QCD resummation in the large-$\beta_0$ limit

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We seek closed expressions for QCD perturbative series.

Problem 1: QCD series are in general divergent.

Problem 2: We don’t know their infinitely many terms!

1. Asymptotic series and the Borel transform

2. The large-$\beta_0$ limit of QCD

In these conditions:

Develop the resummation formalism for...

3. Series without cusp-anomalous dimension

4. Series with cusp-anomalous dimension

5. Phenomenology I: short-distance mass schemes

6. Phenomenology II: jets from massless and massive quarks
Outline

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Perturbative series in QFT can be

- Convergent
- Asymptotic (divergent!)
- Finite convergence radius
- Zero convergence radius

Example: $\sum_{n=0}^{\infty} a_n \alpha^n$, $a_n \sim k b^n \Gamma(n + 1 + c)$

Seems to converge at low $n$ but diverges after $n \sim 20$
We seek closed expressions for perturbative series (resummation).

Is there any way to resum asymptotic series?

Borel resummation
Borel transform improves convergence: \( \alpha_s^{n+1} \rightarrow \frac{u^n}{n!} \)

\[
A(\alpha_s) = \sum_{n=0}^{\infty} a_n \alpha_s^n \quad \rightarrow \quad B[A(\alpha_s)](u) = a_0 + \sum_{n=0}^{\infty} a_n \frac{u^n}{n!}
\]

**Sum up (Borel sum)**

**Invert**

\[
A(\alpha_s) = a_0 + \int_0^\infty du e^{-\alpha_s/u} B[A(\alpha_s)](u)
\]

**Resummed**

**The trick!**

- Convergent series → Finite integral
- Asymptotic series → \( B[A(\alpha_s)](u) \)

has poles
For asymptotic series, the poles of $B[A(\alpha_s)](u)$ in the complex $u$-plane are known as renormalons.

\[ u = \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2, \ldots \]

- **UV renormalons**: $u = -\frac{1}{2}, -1, -\frac{3}{2}, -2, \ldots$ (Alternating sign series (summable))
- **IR renormalons**: $u = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ (Non-alternating sign series (non-summable))

Integration path

Need to be dodged!
Dodge poles by infinitesimally deforming the integration contour

\[
\lim_{\varepsilon \to 0} \left\{ \begin{array}{l}
\int_{u_n - \delta}^{u_n + \delta} dx \frac{1}{(x - u_n)^k} \\
\end{array} \right\} = \begin{cases} 
\mp i \pi, & k = 1 \\
-\frac{2}{(k-1)\delta^{k-1}}, & k \text{ even} \\
0, & k \text{ odd, } k > 1
\end{cases}
\]

Principal value prescription (P.V.)

\[
P.V. \pm \left\{ \int_{u_n - \delta}^{u_n + \delta} dx \frac{1}{(x - u_n)^k} \right\} = \begin{cases} 
\mp i \pi, & k = 1 \\
-\frac{2}{(k-1)\delta^{k-1}}, & k \text{ even} \\
0, & k \text{ odd, } k > 1
\end{cases}
\]

\[
P.V. \{A\} \equiv \frac{1}{2} [P.V. + \{B[A]\} + P.V. - \{B[A]\}]
\]

Estimated resummed value of the series

\[
\delta_A \equiv \frac{1}{2\pi i} [P.V. + \{B[A]\} - P.V. - \{B[A]\}]
\]

Estimated resummation error (\(\sim\) minimal term). Given by sum of residues of IR renormalons.

*This definition ensures \(P.V. \{f\} \in \mathbb{R}\) if \(f \in \mathbb{R}\) and \(P.V. \{f\}\) zero for odd \(f\) along symmetric interval.
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We start with a series in the bare formalism

\[
A_0 = 1 + \sum_{l=1}^{\infty} \left( \frac{g_0^2}{(4\pi)^2} \right)^l \sum_{n=0}^{l-1} a_l, n n_f^n
\]

**Large-\(\beta_0\) counting:** keep track of powers of \(n_f\) by \(\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f\)

\[
A_0 = 1 + \sum_{l=1}^{\infty} \left( \frac{g_0^2}{(4\pi)^2} \right)^l \sum_{n=0}^{l-1} b_l, n \beta_0^n
\]

\[b_{l,i} \equiv \sum_{n=i}^{l-1} \binom{n}{i} (-1)^i \frac{a_l, n}{(\frac{4}{3} T_F)^n} \left( \frac{11}{3} C_A \right)^{n-i}\]

**Large-\(\beta_0\) expansion:** expand assuming \(O(\alpha_s \beta_0) \sim 1\)
Large-$\beta_0$ expansion: expand assuming $O(\alpha_s \beta_0) \sim 1$

The renormalization of $\alpha_s$ is simple when $\beta_0$ is large since from $\beta_n \sim O(\beta_0^n)$ for $n > 0$ one has

$$Z_\alpha = \frac{1}{1 + \alpha_s \beta_0/(4\pi \epsilon)}$$

Performing this substitution and expanding in $1/\beta_0$

$$A_0 = 1 + \frac{1}{\beta_0} \sum_{i=1}^{\infty} c_i \left( \frac{\alpha_s \beta_0}{4\pi} \right)^i + O\left( \frac{1}{\beta_0^2} \right)$$

$$c_i = (-1)^i e^{-i} \sum_{l=1}^{i} \frac{(-1)^l (l)_{i-l} \epsilon^l \mu^{2l\epsilon}}{(i-l)!} b_{l,l-1}$$

