The Renormalization-Group Improved Effective Potential

Aneesh Manohar
(jointly with Emily Nardoni)

23 Mar 2021

Coleman and Weinberg (1973)

Two-loop computation of the CW potential

Ford and Jones (1992), Martin (2002)

RG summation

Bando, Kugo, Maekawa, Nakano (1992)

Casas, Di Clemente, Quiros (1998)

Iso and Kawana (2018)

Okane (2019)

Other approaches

Einhorn and Jones (1984)

Ford (1994)

Nakano and Yoshida (1994)

Many other papers

This talk: E. Nardoni, AM arXiv:2010.15806

Effective Action

The effective action $\Gamma[\hat{\varphi}]$ is an important quantity in QFT

- Determines the ground state of the quantum theory
- Determines the vacuum energy density
- Generates one-particle irreducible diagrams

Formalism developed in Coleman and Weinberg (1973)

To determine the ground state, need to compute $\Gamma[\hat{\varphi}]$ in a systematic expansion, and sum the large logarithms.

Coleman and Weinberg (1973):
Abelian Higgs model with $m_\phi^2 \sim \lambda v^2$, $m_A^2 \sim e^2 v^2$

RG improve by evolving to a scale $\mu \sim v$. Example where quantum effects lead to symmetry breaking.

Logarithms of the form

$$\lambda \ln \frac{m^2}{\mu^2}$$

lead to a breakdown of fixed order perturbation theory. Solution is to pick $\mu \sim m$.

In a multi-scale problem, cannot pick a single μ that works for all masses

$$\lambda \ln \frac{m_H^2}{\mu^2} \qquad \lambda \ln \frac{m_L^2}{\mu^2}$$

Construct a sequence of EFTs, and integrate out heavy scales

Surprisingly, this had not been done (correctly) so far.

$W[J]$

$$Z[J] = e^{\frac{i}{\hbar} W[J]} = \int \prod_a D\varphi^a \; e^{\frac{i}{\hbar}\left(S(\varphi) + \int \mathrm{d}^4x \, J\, \varphi\right)}\,, \qquad S(\varphi) = \int \mathrm{d}^4x \; \mathcal{L}(\varphi)$$

$W[J]$ is the generating function for all connected Feynman diagrams

Define the classical background field $\hat{\varphi}$ as the expectation value of the field φ in the presence of a source $J(x)$,

$$\hat{\varphi}(x) = \langle \varphi(x) \rangle_J = \frac{1}{Z[J]} \int D\varphi \; \varphi(x) \; e^{\frac{i}{\hbar}\left(S(\varphi) + \int \mathrm{d}^4x \, J\, \varphi\right)} = \frac{\delta W[J]}{\delta J(x)}\,.$$

When $J = 0$,

$$\hat{\varphi}(x)|_{J=0} \equiv \hat{\varphi}_0$$

is the expectation value of φ in the true vacuum of the theory.

Effective Action: $\Gamma[\hat{\varphi}]$

The 1PI effective action is the Legendre transform of $W[J]$,

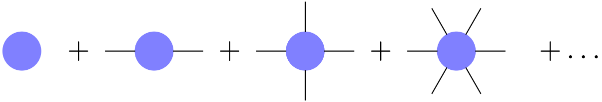
$$\Gamma[\hat{\varphi}] = W[J] - \int \mathrm{d}^4 x \; J(x)\hat{\varphi}(x) \; ,$$

Expand in derivatives:

$$\Gamma[\hat{\varphi}] = -\int \mathrm{d}^4 x \; V_{\rm CW}(\hat{\varphi}) + \int \mathrm{d}^4 x \; \frac{1}{2} Z(\hat{\varphi})(\partial_\mu \hat{\varphi})^2 + \ldots$$

V_{CW} is the Coleman-Weinberg potential $V_{\rm CW}$

The effective action $\Gamma[\hat{\varphi}]$ is given by the sum of 1PI graphs



with $\hat{\varphi}$ on the external lines.

Effective Action

$$\frac{\delta\Gamma[\hat{\varphi}]}{\delta\hat{\varphi}(x)} = -J(x) \quad \Rightarrow \quad \left.\frac{\delta\Gamma[\hat{\varphi}]}{\delta\hat{\varphi}(x)}\right|_{J=0} = 0\,.$$

For constant $\hat{\varphi}$,

$$\left.\frac{\partial V_{\rm CW}(\hat{\varphi})}{\partial\hat{\varphi}}\right|_{\hat{\varphi}_0} = 0\,.$$

- $\hat{\varphi}_0$ is $\langle\varphi\rangle$ in the true ground state with $J = 0$
- $V(\hat{\varphi}_0)$ is the vacuum energy density of the ground state

Coleman-Weinberg

S. Coleman and E. Weinberg, Phys. Rev. D7 (1973) 1888

- Set up the formalism to compute V_{CW}
- Explained the physical significance of V_{CW}
- Computed V_{CW} for $\lambda\phi^4$ theory and the Abelian Higgs model
- Showed how to RG improve V_{CW}
- Wrote down the general one-loop formula for V_{CW}

$$V_{\text{CW}}(\hat{\varphi}) = V_{\text{tree}}(\hat{\varphi}) + \hbar V_{\text{1-loop}}(\hat{\varphi}) + \mathcal{O}(\hbar^2)$$

$$V_{\text{tree}} = V(\hat{\varphi}) + \Lambda$$

Coleman-Weinberg

In the $\overline{\text{MS}}$ scheme:

$$V_{\text{1-loop}} = \frac{1}{64\pi^2}\left\{ \text{Tr}\, W^2 \left[\ln \frac{W}{\mu^2} - \frac{3}{2} \right] - 2\,\text{Tr}\, \left(M_F^\dagger M_F \right)^2 \left[\ln \frac{M_F^\dagger M_F}{\mu^2} - \frac{3}{2} \right] \right.$$
$$\left. + 3\text{Tr}\, M_V^4 \left[\ln \frac{M_V^2}{\mu^2} - \frac{5}{6} \right] \right\} ,$$

Sum over real scalars, Weyl fermions, and real gauge bosons. W is the scalar mass matrix

$$W_{ab}(\hat{\varphi}) = \frac{\partial^2 V(\hat{\varphi})}{\partial \varphi^a \partial \varphi^b}$$

$M_F(\hat{\varphi})$ and $M_V(\hat{\varphi})$ are the fermion and gauge boson mass matrices

Coleman-Weinberg

Logarithms of the form

$$\ln \frac{m^2}{\mu^2}$$

RG improvement by using a scale $\mu \sim m$.

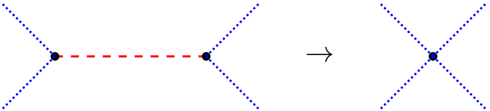
Coleman-Weinberg: RG improvement for a single-scale problem. Want to extend this to multi-scale problems.

Cannot choose a single μ to minimize all the logarithms.

