# A Finite S-matrix


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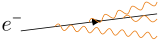
arXiv: 1810.10022 with C. Frye, N. Paul, M. Schwartz, and K. Yan

arXiv: 1906.03271 with M. Schwartz

arXiv: 1911.06821 with M. Schwartz



# Outline

- Introduction: IR divergences
- Ideas for IR finiteness:
  - Cross section method
  - Finite $S$-matrix
  - Coherent states
- Conclusions & Future directions

# The Scattering Matrix ($S$-matrix)

$\langle f|\, S\, |i\rangle$: **Probability amplitude** for measuring a final state $|f\rangle$ given an initial state $|i\rangle$

- Used in most **Quantum Field Theory** calculations.
  - Leads to predictions for **collider experiments**.
  - Standard Model observables computed to high precision.
  - Calculated using **Feynman diagrams**.

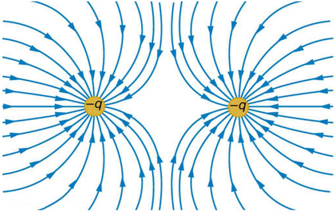
# Problem with $S$-matrix: Infrared Divergences

- Probability of two electrons scattering when calculated naively $= \infty$.
- Problematic since probabilities $p_{fi} \propto |\langle f|S|i\rangle|^2$ should be less than 1.
- **UV divergences** occur at high energies.
  - Remedy using **renormalization**.
- **IR divergences** occur at low energies in theories with massless particles.
  - **No proof of LSZ theorem**.
  - Despite these problems, use $S$-matrix to make predictions.

*Problems provide an opportunity: Explore and gain new insight!*

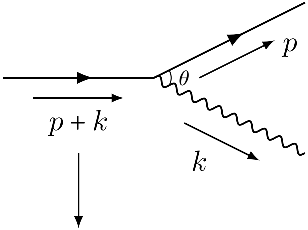
# Problem with $S$-matrix: Infrared Divergences

**Physical Reason:** We are not including the electromagnetic field correctly in scattering calculations.




Lumen Learning

# IR Divergences in QFT




Propagator: $\frac{1}{(p+k)^2} \sim \frac{1}{|k|(1-\cos\theta)}$

$$\text{Singularities:}\quad \left.\begin{array}{ll} |k| \to 0 & \textit{soft} \\ \theta \to 0 & \textit{collinear} \end{array}\right\} \text{ IR divergences}$$

# Ideas for IR Finiteness

1. **Finite cross sections** $\sigma \propto \int |\langle f| S |i\rangle|^2 d\Pi_f$
   - Bloch-Nordsieck theorem
   - KLN theorem
2. **Finite** $\boldsymbol{S}$**-matrix**
3. **Finite scattering amplitudes** $S_{fi} = \langle f| S |i\rangle$

# 1. Finite Cross Sections

# Cross Section Method - Introduction

- Idea: Cross section is **measurable** and needs to be finite.
- Detecting an electron, perhaps a photon with little energy or one close to the electron **escaped detector**.
  - All physical detectors have a finite resolution.
- A sum over all processes consistent with detector measurement should give a finite quantity.

*Need to calculate the same quantity as we measure.*

# Cross Section Method - Introduction

**Physical Motivation:** All physical observables are finite.

**Theoretical Goal:** Find the *minimal set* of Feynman diagrams needed for finiteness.

# Previous Theorems on IR Divergences

1937 — Bloch-Nordsieck theorem

1962-64 — Kinoshita-Lee-Nauenberg (KLN) theorem

2018 — Stronger KLN theorem (Frye, HSH, Paul, Schwartz, Yan)

# Previous Theorems on IR Divergences

**Bloch-Nordsieck (1937):** Soft IR divergences cancel in QED when summing over final state photons with finite energy resolution.

**Doria, Frenkel, Taylor (1980):** Counterexample in QCD: $qq \to \mu\mu qq$ + final state gluons is soft IR divergent at 2-loops.