We take this as our new modified coupling constant

$$\beta \equiv \frac{\alpha_s \beta_0}{4\pi}$$

Coefficients of highest $n_f$ power

Only corrections from gluon propagator corrected with quark loops

The large-$\beta_0$ limit
The effective gluon propagator can be computed in arbitrary gauge with $d = 4 - 2\epsilon$ and naive non-abelianization $n_f \mapsto -\frac{3}{2}\beta_0$:

$$A_0 = 1 + \frac{1}{\beta_0} \sum_{i=1}^{\infty} C_i \left( \frac{\alpha_s \beta_0}{4\pi} \right)^i + O\left( \frac{1}{\beta_0^2} \right)$$

$$C_i = (-1)^i \epsilon^{-i} \sum_{l=1}^{i} \frac{(-1)^l(l)!}{(i-l)!} \epsilon^l \bar{\mu}^{2l} e^{l} n_{l,l-1}$$

The function $\Delta_n^{\mu\nu ab}(k)$ generates the infinitely many $1/\beta_0$ terms.

$$\Delta_n^{\mu\nu ab}(k) \equiv \left[ \frac{g_0^2}{(4\pi)^2-\epsilon} n_f T_F P_B(\epsilon) \right]^n (-i\delta^{ab}) \left[ \frac{g^{\mu\nu}}{(k^2)^{1+n\epsilon}} - \frac{k^\mu k^\nu}{(k^2)^{2+n\epsilon}} (1 - \xi \delta_{n0}) \right]$$

Gluon propagator with momentum shifted by $h \equiv n\epsilon$

$$P_B(\epsilon) \equiv \frac{(-1)^{-\epsilon} 4(1-\epsilon) \Gamma(\epsilon) \Gamma^2(1-\epsilon)}{(2\epsilon - 3) \Gamma(2-2\epsilon)}$$

In practice the insertion of $\Delta_n^{\mu\nu ab}(k)$ is a (shifted) 1-loop computation which we split as:

$$D_{sh}(h) = \left[ \frac{g_0^2}{(4\pi)^2-\epsilon} n_f T_F P_B(\epsilon) \right]^n \left( \frac{g_0}{4\pi} \right)^2 a(h,\epsilon)$$

The function $a(h,\epsilon)$ generates the infinitely many $1/\beta_0$ terms.
All-in all, the expression for our series in the large-\(\beta_0\) limit is (defining \(\delta A_0 \equiv A_0 - 1\)):

\[
\beta_0 A_0 = \sum_{i=1}^{\infty} \beta^i (-1)^i \epsilon^{-i} \sum_{l=1}^{i} \frac{(-1)^l (l - i + l) F^\mu(\epsilon, l \epsilon)}{(i - l)! l}
\]

\[
F^\mu(\epsilon, u) = \left( \frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^u \left[ -\frac{3}{4} e P_B(\epsilon) \right]^u \frac{1}{\epsilon} a(u - \epsilon, \epsilon)
\]

We found closed expressions for all three cases!

\[
F^\mu(\epsilon, u) = \left( \frac{\mu^2}{\omega^2} \right)^u F(\epsilon, u)
\]

where \(\omega\) is an external energy-scale (ex. mass or CM energy).

From here we distinguish when \(u \to 0\)

- **Finite series** (no renormalization) \(\to F(\epsilon, u)\) starts at \(O(u)\).
- **Non-cusp series** (1-loop divergences start as \(1/\epsilon\)) \(\to F(\epsilon, u)\) starts at \(O(1)\)
- **Cusp series** (1-loop divergences start as \(1/\epsilon^2\)) \(\to F(\epsilon, u)\) starts at \(O(1/u)\)

in all cases \(F(\epsilon, u)\) starts at \(O(1)\) when \(\epsilon \to 0\).
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For non-cusp series we have regular $F(\epsilon, u)$:

$$F^\mu(\epsilon, u) \equiv \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \epsilon^i u^j F_{i,j}^\mu, \quad F_{i,0}^\mu \equiv F_{i,0}$$

Plugging it back and after several boring manipulations…

$\mu$-dependent term

$$\beta_0 \delta A_0 = \sum_{i=1}^{\infty} \beta^i \left[ \Gamma(i) \frac{F_{0,i}^\mu}{i} - \frac{(-1)^i F_{i,0}}{i} \right]$$

$\mu$-independent terms

$$- \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} \frac{1}{\epsilon^j} \frac{(-\beta)^{i+j} F_{i,0}}{i+j}$$

Finite term: efficient way of computing all the coefficients of the renormalized series

UV divergences have been separated for simple removal (we discarded $j > 0$ terms)

*In the large-$\beta_0$ limit, multiplicative renormalization reduces to addition:

$$A_0 = Z_A A = 1 + \delta Z_A + \delta A + O(1/\beta_0^2)$$
For non-cusp series we have regular $F(\epsilon, u)$:

$$F^\mu(\epsilon, u) \equiv \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \epsilon^i u^j F^\mu_{i,j}, \quad F^\mu_{i,0} \equiv F_{i,0}$$

