Hadronic contributions to the muon $g - 2$
in a dispersive approach

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Outline

Muon anomaly & dispersion relations

Hadronic vacuum polarization
  Overview & dispersive approach
  $3\pi$ contribution

Hadronic light-by-light scattering
  Overview & dispersive approach
  Pion-pole contribution

Conclusion and outlook
Muon anomaly & dispersion relations

The muon $g_\mu - 2$:

The magnetic dipole moment of a charged lepton ($\ell = e, \mu, \tau$):

$$\mu_\ell = g_\ell \left( \frac{Q_e}{2m_\ell} \right) S$$

- Dirac equation universally predicts $g_\ell = 2$
- Quantum loop corrections lead to a deviation from the classical Dirac value 2

Define the lepton anomalous magnetic moment (lepton anomaly):

$$a_\ell \equiv \frac{g_\ell - 2}{2}$$
The muon $g_\mu - 2$:

- A pure dimensionless number that can be both theoretically calculated and experimentally measured

- Contributions from a heavy scale $M$ usually: $\propto m_\ell^2/M^2$

- $\tau$ is so shortly lived; $a_\mu$ is more sensitive than $a_e$ by a factor of $\sim 4 \times 10^4$

- $a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{Had}}$
  - Hadronic vacuum polarization (HVP)
  - Hadronic light-by-light (HLbL) scattering
Muon anomaly & dispersion relations

QED $\mathcal{O}(\alpha)$:

Feynman diagrams

Feynman, 1948

Schwinger, 1948
Muons anomaly & dispersion relations

QED $\mathcal{O}(\alpha)$:

Feynman diagrams

Electroweak one-loop:

Schwinger, 1948

Feynman, 1948
Muon anomaly & dispersion relations

Standard Model contributions to the muon $g_\mu - 2$:

White paper result by the “Muon $g - 2$ Theory Initiative” (WP 2020)
Aoyama, ..., B.-L. H. et al., 2020

<table>
<thead>
<tr>
<th>Contributions</th>
<th>$a_\mu \times 10^{11}$</th>
<th>$\Delta a_\mu \times 10^{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>QED (5 loops)</td>
<td>116584718.931</td>
<td>0.104</td>
</tr>
<tr>
<td>Electroweak (2 loops)</td>
<td>153.6</td>
<td>1.0</td>
</tr>
<tr>
<td>HVP (3 loops)</td>
<td>6845</td>
<td>40</td>
</tr>
<tr>
<td>HLbL (3 loops)</td>
<td>92</td>
<td>18</td>
</tr>
<tr>
<td>$a_\mu^{SM}$</td>
<td>116591810</td>
<td>43</td>
</tr>
<tr>
<td>$a_\mu^{Exp}$</td>
<td>116592061</td>
<td>41</td>
</tr>
<tr>
<td>$a_\mu^{Exp} - a_\mu^{SM}$</td>
<td>251</td>
<td>59</td>
</tr>
</tbody>
</table>
Muon anomaly & dispersion relations

$g_\mu - 2$ experiment vs theory:

\[ \frac{\Delta g_\mu}{g_\mu} \times 10^{11} \]

- **BNL 2004**
- **Expected Fermilab precision**
- **Fermilab 2021**
- **WP 2020**

2004 2008 2012 2016 2020

4.2σ
Muon anomaly & dispersion relations

Standard Model theory:

Contributions

- QED
- HVP
- HLbL

Error

Uncertainties mainly stem from hadronic contributions.

Future experiments aim for four-fold improvement $\Delta a_\mu \approx 16 \times 10^{-11}$

⇒ With current theory $\sim 5.5\sigma$

Welcome more precise theoretical predictions

⇒ With theory improvements of same precision $\sim 11\sigma$!
Muon anomaly & dispersion relations

Dispersion relations (DR):

- Microcausality ⇒ analyticity

- Unitarity ⇒ $2 \text{Im} M_{fi} = \sum_n \int d\Pi_n M_{fn}^* M_{in}$

- Particle-antiparticle transformation ⇒ crossing

Combining analyticity, unitarity & crossing:

$M(s) = \frac{1}{2\pi i} \int_{s_{th}}^\infty \frac{\text{disc} M(s')}{{s'} - s} \, ds' = \frac{1}{\pi} \int_{s_{th}}^\infty \frac{\text{Im} M(s')}{{s'} - s} \, ds'$
Muon anomaly & dispersion relations