**KLN Theorem (1962-64):** $S$-matrix elements squared are IR finite when summing over **final states and initial states** within some energy window:

$$\sum_{f,i\in[E-E_0,E+E_0]} |\langle f|\, S\, |i\rangle|^2 < \infty$$

# Stronger KLN Theorem

**KLN Theorem (1962-64):** $S$-matrix elements squared are IR finite when summing over **final states and initial states** within some energy window:

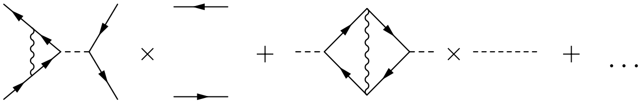
$$\sum_{f,i\in[E-E_0,E+E_0]} |\langle f|\,S\,|i\rangle|^2 < \infty$$

**Stronger KLN Theorem (2018):** $S$-matrix elements squared are IR finite when summing over **final states or initial states:**

$$\sum_f |\langle f|\,S\,|i\rangle|^2 < \infty, \qquad \sum_i |\langle f|\,S\,|i\rangle|^2 < \infty$$

# Stronger KLN Theorem

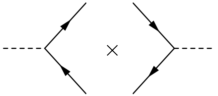
- KLN is a trivial consequence of **unitarity**:
  - Probability of $i \to$ anything is $1 < \infty$
  - Probability of anything $\to f$ is $1 < \infty$
- **KLN requires a term where $f = i \to$ forward scattering**



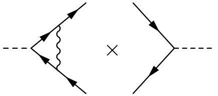
- Works **diagram by diagram**, proof in old-fashioned perturbation theory
  - Fix state and cut up squared diagrams in all possible ways

# $Z \to e^+e^- +$ final state radiation

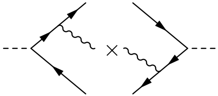
$\mathcal{M} \quad \times \quad \mathcal{M}^\dagger$



$\Gamma = \Gamma_0$ (finite)



$$\frac{\Gamma}{\Gamma_0} \propto -\frac{1}{4\epsilon^2} - \frac{3}{8\epsilon}$$



$$\frac{\Gamma}{\Gamma_0} \propto \frac{1}{4\epsilon^2} + \frac{3}{8\epsilon}$$

$m_e = 0$, Dim reg, CM frame

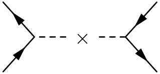
**Soft singularities**

cancel by Bloch-Nordsieck

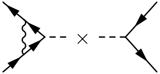
**Collinear singularities**

happen to also cancel

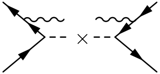
# $e^+e^- \to Z$ + final state radiation



$$\sigma = \sigma_0\, \delta(1-z)\ (\text{finite})$$



$$\frac{\sigma}{\sigma_0} \propto \delta(1-z)\left(-\frac{1}{4\,\epsilon^2} - \frac{3}{8\,\epsilon}\right)$$



$$\frac{\sigma}{\sigma_0} \propto \frac{\delta(1-z)}{4\,\epsilon^2} - \frac{1+z^2}{4\,\epsilon\,[1-z]_+}$$

**Soft singularities**

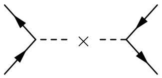
cancel by Bloch-Nordsieck

**Collinear singularities**

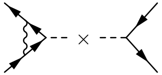
**do not cancel**

$m_e = 0,\ z = \dfrac{m_z^2}{E_{\mathrm{CM}}^2}$, Dim reg, CM frame

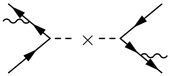
# $e^+e^- \to Z$ + **initial state absorption**



$\sigma = \sigma_0\, \delta(1 - z)$ (finite)



$$\frac{\sigma}{\sigma_0} \propto \delta(1 - z)\left(-\frac{1}{4\,\epsilon^2} - \frac{3}{8\,\epsilon}\right)$$