Plugging it back and after several boring manipulations…

$$\beta_0 \delta A_0 = \sum_{i=1}^{\infty} \beta^i \left[ \Gamma(i) F_{0,i}^\mu - \frac{(-1)^i F_{i,0}}{i} \right] - \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} \frac{1}{\epsilon^{i+j}} (-\beta)^{i+j} \frac{F_{i,0}}{i+j}$$

*In the large-$\beta_0$ limit, multiplicative renormalization reduces to addition:

$$A_0 = Z_A A = 1 + \delta Z_A + \delta A + O(1/\beta_0^2)$$
The magic occurs when one realizes each term in the perturbative sum admits a closed integral form:

$$\sum_{i=1}^{\infty} \beta^i \Gamma(i) F_{0,i}^{\mu} = \sum_{i=1}^{\infty} F_{0,i}^{\mu} \int_0^\infty d\tau \tau^{i-1} e^{-\tau/\beta} = \int_0^\infty d\tau e^{-\tau/\beta} \frac{F^{\mu}(0, \tau) - F(0, 0)}{\tau},$$

$$-\sum_{i=1}^{\infty} \frac{(-\beta)^i}{i} F_{i,0} = \sum_{i=1}^{\infty} F_{i,0} \int_{-\beta}^0 d\tau \tau^{i-1} = \int_{-\beta}^0 d\tau \frac{F(\tau, 0) - F(0, 0)}{\tau},$$

$$-\sum_{i=0}^{\infty} \frac{(-\beta)^{i+j}}{i+j} F_{i,0} = \sum_{i=0}^{\infty} F_{i,0} \int_{-\beta}^0 d\tau \tau^{i+j-1} = \int_{-\beta}^0 d\tau \tau^{j-1} F(\tau, 0).$$
The magic occurs when one realizes each term in the perturbative sum admits a closed integral form:

\[ \sum_{\mu} \frac{1}{\beta_0^j} \int_{-\beta}^{0} d\tau \tau^{j-1} F(\tau,0) \]

Renormalization factor

\[ Z_{\overline{\text{MS}}} = \sum_{j=1}^{\infty} \frac{1}{\epsilon^j} \int_{-\beta}^{0} d\tau \tau^{j-1} F(\tau,0) \]

No UV substractions

\[ \gamma_A(\beta) = \frac{2\beta}{\beta_0} \sum_{i=0}^{\infty} \beta^i (-1)^i F_i,0 = \frac{2\beta}{\beta_0} F(-\beta,0) \]

Anomalous dimension

Closed, non-integral form → unambiguous
Removing the $\mu$ dependence from the Borel integral

Do poles and ambiguities depend on $\mu$?!

In the large-$\beta_0$ limit, the $\beta_{\text{QCD}}$ function acquires a simple form ($\epsilon = 0$):

\[
\beta_{\text{QCD}} = -\frac{\alpha_s^2 \beta_0}{2\pi}
\]

This can be used to solve the running of $\beta \equiv \frac{\alpha_s \beta_0}{4\pi}$

\[
\beta(\mu) = \frac{\beta_{\mu_0}}{1 + 2\beta_{\mu_0} \log\left(\frac{\mu}{\mu_0}\right)} = \frac{1}{2\log\left(\frac{\mu}{\Lambda_{\text{QCD}}}\right)}
\]

where the Landau pole is at the $\mu$-independent position $\Lambda_{\text{QCD}} \equiv \mu e^{-\frac{1}{2\beta(\mu)}}$

\[
\int_0^\infty d\tau e^{-\tau/\beta} \left[ \left(\frac{\mu^2}{\omega^2}\right)^\tau F(0, \tau) - F(0, 0) \right] \tau \]

\[
= \int_0^\infty d\tau e^{-\tau/\beta} \left(\frac{\mu^2}{\omega^2}\right)^\tau F(0, \tau) - F(0, 0) + F(0, 0) \int_0^\infty d\tau \left(\frac{\mu^2}{\omega^2}\right)^\tau - 1
\]

Taylor-expand and integrate

$\mu$-independent!
\[
\beta_0 \delta A = F(0, 0) \log \left( \frac{\beta \omega}{\beta} \right) + \int_0^\infty d\tau \left( \frac{\Lambda_{QCD}}{\omega} \right)^{2\tau} \frac{F(0, \tau) - F(0, 0)}{\tau} + \int_{-\beta}^0 d\tau \frac{F(\tau, 0) - F(0, 0)}{\tau}
\]

Multiplies the residues of \( F(0, \tau) \)

Renormalized series, alternative expression

The entire \( \mu \)-dependence of the series is contained in the unambiguous terms through \( \beta = \beta(\mu) \).

The IR renormalons are the poles of \( F(0, \tau) \) and neither their position nor their residues depend on the unphysical scale \( \mu \), but the latter does depend on \( \omega \).

Each pole’s ambiguity is enhanced by \( \left( \Lambda_{QCD}/\omega \right)^{2\tau} \), with \( \tau \) being the pole’s position in the positive real axis.

The explicit presence of \( \Lambda_{QCD} \) indicates \( \delta_A \) estimates the size of non-perturbative (power) corrections.
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We present an extension of the previous formalism to series with cusp-anomalous dimension.