Pion vector form factor $F_{\pi}^V(s)$:

\[
\text{disc } \left[ \gamma_v^* \pi^+ \right] = \gamma_v^* \pi^+ \pi^+
\]

\[
\text{disc } F_{\pi}^V(s) = 2i \text{ Im } F_{\pi}^V(s) = 2i F_{\pi}^V(s) \sin \delta_1^1(s) e^{-i\delta_1^1(s)} \theta(s - 4M_\pi^2)
\]

Watson’s final-state theorem: phase of $F_{\pi}^V(s)$ is given by $\delta_1^1(s)$

Watson, 1954
Muon anomaly & dispersion relations

Pion vector form factor $F^V_\pi(s)$:

$$\text{disc } \left[ \begin{array}{c} \gamma^*_v \\ \pi^- \\ \pi^+ \end{array} \right] = \left[ \begin{array}{c} \gamma^*_v \\ \pi^- \\ \pi^+ \end{array} \right]$$

$$\text{disc } F^V_\pi(s) = 2i \text{ Im } F^V_\pi(s) = 2i F^V_\pi(s) \sin \delta_1^1(s) e^{-i\delta_1^1(s)} \theta(s - 4M^2_\pi)$$

Watson’s final-state theorem: phase of $F^V_\pi(s)$ is given by $\delta_1^1(s)$  

Watson, 1954

Solution:

$$F^V_\pi(s) = P(s) \Omega(s), \quad \Omega(s) = \exp \left\{ \frac{s}{\pi} \int_{4M^2_\pi}^{\infty} ds' \frac{\delta_1^1(s')}{s'(s' - s)} \right\}$$

- $\Omega(s)$ is the Omnès function  
  
  Omnès, 1958

- $P(s)$ polynomial, $P(0) = 1$ from charge conservation

- $\pi\pi$ $P$-wave phase shift $\delta_1^1(s)$ from Roy equations
Hadronic vacuum polarization

- $m_\mu$ as characteristic scale
  $\Rightarrow$ Not a perturbative QCD problem!
- DR to relate to the observable

$$a_\mu^{\text{HVP}} = \int_{s_{\text{thr}}}^{\infty} ds K(s) \sigma^0(e^+e^- \rightarrow \text{hadrons})$$

- Kinematical function $K(s)$: $K(s) \propto 1/s$ for large $s$
- $\sigma^0(e^+e^- \rightarrow \text{hadrons}) \propto 1/s$ for large $s$
Hadronic vacuum polarization

Comparison between data-driven approaches and lattice:

\[ a_{\mu}^{\text{HVP, LO}} \times 10^{10} \]

- BMW 2020
- KNT 2019
- DHMZ 2019
- WP 2020
Hadronic vacuum polarization

Complete data-driven evaluation needs:

- Exclusive channels $\pi^+\pi^-$, $\pi^+\pi^-\pi^0$, $\pi^0\gamma$, ...

- Inclusive data

- Perturbative QCD

Hadronic cross section:

- Inclusive of final state radiation

- Exempt from initial state radiation & vacuum polarization (VP)
Overview of the $\pi^+\pi^-\pi^0$ channel

- **Second largest** exclusive channel after $\pi^+\pi^-$

- **Large** relative discrepancy between direct data-integration works

<table>
<thead>
<tr>
<th>Davier et al., 2017</th>
<th>Keshavarzi et al., 2018</th>
<th>relative difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$46.20 \pm 1.45$</td>
<td>$47.70 \pm 0.89$</td>
<td>$3.2%$ ($0.7%$ for $\pi^+\pi^-$)</td>
</tr>
</tbody>
</table>

Even lower value $a_{\mu}^{3\pi}|_{\leq 2.0 \text{GeV}} = 44.3(1.5) \times 10^{-10}$!

- **Other independent analyses necessary**

  ⇒ **Dispersive global fit function fulfilling analyticity, unitarity & QCD constraints**
\[ \gamma^* \rightarrow 3\pi \] dispersive representation

The \( \gamma^*(q) \rightarrow \pi^+(p_+)\pi^-(p_-)\pi^0(p_0) \) decay amplitude \( \mathcal{F}(s, t, u; q^2) \):

\[
\langle 0|j_\mu(0)|\pi^+(p_+)\pi^-(p_-)\pi^0(p_0)\rangle = -\epsilon_{\mu\nu\rho\sigma} p_+^\nu p_-^\rho p_0^\sigma \mathcal{F}(s, t, u; q^2)
\]

