \]

- how to satisfy energy-momentum conservation?
option 1: Rambo [Kleiss, Stirling 1986]

- produce $n$ massless 4-momenta
  (energy from finite distribution and momentum in random direction)
- Lorentz transform all momenta to CM frame
- rescale momenta to obtain correct CM energy
- transform to massive momenta
option 1: Rambo [Kleiss, Stirling 1986]

- produce $n$ massless 4-momenta
  (energy from finite distribution and momentum in random direction)
- Lorentz transform all momenta to CM frame
- rescale momenta to obtain correct CM energy
- transform to massive momenta

momenta are evenly distributed for fixed energy
option 2: sequential phase space

- rewrite $n$ particle phase space into 2-particle phase spaces

2-particle phase space known

\[ \int d\Pi_2(P, p_1, p_2) = \frac{1}{(2\pi)^2} \frac{\sqrt{\lambda(P_0^2, m_1^2, m_2^2)}}{8P_0^2} \int d\cos \theta \, d\phi \]

rewrite $n$-particle phase space into 2-particle phase spaces

\[ \int d\Pi_3(P, p_1, p_2, p_3) = \int d\Pi_2(P, p_{12}, p_3) \, d\Pi_2(p_{12}, p_1, p_2) \, dp_{12}^2 \]
option 2: sequential phase space

- rewrite $n$ particle phase space into 2-particle phase spaces

\[
\int d\Pi_2(P, p_1, p_2) = \frac{1}{(2\pi)^2} \frac{\sqrt{\lambda(P_0^2, m_1^2, m_2^2)}}{8P_0^2} \int d\cos\theta\ d\phi
\]

rewrite $n$-particle phase space into 2-particle phase spaces

\[
\int d\Pi_3(P, p_1, p_2, p_3) = \int d\Pi_2(P, p_{12}, p_3) d\Pi_2(p_{12}, p_1, p_2) dp_{12}^2
\]

\[
= \int d\Pi_2(P, p_1, p_{23}) d\Pi_2(p_{23}, p_2, p_3) dp_{23}^2
\]
option 2: sequential phase space

- rewrite \( n \) particle phase space into 2-particle phase spaces

\[
\int d\Pi_2(P, p_1, p_2) = \frac{1}{(2\pi)^2} \frac{\sqrt{\lambda(P_0^2, m_1^2, m_2^2)}}{8P_0^2} \int d\cos \theta \, d\phi
\]

rewrite \( n \)-particle phase space into 2-particle phase spaces

\[
\int d\Pi_3(P, p_1, p_2, p_3) = \int d\Pi_2(P, p_{12}, p_3) \, d\Pi_2(p_{12}, p_1, p_2) \, dp_{12}^2 = \int d\Pi_2(P, p_1, p_{23}) \, d\Pi_2(p_{23}, p_2, p_3) \, dp_{23}^2
\]

example:

\[
\int d\Pi_4(P, p_b, p_{b\gamma}, p_{W^+}, p_{W^-}) = \int d\Pi_2(P, p_t, p_{\ell}) \, d\Pi_2(p_{\ell}, p_b, p_{W^+}) \, d\Pi_2(p_{\ell}, p_b, p_{W^-}) \, dp_{\ell}^2 \, dp_b^2 = \int (d\cos \theta_t \, d\phi_t) \, (d\cos \theta_b \, d\phi_b) \, (d\cos \theta_b \, d\phi_b) \, dp_{\ell}^2 \, dp_b^2 \cdot (\ldots)
\]
Monte Carlo integration error for \(e^+e^- \rightarrow b\bar{b} W^+W^-\)

can we do better?
Where does the large and unstable integration error come from?

- the large function variance leads to a large Monte Carlo integration error
example:

- the large function variance leads to a large Monte Carlo integration error
- it comes from the top quark propagator, which is large for $p_t^2 \approx m_t^2$:

$$\left| \frac{1}{p_t^2 - m_t^2 + i \Gamma_t m_t} \right|^2 = \frac{1}{(p_t^2 - m_t^2)^2 + m_t^2 \Gamma_t^2}$$
Improving Precision: Variable Transformations

Basic Principle: Do a variable transformation to get a function with smaller variance.

$$\int dx \ f(x) = \int dy \ f(y) \cdot (dx(y)/dy) = \int dy \ g(y)$$

![Graphs showing large and small function variance](image)