For such series $F(\epsilon, u)$ starts at $O(1/u)$ so we define $G(\epsilon, u) \equiv uF(\epsilon, u)$

\[
\beta_0 A_0 = \sum_{i=1}^{\infty} \beta_i (-1)^i \epsilon^{-i} \sum_{l=1}^{i} \frac{(-1)^l (l)_{i-l}}{(i-l)!} F^{\mu}(\epsilon, l\epsilon) \quad \frac{1}{l}
\]

\[
\beta_0 \delta A_0 = \sum_{i=1}^{\infty} \beta_i (-1)^i \epsilon^{-i} \sum_{l=1}^{i} \frac{(-1)^l (l)_{i-l}}{(i-l)!} G^{\mu}(\epsilon, l\epsilon) \quad \frac{1}{l^2 \epsilon}
\]

The presence of an extra $l^2$ complicates manipulations, but we do not despair and find...

\[
\beta_0 \delta A_0 = \sum_{i=1}^{\infty} (-\beta)^i \left[ G^{\mu}_{0,i+1} (-1)^i \Gamma(i) - \frac{H_i G_{i+1,0}}{i} - \frac{G^{\mu}_{i+1,1}}{i} \right] - \frac{1}{\epsilon} \sum_{i=1}^{\infty} \frac{(-\beta)^i}{i} \left( H_i G_{i,0} + G^{\mu}_{i-1,1} \right)
\]

\[-\sum_{j=2}^{\infty} \frac{1}{\epsilon^j} \sum_{i=0}^{\infty} \left[ (-\beta)^{i+j} \frac{H_i + j - 1}{i + j - 1} G_{i,0} + (-\beta)^{i+j} \frac{G^{\mu}_{i+1,1}}{i + j} \right] \]

\[G^\mu(\epsilon, u) \equiv \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} G^{\mu}_{i,j} \epsilon^i u^j \quad G(\epsilon, u) \text{ is regular}
\]

\[G^{\mu}_{i,0} \equiv G_{i,0} \]

This term provides the anomalous dimension.

New nuisance! Harmonic numbers.
Again each term admits a closed integral form but it is worth saying this time is much harder to find due to $H_i$...

\[
(-\beta)^i H_i = \int_{-\beta}^{0} d\tau \frac{\tau^i - (-\beta)^i}{\beta + \tau}
\]

Splitting the renormalization equation in a cusp and non-cusp part as

\[
\delta A_0 = \delta A + \delta Z_{A}^{\text{nc}} + \log\left(\frac{\mu^2}{\omega^2}\right)\delta Z_{A}^{\text{cusp}}
\]

\[
\delta Z_{A}^{\text{nc}} = \frac{1}{\epsilon} \int_{-\beta}^{0} d\tau \left[ \frac{dG(\tau, s)}{ds} \right]_{s=0} - \log\left(1 + \frac{\tau}{\beta}\right) \frac{G(\tau, 0) - G(0, 0)}{\tau}
+ \sum_{j=2}^{\infty} \frac{1}{\epsilon^j} \int_{-\beta}^{0} d\tau \tau^{j-1} \left[ \frac{dG(\tau, s)}{ds} \right]_{s=0} - G(\tau, 0) \log\left(1 + \frac{\tau}{\beta}\right)
\]

\[
\delta Z_{A}^{\text{cusp}} = \sum_{i=1}^{\infty} \frac{1}{\epsilon^j} \int_{-\beta}^{0} d\tau \tau^{j-1} G(\tau, 0)
\]
The renormalized cusp-series also has an apparently $\mu$-dependent ambiguous integral that turns out to be $\mu$-independent

$$\beta_0 \delta A = \log \left( \frac{\mu^2}{\omega^2} \right) G_{0,0} + \left[ \frac{G_{0,0}}{\beta_0} - G_{0,1} \right] \log \left( \frac{\beta}{\beta_0} \right)$$

$$+ \int_0^\infty d\tau \left( \frac{\Lambda_{\text{QCD}}}{\omega} \right)^{2\tau} \left[ \frac{G(0, \tau) - G(0, 0)}{\tau^2} - \frac{1}{\tau} \frac{dG(0, s)}{ds} \right]_{s=0}$$

$$+ \int_{-\beta}^0 d\tau \left\{ \frac{1}{\tau} \frac{d[G(\tau, s) - G(0, s)]}{ds} \right\}_{s=0} + \frac{G(\tau, 0) - G(0, 0)}{\tau} \log \left( \frac{\mu^2}{\omega^2} \right)$$

$$- \log \left( 1 + \frac{\tau}{\beta} \right) \left[ \frac{G(\tau, 0) - G(0, 0)}{\tau^2} - \frac{1}{\tau} \frac{dG(s, 0)}{ds} \right]_{s=0}$$

And finally we have closed forms for the anomalous dimension

$$\gamma_A(\beta) = \frac{2}{\beta_0} \int_{-\beta}^0 d\tau \frac{G(\tau, 0) - G(-\beta, 0)}{\beta + \tau} + \frac{2\beta dG(-\beta, s)}{\beta_0 ds} \bigg|_{s=0}$$

$$\Gamma_A(\beta) = \frac{2\beta}{\beta_0} G(-\beta, 0)$$
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We compute the massive quark self-energy with the effective gluon propagator.