\( q = p_+ + p_- + p_0; \quad s, t \& u \) are Mandelstam variables

Decompose into single-variable functions:

\[
\mathcal{F}(s, t, u; q^2) = \mathcal{F}(s, q^2) + \mathcal{F}(t, q^2) + \mathcal{F}(u, q^2)
\]

Normalization from the Wess–Zumino–Witten (WZW) anomaly:

\[
\mathcal{F}(0, 0, 0; 0) = \frac{1}{4\pi^2 F^3_\pi} \equiv F_{3\pi}
\]

\( F_\pi = 92.28(10) \text{ MeV} \): pion decay constant

P. Zyla et al., 2020
\( \gamma^* \rightarrow 3\pi \) dispersive representation

Discontinuity equation:

\[
\text{disc } \mathcal{F}(s, q^2) = 2i(\mathcal{F}(s, q^2) + \hat{\mathcal{F}}(s, q^2))\theta(s - 4M^2) \sin \delta^1(s) e^{-i\delta^1(s)}
\]

- \( \mathcal{F}(s, q^2) \): right-hand cut
- \( \hat{\mathcal{F}}(s, q^2) \): left-hand cut; angular averages of \( \mathcal{F}(t, q^2) \) & \( \mathcal{F}(u, q^2) \)
\[ \gamma^* \rightarrow 3\pi \text{ dispersive representation} \]

A once-subtracted dispersive solution to the discontinuity equation:

\[
\mathcal{F}(s, q^2) = a(q^2) \Omega(s) \left\{ 1 + \frac{s}{\pi} \int_{4M^2_\pi}^{\infty} ds' \frac{\hat{F}(s', q^2) \sin \delta^1_1(s')}{s'(s'-s)|\Omega(s')|} \right\}
\]

\[
\Omega(s) = \exp \left\{ \frac{s}{\pi} \int_{4M^2_\pi}^{\infty} ds' \frac{\delta^1_1(s')}{s'(s'-s)} \right\}
\]

is the Omnès function

Omnès, 1958
\( \gamma^* \rightarrow 3\pi \) dispersive representation

A once-subtracted dispersive solution to the discontinuity equation:

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\]

\[
\Omega(s) = \exp \left\{ \frac{s}{\pi} \int_{4M^2}^{\infty} ds' \frac{\delta_1(s')}{s'(s' - s)} \right\}
\]

is the Omnès function

\( \hat{F}(s, q^2) \) absent:

\[
\mathcal{F}(s, q^2) = \quad \text{wave graph}
\]

\( \hat{F}(s, q^2) \) present:

\[
\mathcal{F}(s, q^2) = \quad \text{wave graph} + \quad \text{crossed-channel graph} + \cdots
\]

- Incorporated crossed-channel interactions
\[ a(q^2) \text{ fit to different } e^+ e^- \rightarrow 3\pi \text{ cross-section data with parameterization:} \]

\[ a(q^2) = \frac{F_{3\pi}}{3} + \frac{q^2}{\pi} \int_{s_{\text{thr}}}^{\infty} ds' \frac{\text{Im} A(s')}{s'(s' - q^2)} + C_n(q^2) \]

\[ A(q^2) = \sum_{V} \frac{c^V}{M_V^2 - q^2 - i\sqrt{q^2 \Gamma_V(q^2)}}, \quad V = \omega, \phi, \omega', \omega'' \]

\[ C_n(q^2) = \sum_{i=1}^{n} c_i (z(q^2)_i - z(0)_i), \quad z(q^2) = \frac{\sqrt{s_{\text{inel}} - s_1} - \sqrt{s_{\text{inel}} - q^2}}{\sqrt{s_{\text{inel}} - s_1} + \sqrt{s_{\text{inel}} - q^2}} \]
\[ a(q^2) \] fit to different \( e^+e^- \rightarrow 3\pi \) cross-section data with parameterization:

\[ a(q^2) = \frac{F_{3\pi}}{3} + \frac{q^2}{\pi} \int_{s_{\text{thr}}}^{\infty} ds' \frac{\text{Im} A(s')}{s'(s' - q^2)} + C_n(q^2) \]

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- \( S' \)-wave cusp eliminated
- **Exact** implementation of \( \gamma^* \rightarrow 3\pi \) anomaly:

\[ \frac{F_{3\pi}}{3} = \frac{1}{\pi} \int_{s_{\text{thr}}}^{\infty} ds' \frac{\text{Im} a(s')}{s'} \]
\[ e^+e^- \rightarrow 3\pi \] cross section data sets

