Groomed heavy jet mass at NNLO+NNNLL accuracy in lepton collisions

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Eötvös University and MTA-DE Particle Physics Research Group

based on arXiv:1603.08927 (PRL), 1606.03453 (PRD), 1807.11472 (PLB), 2002.00942 (PLB), 2002.05730 (PRD)

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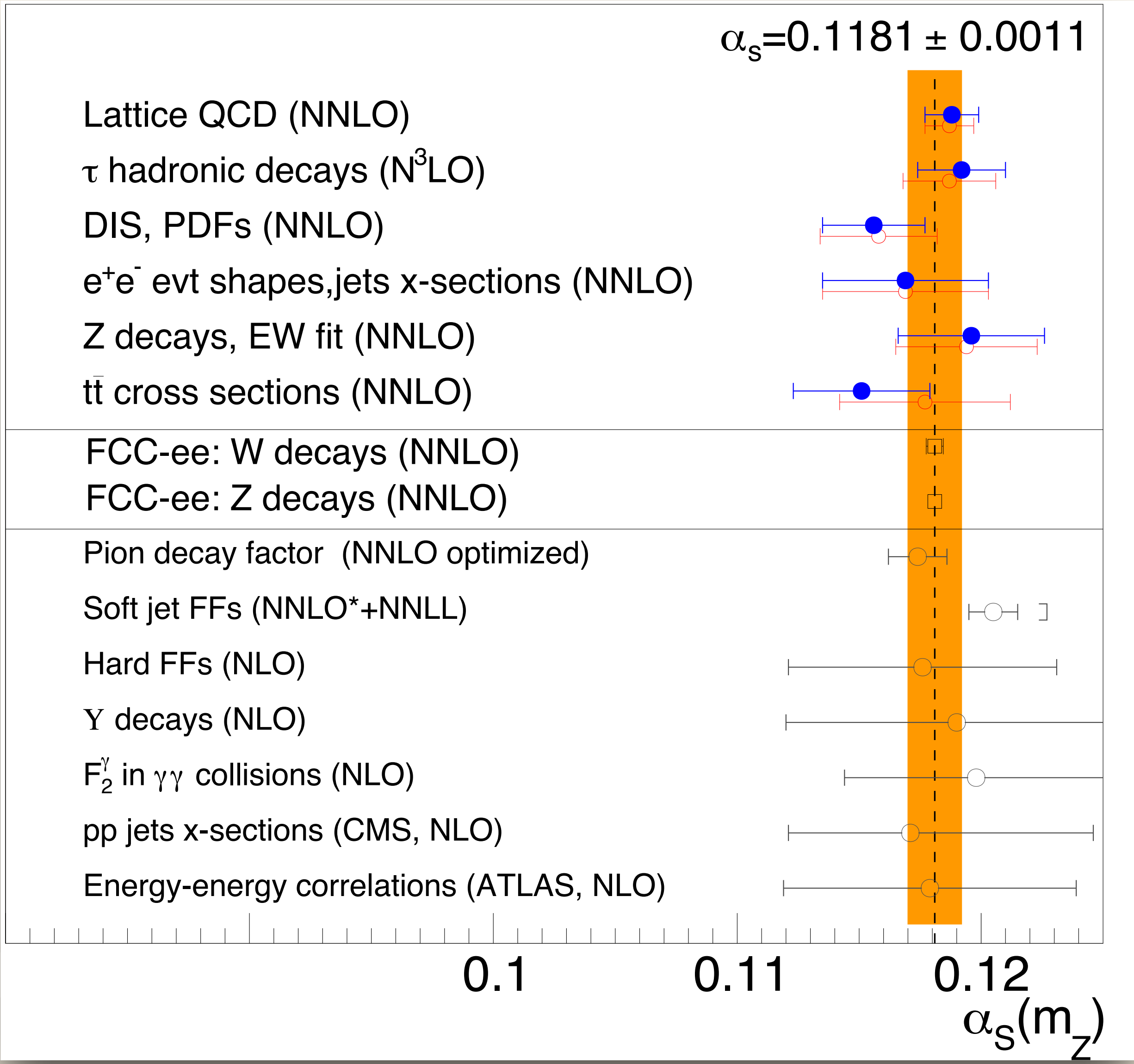
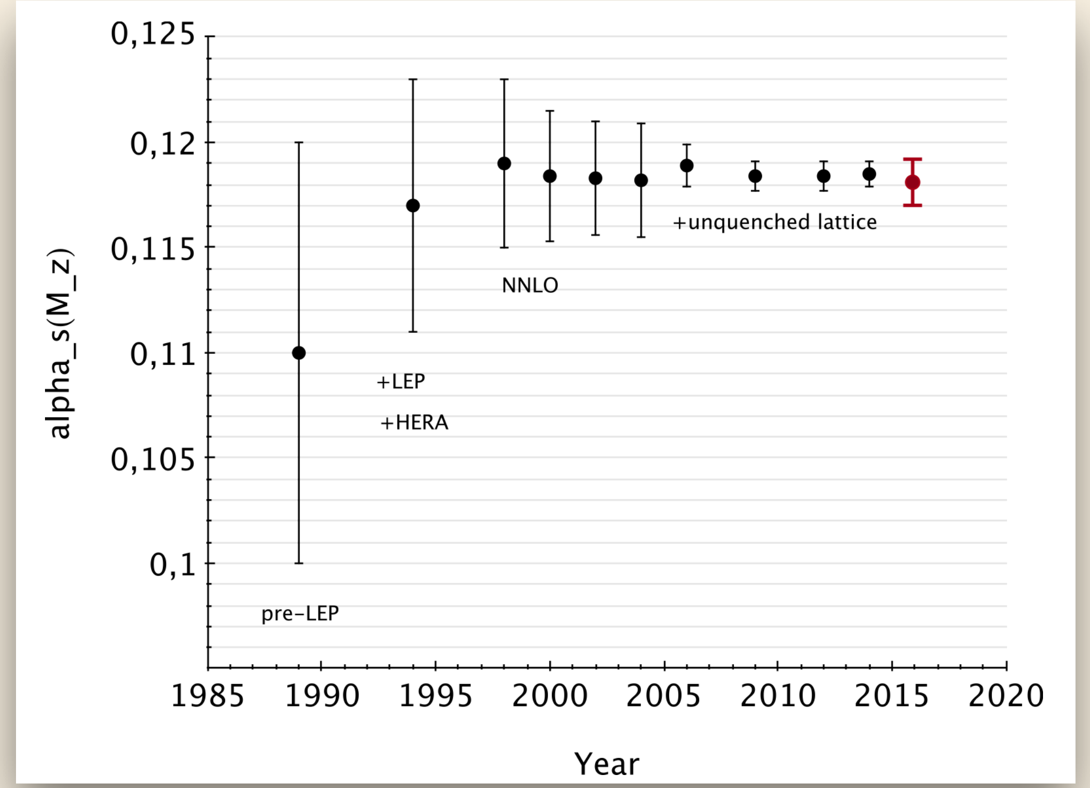
Outline

- Why event shapes in lepton collisions?
- New developments since LEP
- New prospects: groomed event shapes
- Computing mMDT heavy jet mass at NNNLL and NNLO
- Conclusions

Why event shapes in lepton collisions?

summary of α_s determinations:





D. d'Enterria, arXiv: 1806.06156

D. d'Enterria and S. Kluth (eds), arXiv: 1907.01435

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- e^+e^- event shapes, jets have long been considered ideal for measuring α_s

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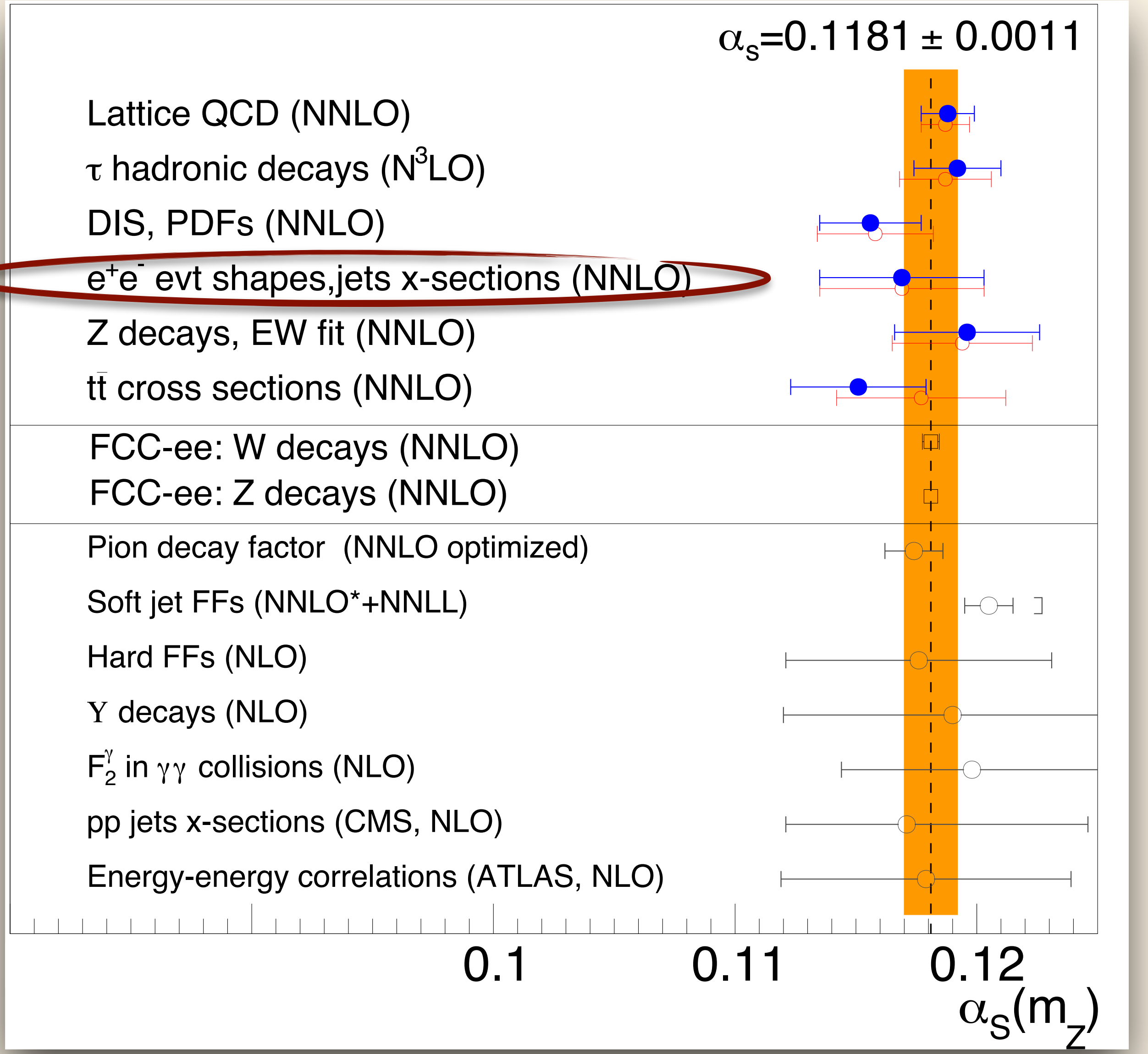