\[ F_{Zm}^{OS}(\epsilon, u) = 2C_F e^{\gamma_E u} (u - 1)(3 - 2\epsilon)\Gamma(1 + u)\Gamma(1 - 2u) \frac{\Gamma(3 - u - \epsilon)}{\Gamma(3 - u - \epsilon)} \left[ \frac{3(1 - \epsilon)\Gamma(1 + \epsilon)\Gamma^2(1 - \epsilon)}{(3 - 2\epsilon)\Gamma(2 - 2\epsilon)} \right] \frac{u}{\epsilon} - 1 \]

Where the energy scale is given by \( \omega = \bar{m}(\mu) \)

This one function generates the large-\( \beta_0 \) series for the mass renormalization factor in the on-shell scheme, which allows us to compute...

- Relation between pole and \( \overline{MS} \) masses \( \rightarrow \overline{MS} \) mass anomalous dimension and running
- MSR mass and its R-anomalous dimension

... all of these perturbatively and in closed form (PV value and ambiguities).
\[ \delta \bar{m} = m_p - \bar{m} = -\delta Z_{m}^{\text{OS}}|_{\text{finite}} = \frac{\bar{m}}{\beta_0} \sum_{i=1}^{\infty} \beta_i \sum_{j=0}^{i} a_{i,j}^{\text{MS}} \log^j \left( \frac{\mu}{\bar{m}} \right) \]

\[ = -\frac{\bar{m}}{\beta_0} \left\{ F_{Z_{m}^{\text{OS}}}^{0,0} \log \left( \frac{\beta m}{\beta} \right) + \int_0^\infty d\tau \left( \frac{\Lambda_{QCD}}{m} \right)^{2\tau} \frac{F_{Z_{m}^{\text{OS}}}(0, \tau) - F_{Z_{m}^{\text{OS}}}(0,0)}{\tau} + \int_{-\beta}^0 d\tau \frac{F_{Z_{m}^{\text{OS}}}^{0,0}(\tau, 0) - F_{Z_{m}^{\text{OS}}}^{0,0}(0,0)}{\tau} \right\} \]

Indeed we don’t cross the poles of \( F(\epsilon, 0) \)

There are poles at all positive half integers. The most severe renormalon lays at \( u = 1/2 \)

The ambiguity \( \delta_{m_p} \) of the pole mass is the sum of the residues and with extra effort it can be also resummed

\[ \delta_m = -\frac{C_F}{2\beta_0} e^{5/6} \Lambda_{QCD} \left[ x^3 + (2 - x^2) \sqrt{4 + x^2} \right] \bigg|_{u=1/2} = -\frac{2C_F}{\beta_0} e^{5/6} \Lambda_{QCD} + O(\Lambda_{QCD}^3) \quad x \equiv \frac{e^{5/6} \Lambda_{QCD}}{\bar{m}} \]

The leading power ambiguity does not depend on \( \bar{m} \)
\[ \delta \bar{m} = m_p - \bar{m} = -\delta Z_{m}^{\text{OS}}|_{\text{finite}} = \frac{\bar{m}}{\beta_0} \sum_{i=1}^{\infty} \beta^i \sum_{j=0}^{i} a_{i,j}^{\text{MS}} \log^j \left( \frac{\mu}{\bar{m}} \right) \]

\[ = -\frac{\bar{m}}{\beta_0} \left\{ F_{0,0}^{Z_{m}^{\text{OS}}} \log \left( \frac{\beta \bar{m}}{\beta} \right) + \int_{0}^{\infty} d\tau \left( \frac{\Lambda_{\text{QCD}}}{\bar{m}} \right)^{2\tau} \frac{F_{Z_{m}^{\text{OS}}}(0, \tau) - F_{0,0}^{Z_{m}^{\text{OS}}}}{\tau} + \int_{-\beta}^{0} d\tau \frac{F_{Z_{m}^{\text{OS}}}(\tau, 0) - F_{0,0}^{Z_{m}^{\text{OS}}}}{\tau} \right\} \]

Closed form results vs partial sum of perturbative series

As series, \( m_p \) is clearly asymptotic

The PV prescription value is \( \mu \)-independent and agrees (within ambiguity) with the “convergent” value of the series.

The fixed order expression is \( \mu \)-dependent and \( \mu \) plays a role in the asymptotic behavior: for lower values the series “converges” faster but the divergent behavior is more pronounced.
The $\overline{\text{MS}}$ anomalous dimension is unambiguous (the $\mu$ derivative cancels the Borel integral) (agrees with the derivation of Palanques-Mestre and Pascual, Grozin)

\begin{align*}
\gamma_{\overline{m}}(\beta) &= -\frac{2\beta}{\beta_0} F_{Z_\overline{m}}^\text{OS}(-\beta, 0) \\
&= -\frac{C_F \beta (3 + 2\beta) \Gamma(4 - 2\beta)}{3 \beta_0 (2 + \beta) \Gamma(1 - \beta) \Gamma(2 + \beta)^3}
\end{align*}