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Region of ( \sqrt{s} ) GeV</th>
<th># data points</th>
<th>Normalization uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>SND 2002</td>
<td>[0.98, 1.38]</td>
<td>67</td>
<td>5.0% or 5.4%</td>
</tr>
<tr>
<td>SND 2003</td>
<td>[0.66, 0.97]</td>
<td>49</td>
<td>3.4% or 4.5%</td>
</tr>
<tr>
<td>SND 2015</td>
<td>[1.05, 1.80]</td>
<td>31</td>
<td>3.7%</td>
</tr>
<tr>
<td>CMD-2 1995</td>
<td>[0.99, 1.03]</td>
<td>16</td>
<td>4.6%</td>
</tr>
<tr>
<td>CMD-2 1998</td>
<td>[0.99, 1.03]</td>
<td>13</td>
<td>2.3%</td>
</tr>
<tr>
<td>CMD-2 2004</td>
<td>[0.76, 0.81]</td>
<td>13</td>
<td>1.3%</td>
</tr>
<tr>
<td>CMD-2 2006</td>
<td>[0.98, 1.06]</td>
<td>54</td>
<td>2.5%</td>
</tr>
<tr>
<td>DM1 1980</td>
<td>[0.75, 1.10]</td>
<td>26</td>
<td>3.2%</td>
</tr>
<tr>
<td>ND 1991</td>
<td>[0.81, 1.39]</td>
<td>28</td>
<td>10% or 20%</td>
</tr>
<tr>
<td>DM2 1992</td>
<td>[1.34, 1.80]</td>
<td>10</td>
<td>8.7%</td>
</tr>
<tr>
<td>BaBar 2004</td>
<td>[1.06, 1.80]</td>
<td>30</td>
<td>all systematics</td>
</tr>
</tbody>
</table>

- Normalization-type systematic uncertainties are assumed to be 100% correlated
- Normalization uncertainties produce a biased fit for an empirical full covariance-matrix minimization
D’Agostini bias & unbiased fits

Simple example of overall normalization uncertainty inducing a bias: D’Agostini, 1994

\[ y_1 = 8.0 \pm 2\% \; \& \; y_2 = 8.5 \pm 2\% \], normalization error of \( \epsilon = 10\% \)

- Covariance matrix

\[
V = \begin{pmatrix}
\sigma_1^2 & 0 \\
0 & \sigma_2^2
\end{pmatrix} + \epsilon^2 \begin{pmatrix}
y_1^2 & y_1 y_2 \\
y_1 y_2 & y_2^2
\end{pmatrix} = \begin{pmatrix}
\sigma_1^2 + \epsilon^2 y_1^2 & \epsilon^2 y_1 y_2 \\
\epsilon^2 y_1 y_2 & \sigma_2^2 + \epsilon^2 y_2^2
\end{pmatrix}
\]

- \( \chi^2 = \Delta^T V^{-1} \Delta \), \( \Delta = \begin{pmatrix}
y_1 - \hat{y} \\
y_2 - \hat{y}
\end{pmatrix} \)

\( \Rightarrow \hat{y} = 7.87 \pm 0.81 < y_1 \; \& \; y_2 \)
D’Agostini bias & unbiased fits

- Happens when the data values are rescaled independently of their errors
- Smaller data points are assigned a smaller uncertainty than larger ones

A better covariance matrix:

\[ V = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} \hat{y}^2 & \hat{y}^2 \\ \hat{y}^2 & \hat{y}^2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 + \varepsilon^2 \hat{y}^2 & \varepsilon^2 \hat{y}^2 \\ \varepsilon^2 \hat{y}^2 & \sigma_2^2 + \varepsilon^2 \hat{y}^2 \end{pmatrix} \]

General iterative solution:  

\[ V_{n+1}(i, j) = V_{\text{stat}}(i, j) + \frac{V_{\text{syst}}(i, j)}{y_i y_j} f_n(x_i) f_n(x_j) \]

NNPDF collaboration, 2010
Fit results

- Resonance parameters $M_\omega, \Gamma_\omega, M_\phi, \Gamma_\phi, c_\omega, c_\phi, c_\omega', c_\omega''$

- Conformal parameters $c_1, c_2, c_3$

- Energy rescaling $\sqrt{s} \to \sqrt{s} + \xi(\sqrt{s} - 3M_\pi)$

\[ \frac{\chi^2}{\text{dof}} = \frac{430.8}{305} = 1.41 \]

- Fit errors are inflated by the scale factor $S = \sqrt{\frac{\chi^2}{\text{dof}}}$
Fit results