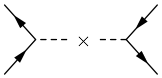
$$\frac{\sigma}{\sigma_0} \propto \delta(1 - z)\left(\frac{1}{4\epsilon^2} + \frac{3}{8\epsilon}\right)$$

Including initial state absorption diagram:

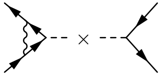
**Soft** and **collinear** singularities cancel

$m_e = 0,\ z = \dfrac{m_z^2}{E_{\mathrm{CM}}^2}$, Dim reg, CM frame

# $e^+e^- \to Z$ + initial state absorption & final state radiation



$$\sigma = \sigma_0\,\delta(1-z)\ \text{(finite)}$$



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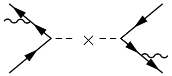

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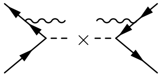


$$\frac{\sigma}{\sigma_0} \propto \frac{\delta(1-z)}{4\,\epsilon^2} - \frac{1+z^2}{4\,\epsilon\,[1-z]_+}$$

**Leftover singularity!**

$m_e = 0,\ \ z = \dfrac{m_z^2}{E_{\mathrm{CM}}^2}$, Dim reg, CM frame

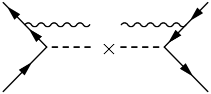
# Cancelling IR singularities

- Which diagrams cancel the leftover singularity?
- **Stronger KLN Theorem (2018):** $S$-matrix elements squared are IR finite when summing over **final states or initial states:**

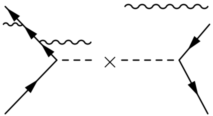
$$\sum_f |\langle f| S |i\rangle|^2 < \infty, \qquad \sum_i |\langle f| S |i\rangle|^2 < \infty$$

- Not including all possible diagrams.

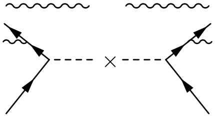
# $e^+e^- \to Z$ + initial state absorption & final state radiation



$$\frac{\sigma}{\sigma_0} \propto \frac{\delta(1-z)}{4\,\epsilon^2} - \frac{1+z^2}{4\,\epsilon\,[1-z]_+}$$



$$\frac{\sigma}{\sigma_0} \propto -\frac{\delta(1-z)}{2\,\epsilon^2} + \frac{6z^2-4z+2}{4\,\epsilon\,[1-z]_+}$$



$$\frac{\sigma}{\sigma_0} \propto \frac{\delta(1-z)}{4\,\epsilon^2} - \frac{5z^2-4z+1}{4\,\epsilon\,[1-z]_+}$$

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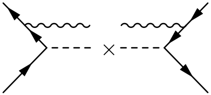
## $e^+e^- \to Z$ + initial state absorption & final state radiation



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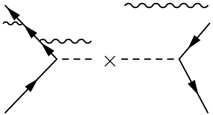
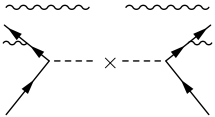


$$\frac{\sigma}{\sigma_0} \propto -\frac{\delta(1-z)}{2\,\epsilon^2} + \frac{6z^2-4z+2}{4\epsilon\,[1-z]_+}$$

**Soft** and **collinear** singularities cancel



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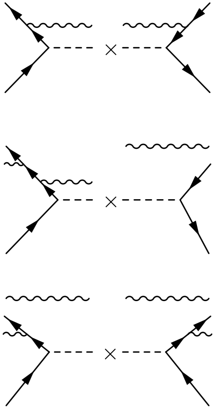
$m_e = 0,\ z = \frac{m_z^2}{E_{\mathrm{CM}}^2}$, Dim reg, CM frame

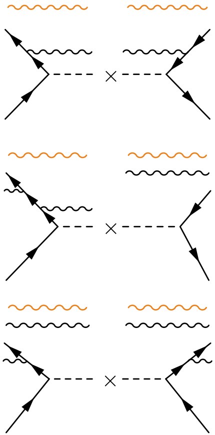
## $e^+e^- \to Z$ + initial state absorption & final state radiation