Convergence radius of $\beta = 2.5$
The MSR mass is obtained from the pole-\overline{MS} relation

\[ \delta m^{\text{MSR}} = m_p - m^{\text{MSR}} = \frac{R}{\beta_0} \sum_{i=1}^{\infty} a_{i,0}^{\text{MS}} \beta_R^i \]

\[ = -\frac{R}{\beta_0} \left\{ \int_0^\infty d\tau \left( \frac{\Lambda_{\text{QCD}}}{R} \right)^{2\tau} \frac{F_{Z^{\text{OS}}}(0,\tau) - F_{Z^{\text{OS}}}(0,0)}{\tau} + \int_{-\beta_R}^0 d\tau \frac{F_{Z^{\text{OS}}}(\tau,0) - F_{Z^{\text{OS}}}(0,0)}{\tau} \right\} \]

and presents the same leading renormalon at 1/2

The R derivative does not cancel the Borel integral, however the 1/2 renormalon does cancel (higher order ones stay)

\[ \beta_0 \gamma_R(\beta_R) = -\frac{d}{dR} [\beta_0 m^{\text{MSR}}(R)] \]

\[ = -\int_0^\infty d\tau \left( \frac{\Lambda_{\text{QCD}}}{R} \right)^{2\tau} \frac{F_{Z^{\text{OS}}}(0,\tau) - F_{Z^{\text{OS}}}(0,0)}{\tau} \]

\[ -\int_{-\beta_R}^0 d\tau \frac{F_{Z^{\text{OS}}}(\tau,0) - F_{Z^{\text{OS}}}(0,0)}{\tau} \]

\[ -2\beta_R[F_{Z^{\text{OS}}}(\tau,0) - F_{Z^{\text{OS}}}(0,0)] \]
The MSR mass is obtained from the pole-\overline{MS} relation

$$\delta m_{\text{MSR}} = m_p - m_{\text{MSR}} = \frac{R}{\beta_0} \sum_{i=1}^{\infty} a_{i,0}^{\overline{\text{MS}}} \beta_R^i$$

The \( R \) derivative does not cancel the Borel integral, however the \( 1/2 \) renormalon does cancel (higher order ones stay)

$$\beta_0 \gamma_R(\beta_R) = -\frac{d}{dR} [\beta_0 m_{\text{MSR}}(R)]$$

$$= -\int_0^\infty \frac{d\tau}{\tau^2} \left( \frac{\Lambda_{\text{QCD}}}{R} \right)^{2\tau} \frac{F_{Z_{m}}(0, \tau) - F_{Z_{m}}(0, 0)}{1 - 2\tau}$$

$$- \int_{-\beta_R}^0 \frac{d\tau}{\tau^2} \frac{F_{Z_{m}}(\tau, 0) - F_{Z_{m}}(0, 0)}{\tau}$$

$$- 2\beta_R [F_{Z_{m}}(-\beta_R, 0) - F_{Z_{m}}(0, 0)]$$

\( n_f = n_l + n_h \)
When expanded in terms of $\alpha_s(\mu)$, the MSR mass acquires powers of $\log(\mu/R)$ and $\mu$ dictates the asymptotic behavior, while $R$ doesn't.

Therefore for a renormalon cancelation between two series both must be expanded in terms of the same $\alpha_s(\mu)$.
Comparison of short-distance mass schemes

The pole mass is computed as $m_p = \bar{m} + \text{P.V.}\{\delta\bar{m}\}$ and its ambiguity is too small to be clearly seen.

The $\bar{MS}$ mass is computed through $\gamma_{\bar{m}}(\alpha_s)$. It grows for $\mu < \bar{m}$ and becomes larger than agrees $m_p$ at $\mu < \bar{m}/2$. It is not ambiguous.

The MSR mass is computed through $\gamma_R(\alpha_s)$. It agrees with the $\bar{MS}$ mass at $\mu = \bar{m}$ and smoothly approaches $m_p$ for $\mu = 0$. Its ambiguity is too small to be shown.
Outline

1. Asymptotic series and the Borel transform
2. The large-$\beta_0$ limit of QCD
3. Series without cusp-anomalous dimension
4. Series with cusp-anomalous dimension
5. Phenomenology I: short-distance mass schemes
6. Phenomenology II: jets from massless and massive quarks
Often, $e^+e^- \rightarrow \text{Hadrons}$ collisions at high energies adopt di-jet configurations:

- High energetic (collinear) radiation travels together in two jets
- Low-energy, soft radiation populates the space between the jets.

In Soft Collinear Effective Theory (SCET) there is a factorization theorem for the event-shape differential cross section

$$\frac{1}{\sigma_0} \frac{d\sigma}{d\hat{e}} = H_Q \times J_n \otimes J_{\bar{n}} \otimes S$$

We present the computation in the large-$\beta_0$ limit of the Hard function (universal QCD matching onto SCET) and the SCET jet function (for hemisphere mass and trust event shapes)