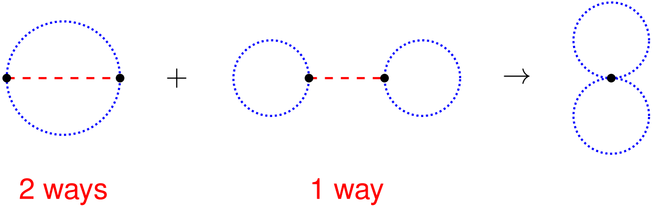
Typical EFT problem where one integrates out heavy particles and constructs a sequence of EFTs.

What is the problem?

Integrate out a heavy particle to get EFT interaction:



Look at loop diagrams contributing to CW potential (in a $\hat{\varphi}$ background)



The second diagram should not be part of the CW potential

Jackiw's method for computing $\Gamma[\hat{\varphi}]$

R. Jackiw, Phys. Rev. D9 (1974) 1686

In the functional integral, let $\varphi(x) = \hat{\varphi}(x) + \varphi_q(x)$

$$Z[J] = e^{\frac{i}{\hbar} W[J]} = \int \prod_a D\varphi_q^a \; e^{\frac{i}{\hbar} \int \mathrm{d}^4 x \left[\mathcal{L}(\hat{\varphi}+\varphi_q) + J\, (\hat{\varphi}+\varphi_q) \right]} ,$$

so that

$$e^{\frac{i}{\hbar}\Gamma[\hat{\varphi}]} = e^{\frac{i}{\hbar}\left(W[J] - \int \mathrm{d}^4 x \, J\, \hat{\varphi}\right)} = \int \prod_a D\varphi_q^a \; e^{\frac{i}{\hbar} \int \mathrm{d}^4 x \left[\mathcal{L}(\hat{\varphi}+\varphi_q) + J\, \varphi_q \right]}$$

We need to adjust $J(x)$ so that

$$\langle \varphi(x) \rangle = \hat{\varphi}(x) \implies \boxed{\langle \varphi_q(x) \rangle = 0} \qquad \text{tadpole condition}$$

Jackiw's method

Wirte φ as the sum of a background (classical) and quantum field:

$$\varphi(x) = \hat{\varphi}(x) + \varphi_q(x)$$

$$\begin{aligned}\mathcal{L}(\hat{\varphi} + \varphi_q) = &\frac{1}{2}(\partial_\mu \varphi_q)^2 - \hat{\Lambda}(\hat{\varphi}) - \hat{\sigma}(\hat{\varphi})\varphi_q - \frac{1}{2}m^2(\hat{\varphi})\hat{\varphi}_q^2 \\ &- \frac{1}{6}\hat{\rho}(\hat{\varphi})\varphi_q^3 - \frac{1}{24}\hat{\lambda}(\hat{\varphi})\varphi_q^4 + \ldots\end{aligned}$$

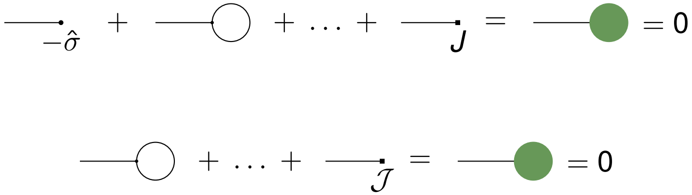
- $\Lambda, \sigma, m^2, \rho, \lambda$, etc. are couplings before the shift
- $\hat{\Lambda}$, etc. are couplings after the shift and depend on $\hat{\varphi}(x)$
- $\hat{\Lambda}(\hat{\varphi}) = V(\hat{\varphi}) + \Lambda$ is the cosmological constant after the shift.

Use a source

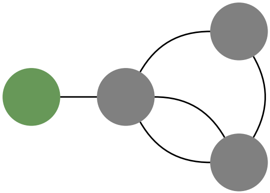
$$J = \hat{\sigma} + \mathcal{J}$$

with $\mathcal{J}$ formally of one-loop order.

Tadpole Condition



This is the same as the 1PI condition



and is another way of proving that $\Gamma[\hat{\varphi}]$ is the sum of 1PI graphs.

Procedure

Use

$$\hat{\mathcal{L}} \equiv \mathcal{L}(\varphi_q + \hat{\varphi}) + J\varphi_q$$
$$= \frac{1}{2}(\partial_\mu \varphi_q)^2 - \hat{\Lambda}(\hat{\varphi}) + \mathcal{J}\varphi_q - \frac{1}{2}m^2(\hat{\varphi})\hat{\varphi}_q^2 - \frac{1}{6}\hat{\rho}(\hat{\varphi})\varphi_q^3 - \frac{1}{24}\hat{\lambda}(\hat{\varphi})\varphi_q^4 + \ldots$$

and treat $\mathcal{J}$ as formally of one-loop order.

- Shift the field
- Drop the linear term in φ_q in $\mathcal{L}$
- Add a source $\mathcal{J}$ for the quantum field

β-functions determined by consistency

$$\left[\frac{\partial}{\partial t} + \beta_i \frac{\partial}{\partial \lambda_i} - \gamma_\varphi \varphi \frac{\partial}{\partial \varphi}\right] V_{\rm CW} = 0\,.$$

λ, m^2, etc. denoted generically as $\{\lambda_i\}$, and satisfy the β-function equations

$$\frac{\mathrm{d}\lambda_i}{\mathrm{d}t} = \beta_i(\{\lambda_i\})\,,$$

where

$$t \equiv \frac{1}{16\pi^2} \ln \frac{\mu}{\mu_0}\,,$$

γ_φ is the anomalous dimension of φ,

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = -\gamma_\varphi\, \varphi\,.$$

Example: Higgs-Yukawa Model

Higgs-Yukawa Model:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + \sum_{k=1}^{N_F} i\,\overline{\psi}_k\, \partial\!\!\!/\, \psi_k - (m_F + g\varphi)\overline{\psi}_k \psi_k - V(\varphi)$$

with scalar potential

$$V(\varphi) = \Lambda + \sigma\varphi + \frac{m_B^2}{2}\varphi^2 + \frac{\rho}{6}\varphi^3 + \frac{\lambda}{24}\varphi^4\,.$$

$$W(\varphi) = \frac{\partial^2 V}{\partial \varphi^2} = m_B^2 + \rho\varphi + \frac{1}{2}\lambda\varphi^2\,, \qquad M_F(\varphi) = m_F + g\varphi\,,$$

are the scalar and Dirac fermion mass matrices.