Fit to $e^+e^- \rightarrow 3\pi$ data up to 1.8 GeV:

- VP removed from the cross section
- Black and Gray bands represent fit and total uncertainties
Fit results

Enlarged $\omega$ and $\phi$ regions:

$$M_\omega = 782.63(3) \text{ MeV},$$
$$M_\phi = 1019.20(2) \text{ MeV},$$

Compared to PDG (VP-subtraction $\Delta M_\omega = -0.13 \text{ MeV}, \Delta M_\phi = -0.26 \text{ MeV}$):

$$M_\omega = 782.65(12) \text{ MeV},$$
$$M_\phi = 1019.461(16) \text{ MeV},$$

• $M_\omega$ consistent with PDG; tension with $\pi\pi$ persists!
Central result for the $3\pi$ contribution to HVP:

$$ a_{\mu}^{3\pi}|_{\leq 1.8 \text{ GeV}} = 46.2(6)(6) \times 10^{-10} = 46.2(8) \times 10^{-10} $$

- Interpolation errors $\Rightarrow$ main discrepancy between different groups

Threshold region $a_{\mu}^{3\pi}|_{\leq 0.66 \text{ MeV}} = 0.02 \times 10^{-10}$

- Twice the estimate from WZW action + vector meson dominance model
  Kuraev and Silagadze, 1995, Ahmedov et al., 2002
Fit results

Update after the current analysis:

\[ a_\mu^{3\pi} \times 10^{10} \text{ below } 1.8 \text{ GeV in WP 2020} \]

<table>
<thead>
<tr>
<th></th>
<th>Davier et al., 2019</th>
<th>Keshavarzi et al., 2019</th>
<th>Hoferichter et al., 2019</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>46.21 ± 1.45</td>
<td>46.63 ± 0.94</td>
<td>46.16 ± 0.82</td>
</tr>
</tbody>
</table>

- Large relative difference resolved

In combination with the \(2\pi\) channel: \[ a_\mu^{2\pi}|_{\leq 1.0 \text{ GeV}} + a_\mu^{3\pi}|_{\leq 1.8 \text{ GeV}} = 541.2(2.7) \times 10^{-10} \]

- 80% of HVP imposing analyticity and unitarity constraints
• Previous estimates based on hadronic models except for lattice

• New model-independent initiatives: employ DR to relate dominant contributions to observables like form factors

Colangelo et al., 2014, Pauk, Vanderhaeghen, 2014

• \(\pi^0\)-pole term is the largest individual contribution to HLbL
Hadronic light-by-light scattering

Comparison between dispersive approach and lattice:

\[ a^{\text{HLbL}}_{\mu} \times 10^{11} \]

- **RBC/UKQCD 2019**
- **Dispersive 2020**
- **WP 2020**
- **Mainz 2021**
A general master formula for the complete HLbL contributions:

\[
a_{\mu}^{\text{HLbL}} = \frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \sqrt{1 - \tau^2} Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)
\]

- \( T_i(Q_1, Q_2, \tau) \): kernel functions
- \( \bar{\Pi}_i(Q_1, Q_2, \tau) \): hadronic scalar functions

The pion pole easily identified with the hadronic functions \( \bar{\Pi}_i \):

\[
\bar{\Pi}_1^{\pi^0\text{-pole}}(Q_1, Q_2, \tau) = -\frac{F_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) F_{\pi^0\gamma^*\gamma^*}(-(Q_1 + Q_2)^2, 0)}{(Q_1 + Q_2)^2 + M_{\pi^0}^2}
\]

\[
\bar{\Pi}_2^{\pi^0\text{-pole}}(Q_1, Q_2, \tau) = -\frac{F_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1 + Q_2)^2) F_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0)}{Q_2^2 + M_{\pi^0}^2}
\]
A 3 dimensional representation for the pion-pole contribution:

\[ a_{\mu}^{\pi^0\text{-pole}} = \left( \frac{\alpha}{\pi} \right)^3 \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \]
\[ \times \left[ w_1(Q_1, Q_2, \tau) \, F_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1 + Q_2)^2) \, F_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0) \right. \]
\[ \left. + \, w_2(Q_1, Q_2, \tau) \, F_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) \, F_{\pi^0\gamma^*\gamma^*}(-(Q_1 + Q_2)^2, 0) \right] \]

- \( w_1(Q_1, Q_2, \tau) \) & \( w_2(Q_1, Q_2, \tau) \): weight functions; concentrated in \( Q_i \leq 0.5 \) GeV
- \( F_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) \): on-shell space-like pion transition form factors
Pion transition form factor