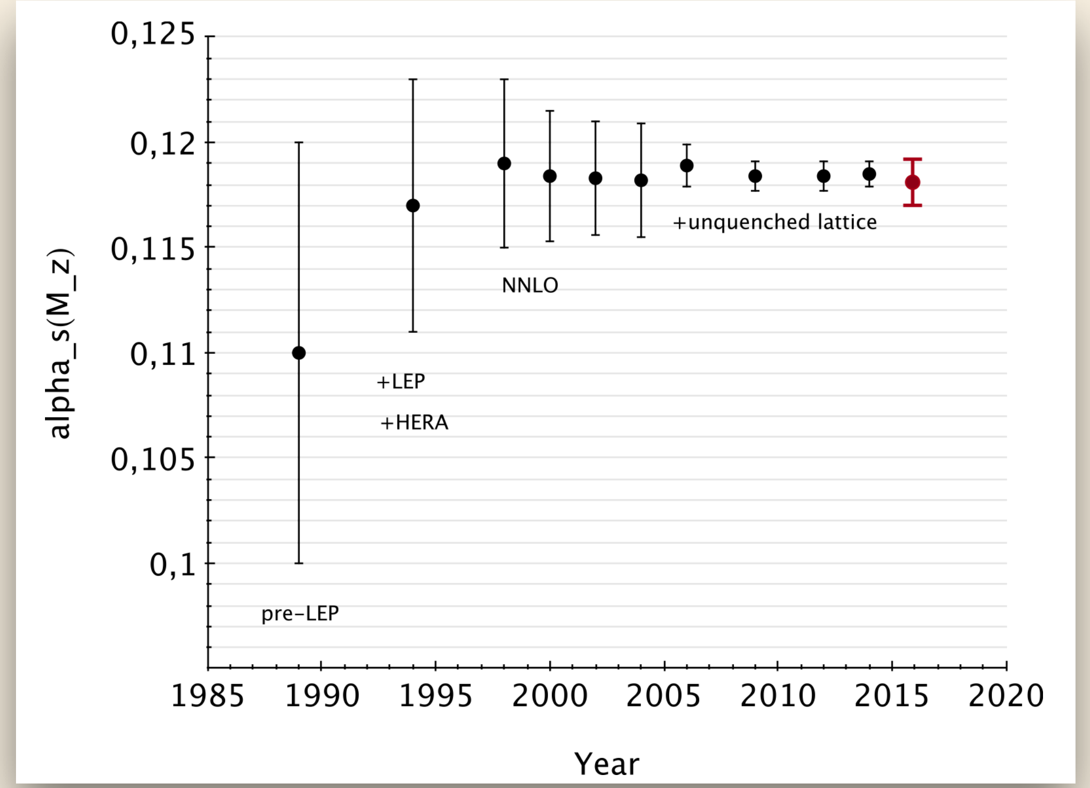

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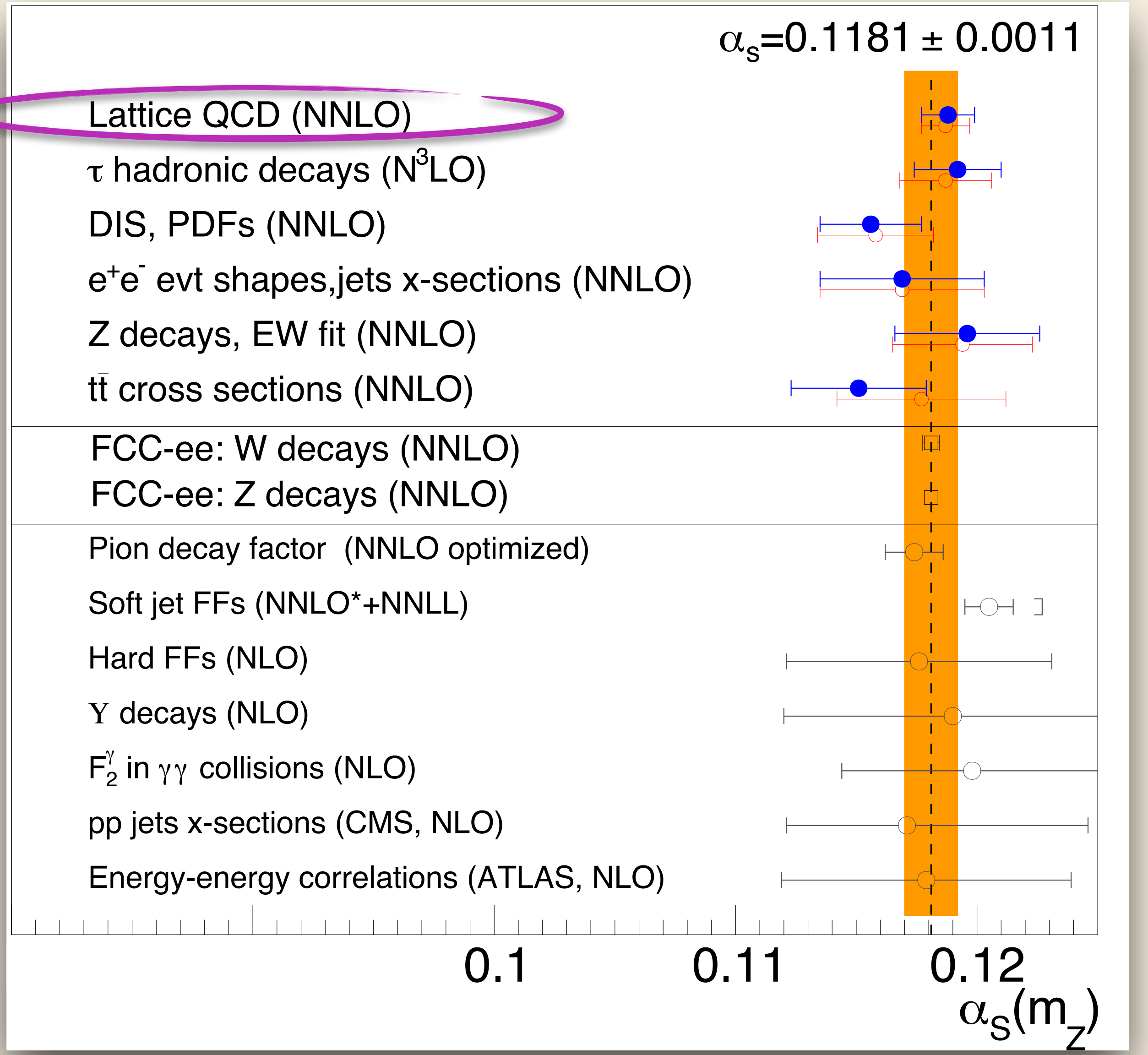

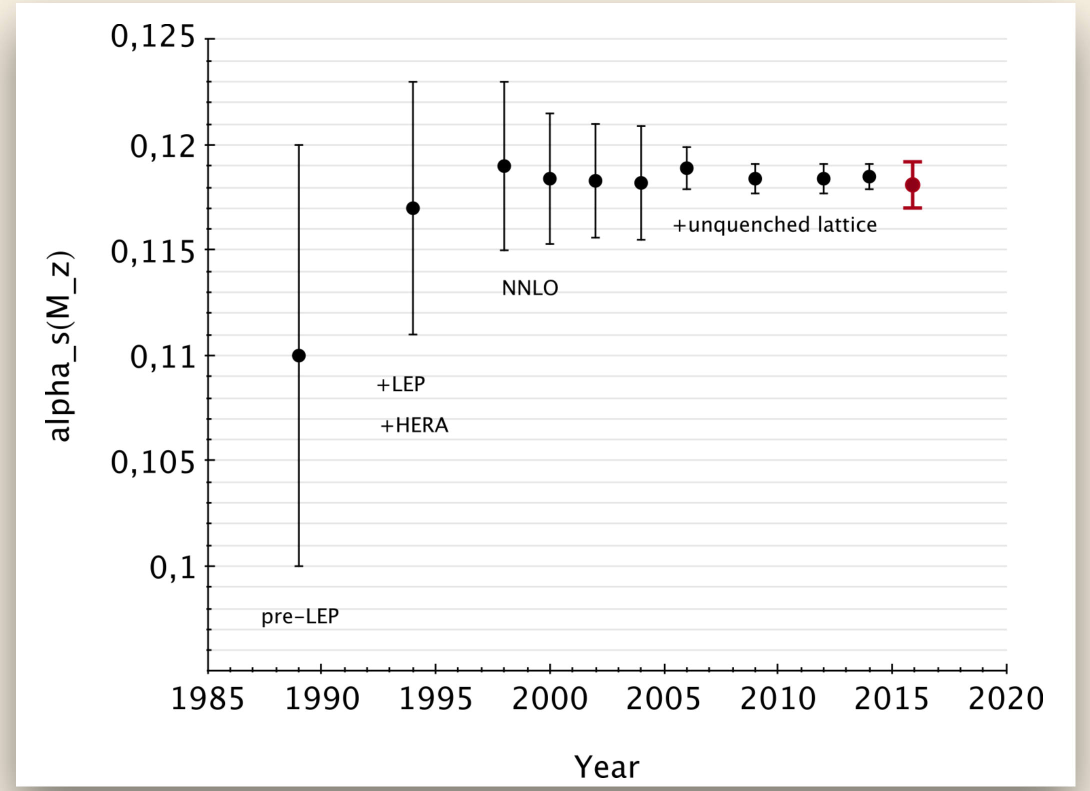

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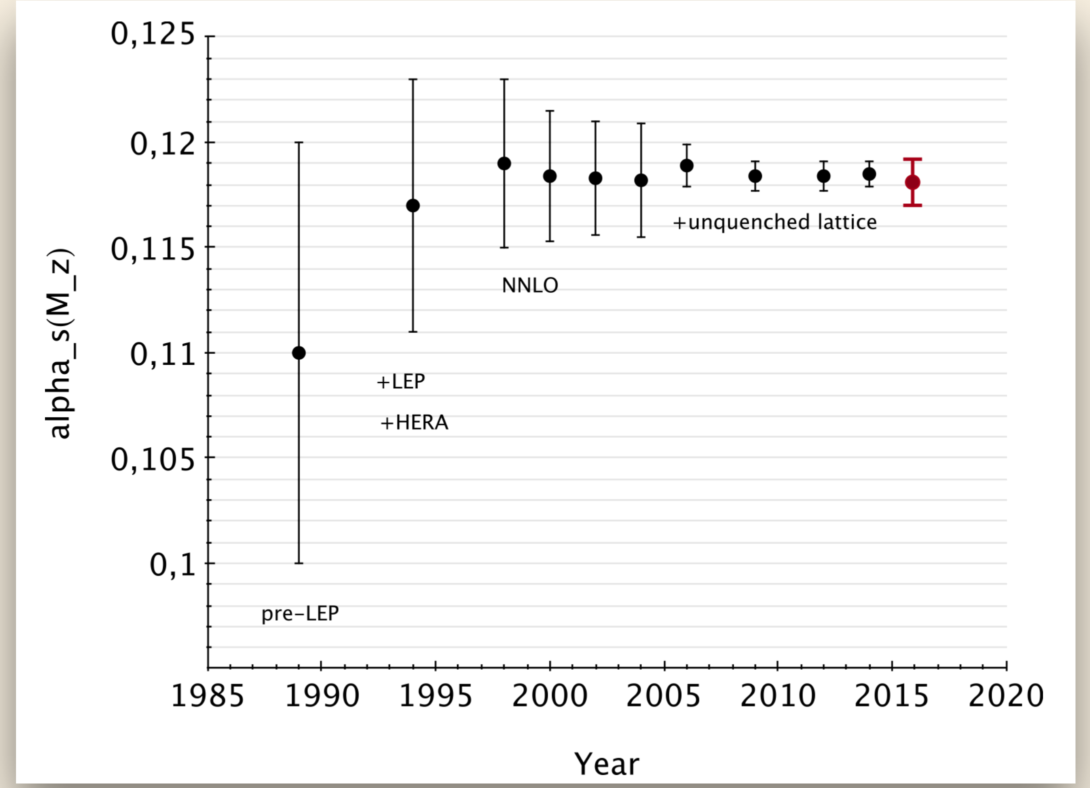
- recent prevailing view: lattice is unbeatable
- yet determination of α_s from experiments remains desirable



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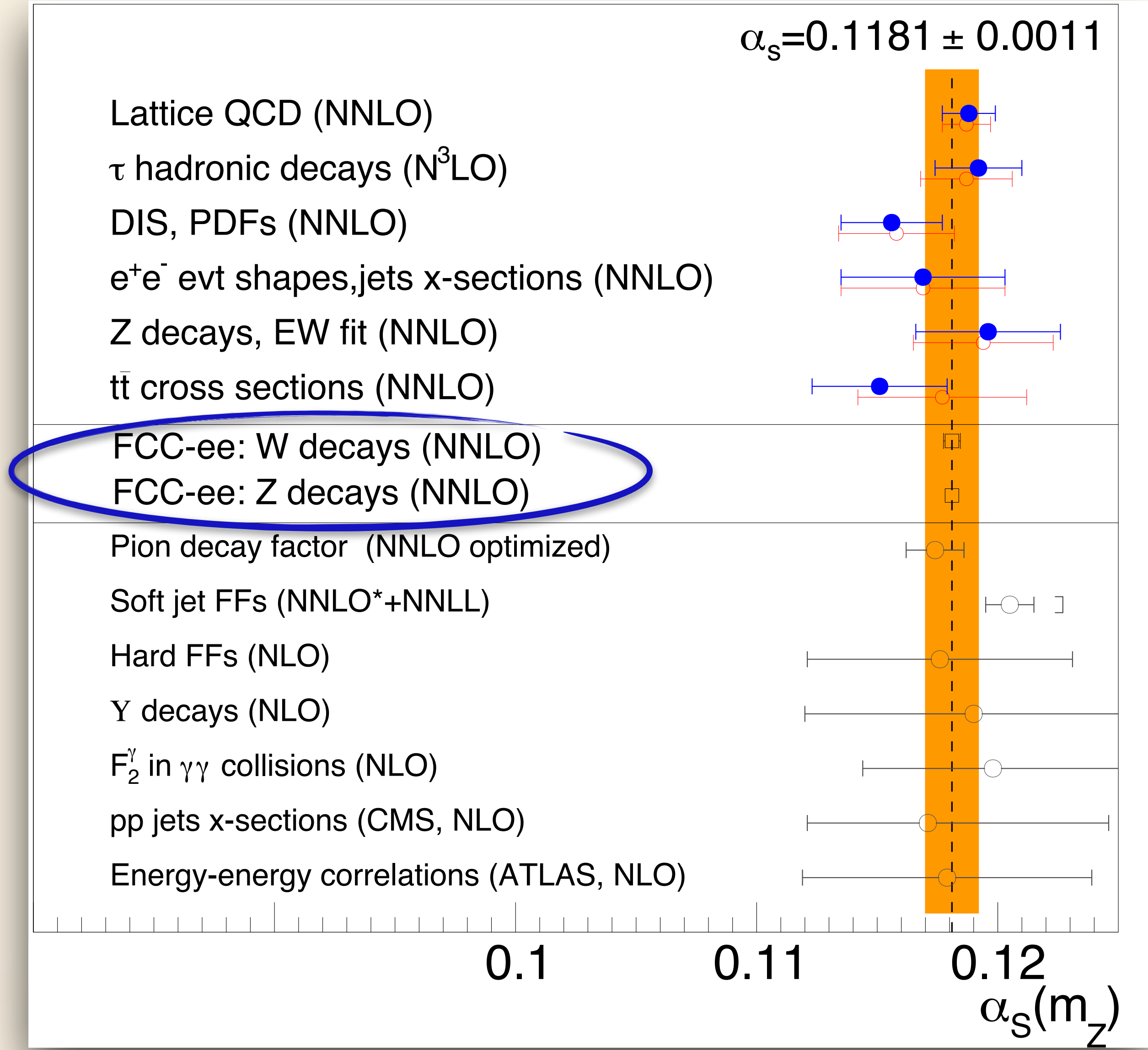

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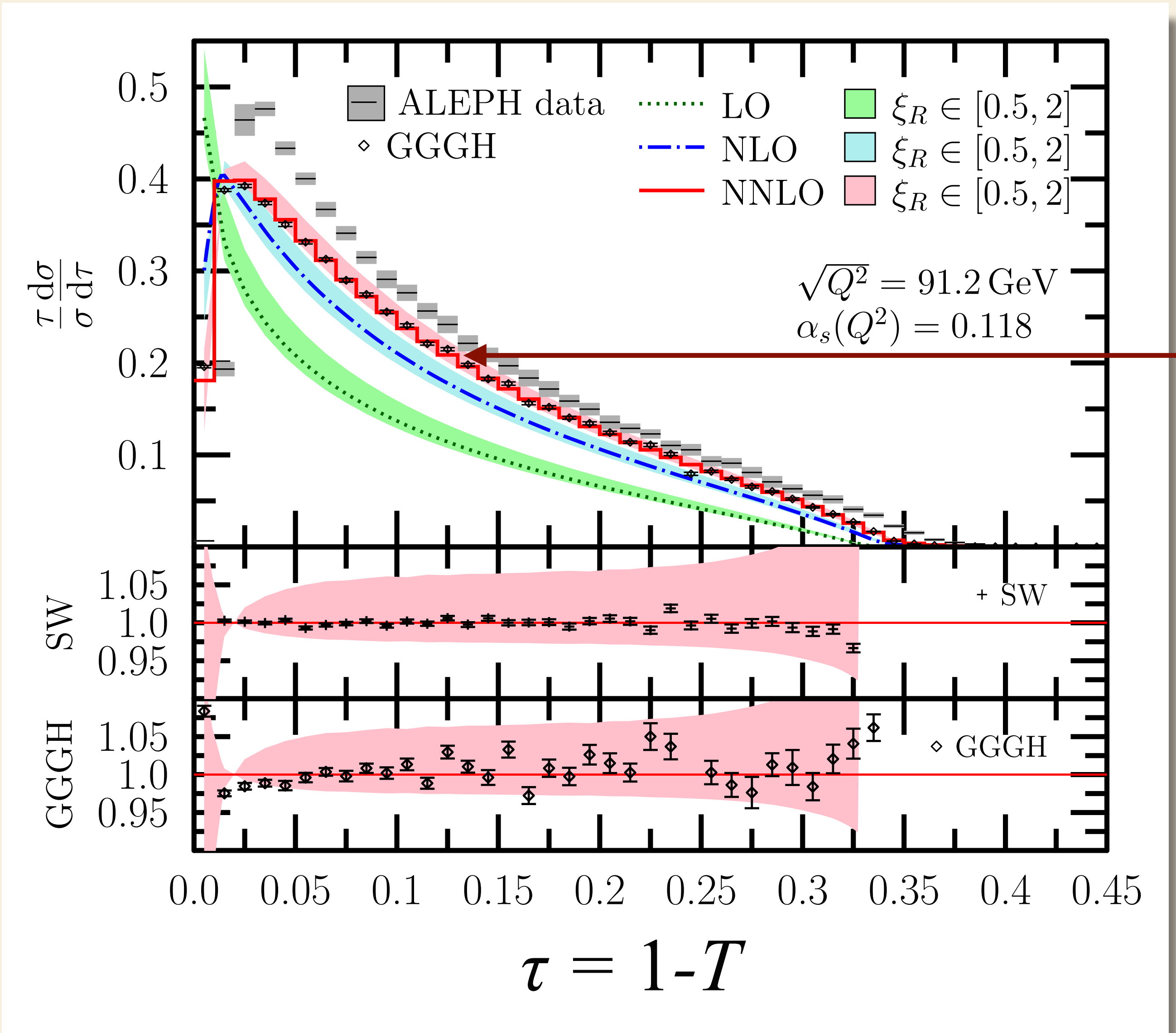
New since LEP