β-functions from One-loop Potential

$$\left[\beta_i \frac{\partial}{\partial g_i} - \gamma_\varphi \varphi \frac{\partial}{\partial \varphi}\right] V_{\text{tree}} + \frac{\partial}{\partial t} V_{1-\text{loop}} = 0$$

$$\implies \left[\beta_i \frac{\partial}{\partial g_i} - \gamma_\varphi \varphi \frac{\partial}{\partial \varphi}\right] V_{\text{tree}} = \frac{1}{2} \text{Tr}\, W^2 - \text{Tr} \left(M_F^\dagger M_F\right)^2 + \frac{3}{2} \text{Tr}\, M_V^4 \,.$$

Equating powers of φ gives one-loop β-functions:

$$\begin{aligned}
\beta_\Lambda &= \frac{1}{2} m_B^4 - 2 N_F m_F^4 \,, \\
\beta_\sigma &= m_B^2 \rho - 8 N_F g m_F^3 + \gamma_\varphi \sigma \,, \\
\beta_{m_B^2} &= \lambda m_B^2 + \rho^2 - 24 N_F g^2 m_F^2 + 2 \gamma_\varphi m_B^2 \,, \\
\beta_\rho &= 3 \lambda \rho - 48 N_F g^3 m_F + 3 \gamma_\varphi \rho \,, \\
\beta_\lambda &= 3 \lambda^2 - 48 N_F g^4 + 4 \gamma_\varphi \lambda \,.
\end{aligned}$$

Can use this for higher dimension operators such as $(H^\dagger H)^3$ in SMEFT

Shifted Lagrangian

Shifted parameters after $\varphi = \hat{\varphi} + \varphi_q$

$$
\begin{aligned}
\hat{m}_F &= m_F + g\hat{\varphi}\,, \\
\hat{g} &= g\,, \\
\hat{\Lambda} &= \Lambda + \sigma\hat{\varphi} + \frac{1}{2}m_B^2\hat{\varphi}^2 + \frac{1}{6}\rho\hat{\varphi}^3 + \frac{1}{24}\lambda\hat{\varphi}^4\,, \\
\hat{\sigma} &= \sigma + m_B^2\hat{\varphi} + \frac{1}{2}\rho\hat{\varphi}^2 + \frac{1}{6}\lambda\hat{\varphi}^3\,, \\
\hat{m}_B^2 &= m_B^2 + \rho\hat{\varphi} + \frac{1}{2}\lambda\hat{\varphi}^2\,, \\
\hat{\rho} &= \rho + \lambda\hat{\varphi}\,, \\
\hat{\lambda} &= \lambda\,.
\end{aligned}
$$

β-functions for the hatted couplings are given by the same functions

$$
\dot{\lambda}_i = \beta_i(\{\lambda_i\}) \qquad\qquad \dot{\hat{\lambda}}_i = \beta_i(\{\hat{\lambda}_i\})
$$

Constraint on β-functions

Shift invariance:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\phi\\ \psi\\ \Lambda\\ \sigma\\ m_B^2\\ \rho\\ \lambda\\ m_F\\ g\end{bmatrix} = c_1\begin{bmatrix}0\\ 0\\ \frac{1}{2}m_B^4\\ m_B^2\rho\\ \lambda m_B^2+\rho^2\\ 3\lambda\rho\\ 3\lambda^2\\ 0\\ 0\end{bmatrix} + c_2\begin{bmatrix}0\\ 0\\ m_F^4\\ 4gm_F^3\\ 12g^2m_F^2\\ 24g^3m_F\\ 24g^4\\ 0\\ 0\end{bmatrix} + c_3\begin{bmatrix}0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ g^2m_F\\ g^3\end{bmatrix} + \begin{bmatrix}\gamma_\phi\\ \gamma_\psi\\ 0\\ \gamma_\phi\sigma\\ 2\gamma_\phi m_B^2\\ 3\gamma_\phi\rho\\ 4\gamma_\phi\lambda\\ 2\gamma_\psi m_F\\ (\gamma_\phi+2\gamma_\psi)\,g\end{bmatrix}$$

The β-functions are given in terms of $c_{1,2,3}$ and $\gamma_{\phi,\psi}$.

Strongly constrains the form of the β-functions

RG Improvement

$$
\begin{array}{c}
 \\ \\ V = \\ \\ \\
\end{array}
\begin{array}{c}
\begin{array}{ccc} \text{LL} & \text{NLL} & \text{NNLL} \end{array} \\
\begin{pmatrix}
\boxed{1} & & \\
\boxed{\lambda \text{L}} & \boxed{\lambda} & \\
\boxed{\lambda^2 \text{L}^2} & \lambda^2 \text{L} & \lambda^2 \\
\boxed{\lambda^3 \text{L}^3} & \lambda^3 \text{L}^2 & \ldots \\
\vdots & &
\end{pmatrix}
\end{array}
\begin{array}{l}
\\ \text{tree} \\ \text{1-loop} \\ \text{2-loop} \\ \text{3-loop} \\ \\
\end{array}
$$

RG improvement sums the λL series because the β-functions being integrated do not contain logarithms.

We can check our method because the explicit two-loop results are known.

LL series:	tree-level matching and one-loop running
LL + NLL series:	one-loop matching and two-loop running

We will compute the boxed terms to compare with known results.

Requires in addition to the LL terms, the one-loop matching for V_{CW}.

The two-loop $(\lambda L)^2$ term is obtained by integrating one-loop results, and is a non-trivial check of the method

V_{CW} is the cosmological constant after everything has been integrated out.

Example: Two Scalar Fields in the Broken Phase

Consider a theory with two scales

- χ_q the heavy field
- ϕ_q the light field
- $v_\chi \gg v_\phi$
- $z \ll 1 \sim v_\phi / v_\chi$ is the expansion parameter

$$
\begin{aligned}
\mathcal{L} = & \frac{1}{2}(\partial_\mu \chi)^2 + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda_\phi}{24}(\phi^2 - v_\phi^2)^2 - \frac{\lambda_\chi}{24}(\chi^2 - v_\chi^2)^2 \\
& - \frac{\lambda_3}{4}(\phi^2 - v_\phi^2)(\chi^2 - v_\chi^2) - \Lambda \,,
\end{aligned}
$$

Compute $V(\hat{\chi}, \hat{\phi})$, where $\hat{\chi} = \langle \chi(\mu_0) \rangle$, etc.