(Fleming, Hoang, Mantry, Stewart)
(Bauer, Fleming, Lee, Sterman)
The hard function $H_Q(Q, \mu)$ is the modulus squared of the QCD matching coefficient onto SCET, $C_H(Q^2 + i0^+, \mu)$

on-shell scheme and dim.reg. → all self-energy and SCET diagrams vanish

→ $C_H$ = massless quark vector form factor

$$G_{CH}(\epsilon, u) = 2C_F e^{\gamma_E u}[(u - 1)\epsilon^2 + (u^2 - 2u + 3)\epsilon - 2]\frac{\Gamma(1 + u)\Gamma^2(1 - u)}{\Gamma(3 - u - \epsilon)} \left[\frac{3(\epsilon - 1)\Gamma^2(1 - \epsilon)\Gamma(1 + \epsilon)}{(2\epsilon - 3)\Gamma(2 - 2\epsilon)}\right]^{\frac{u-1}{\epsilon}}$$

On the way to the hard function we can use $G_{CH}$ to compute the anomalous dimension

$$\Gamma_{cusp}(\beta) = \frac{2C_F}{3\pi} \frac{\sin(\pi\beta)}{\beta_0 \Gamma(2 + \beta)^2}$$

(agree with Scimemi and Vladimirov)

Our results reproduce the leading flavour known results in full QCD up to $O(\alpha_s^4)$
The hard function $H_Q(Q, \mu)$ is the modulus squared of the QCD matching coefficient $C_H(Q^2 + i0^+, \mu)$

$$G_H(\epsilon, u) = 2\cos(\pi u)G_C(\epsilon, u)$$

<table>
<thead>
<tr>
<th>$G_H(\epsilon, 0)$</th>
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<tbody>
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<td>$(2n + 1)/2, \ n = 2, 3, 4...$</td>
<td>simple</td>
<td>No</td>
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<td>$1, 2, \ n = 3, 4, 5...$</td>
<td>double and simple</td>
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Again we don’t cross the poles of $G_H(\epsilon, 0)$

This time the two first renormalons at $u = 1$ and 2 are two double poles

The ambiguity can again be resumed

We observe logarithmic enhancement in the ambiguities corresponding to double poles

Double poles signal anomalous dimension with $n_f$ dependence at leading order for dimension 2 and 4 operators in OPE.
The hard function $H_Q(Q, \mu)$ is the modulus squared of the QCD matching coefficient $C_H(Q^2 + i0^+, \mu)$

$$G_H(\epsilon, u) = 2\cos(\pi u)G_{C_H}(\epsilon, u)$$

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Although it is small even for the smallest $Q$ applicable in SCET.
The hard function $H_Q(Q, \mu)$ is the modulus squared of the QCD matching coefficient $C_H(Q^2 + i0^+, \mu)$

$$G_H(\epsilon, u) = 2\cos(\pi u) G_{C_H}(\epsilon, u)$$

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Perturbatively one can decide whether to sum up factors of $\pi$.

$$\mu_0 = -iQ \quad \mu_0 = Q$$

We observe faster convergence faster convergence when $\pi$-resummation is included.
The relevant diagrams to compute the jet function at $O(1/\beta_0)$ are

$$G_j(\epsilon, u) = 2C_F[(u - 2)\epsilon - 3u + 4] \frac{\Gamma(2 - \epsilon)\Gamma(1 - u)}{\Gamma(1 + u - \epsilon)\Gamma(3 - u - \epsilon)} \left[ \frac{3(1 - \epsilon)\Gamma(1 + \epsilon)\Gamma^2(1 - \epsilon)}{(3 - 2\epsilon)\Gamma(2 - 2\epsilon)} \right]^\frac{u}{\epsilon - 1}$$

$$\omega = -ie^{-\gamma_E / y}$$

We take the Fourier transform w.r.t. $p^2 = s$ to avoid distributions.

We recover the universal cusp anomalous dimension (cross-check)

We compute $\gamma_J(\alpha_s)$,

And by consistency we predict $\gamma_s(\alpha_s)$

Perturbatively, we find agreement up to $O(\alpha_s^3)$ with the leading flavour structure in full QCD.
\[ G_j(\epsilon, u) = 2C_F[(u - 2)\epsilon - 3u + 4] \frac{\Gamma(2 - \epsilon)\Gamma(1 - u)}{\Gamma(1 + u - \epsilon)\Gamma(3 - u - \epsilon)} \left[ \frac{3(1 - \epsilon)\Gamma(1 + \epsilon)\Gamma^2(1 - \epsilon)}{(3 - 2\epsilon)\Gamma(2 - 2\epsilon)} \right]^{u-1} \]

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Again we don’t cross the poles of \( G_j(\epsilon, 0) \)

There are only two (simple) poles at \( u = 1 \) and 2

There are only two contributions to the ambiguity

\[ \delta J = -\frac{2C_F}{\beta_0} \left[ iye^{5/3}\Lambda_{QCD}^2 + \left( \frac{1}{2} iye^{5/3}\Lambda_{QCD}^2 \right)^2 \right] \]

For real \( y \)

\( \text{Re}[\tilde{J}] \) is free from \( u = 1 \) renormalon

\( \text{Im}[\tilde{J}] \) is free from \( u = 2 \) renormalon
There are only two (simple) poles at \( u = 1 \) and 2. Again we don’t cross the poles of \( G_j(\epsilon, 0) \)

\[
G_j(\epsilon, u) = 2C_F[(u-2)\epsilon - 3u + 4] \frac{\Gamma(2-\epsilon)\Gamma(1-u)}{\Gamma(1+u-\epsilon)\Gamma(3-u-\epsilon)} \left[ \frac{3(1-\epsilon)\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{(3-2\epsilon)\Gamma(2-2\epsilon)} \right]^{\frac{u}{\epsilon}-1}
\]
When jets are produced by heavy quarks there is an extra energy scale involved: the quark’s mass $m$.