Impact of corrections at NNLO



fixed-order PT is insufficient to describe data

$$T = \max_{\vec{n}} \left(\frac{\sum_i |\vec{n} \cdot \vec{p}_i|}{\sum_i |\vec{p}_i|} \right)$$

$$\frac{\tau}{\sigma}\frac{d\sigma}{d\tau} = \left(\frac{\alpha_s}{2\pi}\right) A(\tau) + \left(\frac{\alpha_s}{2\pi}\right)^2 B(\tau) + \left(\frac{\alpha_s}{2\pi}\right)^3 C(\tau)$$

A, B and C computed with MCCSM (=Monte Carlo for the CoLoRFulNNLO Subtraction Method)

V. Del Duca et al, arXiv:1603.08927

Causes of failure

1. QCD radiative corrections are large
2. fixed-order perturbation theory fails when logarithms become large → we need
    A. resummation of such logarithmic terms at all orders
    B. matching of fixed order and resummed predictions

An example of analytic structure of the perturbative expansion

$$\boxed{\frac{\tau}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \left(\frac{\alpha_s}{2\pi}\right) A(\tau) + \left(\frac{\alpha_s}{2\pi}\right)^2 B(\tau) + \left(\frac{\alpha_s}{2\pi}\right)^3 C(\tau)}$$

$$L = -\ln\tau$$

$$\begin{aligned}
A(\tau) &= \boxed{A_1 L + A_0}, \\
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LL NLL $\mathrm{N^2LL}$ $\mathrm{N^3LL}$ …

for $L \sim 1/\alpha_s$ we need resummation of logarithmic terms at all orders

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LL NLL N^2LL N^3LL …

matching predictions at fixed order with resummed has to avoid double counting — achieved by removing coefficients known analytically (precisely)

→ need coefficients in fixed order also precisely

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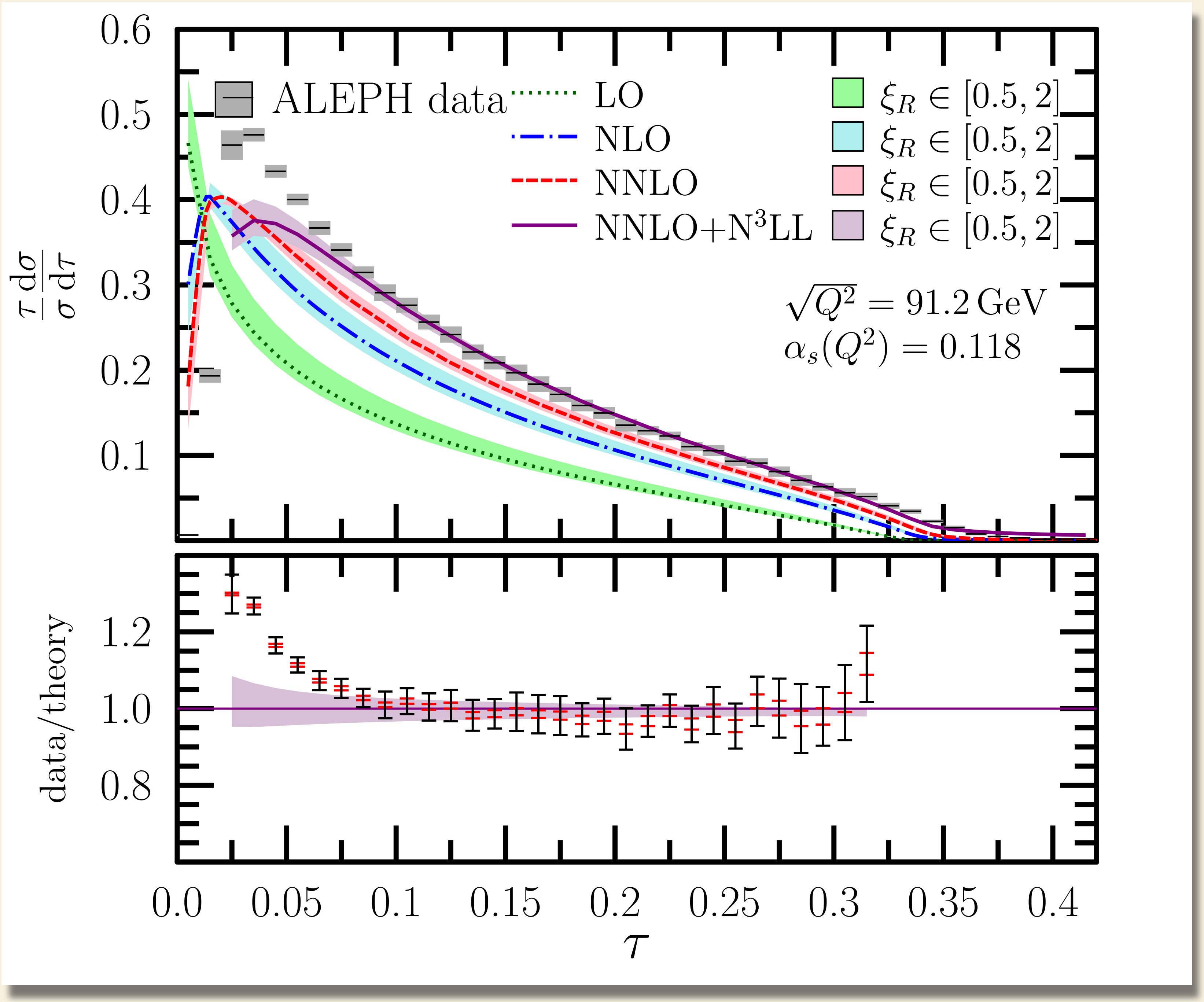
some precise predictions are available:

– NNLO+N^{3}LL for 1-T, C-parameter & heavy jet mass (ρ)
– NNLO+N^{2}LL for broadenings and EEC

Matching NNLO with N^{3}LL



Works for $\tau > 0.1$, fails in peak regions

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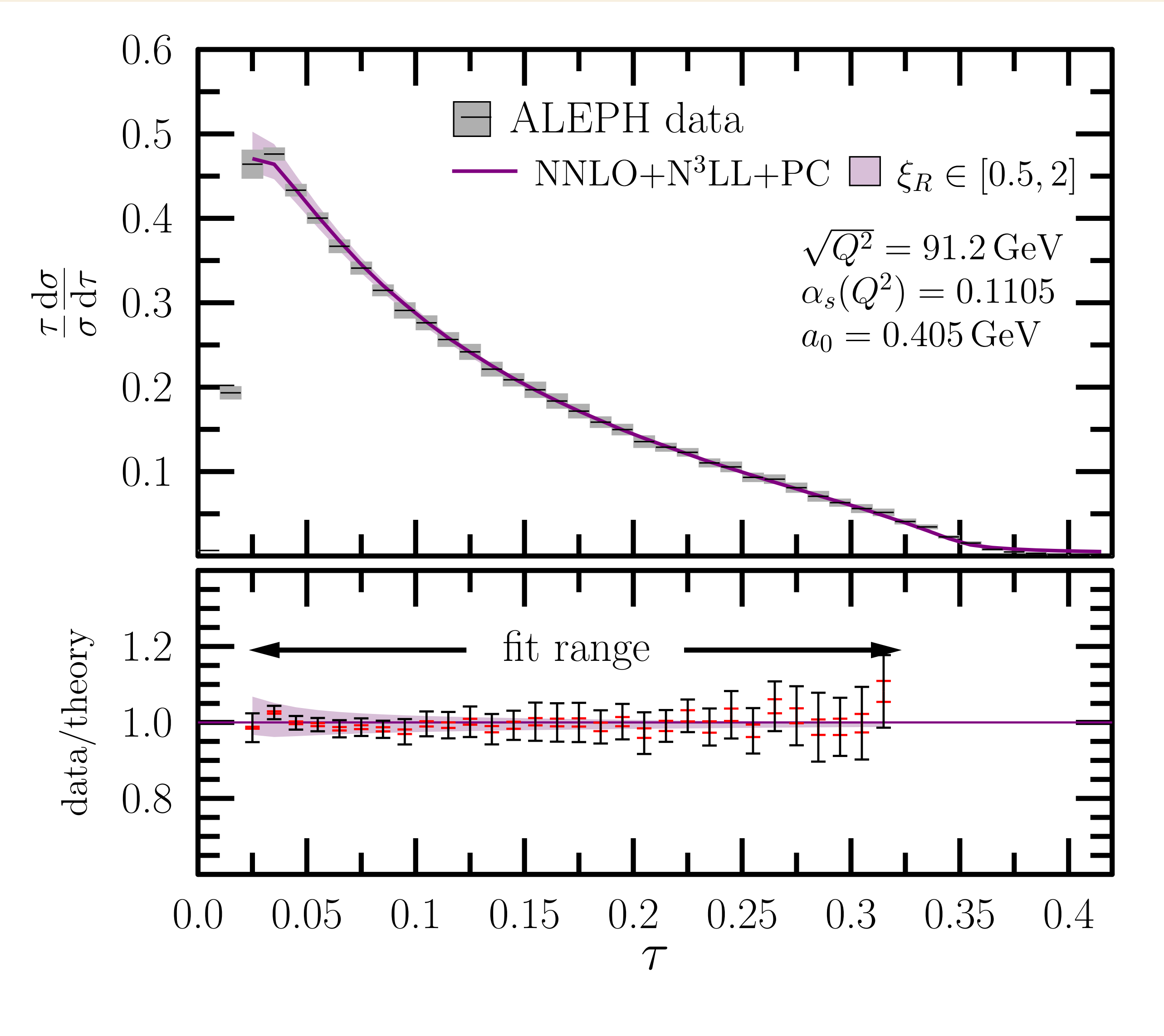
3. hadronization corrections are
   A. large, especially for small values of the event shape, i.e near the peak
   B. not well understood from first principles

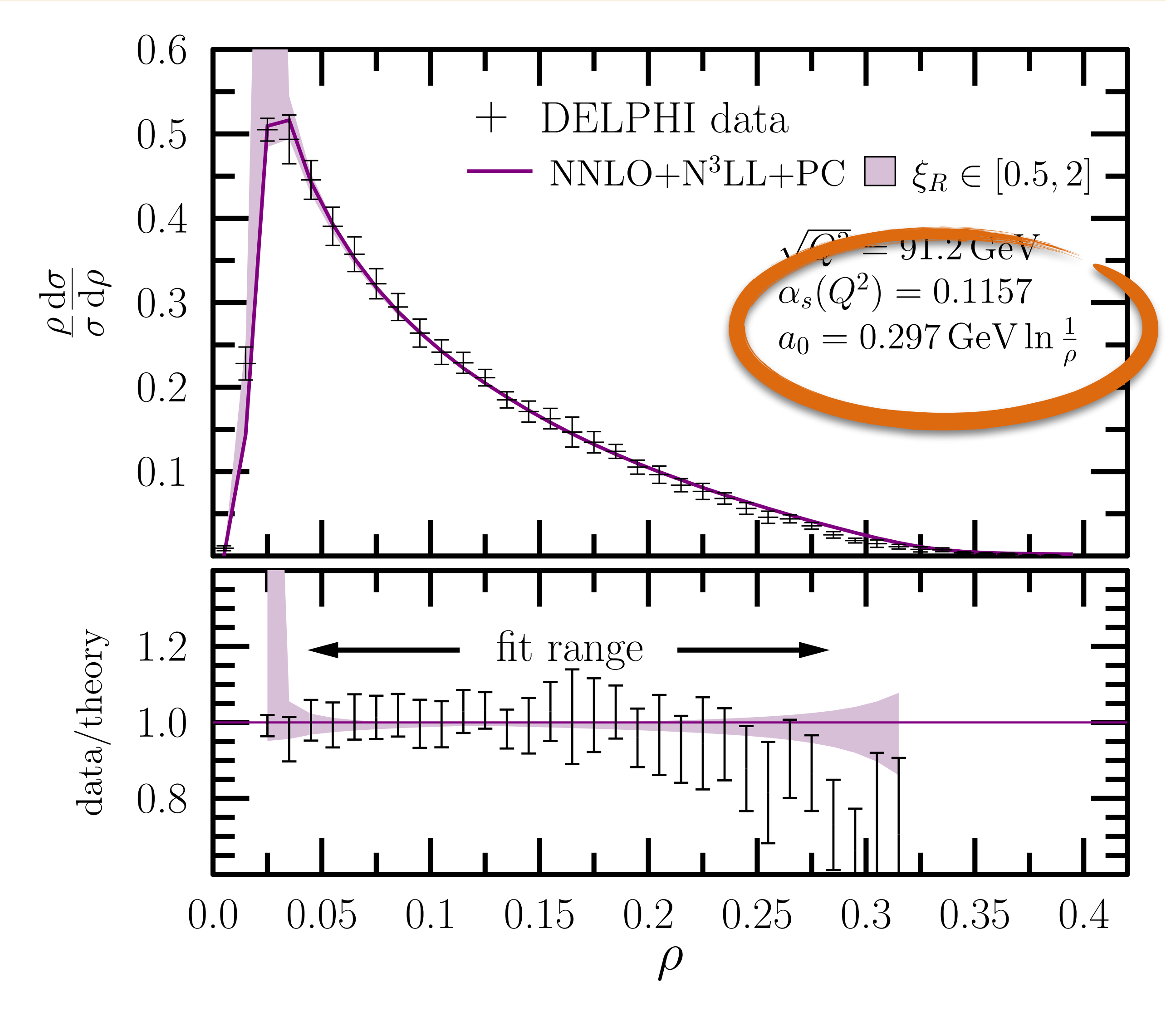
two options:

– estimate of hadronisation using modern MC tools
– use analytic model for power corrections

both have their caveats

Fit data on thrust and heavy jet mass with NNLO+N^{3}LL+PC





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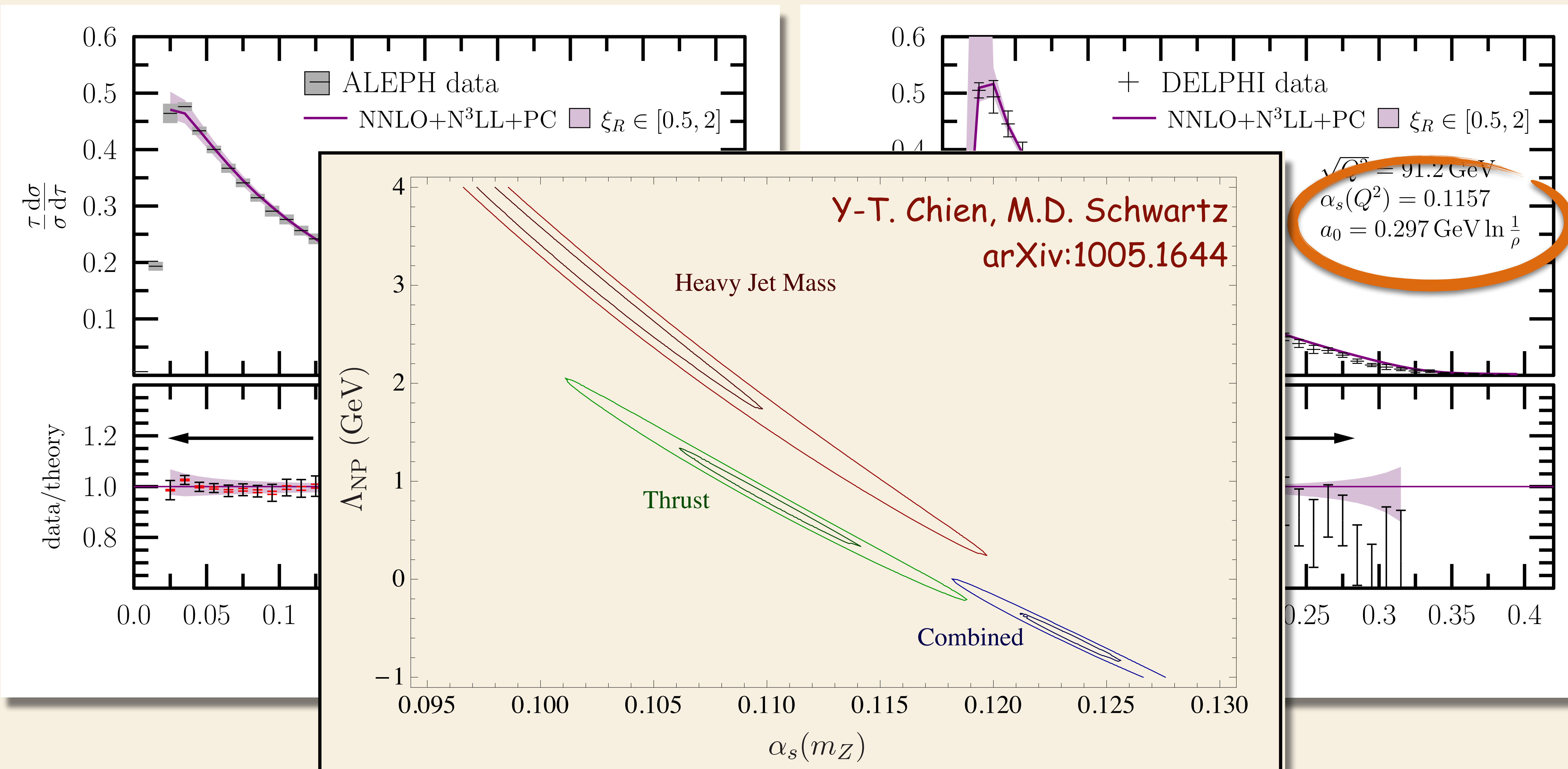

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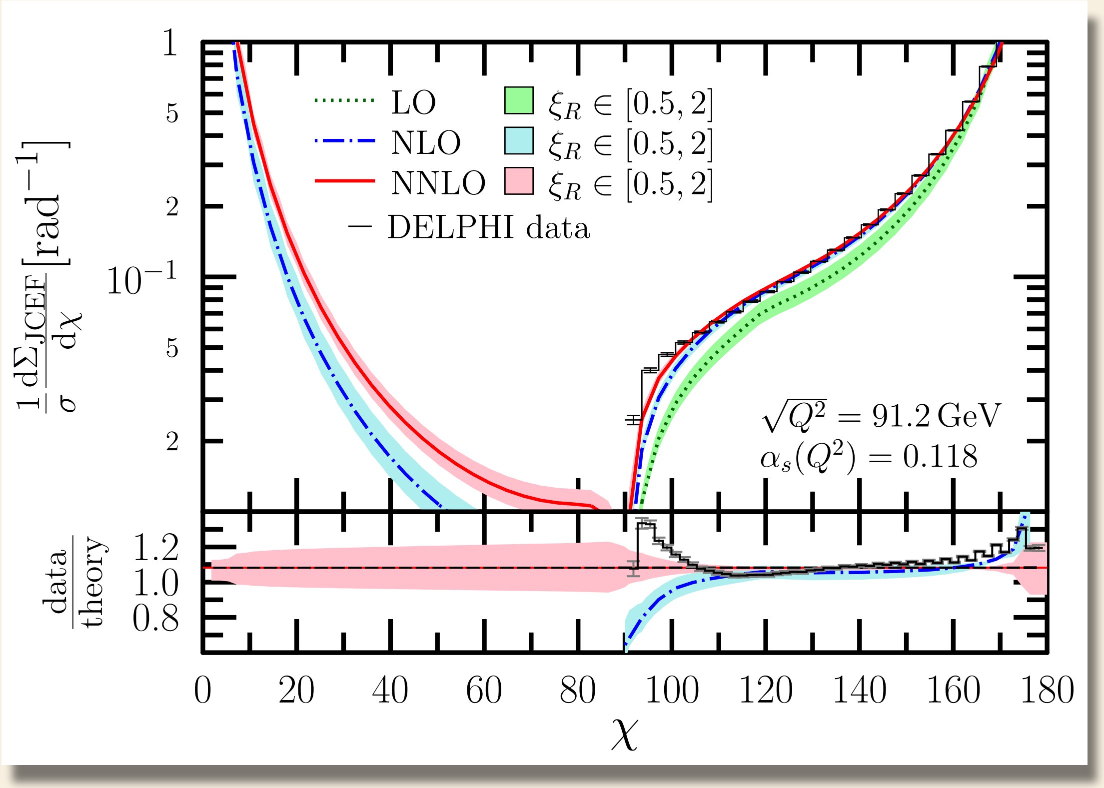
4. the two-types of corrections are strongly correlated for analytic models of hadronisation
5. Monte Carlo estimates are model dependent

How to improve?

✓ Find observable quantities with small perturbative and hadronisation corrections:

motto: "large uncertainty in small quantity is small uncertainty"

jet cone energy fraction:
$$\frac{d\Sigma_{\text{JCEF}}}{d\cos\chi} = \sum_i \int \frac{E_i}{Q} d\sigma_{e^+e^-\to i+X}\,\delta\left(\cos\chi - \frac{\vec{p}_i\cdot\vec{n}_T}{|\vec{p}_i|}\right)$$



V. Del Duca et al, arXiv:1606.03453

How to improve?

- ✓ Correct for hadronisation, 2nd option:
  - – estimate of hadronisation using modern MC tools
- ✓ Find observable quantities with small perturbative and hadronisation corrections:

motto: "large uncertainty in small quantity is small uncertainty"

- – precluster hadrons and compute shapes from jets

Decamp et al [ALEPH], Phys.Lett. B257 (1991) 479-491

- – groomed event shapes, designed to reduce contamination from non-perturbative effects

Groomed heavy jet mass

mMDT grooming algorithm

1. Divide the final state of an e+e− → hadrons event into two hemispheres in any infrared and collinear safe way.
2. In each hemisphere, run the Cambridge/Aachen jet algorithm to produce an angular-ordered pairwise clustering history of particles.
3. Undo the last step of the clustering for the one hemisphere, and split it into two particles; check if these particles pass the mass drop condition, which is defined for e^+e^- collisions as:

$$\frac{\min[E_i, E_j]}{E_i + E_j} > z_{\mathrm{cut}}$$

   where E_i and E_j are the energies of the two particles

3. If the splitting fails this condition, the softer particle is dropped and the groomer continues to the next step in the clustering at smaller angle.
4. If the splitting passes this condition the procedure ends and any observable can be measured in the remaining hemispheres

Resummation formula

Factorization formula for $\tau_{\mathrm{L}}, \tau_{\mathrm{R}} \ll z_{\mathrm{cut}} \ll 1$ $\qquad \tau_i = \frac{m_i^2}{E_i^2}$

$$\frac{1}{\sigma_0}\frac{\mathrm{d}^2\sigma}{\mathrm{d}\tau_{\mathrm{L}}\,\mathrm{d}\tau_{\mathrm{R}}} = H(Q^2)S(z_{\mathrm{cut}})\left[J(\tau_{\mathrm{L}}) \otimes S_c(\tau_{\mathrm{L}}, z_{\mathrm{cut}})\right]\left[J(\tau_{\mathrm{R}}) \otimes S_c(\tau_{\mathrm{R}}, z_{\mathrm{cut}})\right]$$

C. Frye et al, arXiv: 1603.09338

Convolutions —
true product for Laplace transforms:

$$\frac{\sigma(\nu_{\mathrm{L}}, \nu_{\mathrm{R}})}{\sigma_0} = H(Q^2)S(z_{\mathrm{cut}})\tilde{J}(\nu_{\mathrm{L}})\tilde{S}_c(\nu_{\mathrm{L}}, z_{\mathrm{cut}})\tilde{J}(\nu_{\mathrm{R}})\tilde{S}_c(\nu_{\mathrm{R}}, z_{\mathrm{cut}})$$

Modified mass drop tagger groomed heavy jet mass:

$$\frac{1}{\sigma_0}\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} \equiv \int \mathrm{d}\tau_{\mathrm{L}}\,\mathrm{d}\tau_{\mathrm{R}}\,\frac{1}{\sigma_0}\frac{\mathrm{d}^2\sigma}{\mathrm{d}\tau_{\mathrm{L}}\,\mathrm{d}\tau_{\mathrm{R}}}\left[\Theta(\tau_{\mathrm{L}} - \tau_{\mathrm{R}})\,\delta(\rho - \tau_{\mathrm{L}}) + \Theta(\tau_{\mathrm{R}} - \tau_{\mathrm{L}})\,\delta(\rho - \tau_{\mathrm{R}})\right]$$

Resummation by RGE

Resummation is made possible by the RGEs:

$$\mu \frac{\partial \tilde{F}}{\partial \mu} = \left(d_F \Gamma_{\text{cusp}} \log \frac{\mu^2}{\mu_F^2} + \gamma_F \right) \tilde{F} \,, \quad (\tilde{F} = H \,,\ S \,,\ \tilde{J} \,,\ \tilde{S}_c)$$

The factorization theorem successfully resums all logarithms of both ρ and z_{cut} simultaneously, to leading power in the limit where $\rho \ll z_{\text{cut}} \ll 1$ in the exponent of the cross section cumulative in the mass ρ

Thus N^{n}LL refers to the resummation of all terms of the form

$$\alpha_s^m \log^{m+1-n} \rho,\ \alpha_s^m \log^{m+1-n} z_{\text{cut}},\ \alpha_s^m \log^{m+1-n} \frac{\rho}{z_{\text{cut}}}$$

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order of ingredients needed for N^{n}LL resummation

	Γ_{cusp}	γ_F	β	c_F
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NLL	α_s^2	α_s	α_s^2	-
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not yet known: $c_{S_c}^{(2)}$

known | known partially

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$\gamma_S^{(2)}\,,\ \gamma_{S_c}^{(2)}$

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Extraction of constants

Missing constants

Constraint on the non-cusp anomalous dimensions:

$$0 = \gamma_H + \gamma_S + 2\gamma_J + 2\gamma_{S_c}$$

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We use fixed-order code EVENT2 to determine

$$c_{S_c}^{(2)} = \left(\frac{\alpha_s}{4\pi}\right)^2 \left[C_F^2\,(22 \pm 4) + C_F C_A\,(41 \pm 1) + C_F T_R n_f\,(14.4 \pm 0.1)\right]$$