Work to order z^4.

$$\mathcal{L}(\phi, \chi, \mu_0) \qquad \lambda_i(\mu_0),\ \Lambda(\mu_0)$$

$\lambda(\mu_0)$

$\downarrow$ $\phi = \hat{\phi} + \phi_q$, $\chi = \hat{\chi} + \chi_q$, drop linear

$$\hat{\mathcal{L}}(\phi_q, \chi_q, \mu_0) \qquad \hat{\lambda}_i(\hat{\phi}, \hat{\chi}, \mu_0),\ \hat{\Lambda}(\hat{\varphi}, \hat{\chi}, \mu_0)$$

$\hat{\lambda}(\mu_0)$

$\downarrow$ $\mu_0 \to \mu_H$

$$\hat{\mathcal{L}}(\phi_q, \chi_q, \mu_H) \qquad \hat{\lambda}_i(\hat{\phi}, \hat{\chi}, \mu_H),\ \hat{\Lambda}(\hat{\phi}, \hat{\chi}, \mu_H)$$

$\hat{\lambda}(\mu_H)$

$\downarrow$ integrate out χ_q

$$\mathcal{L}_{\text{EFT}}(\phi_q, \mu_H) \qquad \widetilde{\lambda}_i(\hat{\phi}, \hat{\chi}, \mu_H),\ \widetilde{\Lambda}(\hat{\phi}, \hat{\chi}, \mu_H)$$

$\widetilde{\lambda}(\mu_H)$

$\downarrow$ $\mu_H \to \mu_L$

$$\mathcal{L}_{\text{EFT}}(\phi_q, \mu_L) \qquad \widetilde{\lambda}_i(\hat{\phi}, \hat{\chi}, \mu_L),\ \widetilde{\Lambda}(\hat{\phi}, \hat{\chi}, \mu_L)$$

$\widetilde{\lambda}(\mu_L)$

$\downarrow$ integrate out ϕ_q

$$\langle \chi_q \rangle \overset{!}{=} 0 \longrightarrow V_{\text{CW}}(\hat{\phi}, \hat{\chi}, \mu_L)$$

all fields integrated out

Mass Hierarchy

$$W_{\chi\chi} = \underbrace{\frac{1}{3}\lambda_\chi v_\chi^2}_{\boxed{1}} + \underbrace{\frac{1}{2}\lambda_\chi\left(\hat{\chi}^2 - v_\chi^2\right) + \frac{1}{2}\lambda_3\left(\hat{\phi}^2 - v_\phi^2\right)}_{\boxed{z^2}},$$

$$W_{\phi\phi} = \underbrace{\frac{1}{6}\lambda_\phi\left(3\hat{\phi}^2 - v_\phi^2\right) + \frac{1}{2}\lambda_3\left(\hat{\chi}^2 - v_\chi^2\right)}_{\boxed{z^2}},$$

$$W_{\phi\chi} = W_{\chi\phi} = \underbrace{\lambda_3\hat{\phi}\hat{\chi}}_{\boxed{z}}.$$

For a mass hierarchy:

$$\hat{\chi}^2 - v_\chi^2 \sim z^2, \qquad \hat{\phi}^2 \sim v_\phi^2 \sim z^2$$

so that

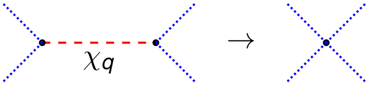
$$W_{\chi\chi} \sim 1, \qquad W_{\phi\phi} \sim z^2, \qquad W_{\phi\chi} = W_{\chi\phi} \sim z.$$

The cosmological constant is

$$\hat{\Lambda} = \frac{\lambda_\phi}{24}(\hat{\phi}^2 - v_\phi^2)^2 + \frac{\lambda_\chi}{24}(\hat{\chi}^2 - v_\chi^2)^2 + \frac{\lambda_3}{4}(\hat{\phi}^2 - v_\phi^2)(\hat{\chi}^2 - v_\chi^2) + \Lambda\,.$$

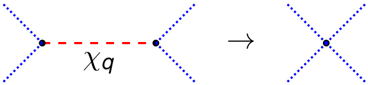
in shifted theory.

Tree-level matching by integrating out χ_q using shifted Lagrangian with sources $\mathcal{J}_\chi$ and $\mathcal{J}_\phi$:



Couplings in EFT after χ_q integrated out denoted $\widetilde{\lambda}$, etc.

EFT parameters depend on $\mathcal{J}_\phi$ and $\mathcal{J}_\chi$.



X_1 is the linear term in χ_q

$$\hat{\mathcal{L}} \supset -X_1(\phi_q)\chi_q \qquad X_1(\phi_q) = \lambda_3\hat{\chi}\hat{\phi}\phi_q + \frac{1}{2}\lambda_3\hat{\chi}\phi_q^2 - \mathcal{J}_\chi \,.$$

$X_1 \sim z^2$.

Integrating out χ_q gives a quadratic term in the EFT

$$\mathcal{L}_{\text{EFT}} \supset \frac{1}{2}X_1\frac{1}{W_{\chi\chi}-p^2}X_1 = \underbrace{\frac{1}{2}X_1\frac{1}{W_{\chi\chi}}X_1}_{z^4} + \underbrace{\frac{1}{2}X_1\frac{p^2}{W_{\chi\chi}^2}X_1}_{z^6} + \ldots$$

Compute V_{CW} to order z^4:

$$\widetilde{\lambda} \overset{\mu=\mu_H}{=} \lambda_\phi - \frac{9\lambda_3^2\hat{\chi}^2}{\lambda_\chi v_\chi^2} + \mathcal{O}(z^2)\,,$$

$$\widetilde{\rho} \overset{\mu=\mu_H}{=} \hat{\phi}\left[\lambda_\phi - \frac{9\lambda_3^2\hat{\chi}^2}{\lambda_\chi v_\chi^2}\right] + \mathcal{O}(z^3) = \widetilde{\lambda}\hat{\phi} + \mathcal{O}(z^3)\,,$$

$$\widetilde{m}^2 \overset{\mu=\mu_H}{=} \frac{1}{6}\lambda_\phi\left(3\hat{\phi}^2 - v_\phi^2\right) + \frac{1}{2}\lambda_3\left(\hat{\chi}^2 - v_\chi^2\right) - \frac{3\lambda_3^2\hat{\chi}^2\hat{\phi}^2}{\lambda_\chi v_\chi^2} + \frac{3\lambda_3\hat{\chi}\,\mathcal{J}_\chi}{\lambda_\chi v_\chi^2} + \mathcal{O}(z^4)\,,$$

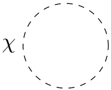
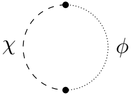
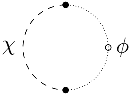
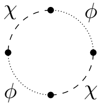
$$\equiv \widetilde{m}_-^2 + \frac{3\lambda_3\hat{\chi}\,\mathcal{J}_\chi}{\lambda_\chi v_\chi^2} + \mathcal{O}(z^4)$$

$$\widetilde{\sigma} \overset{\mu=\mu_H}{=} -\mathcal{J}_\phi + \frac{3\lambda_3\hat{\chi}\hat{\phi}\,\mathcal{J}_\chi}{\lambda_\chi v_\chi^2} + \mathcal{O}(z^5)$$

$$\widetilde{\Lambda} \overset{\mu=\mu_H}{=} \hat{\Lambda} + V_{\text{match}} - \frac{3\left(\mathcal{J}_\chi\right)^2}{2\lambda_\chi v_\chi^2} + \mathcal{O}(z^6)$$

EFT couplings depend on the sources $\mathcal{J}_\chi$ and $\mathcal{J}_\phi$ which have not yet been determined