- Defined by the matrix element of two electromagnetic currents \( j_\mu(x) \)

\[
i \int d^4 x \ e^{i q_1 \cdot x} \ 0 \left\langle T \{ j_\mu(x) j_\nu(0) \} \right\rangle \pi^0(q_1 + q_2) = -\epsilon_{\mu\nu\rho\sigma} q_1^\rho q_2^\sigma F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)
\]

- Normalization fixed by the Adler–Bell–Jackiw anomaly:

\[
F_{\pi^0\gamma^*\gamma^*}(0, 0) = \frac{1}{4\pi^2 F_\pi} \equiv F_{\pi\gamma\gamma}
\]
Pion transition form factor

• Defined by the matrix element of two electromagnetic currents \( j_\mu(x) \)

\[
i \int d^4x \, e^{i q_1 \cdot x} \langle 0 | T \{ j_\mu(x) j_\nu(0) \} | \pi^0(q_1 + q_2) \rangle
= -\epsilon_{\mu\nu\rho\sigma} q_1^\rho q_2^\sigma F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)
\]

• Normalization fixed by the Adler–Bell–Jackiw anomaly:

\[
F_{\pi^0\gamma^*\gamma^*}(0, 0) = \frac{1}{4\pi^2 F_\pi} \equiv F_{\pi\gamma\gamma}
\]

We build the form factor representation:

\[
F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = F_{\pi^0\gamma^*\gamma^*}\text{disp}(q_1^2, q_2^2) + F_{\pi^0\gamma^*\gamma^*}\text{eff}(q_1^2, q_2^2) + F_{\pi^0\gamma^*\gamma^*}\text{asym}(q_1^2, q_2^2)
\]

• Reconstructed from all lowest-lying singularities
• Fulfills the asymptotic constraints at \( \mathcal{O}(1/Q^2) \)
• Suitable for \( a_\mu \) loop-integral evaluation
Pion transition form factor

Dispersive reconstruction from the lowest-lying hadronic intermediate states:

\[ F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = F_{vs}(q_1^2, q_2^2) + F_{vs}(q_2^2, q_1^2) \]
Pion transition form factor

Dispersive reconstruction from the lowest-lying hadronic intermediate states:

\[ F_{\pi^0 \gamma^* \gamma^*}(q_1^2, q_2^2) = F_{vs}(q_1^2, q_2^2) + F_{vs}(q_2^2, q_1^2) \]

Isovector photon: 2 pions
- \( \gamma_v^* \rightarrow \pi^+ \pi^- \rightarrow \gamma_s^* \pi^0 \)
- \( \text{disc} \propto \text{pion vector form factor} \times \gamma_s^* \rightarrow 3\pi \text{ amplitude} \)

Isoscalar photon: 3 pions
- \( \gamma_s^* \rightarrow \pi^+ \pi^- \pi^0 \rightarrow \gamma_v^* \pi^0 \)
- Dominated by resonances \( \omega, \phi, \omega', \& \omega'' \)
Pion transition form factor

Dispersive reconstruction from the lowest-lying hadronic intermediate states:

\[ F_{\pi^0 \gamma^* \gamma^*}(q_1^2, q_2^2) = F_{vs}(q_1^2, q_2^2) + F_{vs}(q_2^2, q_1^2) \]

Building blocks of the dispersive treatment:

- Pion vector form factor \( F_{\pi}^V(s) \)
- Partial wave amplitude \( f_1(s, q^2) \) for the \( \gamma_s^*(q) \rightarrow \pi^+\pi^-\pi^0 \) reaction
Fit $a(q^2)$ to different $e^+e^- \to 3\pi$ cross-section data with:

- $S$-wave cusp eliminated
- Exact implementation of $\gamma_s^*(q) \to 3\pi$ anomaly

$$\frac{F_{3\pi}}{3} = \frac{1}{\pi} \int_{s_{\text{thr}}}^{\infty} ds' \frac{\text{Im} a(s')}{s'}$$

- Asymptotic behavior of $C_n(q^2)$ controlled
Pion transition form factor

6 (7) parameters $c_\omega$, $c_\phi$, $c_\omega'$, $c_\omega''$, $c_1$, $c_2$ & $(c_3)$ fit to $e^+e^- \rightarrow 3\pi$ data:

- Substantially improved above the $\phi$ peak
Double-spectral representation of the form factor:

\[
F_{\pi^0\gamma^*\gamma^*}^{\text{disp}}( -Q_1^2, -Q_2^2) = \frac{1}{\pi^2} \int_{\frac{4M_{\pi}^2}{s}}^{s_{iv}} dx \int_{s_{\text{thr}}}^{s_{is}} dy \rho_{\text{disp}}(x, y) \frac{\rho_{\text{disp}}(x, y)}{(x + Q_1^2)(y + Q_2^2)},
\]

\[
\rho_{\text{disp}}(x, y) = \frac{q_{\pi}^3(x)}{12\pi \sqrt{x}} \text{Im} \left[ (F_{\pi}^V(x))^* f_1(x, y) \right] + [x \leftrightarrow y]
\]

\[
f_1(s, q^2) = \mathcal{F}(s, q^2) + \hat{\mathcal{F}}(s, q^2): \ \gamma_s^*(q) \rightarrow 3\pi \ P\text{-wave}
\]
Pion transition form factor

Effective pole term:

\[ F_{\pi^0 \gamma^* \gamma^*}^{\text{disp}}(q_1^2, q_2^2) \] fulfills the chiral anomaly \( F_{\pi \gamma \gamma} \) by around 90%

⇒ Introduce an effective pole term

\[ F_{\pi^0 \gamma^* \gamma^*}^{\text{eff}}(q_1^2, q_2^2) = \frac{g_{\text{eff}}}{4\pi^2 F_\pi} \frac{M_{\text{eff}}^4}{(M_{\text{eff}}^2 - q_1^2)(M_{\text{eff}}^2 - q_2^2)} \]

\( g_{\text{eff}} \) fixed by fulfilling the chiral anomaly

\( g_{\text{eff}} \sim 10\% \), ⇒ small

\( M_{\text{eff}} \) fit to singly-virtual data excluding BaBar above 5 GeV²

Gronberg et al., 1998, Aubert et al., 2009, Uehara et al., 2012

\( M_{\text{eff}} \sim 1.5-2 \text{ GeV} \), ⇒ reasonable
Asymptotically, $F_{\pi^0 \gamma^* \gamma^*}$ should fulfill

$$F_{\pi^0 \gamma^* \gamma^*}(q_1^2, q_2^2) = -\frac{2F_\pi}{3} \int_0^1 dx \frac{\phi_\pi(x)}{xq_1^2 + (1 - x)q_2^2} + O\left(\frac{1}{q_i^4}\right),$$

Pion distribution amplitude $\phi_\pi(x) = 6x(1 - x) + \cdots$

Brodsky–Lepage (BL) limit:

$$\lim_{Q^2 \to \infty} F_{\pi^0 \gamma^* \gamma^*(-Q^2, 0)} = \frac{2F_\pi}{Q^2}$$

Operator product expansion (OPE):

$$\lim_{Q^2 \to \infty} F_{\pi^0 \gamma^* \gamma^*(-Q^2, -Q^2)} = \frac{2F_\pi}{3Q^2}$$
Pion transition form factor

Rewrite the asymptotic form into a double-spectral representation:

\[ F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{1}{\pi^2} \int_0^\infty \int_0^\infty dx dy \frac{\rho^{\text{asym}}(x, y)}{(x - q_1^2)(y - q_2^2)}, \]

\[ \rho^{\text{asym}}(x, y) = -2\pi^2 F_\pi xy\delta''(x - y) \]

Decomposition of the pion-transition form factor:

\[ F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{1}{\pi^2} \int_0^{s_m} dx \int_0^{s_m} dy \frac{\rho^{\text{disp}}(x, y)}{(x - q_1^2)(y - q_2^2)} + \frac{1}{\pi^2} \int_{s_m}^\infty dx \int_0^\infty dy \frac{\rho^{\text{asym}}(x, y)}{(x - q_1^2)(y - q_2^2)} \]

\[ + \frac{1}{\pi^2} \int_0^{s_m} dx \int_{s_m}^\infty dy \frac{\rho(x, y)}{(x - q_1^2)(y - q_2^2)} + \frac{1}{\pi^2} \int_{s_m}^\infty dx \int_0^{s_m} dy \frac{\rho(x, y)}{(x - q_1^2)(y - q_2^2)} \]

- \( s_m \): continuum threshold
- \( \rho(x, y) \) not known rigorously, \( \rho^{\text{asym}}(x, y) \) applied in mixed regions vanishes

\[ \Rightarrow \text{All constraints can be fulfilled discarding mixed regions} \]
Pion transition form factor