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and MCCSM to find

$$\gamma_S^{(2)} = \left(\frac{\alpha_s}{4\pi}\right)^3 \left[-11600 \pm 2000\right] \qquad (n_f = 5)$$

for mMDT

mMDT groomed heavy jet mass at N^{3}LL

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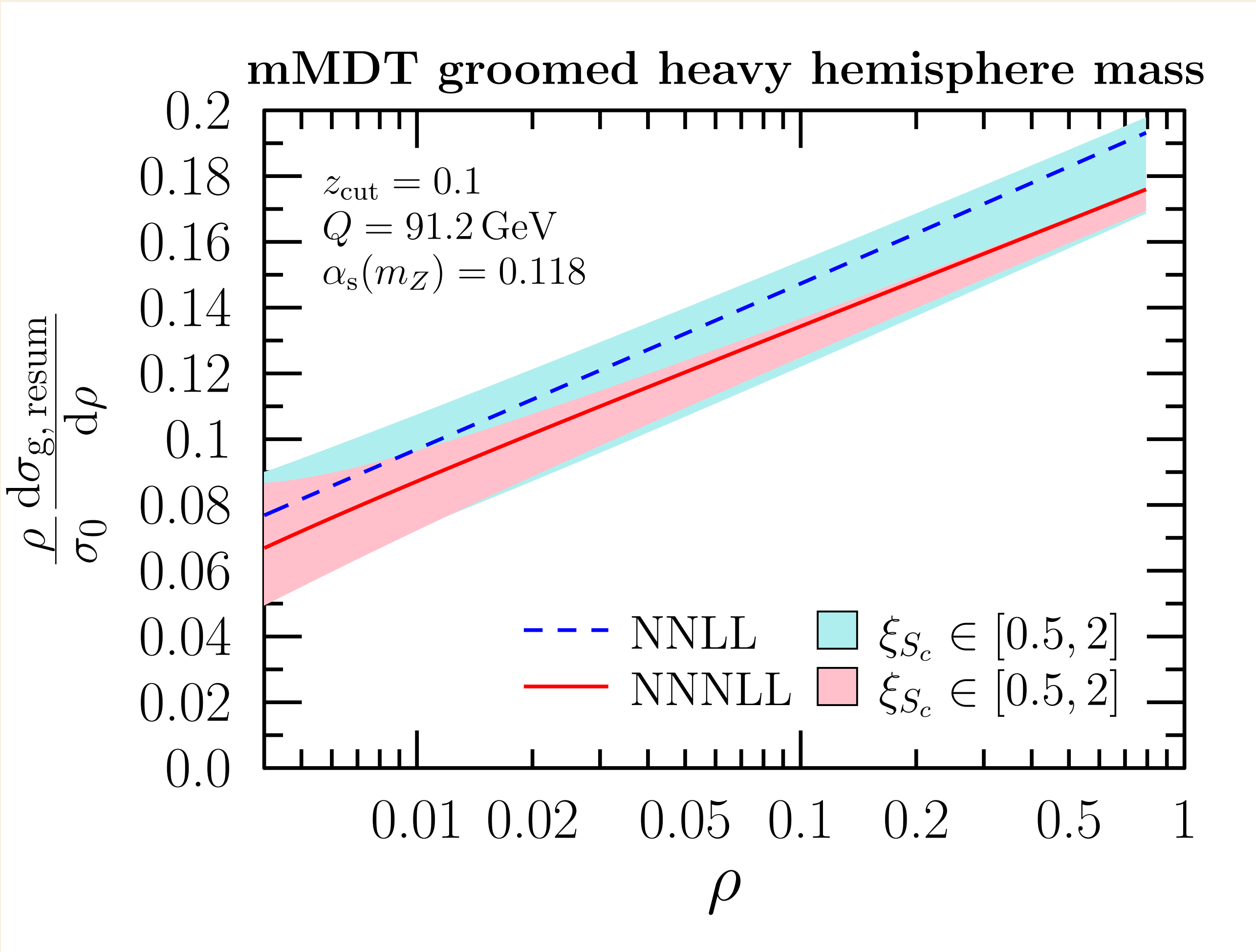

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new in this plot:

$c_{S_c}^{\rm mMDT}$ and $\gamma_S^{\rm mMDT}$

known

Technical details

Kinematics

For $z_{\rm cut} \ll 1$ only soft particles are groomed away,

hence the mMDT constraint is (E_H: hemisphere energy)

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Contribution to ρ of a soft particle, just passing mMDT, is

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mMDT grooming acts where $\rho < 2z_{\rm cut}$

Extraction of two-loop constants

The leading-power (LP) differential cross section for $\rho \to 0$

$$\frac{\mathrm{d}\sigma_{\mathrm{g,LP}}}{\mathrm{d}\rho} = D_{\delta,\mathrm{g}}\,\delta(\rho) + \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}$$

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σ^{sing} is defined to integrate to 0 on $[0,1]$, hence the total sec is

$$\sigma_{\mathrm{tot}} = D_{\delta,\mathrm{g}} + \int_0^1 \mathrm{d}\rho\,\left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right)$$

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integrate numerically

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known contains $c_{S_c}^{(2)}$ difficult to integrate numerically

need a better strategy: will be achieved through steps of identities

Extraction of two-loop constants

We want numerical integrals in the region where grooming acts

$$\int_0^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right) = \int_0^{2z_{\mathrm{cut}}} \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right) + \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \, \frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \, \frac{d\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}$$

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the integral of $\mathrm{d}\sigma_{\mathrm{g}}/\mathrm{d}\rho$ can be rewritten into

$$\underline{\int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \frac{d\sigma_{\mathrm{g}}}{\mathrm{d}\rho}} = \int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho}\Theta(1-\rho) - \frac{\mathrm{d}\sigma}{\mathrm{d}\rho}\right) + \int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho \frac{\mathrm{d}\sigma}{\mathrm{d}\rho}$$

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mMDT has effect near z_{cut}, so can drop the Θ function

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mMDT has effect near z_{cut}, so can drop the Θ function

upper limit is 4 in the ungroomed xsec because there ρ is normalized to the cm instead of the hemisphere energy,

(yet the integrand of the 1st integral vanishes below 1)

Extraction of two-loop constants

Resume:

$$\int_0^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho} \right) = \int_0^{2z_{\mathrm{cut}}} \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho} \right) + \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \, \frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \, \frac{d\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}$$

$$\int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \, \frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} = \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma}{\mathrm{d}\rho} \right) + \int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho \, \frac{\mathrm{d}\sigma}{\mathrm{d}\rho}$$

Up to power corrections, the ungroomed xsec is

$$\int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho \, \frac{\mathrm{d}\sigma}{\mathrm{d}\rho} = \int_0^4 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho} \right) + \mathcal{O}\left(z_{\mathrm{cut}}\right) + \int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho \, \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho}$$

Extraction of two-loop constants

Resume:

$$\int_0^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right) = \int_0^{2z_{\mathrm{cut}}} \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right) + \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \, \frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \, \frac{d\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}$$

$$\int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \, \frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} = \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma}{\mathrm{d}\rho}\right) + \int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho \, \frac{\mathrm{d}\sigma}{\mathrm{d}\rho}$$

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$$\int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho \, \frac{\mathrm{d}\sigma}{\mathrm{d}\rho} = \int_0^4 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho}\right) + \mathcal{O}\left(z_{\mathrm{cut}}\right) + \int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho \, \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho}$$

Collecting all steps we find

$$\begin{aligned}\sigma_{\mathrm{tot}} = D_{\delta,\mathrm{g}} &+ \int_0^{2z_{\mathrm{cut}}} \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right) + \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma}{\mathrm{d}\rho}\right) \\ &+ \int_0^4 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho}\right) + \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right) + \int_1^4 \mathrm{d}\rho \, \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho} + \mathcal{O}\left(z_{\mathrm{cut}}\right)\end{aligned}$$

Extraction of two-loop constants

Resume:

$$\int_0^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right) = \underline{\int_0^{2z_{\mathrm{cut}}} \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right)} + \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \frac{d\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}$$

$$\int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} = \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma}{\mathrm{d}\rho}\right) + \int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho \frac{\mathrm{d}\sigma}{\mathrm{d}\rho}$$

Up to power corrections, the ungroomed xsec is

$$\int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho \frac{\mathrm{d}\sigma}{\mathrm{d}\rho} = \int_0^4 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho}\right) + \mathcal{O}\left(z_{\mathrm{cut}}\right) + \int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho}$$

Collecting all steps we find

$$\begin{aligned}\sigma_{\mathrm{tot}} = D_{\delta,\mathrm{g}} &+ \underline{\int_0^{2z_{\mathrm{cut}}} \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right)} + \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma}{\mathrm{d}\rho}\right) \\ &+ \int_0^4 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho}\right) + \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right) + \int_1^4 \mathrm{d}\rho \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho} + \mathcal{O}\left(z_{\mathrm{cut}}\right)\end{aligned}$$

Extraction of two-loop constants

Resume:

$$\int_0^1 \mathrm{d}\rho\left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho}-\frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right)=\int_0^{2z_{\mathrm{cut}}} \mathrm{d}\rho\left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho}-\frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right)+\int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho\,\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho}-\int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho\,\frac{d\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}$$

$$\int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho\,\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho}=\int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho\left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho}-\frac{\mathrm{d}\sigma}{\mathrm{d}\rho}\right)+\int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho\,\frac{\mathrm{d}\sigma}{\mathrm{d}\rho}$$

Up to power corrections, the ungroomed xsec is

$$\int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho\,\frac{\mathrm{d}\sigma}{\mathrm{d}\rho}=\int_0^4 \mathrm{d}\rho\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\rho}-\frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho}\right)+\mathcal{O}\left(z_{\mathrm{cut}}\right)+\int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho\,\frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho}$$

Collecting all steps we find

$$\begin{aligned}\sigma_{\mathrm{tot}}=D_{\delta,\mathrm{g}}&+\int_0^{2z_{\mathrm{cut}}} \mathrm{d}\rho\left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho}-\frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right)+\int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho\left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho}-\frac{\mathrm{d}\sigma}{\mathrm{d}\rho}\right)\\&+\int_0^4 \mathrm{d}\rho\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\rho}-\frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho}\right)+\int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho\left(\frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho}-\frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right)+\int_1^4 \mathrm{d}\rho\,\frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho}+\mathcal{O}\left(z_{\mathrm{cut}}\right)\end{aligned}$$

Extraction of two-loop constants

Resume:

$$\int_0^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right) = \underline{\int_0^{2z_{\mathrm{cut}}} \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right)} + \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \, \frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \underline{\int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \, \frac{d\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}}$$

$$\int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \, \frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} = \underline{\int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma}{\mathrm{d}\rho}\right)} + \int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho \, \frac{\mathrm{d}\sigma}{\mathrm{d}\rho}$$

Up to power corrections, the ungroomed xsec is

$$\int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho \, \frac{\mathrm{d}\sigma}{\mathrm{d}\rho} = \int_0^4 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho}\right) + \mathcal{O}\left(z_{\mathrm{cut}}\right) + \int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho \, \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho}$$

Collecting all steps we find

$$\begin{aligned}\sigma_{\mathrm{tot}} = D_{\delta,\mathrm{g}} &+ \underline{\int_0^{2z_{\mathrm{cut}}} \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right)} + \underline{\int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma}{\mathrm{d}\rho}\right)} \\ &+ \int_0^4 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho}\right) + \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho} - \underline{\frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}}\right) + \int_1^4 \mathrm{d}\rho \, \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho} + \mathcal{O}\left(z_{\mathrm{cut}}\right)\end{aligned}$$

Extraction of two-loop constants

Resume:

$$\int_0^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right) = \underline{\int_0^{2z_{\mathrm{cut}}} \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right)} + \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \underline{\int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \frac{d\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}}$$

$$\int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} = \underline{\int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma}{\mathrm{d}\rho}\right)} + \int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho \frac{\mathrm{d}\sigma}{\mathrm{d}\rho}$$

Up to power corrections, the ungroomed xsec is

$$\int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho \frac{\mathrm{d}\sigma}{\mathrm{d}\rho} = \underline{\int_0^4 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho}\right)} + \mathcal{O}\left(z_{\mathrm{cut}}\right) + \int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho}$$

Collecting all steps we find

$$\sigma_{\mathrm{tot}} = D_{\delta,\mathrm{g}} + \underline{\int_0^{2z_{\mathrm{cut}}} \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right)} + \underline{\int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma}{\mathrm{d}\rho}\right)}$$
$$+ \underline{\int_0^4 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho}\right)} + \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho} - \underline{\frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}}\right) + \int_1^4 \mathrm{d}\rho \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho} + \mathcal{O}\left(z_{\mathrm{cut}}\right)$$

Extraction of two-loop constants

Resume:

$$\int_0^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right) = \int_0^{2z_{\mathrm{cut}}} \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right) + \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \frac{d\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}$$

$$\int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} = \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma}{\mathrm{d}\rho}\right) + \int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho \frac{\mathrm{d}\sigma}{\mathrm{d}\rho}$$

Up to power corrections, the ungroomed xsec is

$$\int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho \frac{\mathrm{d}\sigma}{\mathrm{d}\rho} = \int_0^4 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho}\right) + \mathcal{O}\left(z_{\mathrm{cut}}\right) + \int_{2z_{\mathrm{cut}}}^4 \mathrm{d}\rho \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho}$$

Collecting all steps we find

$$\begin{aligned}\sigma_{\mathrm{tot}} = D_{\delta,\mathrm{g}} &+ \int_0^{2z_{\mathrm{cut}}} \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right) + \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\mathrm{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma}{\mathrm{d}\rho}\right) \\ &+ \int_0^4 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho}\right) + \int_{2z_{\mathrm{cut}}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\mathrm{g}}^{\mathrm{sing}}}{\mathrm{d}\rho}\right) + \int_1^4 \mathrm{d}\rho \frac{\mathrm{d}\sigma^{\mathrm{sing}}}{\mathrm{d}\rho} + \mathcal{O}\left(z_{\mathrm{cut}}\right)\end{aligned}$$

Extraction of two-loop constants

For σ_{tot} we have from previous page:

$$\sigma_{\text{tot}} = D_{\delta,\text{g}} + \int_0^{2z_{\text{cut}}} \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\text{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\text{g}}^{\text{sing}}}{\mathrm{d}\rho}\right) + \int_{2z_{\text{cut}}}^{1} \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\text{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma}{\mathrm{d}\rho}\right)$$
$$+ \int_0^4 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma^{\text{sing}}}{\mathrm{d}\rho}\right) + \int_{2z_{\text{cut}}}^{1} \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma^{\text{sing}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\text{g}}^{\text{sing}}}{\mathrm{d}\rho}\right) + \int_1^4 \mathrm{d}\rho \, \frac{\mathrm{d}\sigma^{\text{sing}}}{\mathrm{d}\rho} + \mathcal{O}\left(z_{\text{cut}}\right)$$

Extraction of two-loop constants

For σ_{tot} we have from previous page:

$$\sigma_{\text{tot}} = D_{\delta,\text{g}} + \int_0^{2z_{\text{cut}}} \text{d}\rho \left(\frac{\text{d}\sigma_{\text{g}}}{\text{d}\rho} - \frac{\text{d}\sigma_{\text{g}}^{\text{sing}}}{\text{d}\rho} \right) + \int_{2z_{\text{cut}}}^{1} \text{d}\rho \left(\frac{\text{d}\sigma_{\text{g}}}{\text{d}\rho} - \frac{\text{d}\sigma}{\text{d}\rho} \right)$$
$$+ \int_0^4 \text{d}\rho \left(\frac{\text{d}\sigma}{\text{d}\rho} - \frac{\text{d}\sigma^{\text{sing}}}{\text{d}\rho} \right) + \int_{2z_{\text{cut}}}^{1} \text{d}\rho \left(\frac{\text{d}\sigma^{\text{sing}}}{\text{d}\rho} - \frac{\text{d}\sigma_{\text{g}}^{\text{sing}}}{\text{d}\rho} \right) + \int_1^4 \text{d}\rho \, \frac{\text{d}\sigma^{\text{sing}}}{\text{d}\rho} + \mathcal{O}\left(z_{\text{cut}}\right)$$

but it can also be expressed with the ungroomed distribution:

$$\sigma_{\text{tot}} = D_{\delta} + \int_0^4 \text{d}\rho \left(\frac{\text{d}\sigma}{\text{d}\rho} - \frac{\text{d}\sigma^{\text{sing}}}{\text{d}\rho} \right)$$