**No reason to stop at**

**1 disconnected photons**

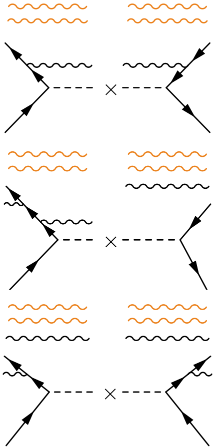
## $e^+e^- \to Z$ + initial state absorption & final state radiation



**No reason to stop at**

**2 disconnected photons**

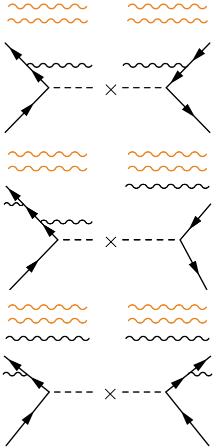
# $e^+e^- \to Z$ + initial state absorption & final state radiation



**No reason to stop at**

**$n$ disconnected photons**

# $e^+e^- \to Z$ + initial state absorption & final state radiation



**Soft** and **collinear** singularities

cancel in each triplet of diagrams

$$\sum_m \sigma_{m,n} \propto -\frac{(1-z)^3}{z^2 n^4} + \mathcal{O}\left(\frac{1}{n^6}\right)$$

$m:$ No. of initial state photons

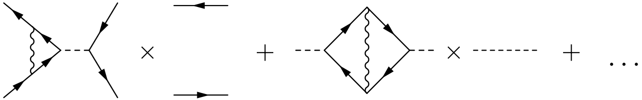
$n:$ No. of final state photons

$z = m_z^2/E_{CM}^2$

# KLN Theorem Interpretation

Why did it work to sum over disconnected photons?

**KLN requires a term where $f = i \rightarrow$ forward scattering**



The IR singularity cancellation worked only since the forward scattering diagrams

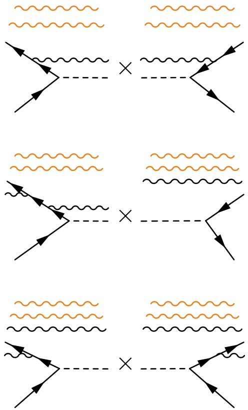
$Z + n\,\gamma \rightarrow Z + n\,\gamma$ are finite for any $n$

# 3 WAYS OF MAKING $e^+e^- \to Z$ FINITE:

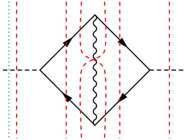
1. With infinite number disconnected photons, but no forward scattering.
2. Initial state sum, including forward scattering.
3. Final state sum, including forward scattering.