V_{match}

	Graph	S	Integrand	V
1		$\frac{1}{2}$	$-\ln(p^2 - W_{\chi\chi})$	$\frac{1}{32\pi^2} W_{\chi\chi}^2 \left[-\frac{1}{2\epsilon} + \frac{1}{2}\ln\frac{W_{\chi\chi}}{\mu^2} - \frac{3}{4}\right]$
z^2		$\frac{1}{2}$	$\frac{W_{\chi\phi}W_{\phi\chi}}{(p^2 - W_{\chi\chi})p^2}$	$\frac{1}{32\pi^2} W_{\chi\phi}W_{\phi\chi}\left[-\frac{1}{\epsilon} + \ln\frac{W_{\chi\chi}}{\mu^2} - 1\right]$
z^4		$\frac{1}{2}$	$\frac{W_{\chi\phi}W_{\phi\chi}}{(p^2 - W_{\chi\chi})p^2}\frac{W_{\phi\phi}}{p^2}$	$\frac{1}{32\pi^2}\frac{W_{\chi\phi}W_{\phi\chi}W_{\phi\phi}}{W_{\chi\chi}}\left[-\frac{1}{\epsilon} + \ln\frac{W_{\chi\chi}}{\mu^2} - 1\right]$
z^4		$\frac{1}{4}$	$\frac{W_{\chi\phi}^2 W_{\phi\chi}^2}{(p^2 - W_{\chi\chi})^2 p^4}$	$\frac{1}{32\pi^2}\frac{W_{\chi\phi}^2 W_{\phi\chi}^2}{W_{\chi\chi}^2}\left[\frac{1}{2\epsilon} - \frac{1}{2}\ln\frac{W_{\chi\chi}}{\mu^2} + 1\right]$

Full theory graphs expanded in the low scale, i.e. in z

$$V_{\text{match}}(\mu_H) = \frac{1}{64\pi^2}\Bigg\{ \left[\ln\frac{W_{\chi\chi}}{\mu_H^2} - \frac{3}{2}\right] W_{\chi\chi}^2 + \left[\ln\frac{W_{\chi\chi}}{\mu_H^2} - 1\right] 2 W_{\chi\phi} W_{\phi\chi}$$
$$+ \left[\ln\frac{W_{\chi\chi}}{\mu_H^2} - 1\right] \frac{2 W_{\chi\phi} W_{\phi\chi} W_{\phi\phi}}{W_{\chi\chi}} - \left[\ln\frac{W_{\chi\chi}}{\mu_H^2} - 2\right] \frac{W_{\chi\phi}^2 W_{\phi\chi}^2}{W_{\chi\chi}^2} \Bigg\},$$

where the first term is order 1, the second term is order z^2, and the last two terms are order z^4.

Run the EFT Lagrangian down from μ_H to μ_L using the EFT β-functions.

Compute Coleman-Weinberg potential at μ_L (or equivalently, integrate out ϕ_q at μ_L):

$$V_{\text{CW}} = \widetilde{\Lambda}(\mu_L) + \frac{1}{64\pi^2}\widetilde{m}^4(\mu_L)\left[\ln\frac{\widetilde{m}^2(\mu_L)}{\mu_L^2} - \frac{3}{2}\right],$$

RG Evolution

$$V(\varphi) = \Lambda + \sigma\varphi + \frac{m^2}{2}\varphi^2 + \frac{\rho}{6}\varphi^3 + \frac{\lambda}{24}\varphi^4$$

$$\begin{aligned} &\gamma_\varphi = 0\,, && \beta_\Lambda = \frac{1}{2}m^4\,, && \beta_\sigma = m^2\rho\,, \\ &\beta_{m^2} = \lambda m^2 + \rho^2\,, && \beta_\rho = 3\lambda\rho\,, && \beta_\lambda = 3\lambda^2\,, \end{aligned}$$

with solution

$$\begin{aligned} \lambda(\mu) &= \lambda(\mu_0)\,\eta^{-1}\,, \\ \rho(\mu) &= \rho(\mu_0)\,\eta^{-1}\,, \\ m^2(\mu) &= m^2(\mu_0)\left\{\eta^{-1/3}\left[1 - \frac{1}{2}\xi\right] + \frac{1}{2}\xi\eta^{-1}\right\}\,, \\ \sigma(\mu) &= \sigma(\mu_0) + \frac{m^2(\mu_0)\rho(\mu_0)}{\lambda(\mu_0)}\left\{\left(\frac{1}{3}\xi - 1\right) + \frac{1}{6}\xi\eta^{-1} + \left(1 - \frac{1}{2}\xi\right)\eta^{-1/3}\right\}\,, \\ \Lambda(\mu) &= \Lambda(\mu_0) + \frac{m^4(\mu_0)}{2\lambda(\mu_0)}\left\{\frac{1}{3}\left(3 - 6\xi + 2\xi^2\right) - \frac{1}{2}\xi\,(\xi - 2)\,\eta^{-1/3} - \frac{1}{4}\,(\xi - 2)^2\,\eta^{1/3} + \frac{1}{12}\xi^2\eta^{-1}\right\}, \end{aligned}$$

where

$$t = \frac{1}{16\pi^2}\ln\frac{\mu}{\mu_0}\,, \qquad \eta = 1 - 3\lambda(\mu_0)t\,, \qquad \xi = \frac{\rho^2(\mu_0)}{\lambda(\mu_0)m^2(\mu_0)}\,.$$

Tadpole Condition

Fix $\mathcal{J}_\chi$ and $\mathcal{J}_\phi$ by

$$\langle \chi_q(\mu_0) \rangle = 0 \qquad\qquad \langle \phi_q(\mu_0) \rangle = 0$$

We are in the EFT where χ_q has been integrated out. But we started with

$$\int \mathcal{D}\chi_q \, \mathcal{D}\phi_q \, e^{i\left[S+\int \mathcal{J}_\chi \chi_q + \mathcal{J}_\phi \phi_q\right]}$$

$$\frac{\delta}{\delta \mathcal{J}_\chi} \implies \langle \chi_q \rangle$$

and we have computed the EFT keeping $\mathcal{J}_\chi$, $\mathcal{J}_\phi$,

$$0 = \frac{\delta}{\delta \mathcal{J}_\chi} \int \mathcal{D}\phi_q \, e^{iS_{EFT}} \implies \langle \chi_q \rangle = 0$$

χ_q tadpole in EFT

We can compute the tadpole of the heavy field using the EFT

$$\chi_q(\mu_H) \to -\frac{\lambda_3(\mu_H)\hat{\chi}}{2m_\chi^2(\mu_H)}\left[\phi_q^2\right]_{\mu_H} - \frac{\lambda_3(\mu_H)\hat{\chi}\hat{\phi}}{m_\chi^2(\mu_H)}\phi_q(\mu_H) + \frac{\mathcal{J}_\chi(\mu_H)}{m_\chi^2(\mu_H)}\,,$$

and the tadpole condition becomes

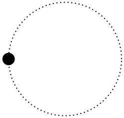
$$\mathcal{J}_\chi(\mu_H) = \frac{1}{2}\lambda_3(\mu_H)\hat{\chi}\left\langle\left[\phi_q^2\right]\right\rangle_{\mu_H}$$

since $\langle\phi_q\rangle = 0$.