Extraction of two-loop constants

For $\sigma_{\rm tot}$ we have from previous page:

$$\sigma_{\rm tot} = D_{\delta,{\rm g}} + \int_0^{2z_{\rm cut}} {\rm d}\rho \left(\frac{{\rm d}\sigma_{\rm g}}{{\rm d}\rho} - \frac{{\rm d}\sigma_{\rm g}^{\rm sing}}{{\rm d}\rho}\right) + \int_{2z_{\rm cut}}^{1} {\rm d}\rho \left(\frac{{\rm d}\sigma_{\rm g}}{{\rm d}\rho} - \frac{{\rm d}\sigma}{{\rm d}\rho}\right)$$
$$+ \underline{\int_0^4 {\rm d}\rho \left(\frac{{\rm d}\sigma}{{\rm d}\rho} - \frac{{\rm d}\sigma^{\rm sing}}{{\rm d}\rho}\right)} + \int_{2z_{\rm cut}}^{1} {\rm d}\rho \left(\frac{{\rm d}\sigma^{\rm sing}}{{\rm d}\rho} - \frac{{\rm d}\sigma_{\rm g}^{\rm sing}}{{\rm d}\rho}\right) + \int_1^4 {\rm d}\rho \,\frac{{\rm d}\sigma^{\rm sing}}{{\rm d}\rho} + \mathcal{O}\left(z_{\rm cut}\right)$$

but it can also be expressed with the ungroomed distribution:

$$\sigma_{\rm tot} = D_\delta + \underline{\int_0^4 {\rm d}\rho \left(\frac{{\rm d}\sigma}{{\rm d}\rho} - \frac{{\rm d}\sigma^{\rm sing}}{{\rm d}\rho}\right)}$$

Extraction of two-loop constants

For σ_{tot} we have from previous page:

$$\sigma_{\text{tot}} = D_{\delta,\text{g}} + \int_0^{2z_{\text{cut}}} \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\text{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\text{g}}^{\text{sing}}}{\mathrm{d}\rho} \right) + \int_{2z_{\text{cut}}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\text{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma}{\mathrm{d}\rho} \right)$$
$$+ \underline{\int_0^4 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma^{\text{sing}}}{\mathrm{d}\rho} \right)} + \int_{2z_{\text{cut}}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma^{\text{sing}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\text{g}}^{\text{sing}}}{\mathrm{d}\rho} \right) + \int_1^4 \mathrm{d}\rho \, \frac{\mathrm{d}\sigma^{\text{sing}}}{\mathrm{d}\rho} + \mathcal{O}(z_{\text{cut}})$$

but it can also be expressed with the ungroomed distribution:

$$\sigma_{\text{tot}} = D_\delta + \underline{\int_0^4 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma^{\text{sing}}}{\mathrm{d}\rho} \right)}$$

so finally

$$D_\delta = D_{\delta,\text{g}} + \int_0^{2z_{\text{cut}}} \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\text{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\text{g}}^{\text{sing}}}{\mathrm{d}\rho} \right) + \int_{2z_{\text{cut}}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\text{g}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma}{\mathrm{d}\rho} \right)$$
$$+ \int_{2z_{\text{cut}}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma^{\text{sing}}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\text{g}}^{\text{sing}}}{\mathrm{d}\rho} \right) + \int_1^4 \mathrm{d}\rho \, \frac{\mathrm{d}\sigma^{\text{sing}}}{\mathrm{d}\rho} \, .$$

Extraction of two-loop constants

For $\sigma_{\rm tot}$ we have from previous page:

$$\sigma_{\rm tot} = D_{\delta,\rm g} + \int_0^{2z_{\rm cut}} \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\rm g}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\rm g}^{\rm sing}}{\mathrm{d}\rho}\right) + \int_{2z_{\rm cut}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\rm g}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma}{\mathrm{d}\rho}\right)$$
$$+ \underline{\int_0^4 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma^{\rm sing}}{\mathrm{d}\rho}\right)} + \int_{2z_{\rm cut}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma^{\rm sing}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\rm g}^{\rm sing}}{\mathrm{d}\rho}\right) + \int_1^4 \mathrm{d}\rho \frac{\mathrm{d}\sigma^{\rm sing}}{\mathrm{d}\rho} + \mathcal{O}\left(z_{\rm cut}\right)$$

but it can also be expressed with the ungroomed distribution:

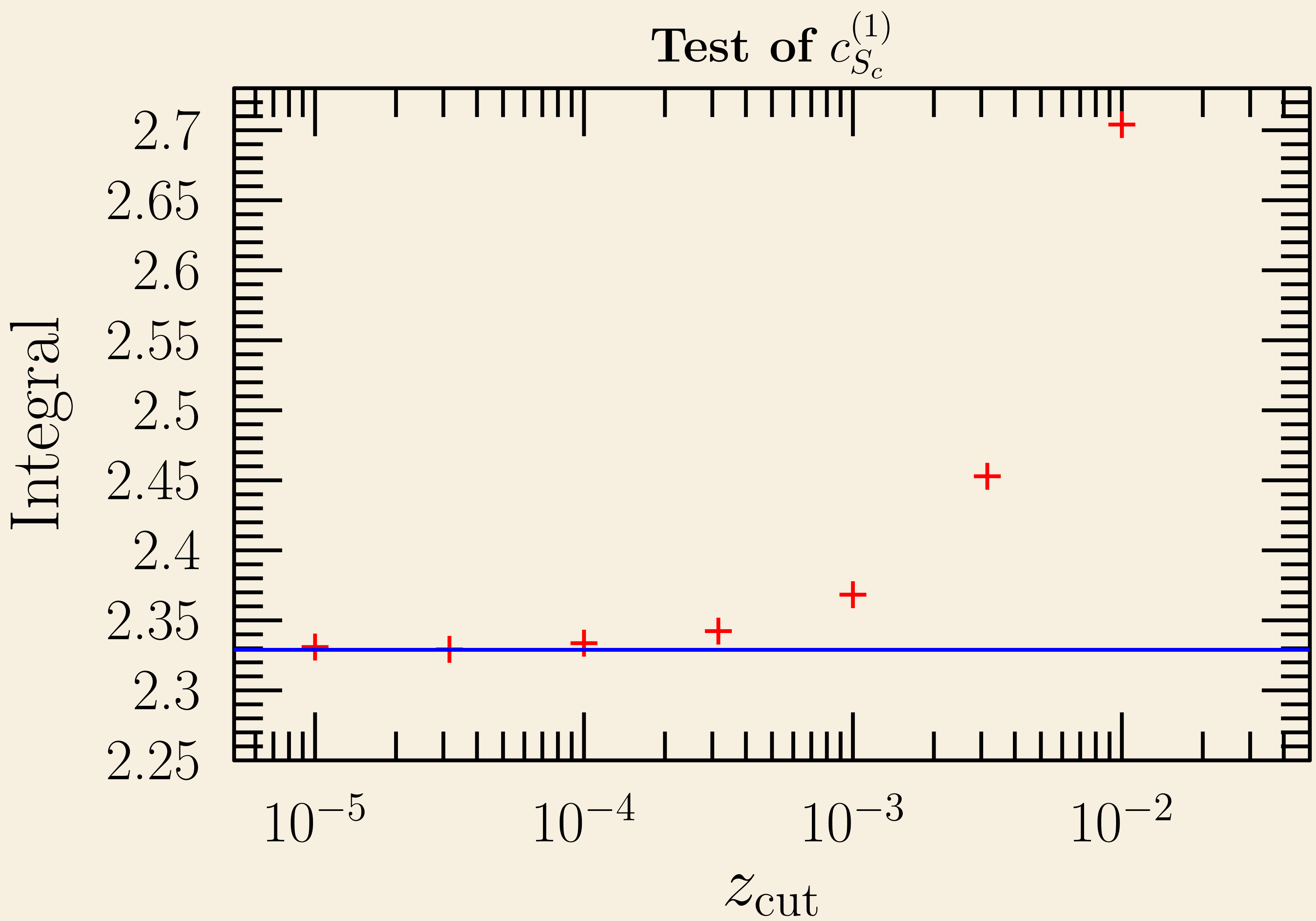
$$\sigma_{\rm tot} = D_\delta + \underline{\int_0^4 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma^{\rm sing}}{\mathrm{d}\rho}\right)}$$

so finally

$$\boxed{D_\delta} = D_{\delta,\rm g} + \boxed{\int_0^{2z_{\rm cut}} \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\rm g}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\rm g}^{\rm sing}}{\mathrm{d}\rho}\right) + \int_{2z_{\rm cut}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma_{\rm g}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma}{\mathrm{d}\rho}\right)}$$
$$\boxed{+ \int_{2z_{\rm cut}}^1 \mathrm{d}\rho \left(\frac{\mathrm{d}\sigma^{\rm sing}}{\mathrm{d}\rho} - \frac{\mathrm{d}\sigma_{\rm g}^{\rm sing}}{\mathrm{d}\rho}\right) + \int_1^4 \mathrm{d}\rho \frac{\mathrm{d}\sigma^{\rm sing}}{\mathrm{d}\rho}\,.}$$

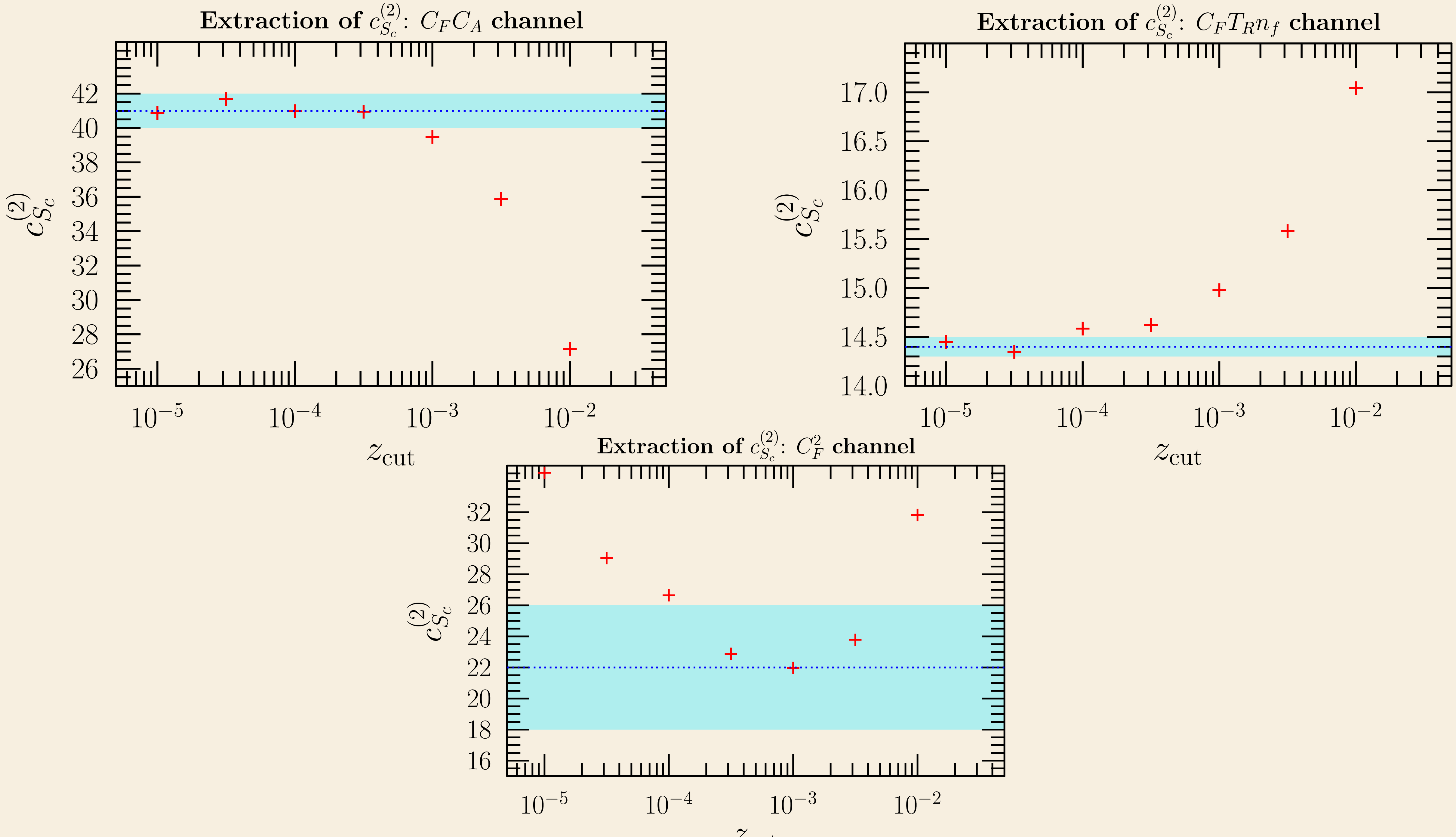
Validation at one loop

At one loop the integral can be computed numerically, but also known analytically:



Fits at two loops

At two loops the integral computed numerically with EVENT2, can be fitted (separately for each color channel)