# Making $e^+e^- \to Z$ finite: 1. Disconnected photons

Sum of triplets of diagrams are IR finite



# Making $e^+e^- \to Z$ finite: 2. Initial state sum



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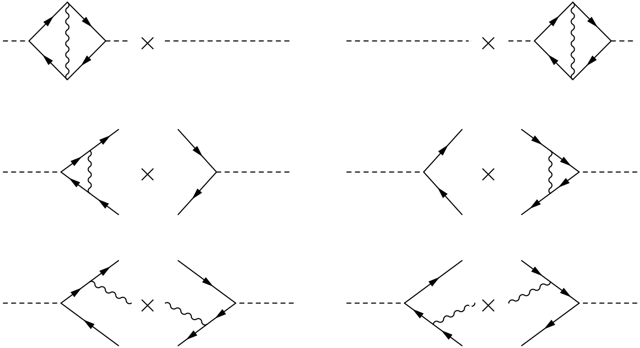
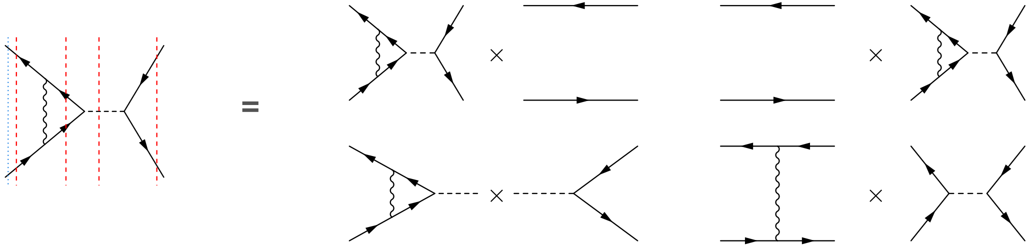
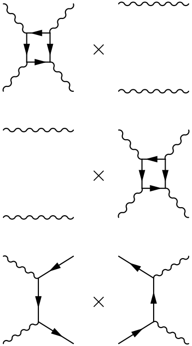


# Making $e^+e^- \to Z$ finite: 3. Final state sum



# $\gamma\gamma \to \gamma\gamma$ SCATTERING



$$\sigma \propto \frac{1}{2\,\epsilon}$$

$$\sigma \propto \frac{1}{2\,\epsilon}$$

$$\sigma \propto -\frac{1}{\epsilon}$$

**Rate to produce no charged particles in photon collisions is not IR safe**

$m_e = 0$, Dim reg, CM frame

# Summary

- Need **forward scattering** and **disconnected diagrams** in KLN theorem.
- 3 ways of making $e^+e^- \to Z$ finite:
  - With infinite number disconnected photons, but no forward scattering.
  - Initial state sum, including forward scattering.
  - Final state sum, including forward scattering.
- IR divergence in $\gamma\gamma \to \gamma\gamma$ scattering is cancelled by $\gamma\gamma \to e^+e^-$.
- Need a **revised understanding** of what is observable.

# Conclusion of Cross Section Method

$$\sum_f |\langle f|\, S\, |i\rangle|^2 \propto 1 < \infty$$

**Conclusion:** KLN theorem = unitarity.

If we sum over **all possible diagrams** we get 1 by unitarity, and 1 is IR finite.

Not closer to finding the **minimal set of diagrams** needed for IR finiteness.

**Need new ideas beyond the cross section method.**

# 2. A Finite $S$-matrix

# The Scattering Matrix ($S$-matrix)

- Properties extensively studied.
  - How to **encode its content**? *Spinors, twistors, amplituhedron?*
  - What are its **symmetries**? *Lorentz invariance, Dual conformal invariance?*
  - What **constraints** can we impose? *Steinmann relations, limits?*
- Still, **the $S$-matrix does not exist** in theories with massless particles.
  - Divergent in perturbation theory.
  - Zero non-perturbatively.

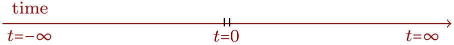
# The Scattering Matrix ($S$-matrix)

Why are our previous calculations valuable?

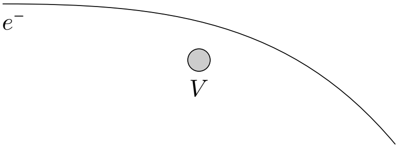
What is the **fundamental object** we should calculate?

What do we gain from a **firmer mathematical ground?**

# What is Scattering?




| $e^{-iH_0t}\,\lvert i\rangle$ | $e^{-iHt}$ | $e^{-iH_0t}\,\lvert f\rangle$ |
|---|---|---|
| $\downarrow$ | $\downarrow$ | $\downarrow$ |
| freely evolving state | scattering, | freely evolving state |
| in far past | evolution with $H$ | in far future |