RG Improve the Tadpole

Need to compute the RG improved tadpole. $\left[\phi_q^2\right]_{\mu_L}$ has no large logarithms since $\mu_L \sim m_\phi$.

$$\left\langle \left[\phi_q^2\right] \right\rangle_\mu = \frac{1}{16\pi^2}\, \widetilde{m}^2 \left[\ln \frac{\widetilde{m}^2}{\mu^2} - 1 \right] ,$$



RG equations in the EFT relate $\left[\phi_q^2\right]_{\mu_L}$ and $\left[\phi_q^2\right]_{\mu_H}$ by integrating operator anomalous dimension

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\phi_q^2\right] = -\widetilde{\lambda} \left[\phi_q^2\right] - 2\widetilde{\rho}\,\phi_q - 2\widetilde{m}^2 ,$$

This gives the RG improved tadpole

The result is the RG-improved Coleman-Weinberg effective potential, given by substituting the RG improved tadpole into the formula for $\widetilde{\Lambda}(\mu_L)$.

The explicit form is complicated because the solution of the RG equations is complicated.

Need the full EFT Lagrangian, not just $\widetilde{\Lambda}$ because the running of $\widetilde{\Lambda}$ depends on all the couplings

Expand and identify the LL terms to compare with existing calculations

Expand in

$$t = \frac{1}{16\pi^2} \ln \frac{\mu_L}{\mu_H}$$

Expansion in t

RG evolution of the cosmological constant:

$$\widetilde{\Lambda}(\mu_L) = \widetilde{\Lambda}(\mu_H) + \frac{1}{2}\widetilde{m}^4(\mu_H)t + \frac{1}{2}\widetilde{m}^2(\mu_H)\left[\widetilde{\lambda}(\mu_H)\widetilde{m}^2(\mu_H) + \widetilde{\rho}^2(\mu_H)\right] t^2 + \ldots$$

Rewriting $\widetilde{m}^2$ in terms of $\widetilde{m}_-^2$ and $\mathcal{J}_\chi(\mu_H)$ using the matching condition:

$$\widetilde{m}^2 = \underbrace{\frac{1}{6}\lambda_\phi\left(3\hat{\phi}^2 - v_\phi^2\right) + \frac{1}{2}\lambda_3\left(\hat{\chi}^2 - v_\chi^2\right) - \frac{\lambda_3^2\hat{\chi}^2\hat{\phi}^2}{m_\chi^2}}_{\widetilde{m}_-^2} + \frac{\lambda_3\hat{\chi}\mathcal{J}_\chi}{m_\chi^2}$$

$$\equiv \widetilde{m}_-^2 + \frac{\lambda_3\hat{\chi}\mathcal{J}_\chi}{m_\chi^2}$$

and using the matching condition for Λ:

$$\widetilde{\Lambda}(\mu_H) = \hat{\Lambda} + V_{\text{match}} - \frac{(\mathcal{J}_\chi)^2}{2m_\chi^2} + \mathcal{O}(z^6)\,.$$

Expand in t

Logarithmic terms:

$$\widetilde{\Lambda}(\mu_L) = \frac{1}{2}\widetilde{m}_-^4(\mu_H)t + \frac{1}{2}\widetilde{m}_-^2(\mu_H)\left[\widetilde{\lambda}(\mu_H)\widetilde{m}_-^2(\mu_H) + \widetilde{\rho}^2(\mu_H)\right]t^2 + \underbrace{\frac{\widetilde{m}_-^2(\mu_H)\lambda_3(\mu_H)\hat{\chi}}{m_\chi^2(\mu_H)}\mathcal{J}_\chi(\mu_H)t}_{\boxed{t^2}} - \underbrace{\frac{(\mathcal{J}_\chi(\mu_H))^2}{2m_\chi^2(\mu_H)}}_{\boxed{t^2}} + \ldots$$

LL Tadpole:

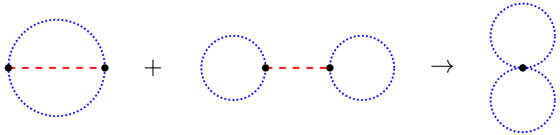
$$\mathcal{J}_\chi(\mu_H) \approx \underbrace{\lambda_3(\mu_H)\hat{\chi}\,\widetilde{m}_-^2(\mu_H)t}_{\boxed{t}} \;.$$

(remember $\mathcal{J}$ is one-loop, so starts at order t)

$$\widetilde{\lambda} = \lambda_\phi - 3\frac{\lambda_3^2\hat{\chi}^2}{m_\chi^2} \qquad\qquad \widetilde{\rho} = \widetilde{\lambda}\hat{\phi}$$

$$\begin{aligned}\widetilde{\Lambda}(\mu_L) &= \frac{1}{2}\widetilde{m}_-^4(\mu_H)t + \frac{1}{2}\left\{\left[\lambda_\phi - 3\frac{\lambda_3^2\hat{\chi}^2}{m_\chi^2}\right]\widetilde{m}_-^4 + \left[\lambda_\phi - 3\frac{\lambda_3^2\hat{\chi}^2}{m_\chi^2}\right]^2\hat{\phi}^2\widetilde{m}_-^2 + \frac{\lambda_3^2\hat{\chi}^2}{m_\chi^2}\widetilde{m}_-^4\right\}t^2 \\ &= \frac{1}{2}\widetilde{m}_-^4(\mu_H)t + \frac{1}{2}\left\{\left[\lambda_\phi - 2\frac{\lambda_3^2\hat{\chi}^2}{m_\chi^2}\right]\widetilde{m}_-^4 + \left[\lambda_\phi - 3\frac{\lambda_3^2\hat{\chi}^2}{m_\chi^2}\right]^2\hat{\phi}^2\widetilde{m}_-^2\right\}t^2 .\end{aligned}$$

The t^2 term is multiplied by $2/3$.