Extraction of three-loop non-cusp anomalous dimension

the formal expansion of the mMDT groomed distribution for $\rho \ll z_{\rm cut} \ll 1$:

$$\frac{d\sigma_{\rm g,LP}}{d\rho} = \delta(\rho) D_{\delta,\rm g} + \frac{\alpha_s}{2\pi}(D_{A,\rm g}(\rho))_+ + \left(\frac{\alpha_s}{2\pi}\right)^2 (D_{B,\rm g}(\rho))_+ + \left(\frac{\alpha_s}{2\pi}\right)^3 (D_{C,\rm g}(\rho))_+$$

mMDT grooming removes double logarithms in ρ to all orders:

$$D_{\delta,g} = c_\delta(z_{\rm cut})$$

$$\rho D_{A,g} = c_A(z_{\rm cut})$$

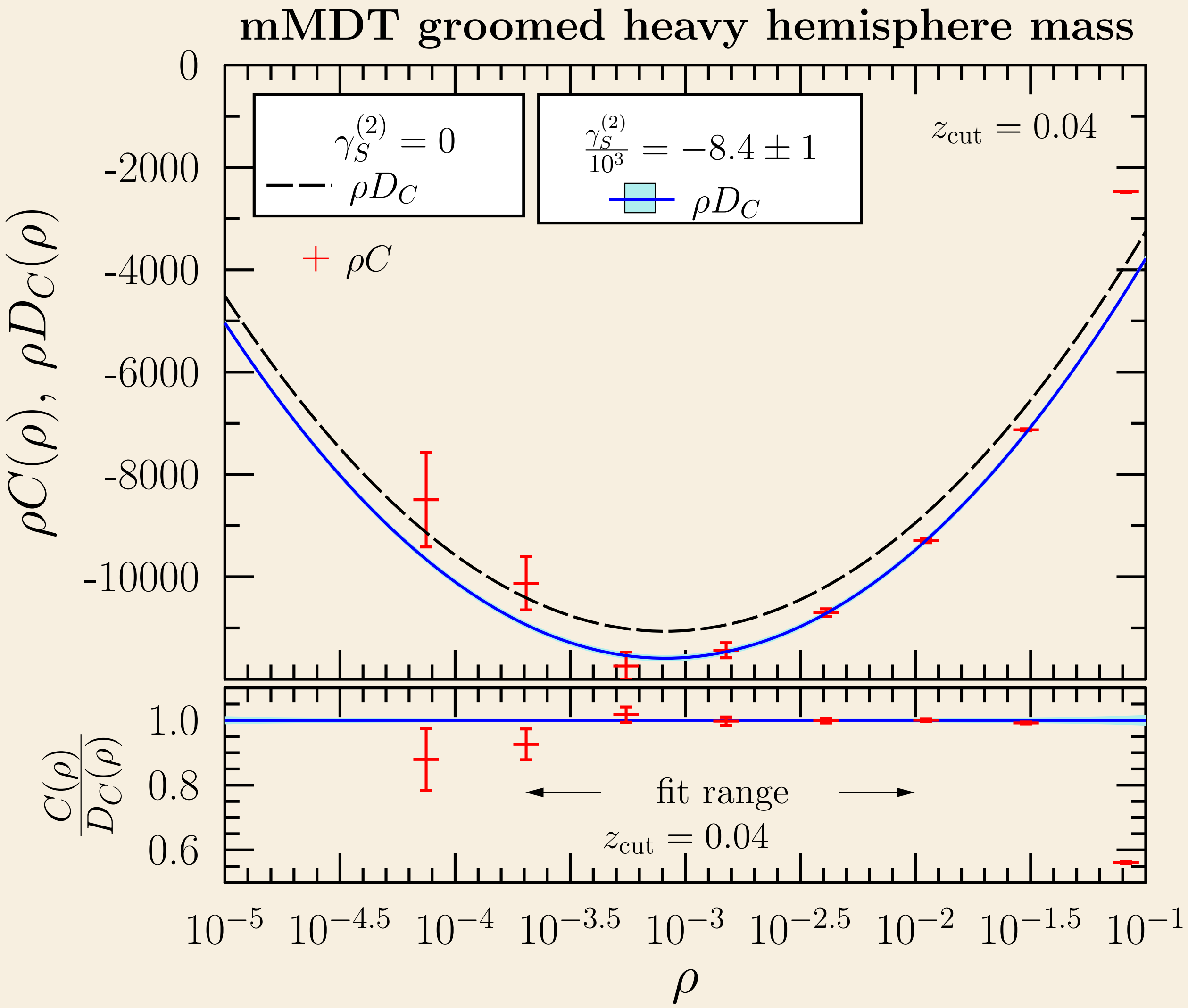
$$\rho D_{B,g} = b_B(z_{\rm cut}) \log\rho + c_B(z_{\rm cut})$$

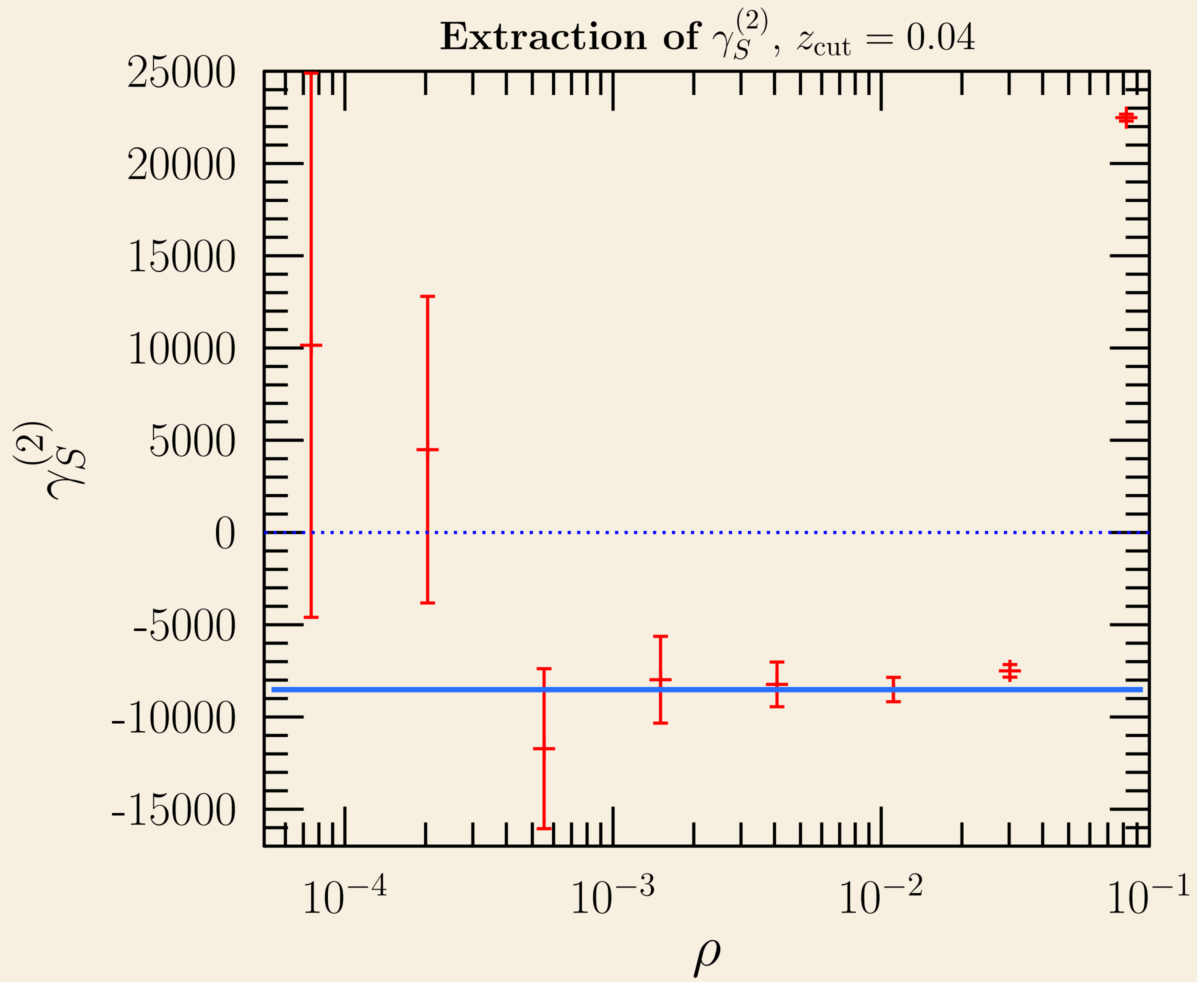
$$\rho D_{C,g} = a_C(z_{\rm cut}) \log^2\rho + b_C(z_{\rm cut}) \log\rho + c_C(z_{\rm cut})$$

can be computed by MCCSM

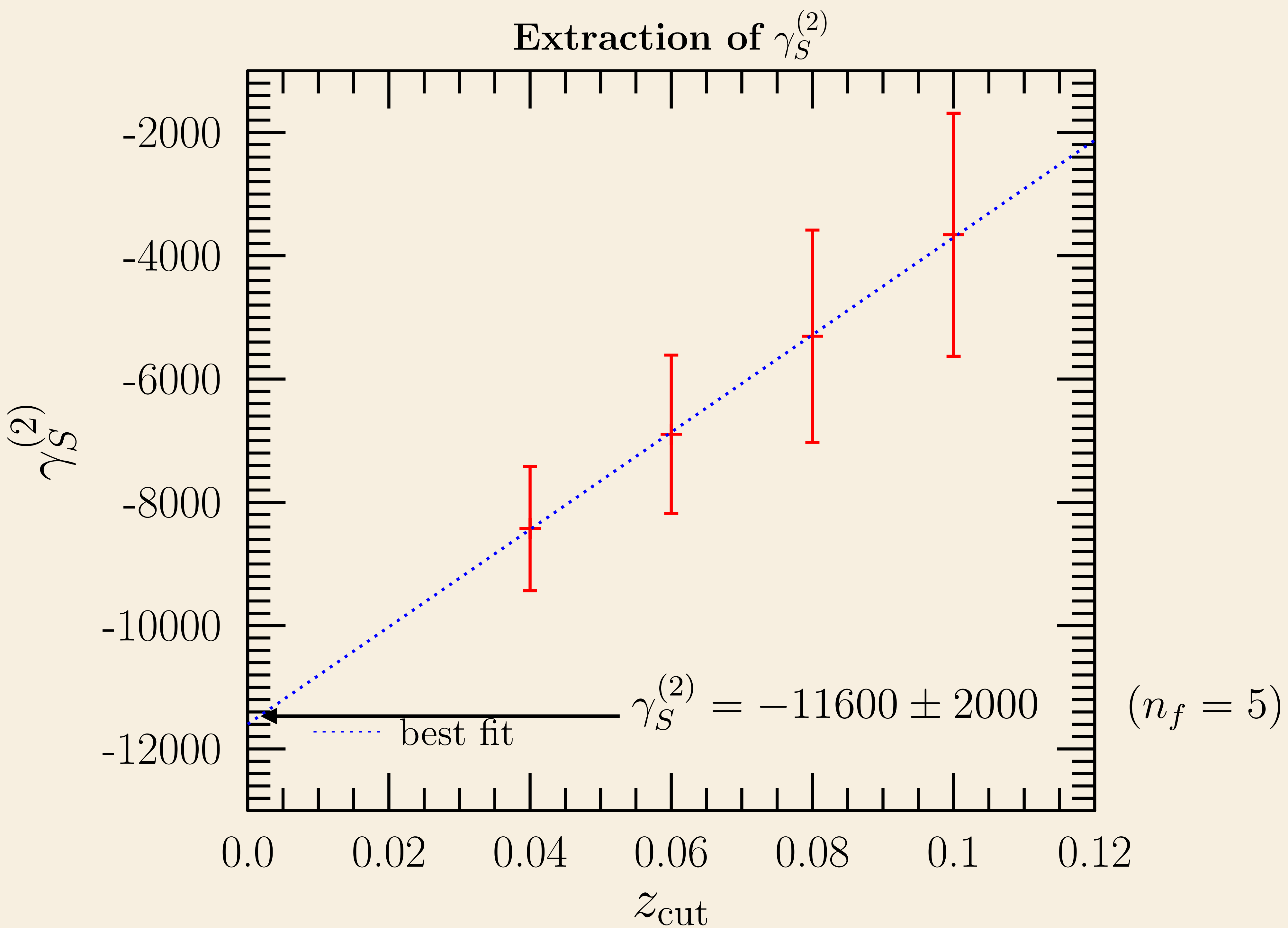
$$c_C(z_{\rm cut}) \xrightarrow{z_{\rm cut} \to 0} \gamma_S^{(2)}/16 - 1944.97$$

Fit a parabola in log ρ for fixed z_{cut}





Extrapolation of constant term to $z_{\mathrm{cut}} = 0$



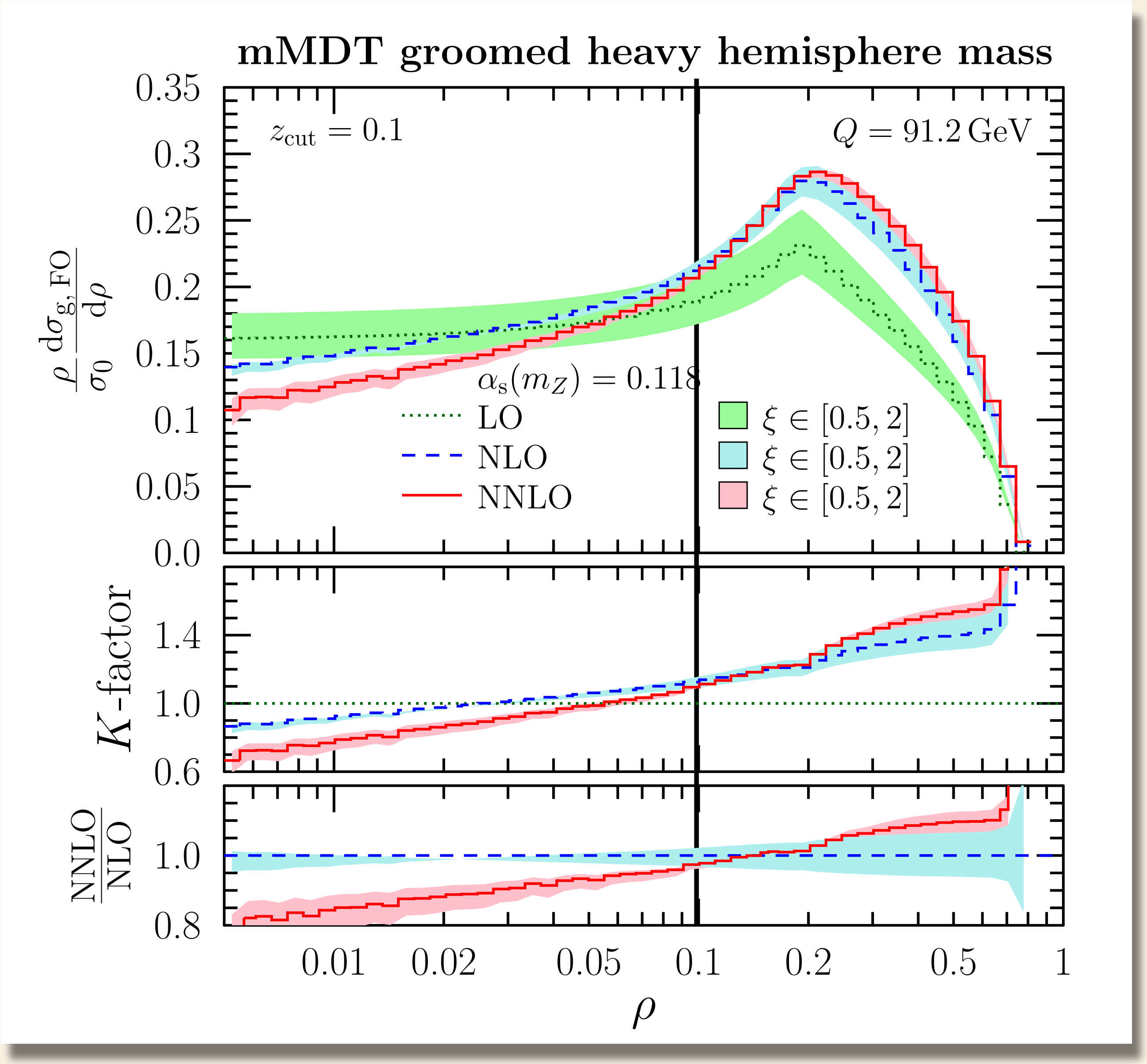
Matching to fixed order

mMDT groomed heavy jet mass

$$\rho\frac{\mathrm{d}\sigma_{\mathrm{g,NNLO}}}{\mathrm{d}\rho} = \frac{\alpha_s}{2\pi}A_{\mathrm{g}} + \left(\frac{\alpha_s}{2\pi}\right)^2\left[B_{\mathrm{g}} + A_{\mathrm{g}}\beta_0\log\frac{\mu}{Q}\right]$$
$$+\left(\frac{\alpha_s}{2\pi}\right)^3\left[C_{\mathrm{g}} + 2B_{\mathrm{g}}\beta_0\log\frac{\mu}{Q} + A_{\mathrm{g}}\left(\frac{\beta_1}{2}\log\frac{\mu}{Q} + \beta_0^2\log^2\frac{\mu}{Q}\right)\right]$$