In this case one can match SCET onto two copies of bHQET to sum up the new logs:

$$\frac{1}{\sigma_0} \frac{d\sigma}{de} = H_Q \times H_m \times B_n \otimes B_{\bar{n}} \otimes S$$

We present the computation of the additional mass-scale hard function and the bHQET jet function for hemisphere mass in the large-$\beta_0$ limit.
The mass-scale hard function $H_m(m, \mu)$ is the modulus squared of the massive SCET matching coefficient onto bHQET: $C_m(m, \mu)$ on-shell scheme and dim.reg. → all bHQET diagrams vanish.

For the self-energy diagram we need the wave-function renormalization $Z_\xi^{0S}$, which we obtain from quark’s self energy computation in QCD.

\[
G_{C_m}(\epsilon, u) = 4 C_F e^{\gamma_E u} u^2 (1 + u - \epsilon) [ (2u^2 - 2u + 1) \epsilon - 3u^2 + 4u - 2] \frac{\Gamma(u) \Gamma(-2u)}{\Gamma(3 - u - \epsilon)} \\
\left[ \frac{3(1 - \epsilon) \Gamma(1 + \epsilon) \Gamma^2(1 - \epsilon)}{(3 - 2\epsilon) \Gamma(2 - 2\epsilon)} \right]^{\frac{u}{\epsilon} - 1}.
\]

Again we use this to compute the anomalous dimension.

Once more we recover the cusp anomalous dimension and find full agreement up to $O(\alpha_s^3)$ for the non-cusp part (unambiguous).
We analyze the poles of $G_{Hm}(\epsilon, u) \equiv 2G_{Cm}(\epsilon, u)$ and find...

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<td>Yes</td>
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Again we don't cross the poles of $G_j(\epsilon, 0)$

There are poles at $u = 1$ and $2$ and all the positive half-integers

The leading renormalon for $H_m$ lays then at $u = 1/2$ and its ambiguity is

$$\delta_{H_m} = -\frac{6\epsilon_5^{5/6}C_F}{\Lambda_{QCD}} \frac{\Lambda_{QCD}}{m}$$

This is three times higher than the pole's mass ambiguity: $$\delta_{m_p} = -\frac{2C_F}{\beta_0} \epsilon^{5/6} \Lambda_{QCD}$$

Therefore the combination $H_m/m_p^3$ is free from the leading ambiguity.
Therefore the combination $H_m/m_p^3$ is free from the leading ambiguity...

...when both series are expanded (left) in terms of the same $\alpha_s(\mu_m)$ (right)

**Note:** this renormalon affects the norm of the distribution and might lead to bad convergence of the distribution if not properly accounted for.
The relevant diagrams to compute the jet function for hemisphere masses at $O(1/\beta_0)$ are

We find

$$G_{\tilde{B}}(\epsilon, u) = 2C_F \frac{e^{-u\gamma_E(1-u)}\Gamma(1-u)}{(1-2u)\Gamma(1+u-\epsilon)} \left[ \frac{3(1-\epsilon)\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{(3-2\epsilon)\Gamma(2-2\epsilon)} \right]^{\frac{u}{\epsilon}-1} \omega = -ie^{-\gamma_E}/x$$

where we have taken the Fourier transform w.r.t. $\hat{s} = (s - m^2)/m$ to avoid distributions

We recover the universal cusp anomalous dimension (cross-check)

We compute the non-cusp, unambiguous anomalous dimension

Again, for both we find agreement up to $O(\alpha_s^3)$ with the leading flavour structure in full QCD
The leading renormalon lays at $u = 1/2$.

This time the leading ambiguity is twice that of the pole mass (except for a factor of $ix$):

$$\delta_\tilde{B} = -\frac{4C_F e^{5/6}}{\beta_0} ix \Lambda_{QCD}$$

$$\delta_{mp} = -\frac{2C_F e^{5/6}}{\beta_0} \Lambda_{QCD}$$

Therefore the combination $\tilde{B}(x)e^{-2ixmp}$ is free from the leading renormalon.
Therefore the combination $\hat{B}(x)e^{-2ixm_p}$ is free from the leading renormalon.

Expanding $m_p$ in terms of the $\overline{\text{MS}}$ mass breaks bHQET power counting since $\delta \bar{m} \propto \bar{m}$.

Use instead the MSR mass in an expansion in powers of $\alpha_s(\mu)$ with $\mu \sim R \sim 1/x$ to avoid large logs.

We used complex $x$ since for real $x$ the real part is free from the $1/2$ renormalon.
Conclusions

• In the large-$\beta_0$ limit we completely know QCD perturbative series, and we can resum them

  • We derived a formalism that recovers the known closed expressions for finite, non-cusp and extended it to cusp series and their anomalous dimensions

  • From these expressions we can study their asymptotic behaviour and estimate the size of non-perturbative power corrections

• We computed SCET and bHQET matrix elements (all divergent) and their anomalous dimensions (all finite)
Thanks for your attention