C. Ford and D. Jones, PLB274 (1992) 409
C. Ford, I. Jack, and D. Jones, NPB387 (1992) 373
S. Martin, PRD65 (2002) 116003

Fixed Order Result

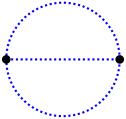
in terms of mass eigenstates

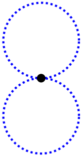
$$V = V^{(0)} + \frac{1}{16\pi^2} V^{(1)} + \frac{1}{(16\pi^2)^2} V^{(2)} + \dots$$

$$V^{(1)} = \frac{1}{4} \sum_r m_r^4 \left(\ln \frac{m_r^2}{\mu^2} - \frac{3}{2} \right)$$

$$V^{(2)} = \frac{1}{12} \lambda_{ijk} \lambda_{ijk} f_{SSS}(m_i^2, m_j^2, m_k^2) + \frac{1}{8} \lambda_{iijj} f_{SS}(m_i^2, m_j^2)$$

$$\lambda_{ijk} = \frac{\partial^3 V}{\partial\phi_i \partial\phi_j \partial\phi_k} \qquad \lambda_{ijkr} = \frac{\partial^4 V}{\partial\phi_i \partial\phi_j \partial\phi_k \partial\phi_r}$$





Fixed Order Result

$$f_{SS}(m_1^2, m_2^2) = \left[m_1^2\left(\ln\frac{m_1^2}{\mu^2} - 1\right)\right]\left[m_2^2\left(\ln\frac{m_2^2}{\mu^2} - 1\right)\right].$$

$$f_{SSS}(m_1^2, m_1^2, m_2^2) = -\frac{\Delta+2}{2} m_1^2 \Big\{ -5 + \frac{2\Delta \ln \Delta}{\Delta+2}\left[2 - \ln\frac{m_1^2}{\mu^2}\right] + 4\ln\frac{m_1^2}{\mu^2} \\ - \ln^2\frac{m_1^2}{\mu^2} - 8\Omega(\Delta)\Big\},$$

which is given in terms of $\Delta \equiv m_2^2/m_1^2$.

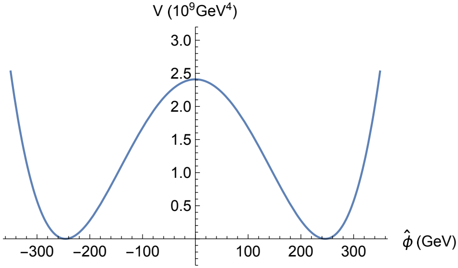
Fixed Order Result

$$\Omega(\Delta) = \frac{\sqrt{\Delta(\Delta-4)}}{\Delta+2} \int_0^{\alpha} \ln(2\cosh x)\, \mathrm{d}x\ , \qquad \cosh\alpha = \frac{1}{2}\sqrt{\Delta}\,,$$

for $\Delta \geq 4$, and by

$$\Omega(\Delta) = \frac{\sqrt{\Delta(4-\Delta)}}{\Delta+2} \int_0^{\theta} \ln(2\sin x)\, \mathrm{d}x\ , \qquad \sin\theta = \frac{1}{2}\sqrt{\Delta}\,,$$

Numerics



$$\frac{\lambda_\phi}{16\pi^2} = 0.1\,, \qquad \frac{\lambda_\chi}{16\pi^2} = 0.25\,, \qquad \frac{\lambda_3}{16\pi^2} = 0.04\,,$$

at the scale μ_H and

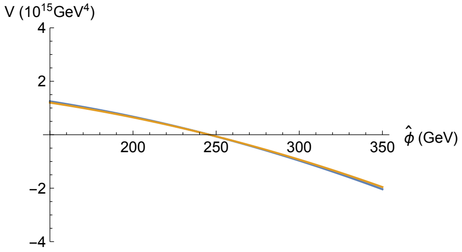
$$v_\phi = 246\,\text{GeV}\,, \qquad v_\chi = 500\,\text{TeV}\,,$$

$V_{\text{tree}} \sim 10^9\,\text{GeV}^4$

$$
\begin{aligned}
V_{\text{match}}(\mu_H) = \frac{1}{64\pi^2}\Bigg\{ \\
&\left[\ln \frac{W^{(0)}_{\chi\chi}}{\mu_H^2} - \frac{3}{2}\right]\left[W^{(0)}_{\chi\chi}\right]^2 && + \left[\ln \frac{W^{(0)}_{\chi\chi}}{\mu_H^2} - 1\right] 2 W^{(0)}_{\chi\chi} W^{(2)}_{\chi\chi} && + \left[\ln \frac{W^{(0)}_{\chi\chi}}{\mu_H^2}\right]\left[W^{(2)}_{\chi\chi}\right]^2 \\
&+0 && + \left[\ln \frac{W^{(0)}_{\chi\chi}}{\mu_H^2} - 1\right] 2 W_{\phi\chi} W_{\chi\phi} && + \frac{2 W_{\phi\chi} W_{\chi\phi} W^{(2)}_{\chi\chi}}{W^{(0)}_{\chi\chi}} \\
&+0 && +0 && + \left[\ln \frac{W^{(0)}_{\chi\chi}}{\mu_H^2} - 1\right] \frac{2 W_{\phi\chi} W_{\chi\phi} W_{\phi\phi}}{W^{(0)}_{\chi\chi}} \\
&+0 && +0 && - \left[\ln \frac{W^{(0)}_{\chi\chi}}{\mu_H^2} - 2\right] \left(\frac{W_{\phi\chi} W_{\chi\phi}}{W^{(0)}_{\chi\chi}}\right)^2 \Bigg\} \\
&1 && z^2 && z^4
\end{aligned}
$$

The order 1 term gives a constant shift of $\simeq -2.6 \times 10^{22}\,\text{GeV}^4$, about 13 orders of magnitude larger than the tree-level contribution.

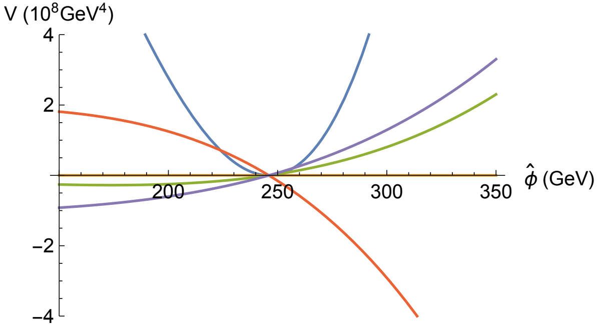
z^2 terms



The order z^2 contributions. The blue and orange curves are from the first and second rows. The two values happen to be nearly equal for our particular choice of input parameters.

$$V \simeq \times 10^{15}\,\text{GeV}^4$$

z^4 terms



The tree-level potential and the order z^4 contributions The blue curve is the tree-level potential, and the orange, green, red and purple curves are the z^4 terms in the four rows

The instability is related to the cosmological constant and hierarchy problem.

Need to make sure the low-energy particle ϕ remains light.