This defines the asymptotic contribution:

\[ F^\text{asym}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = 2F_\pi \int_{s_m}^\infty dx \frac{q_1^2 q_2^2}{(x - q_1^2)^2(x - q_2^2)^2} \]

- Does not contribute for the singly-virtual kinematics
- Restores the asymptotics for singly-/doubly-virtual kinematics

The final representation:

\[ F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = F^\text{disp}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) + F^\text{eff}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) + F^\text{asym}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \]
Numerical results

Uncertainty estimates:

- The uncertainty in $F_{\pi\gamma\gamma}$ at 0.8% from PrimEx II
  - Varying the coupling $g_{\text{eff}}$

- Dispersive uncertainties estimated by
  - Varying the cutoffs between 1.8 and 2.5 GeV
  - Different $\pi\pi$ phase shifts
  - Different representations of $F^V_\pi(s)$
  - Different conformal polynomial fits to $e^+e^- \rightarrow 3\pi$

Larin et al., 2020
Caprini et al., 2012, García-Martín et al., 2011, Schneider et al., 2012
Numerical results

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    Caprini et al., 2012, García-Martín et al., 2011,
    Schneider et al., 2012
  ▶ Different representations of $F_V^\pi(s)$
  ▶ Different conformal polynomial fits to $e^+e^- \rightarrow 3\pi$

• BL limit uncertainty by $^{+20}_{-10}\%$
  ▶ Varying the mass parameter $M_{\text{eff}}$

  Aubert et al., 2009, Uehara et al., 2012
  ▶ Completely covers $3\sigma$ band

• Asymptotic part $s_m = 1.7(3)\,\text{GeV}^2$
  ▶ Expected from light-cone sum rules

  Khodjamirian, 1999, Agaev et al., 2011,
  Mikhailov et al., 2016
Numerical results

Singly-virtual space-like transition form factor in $Q^2 F_{\pi^0 \gamma^* \gamma^*}(-Q^2, 0)$:

\[ Q^2 F_{\pi^0 \gamma^* \gamma^*}(-Q^2, 0) \]
Numerical results

Singly-virtual space-like transition form factor in $Q^2 F_{\pi^0,\gamma^*\gamma^*}(-Q^2, 0)$:

![Graph showing the numerical results](image_url)
Numerical results

Diagonal form factor in $Q^2 F_{\pi^0 \gamma^* \gamma^*}(-Q^2, -Q^2)$ in comparison to lattice:

Gérardin et al., 2019
Numerical results

Diagonal form factor in $Q^2 F_{\pi^0 \gamma^* \gamma^*}(-Q^2, -Q^2)$ in comparison to LMD+V fit to lattice:

Gérardin et al., 2016
Numerical results

\[(Q_1^2 + Q_2^2) F_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\] as a function of \(Q_1^2\) and \(Q_2^2\):

- \(1/Q_i^2\) behavior in the entire domain of space-like virtualities
  ⇒ Hard to obtain in resonance models
Numerical results

Pion-pole contribution to $a_\mu$ from the final representation:

$$a_{\mu}^{\pi^0\text{-pole}} = 63.0(0.9)F_{\pi\gamma\gamma}(1.1)\text{disp}(2.2)\text{BL}(0.6)_{\text{asym}} \times 10^{-11}$$

$$= 63.0^{+2.7}_{-2.1} \times 10^{-11}$$

- First complete data-driven determination
- Fully controlled uncertainty estimates

Comparison to lattice:

- Lattice 2016: $(65.0 \pm 8.3) \times 10^{-11}$  
  [Gérardin et al., 2016]
- Lattice 2019: $(59.7 \pm 3.6) \times 10^{-11}$, changes to $(62.3 \pm 2.3) \times 10^{-11}$ after fixing normalization from experiment  
  [Gérardin et al., 2019]
Conclusion and outlook

- Hadronic vacuum polarization: $3\pi$
  - Independent dispersive analysis for the $3\pi$ channel
  - Resolved main tension in $3\pi$, reaffirmed tension in $a_\mu$

- Hadronic light-by-light scattering: pion pole
  - Dispersive reconstruction of the pion transition form factor
  - Data-driven determination of $a_{\mu}^{\pi^0}$-pole with carefully estimated improvable uncertainties

- Similar analysis done for $\pi^0\gamma$  
  - $\pi^0 \rightarrow e^+e^-$

- $\eta$ and $\eta'$ pole contributions

Hoid et al. 2020

Hoferichter et al. 2021

S. Holz et al.
Much obliged for your attention!