- large function variance • large Monte Carlo integration error
- small function variance • small Monte Carlo integration error
**Improving Precision: Variable Transformations**

Basic Principle: Do a variable transformation to get a function with smaller variance.

\[
\int dx \, f(x) = \int dy \, f(y) \cdot \frac{dx(y)}{dy}
\]

\[
= \int dy \, g(y)
\]

method 1: function known (example resonant propagators)
method 2: function unknown (Vegas)
method 1: function known

example: resonant propagator \( f(p^2) = \frac{1}{(p^2 - m^2)^2 + m^2 \Gamma^2} \)

variable transformation:

\[
\int dp^2 \ f(p^2) = \int d\theta \ f(p^2(\theta)) \cdot (dp^2(\theta)/d\theta)
= \int d\theta \ \frac{(p^2(\theta) - m^2)^2 + \Gamma^2 m^2}{\Gamma m} f(p^2(\theta))
= \int d\theta \ \frac{1}{\Gamma m}
\]

\[\theta(p^2) = \arctan \left( \frac{p^2 - m^2}{\Gamma m} \right)\]
\[p^2(\theta) = m^2 + \Gamma m \cdot \tan(\theta)\]
method 1: function known

example: resonant propagator \[ f(p^2) = \frac{1}{(p^2-m^2)^2+m^2\Gamma^2} \]

variable transformation:

\[
\int dp^2 \ f(p^2) = \int d\theta \ f(p^2(\theta)) \cdot (dp^2(\theta)/d\theta) \\
= \int d\theta \ \frac{(p^2(\theta) - m^2)^2 + \Gamma^2m^2}{\Gamma m} \ f(p^2(\theta)) \\
= \int d\theta \ \frac{1}{\Gamma m}
\]

\[ \theta(p^2) = \arctan \left( \frac{p^2 - m^2}{\Gamma m} \right) \]

\[ p^2(\theta) = m^2 + \Gamma m \cdot \tan(\theta) \]
method 1: function known

example: \( e^+ e^- \rightarrow b\bar{b} W^+ W^- \)
method 1: function known

example: \( e^+ e^- \rightarrow b \bar{b} W^+ W^- \)
method 2: function unknown (Vegas)

example: 1) separate integration region into 2 intervals of different size

\[ \int_0^1 dx f(x) = \int_0^{x_1} dx f(x) + \int_{x_1}^1 dx f(x) \]
method 2: function unknown (Vegas)

example:  
1) separate integration region into 2 intervals of different size  
2) make intervals of same size by stretching one interval and compressing the other

\[ \int_0^1 dx \, f(x) = \int_0^{x_1} dx \, f(x) + \int_{x_1}^1 dx \, f(x) \]
method 2: function unknown (Vegas)

example: 
1) separate integration region into 2 intervals of different size
2) make intervals of same size by stretching one interval and compressing the other
3) add Jacobian factor

variable transformation: 
\[ x(y) = \begin{cases} 
0 + \Delta x_0 \cdot 2y & 0 < y < 1/2 \\
x_1 + \Delta x_1 \cdot (2y - 1) & 1/2 < y < 1 
\end{cases} \quad \Delta x_0 = x_1 - 0, \quad \Delta x_1 = 1 - x_1 \]

\[
\int_0^1 dx \, f(x) = \int_0^{x_1} dx \, f(x) + \int_{x_1}^1 dx \, f(x) = \int_0^{1/2} dy \, f(x(y)) \cdot J(y) + \int_{1/2}^1 dy \, f(x(y)) \cdot J(y)
\]
method 2: function unknown (Vegas)

example:  
1) separate integration region into 2 intervals of different size
2) make intervals of same size by stretching one interval and compressing the other
3) add Jacobian factor

variable transformation: 
\[ x(y) = \begin{cases} 
0 + \Delta x_0 \cdot 2y & 0 < y < 1/2 \\
x_1 + \Delta x_1 \cdot (2y - 1) & 1/2 < y < 1 ,
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in practice:

- take $N$ intervals
- calculate contribution to variance in each interval
- change size of interval accordingly
- repeat
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\[ e^+ e^- \rightarrow b \bar{b} \ W^+ W^- \]
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in practice:

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\[ e^+ e^- \rightarrow b \bar{b} W^+ W^- \]

caveat: cannot adapt to diagonal structures
• discussed how to use the structure of one diagram for integration