A simple solution is to match at the scale

$$\mu_H^2 = \frac{1}{e} W_{\chi\chi}^{(0)}$$

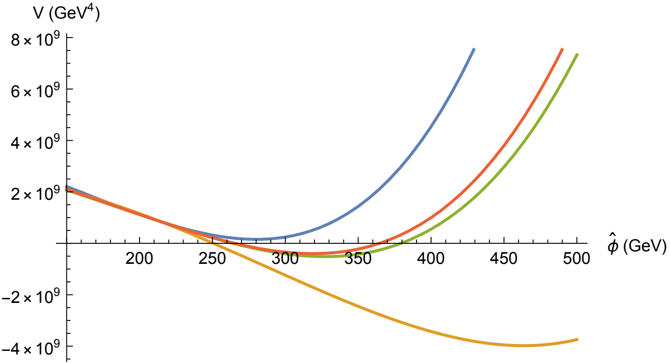
One-loop Matching for V_{CW}

$$
V_{\text{match}}(\mu_H) = \frac{1}{64\pi^2} \begin{array}{c}
\left\{ \begin{array}{ccc}
-\frac{1}{2}\left[W^{(0)}_{\chi\chi}\right]^2 & +0 & +\left[W^{(2)}_{\chi\chi}\right]^2 \\
+0 & +0 & +\frac{2W_{\phi\chi}W_{\chi\phi}W^{(2)}_{\chi\chi}}{W^{(0)}_{\chi\chi}} \\
+0 & +0 & +0 \\
+0 & +0 & +\left(\frac{W_{\phi\chi}W_{\chi\phi}}{W^{(0)}_{\chi\chi}}\right)^2
\end{array} \right\} \\
\begin{array}{ccc} 1 & z^2 & z^4 \end{array}
\end{array} .
$$

There are no z^2 corrections. Radiative corrections to V_{CW} are under control.

A large shift in the cosmological constant remains (independent of $\hat{\chi}$ and $\hat{\phi}$)

RG improved V_{CW}



blue:	tree-level potential
orange:	fixed order potential at one-loop
green:	RG improved potential neglecting the tadpole
red:	RG improved potential

$\hat{\chi}$ equal to its value at the minimum.

Previous Work

- $O(N)$ model
- two scalar fields in the unbroken phase
- two scalar fields in the broken phase
- Higgs-Yukawa model with $m_F \gg m_B$ or $m_B \gg m_F$

In all cases our results agree with explicit two-loop calculations.

Higgs-Yukawa model:
J. Casas, V. Di Clemente, M. Quiros, NPB553 (1993) 405,

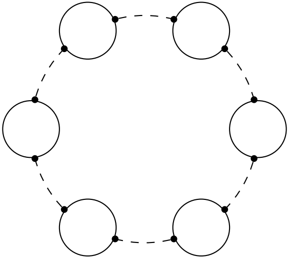
M. Bando, T. Kugo, N. Maekawa, and H. Nakano, Prog. Theo. Phys. 90 (1993) 405

$$
\begin{aligned}
\dot{\phi} &= -2g^2 N \theta_F \phi = -\gamma_\phi \theta_F \phi \,, \\
\dot{g} &= 3\theta_B \theta_F g^3 + \gamma_\phi \theta_F g \,, \\
\dot{m}_B^2 &= \lambda m_B^2 \theta_B + 2\gamma_\phi \theta_F m_B^2 \,, \\
\dot{\lambda} &= 3\lambda^2 \theta_B - 48 g^4 N \theta_F + 4\gamma_\phi \theta_F \lambda \,, \\
\dot{\Lambda} &= \frac{1}{2} m_B^4 \theta_B \,.
\end{aligned}
$$

Goldstone Boson IR Divergence Problem

S. P. Martin, PRD89 (2014) 013003, S. P. Martin, PRD90 (2014) 016013,

J. Elias-Miro, J. Espinosa, and T. Konstandin, JHEP 08 (2014) 034



top quarks and φ^+ and φ_Z, which are propagating degrees of freedom in R_ξ gauge.

IR divergence in the SM model at three-loops and beyond

$$\Sigma_{\varphi\varphi}(p^2) = -\frac{y^2}{8\pi^2} m_F^2 \ln \frac{m_F^2}{\mu^2}$$

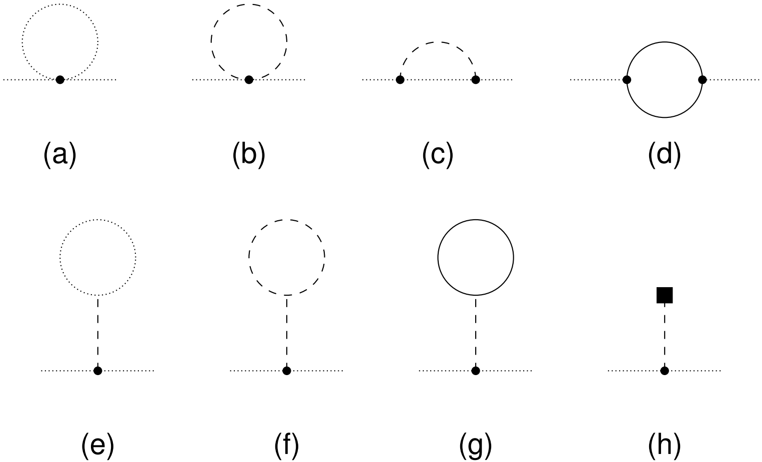
to the φ mass. At ℓ loop order — i.e. with $\ell - 1$ fermion loops, get

$$V \sim (y^2)^{\ell-1} \left(m_F^2\right)^{\ell-1} W_{\varphi\varphi}^{3-\ell} \left[\ln \frac{W_{\varphi\varphi}}{m_F^2} + \ldots\right],$$

and leads to an infrared divergence as $W_{\varphi\varphi} \to 0$ for $l \geq 3$ loops. This is the Goldstone boson infrared divergence problem.

Expansion in m_F^2/m_φ^2

EFT says to use the one-loop corrected mass



One-loop contributions to the φ two-point function. The dotted lines are φ, the dashed lines are h_q and the solid line is the fermion ψ.

$\Gamma[\hat{\varphi}]$ includes some 1PI graphs

$$\Sigma_{\varphi\varphi}(p^2) = \frac{1}{16\pi^2}\left\{-\lambda\overline{A}_0(W_{hh}) - 4\lambda^2\hat{h}^2\overline{B}_0(0, W_{hh}, 0) + 2y^2\overline{A}_0(m_F^2)\right\}$$
$$+ \frac{1}{16\pi^2}\left[\frac{6\lambda^2\hat{h}^2}{W_{hh}}\overline{A}_0(W_{hh}) - 4\sqrt{2}\frac{\lambda\hat{h}}{W_{hh}}ym_F\overline{A}_0(m_F^2)\right] + \frac{2\lambda\hat{h}\mathcal{J}_h}{W_{hh}} + \mathcal{O}\left(z^2\right) ,$$

$$\Sigma_{\varphi\varphi}(p^2 = 0) = \frac{\mathcal{J}_h}{v} .$$

$$i\,\Gamma_{\varphi\varphi}(0) = W_{\varphi\varphi} + \Sigma_{\varphi\varphi}(p^2 = 0) = \frac{\hat{\sigma}}{v} + \frac{\mathcal{J}_h}{v} = \frac{J}{v}.$$

Consequently, at the minimum of the potential where $J = 0$, at one-loop order

$$\Gamma_{\varphi\varphi}(p = 0) = 0\,,$$

Conclusions

- Formalism to systematically improve the CW potential
- The $(\lambda L)^2$ agrees with existing two-loop fixed order calculations
- The formalism applies also to
  - gauge case
  - higher orders

  without change