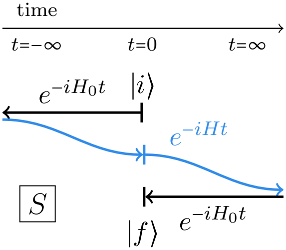

$\langle f|\,S\,|i\rangle$: **Probability amplitude**

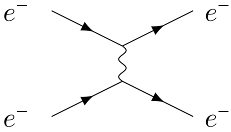
for measuring $|f\rangle$ given $|i\rangle$

# What is Scattering?

$S$-matrix: **Probability amplitude** for measuring $|f\rangle$ given $|i\rangle$

$$S_{fi} = \lim_{t_\pm \to \pm\infty} \langle f| e^{iH_0 t_+} e^{-iHt_+} e^{iHt_-} e^{-iH_0 t_-} |i\rangle$$



# Traditional Definition of $S$-matrix

$$S_{fi} = \lim_{t_\pm \to \pm\infty} \langle f|\, e^{iH_0 t_+} e^{-iHt_+} e^{iHt_-} e^{-iH_0 t_-} \,|i\rangle$$

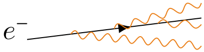
Free Theory: $S = \mathbb{1}$ $\quad S_{fi} = \langle f|i\rangle$ ✓

QM, short range potential: ✓

Const. potential $\quad H = H_0 + V_0$: $\quad S_{fi} = \langle f|i\rangle \lim_{T\to\infty} e^{-2iV_0 T}$ ?

QED: $S = \mathbb{1} - \frac{\alpha}{\epsilon^2} + \cdots = -\infty?$

$S = \exp\left\{-\frac{\alpha}{\epsilon^2}\right\} = 0?$



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QED: $\quad S = \mathbb{1} - \frac{\alpha}{\epsilon^2} + \cdots = -\infty?$

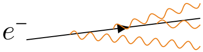
$$S = \exp\left\{-\frac{\alpha}{\epsilon^2}\right\} = 0?$$

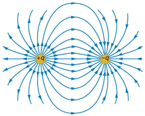
Interactions do not vanish as $t \to \pm\infty$ in QED

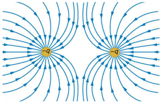
Must **redefine** $S$-matrix in theories with long range interactions

# Modify $S$-matrix to $S_H$

**Recall:** Interactions do not vanish as $t \to \pm\infty$ in QED.



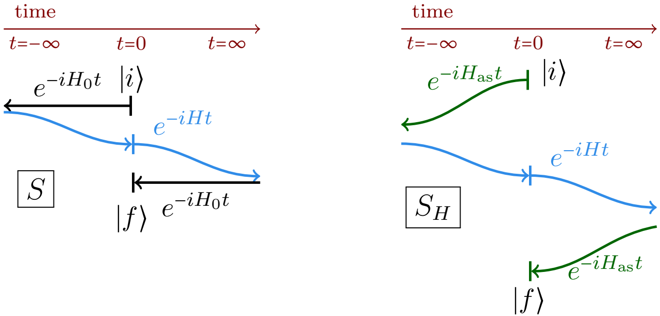


**Redefine** $S$-matrix in theories with long range interactions:

$$S_{fi} = \lim_{t_\pm \to \pm\infty} \langle f | \, e^{iH_0 t_+} e^{-iHt_+} e^{iHt_-} e^{-iH_0 t_-} \, | i \rangle$$

$$\to S^H_{fi} = \lim_{t_\pm \to \pm\infty} \langle f | \, e^{iH_{\rm as} t_+} e^{-iHt_+} e^{iHt_-} e^{-iH_{\rm as} t_-} \, | i \rangle$$

# Modify $S$-matrix to $S_H$



# Questions

$$S^H_{fi} = \lim_{t_\pm \to \pm\infty} \langle f | \, e^{iH_{\rm as} t_+} e^{-iHt_+} e^{iHt_-} e^{-iH_{\rm as} t_-} \, | i \rangle$$

(i) How to pick $H_{\rm as}$?

- Criteria: IR finite, easy to calculate, useful in practice, consistent with every measurement to date.