• what about all other 61 diagram structures?
separate into parts, where the resonant propagators are coming mainly from one diagram:

\[
\int d\Pi |\mathcal{M}_1 + \cdots + \mathcal{M}_n|^2 = \int d\Pi |\mathcal{M}|^2
= \int d\Pi |\mathcal{M}|^2 \frac{|\mathcal{M}_1|^2 + \cdots + |\mathcal{M}_n|^2}{|\mathcal{M}_1|^2 + \cdots + |\mathcal{M}_n|^2}
= \int d\Pi^{(1)} |\mathcal{M}_1|^2 \frac{|\mathcal{M}|^2}{\sum_{i=1}^n |\mathcal{M}_i|^2} + \cdots + \int d\Pi^{(n)} |\mathcal{M}_n|^2 \frac{|\mathcal{M}|^2}{\sum_{i=1}^n |\mathcal{M}_i|^2}
\]
separate into parts, where the resonant propagators are coming mainly from one diagram:

\[
\int d\Pi |M_1 + \cdots + M_n|^2 = \int d\Pi |M|^2 \\
= \int d\Pi |M|^2 \frac{|M_1|^2 + \cdots + |M_n|^2}{|M_1|^2 + \cdots + |M_n|^2} \\
= \int d\Pi^{(1)} |M_1|^2 \frac{|M|^2}{\sum_{i=1}^n |M_i|^2} + \cdots + \int d\Pi^{(n)} |M_n|^2 \frac{|M|^2}{\sum_{i=1}^n |M_i|^2}
\]

adapt phase space structures to diagrams ("integration channels"):

\[
\int d\Pi^{(1)} = \int d\Pi_2(P, p_t, p_f) \ d\Pi_2(p_t, p_b, p_{W^+}) \ d\Pi_2(p_{W^+} p_{W^-}) \ dp_t^2 \ dp_{W^+}^2 \\

\int d\Pi^{(n)} = \int d\Pi_2(P, p_{W^-}, p_{W_{\text{intermediate}}}) \ d\Pi_2(p_{W_{\text{intermediate}}}, p_{W^+}, p_Z) \ d\Pi_2(p_Z, p_b, p_{\bar{b}}) \ dp_{W_{\text{intermediate}}}^2 \ dp_{Z}^2
\]
multiply with factors $\alpha_1, \ldots, \alpha_n$ (= “channel weights”)

\[
\int d\Pi |M_1 + \cdots + M_n|^2 = \int d\Pi |M|^2 \\
= \int d\Pi |M|^2 \frac{\alpha_1 |M_1|^2 + \cdots + \alpha_n |M_n|^2}{\alpha_1 |M_1|^2 + \cdots + \alpha_n |M_n|^2} \\
= \alpha_1 \int d\Pi^{(1)} |M_1|^2 \frac{|M|^2}{\sum_{i=1}^{n} \alpha_i |M_i|^2} + \cdots + \alpha_n \int d\Pi^{(n)} |M_n|^2 \frac{|M|^2}{\sum_{i=1}^{n} \alpha_i |M_i|^2}
\]

→ channel weights are adapted to minimize the integration error
Comparison of methods and generators:
(each point evaluated with 300 000 phase space points)

\[ e^+e^- \rightarrow b\bar{b}W^+W^- \]
Project

- want to improve Monte Carlo event generation for top quark production:
  - NLL threshold corrections for the differential cross section for top quark pair production at the threshold
  - parton showers off intermediate top quarks
  - ....

- we are building an NLO Monte Carlo for $e^+e^- \rightarrow b\bar{b}W^+W^-$ with resonant subtraction
Conclusion

- Matrix elements can be efficiently calculated with helicity amplitudes.
- Phase space integration usually uses multiple integration channels.
- Each channel uses variable transformations to reduce the integration error.
- We have finished a LO C++ Monte Carlo for $e^+e^- \rightarrow b\bar{b}W^+W^-$ and are now building an NLO code with resonant subtraction.
Conclusion

- matrix elements can be efficiently calculated with helicity amplitudes
- phase space integration usually uses multiple integration channels
- each channel uses variable transformations to reduce the integration error
- we have finished a LO C++ Monte Carlo for $e^+e^- \rightarrow b \bar{b} W^+W^-$ and are now building an NLO code with resonant subtraction

Thank you!