A, B and C are computed with MCCSM (=Monte Carlo for the CoLoRFulNNLO Subtraction Method)

Converges for ρ > 0.1,
cannot be trusted for ρ < 0.1

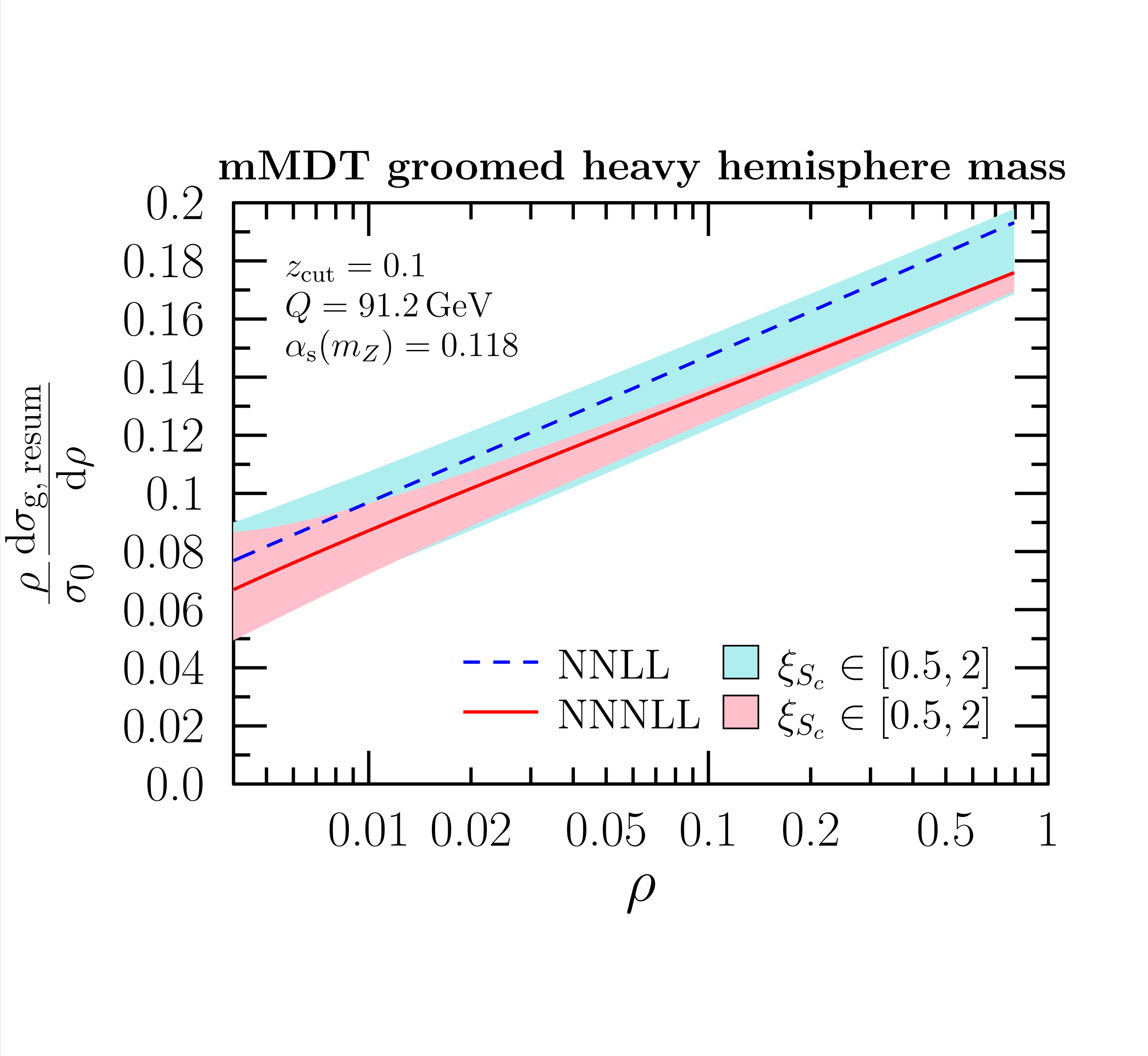


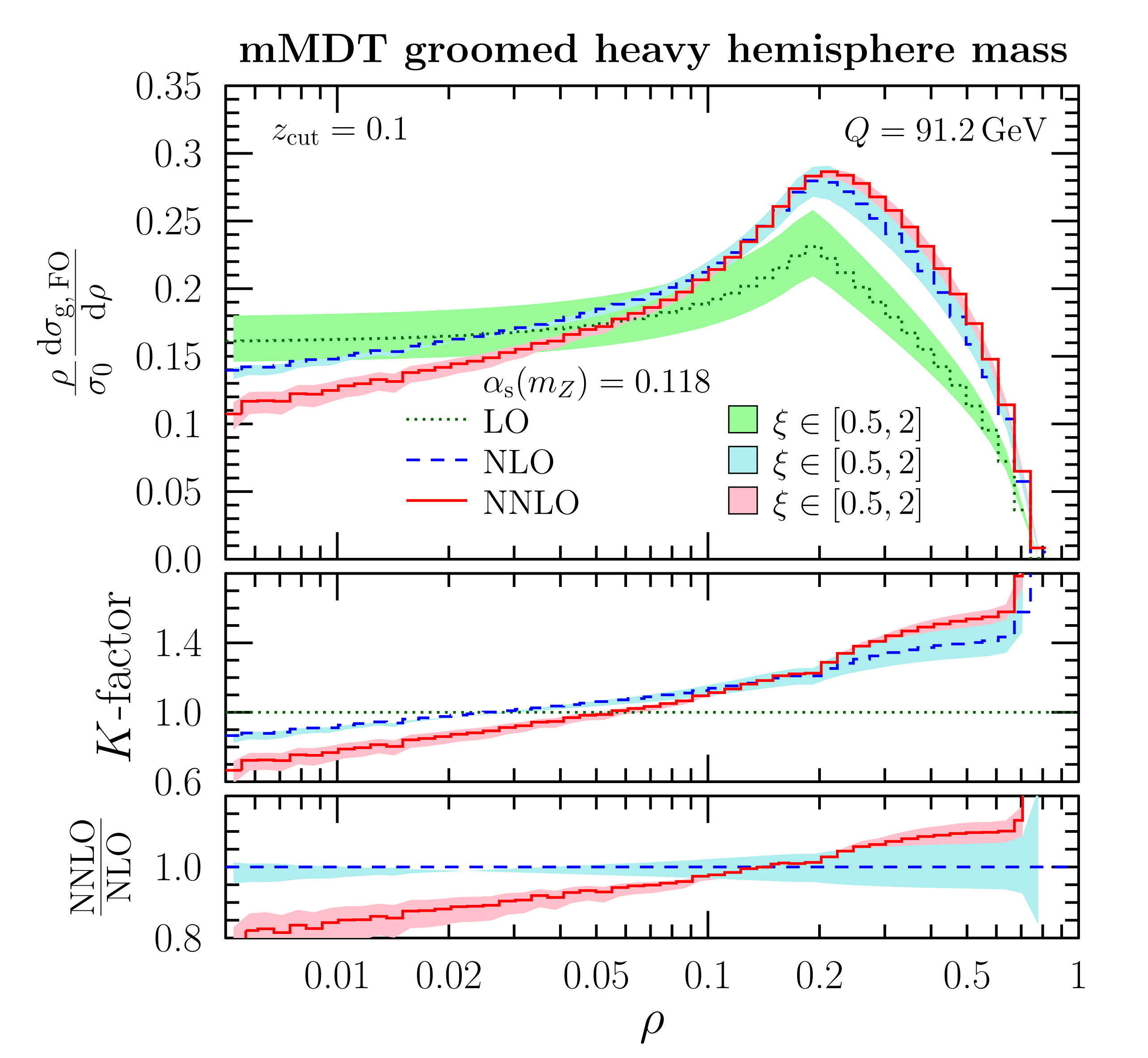
A. Kardos et al, arXiv: 1807.11472

mMDT groomed heavy jet mass

N^3LL can be matched to N^2LO additively by subtracting the expansion of N^3LL through $O(\alpha_s^3)$

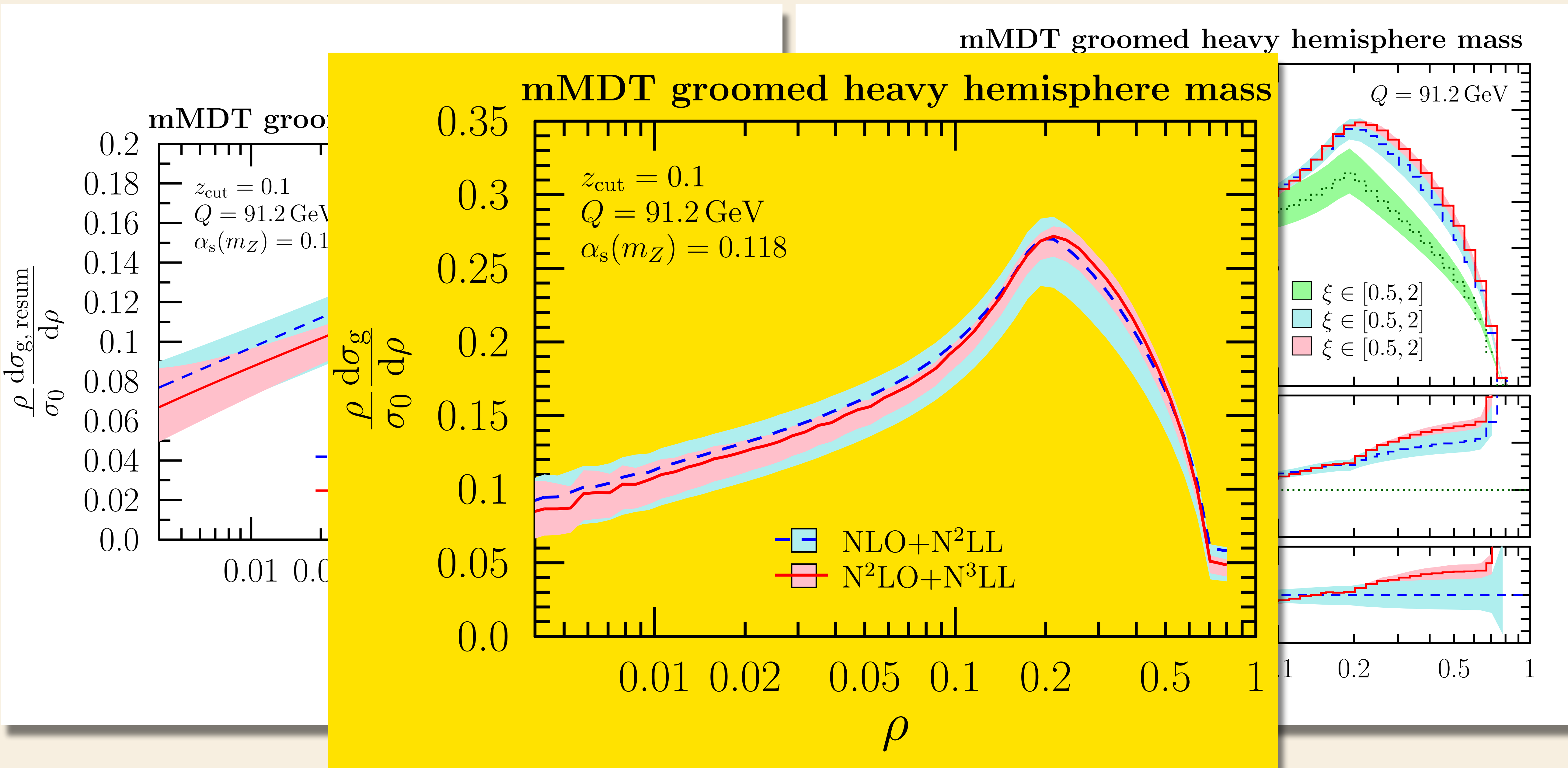
$$\frac{\rho}{\sigma_0}\frac{d\sigma_{g,\mathrm{FO+res}}}{d\rho} = \frac{\rho}{\sigma_0}\left(\frac{d\sigma_{g,\mathrm{N^3LL}}}{d\rho} + \frac{d\sigma_{g,\mathrm{N^2LO}}}{d\rho} - \frac{d\sigma_{g,\mathrm{LP}}}{d\rho}\right)$$





$$\frac{d\sigma_{g,\mathrm{LP}}}{d\rho} = \delta(\rho)D_{\delta,g} + \frac{\alpha_s}{2\pi}(D_{A,g}(\rho))_+ + \left(\frac{\alpha_s}{2\pi}\right)^2 (D_{B,g}(\rho))_+ + \left(\frac{\alpha_s}{2\pi}\right)^3 (D_{C,g}(\rho))_+$$

mMDT groomed heavy jet mass



Conclusions

✓ Precise determination of the strong coupling using hadronic final states in electron-positron annihilation requires

careful selection of observables with small perturbative and non-perturbative corrections (and data — not discussed here)

✓ MCCSM was used to compute differential distributions for groomed event shapes — mMDT groomed heavy jet mass among others

✓ Our predictions

- show good perturbative stability for $\rho > 10^{-1}$ (smaller scale dependence than un-groomed event shapes)
- are stable numerically to $\rho \sim 10^{-4}$
- were used to extract unknown constants needed for NNNLL resummation and matching

✓ NNLO+NNNLL additive matching is made possible the first time