(ii) How to calculate matrix elements of $S_H$?

(iii) How to interpret $S_H$?

# Choice of $H_{\text{as}}$

(i) *How to pick $H_{as}$?*

- Use **factorization**, and techniques from Soft-Collinear Effective Theory (SCET):

$$H_{\text{as}} = H_{SCET}$$

- IR finite by construction due to **universality of IR divergences**.
- States **evolve independently of how they scatter**.
- New UV divergences dealt with using renormalization.
- No scales, most integrals are zero in dim reg.

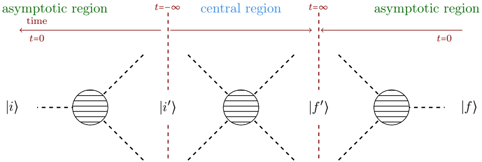
# Three Part Calculation

(ii) *How to calculate matrix elements of $S_H$?*

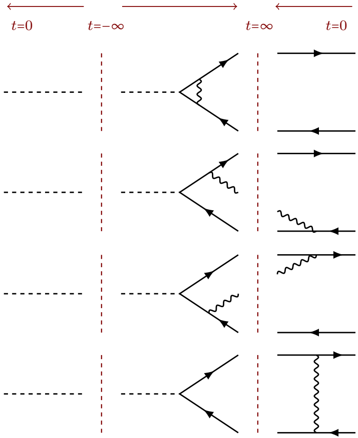
- Calculation trick in perturbation theory:

$$S^H_{fi} = \int d\Pi'_f \int d\Pi'_i \underbrace{\langle f | \Omega^{\mathrm{as}}_+ | f' \rangle}_{\text{TOPT rules}} \underbrace{\langle f' | S | i' \rangle}_{\text{usual Feynman rules}} \underbrace{\langle i' | \Omega^{\mathrm{as}}_+ | i \rangle}_{\text{TOPT rules}}$$

- Calculations split into three parts:



# EXAMPLE: $Z \to e^+ \ e^-$ FOR $H_{AS} = H_{SCET}$




$$= \mathcal{M}_0 \frac{\alpha}{4\pi}\left[\frac{1}{\epsilon_{\mathrm{UV}}} - \frac{2}{\epsilon_{\mathrm{IR}}^2} - \frac{4+2L}{\epsilon_{\mathrm{IR}}} - 8 + \frac{\pi^2}{6} - L^2 + 3L\right]$$

$$= \mathcal{M}_0 \frac{\alpha}{4\pi}\left[\frac{2}{\epsilon_{\mathrm{IR}}^2} + \frac{4+2L}{\epsilon_{\mathrm{IR}}} - \frac{2}{\epsilon_{\mathrm{UV}}^2} - \frac{4+2L}{\epsilon_{\mathrm{UV}}}\right]$$

$$\langle e^+ e^- | S_H | Z\rangle^{\overline{\mathrm{MS}}} = \mathcal{M}_0 + \mathcal{M}_0 \frac{\alpha}{4\pi}\left[-8 + \frac{\pi^2}{6} - L^2 + 3L\right]$$

$m_e$=0, $L$=ln $\frac{-E_{CM}^2}{\mu^2}$, $\mathcal{M}_0$: LO matrix element, Dim reg, CM frame

# Interpretation of $S_H$

(iii) *How to interpret* $S_H$*?*

a. Wilson coefficients in Soft-Collinear Effective Theory (SCET)

b. Remainder functions in $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory (SYM)

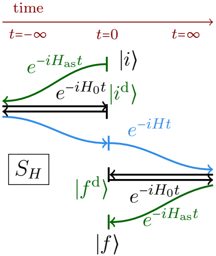
c. Dressed states / Coherent states

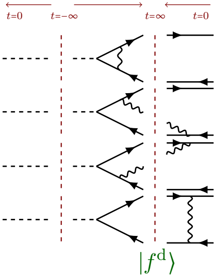
# 3. Finite Scattering Amplitudes

## C. Coherent States

- Arise as intermediate steps in $S_H$ calculations:

$$S^H_{fi} = \sum_{f'} \sum_{i'} \underbrace{\langle f | \, \Omega^{\mathrm{as}}_+ \, | f' \rangle}_{\langle f^{\mathrm{d}} |} \langle f' | \, S \, \underbrace{| i' \rangle \langle i' | \, \Omega^{\mathrm{as}}_+ \, | i \rangle}_{| i^{\mathrm{d}} \rangle}$$



## C. Coherent States

- Arise as intermediate steps in $S_H$ calculations:

$$S^H_{fi} = \sum_{f'} \sum_{i'} \underbrace{\langle f | \Omega^{\text{as}}_{+} | f' \rangle \langle f' |}_{\langle f^{\mathrm{d}} |} S \underbrace{| i' \rangle \langle i' | \Omega^{\text{as}}_{+} | i \rangle}_{| i^{\mathrm{d}} \rangle}$$

**Mathematically the same as the finite $S$-matrix**

# Future Directions: Analytic Structure of $S_H$

*We have explored:*

$S_H$ provides an alternative definition of familiar QFT objects.

*New goal:*

Examine properties of $S_H$, e.g. using **bootstrapping** methods.

*Tools needed:*

Better handle on **analytic structure** of amplitudes.

# Conclusions of Finite $S$-matrix Method

$\boldsymbol{S_H}$: "hard" $S$-matrix defined by exploiting universality of asymptotic interactions in theories with massless particles.

- Encodes **hard dynamics** of scattering processes.
- Interpretations:
  a. Wilson coefficients
  b. $\mathcal{N} = 4$ remainder functions
  c. Coherent states
- *Explore* ***analytic structure*** *of* $S$ *and* $S_H$.

# Future directions

- Extend Steinmann relations.
- Apply new results to **bootstrapping** finite $S$-matrix?
- Extend to **massless particles**?
- More general proof of **Steinmann**?

# Conclusions

- **IR divergences** remain a problem in QFT
- Explored **three solutions**:
    1. Finite cross sections: *Sum over all diagrams for finiteness.*
    2. Finite $S$-matrix: *Encodes hard dynamics of scattering processes.*
    3. Finite scattering amplitudes (Coherent states): *Same as Finite $S$-matrix.*
- Future directions: Explore **analytic structure**.

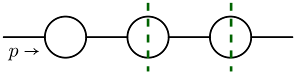
Thanks!

# Previous Results: Extend Cutting Rules

- **Traditional Cutting Rules**: (Cutkosky 1960)

$$\mathrm{Disc}\mathcal{M} = \mathcal{M}|_{+i\varepsilon} - \mathcal{M}|_{-i\varepsilon} \quad = \quad \sum \mathrm{Cut}\mathcal{M}$$

- **Extended Cutting Rules:** (Bourjaily, HH, McLeod, Schwartz, Vergu 2020)

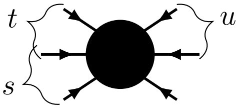



$$\mathrm{Disc}^2\mathcal{M} \sim \sum \mathrm{Cut}^2\mathcal{M} + \sum \mathrm{Cut}^3\mathcal{M} + \cdots$$

# Steinmann Relations

**$\mathcal{M}$ cannot have sequential discontinuities**

**in partially overlapping channels**




- $\mathcal{M}$ cannot contain $\ln(s)\ln(t)$ but can contain $\ln(s)\ln(u)$.
- Important for **bootstrapping** amplitudes.
- Proofs: Steinmann 1960; Bourjaily, HH, McLeod, Schwartz, Vergu 2020.

# Steinmann Relations

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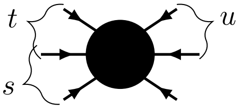


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**New Goal: Find stronger constraints**

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