Open loops to trees in four space-time dimensions

Germán Rodrigo
- all options aimed at \textit{attobarn}^{-1} physics
- requires to go \textit{far beyond} NNLO for theory
- even conservative estimates \textit{not reachable with current techniques}
The difficulties to reach higher orders arise because we have defined Quantum Field Theory \textbf{not in the optimal way}.

\textbf{Math}

\begin{align*}
y \quad & \quad x \\
\end{align*}

\textbf{Physics}

\begin{align*}
x \quad & \quad t \\
\end{align*}

Find the Slope

*“Mathematics is the language of the Universe”, attributed to \textbf{Galileo Galilei} 1564-1642*
QFT = Quantum Mechanics + space-time

- Loops encode quantum fluctuations at infinite energy (zero distance): SM/BSM extrapolated at energies \( \gg M_{\text{Plank}} \)
- QED/QCD massless gauge bosons/quarks: quantum state with \( N \) partons \( \neq \) quantum state with zero energy emission (infinite distance) of extra partons
- Partons can be emitted in exactly the same direction (zero distance)
Dimensional Regularization DREG

\[ d = 4 - 2\epsilon \]

- Qualitative interpretation: A way to give some extra space to the colliding particles
- Intrinsic infinities appear as poles in the extra dimensions: \( 1/\epsilon \) after integration over loop momenta and phase-space
- Different quantum fluctuations should contribute with poles of opposite sign, such that theoretical predictions for physical observables remain finite
- Mathematically well-defined but unphysical, and difficult to implement at higher orders
OUTLINE | AIMED FULLY LOCAL AND IN THE PHYSICAL FOUR DIMENSIONS

**ASYMPTOTIC EXPANSIONS**
LTD works in the Euclidean space of the loop three-momenta where the hierarchy of scales is well defined.

**LOOP-TREE DUALITY (LTD)**
open loop amplitudes to non-disjoint trees by introducing a number of on-shell conditions equal to the number of loops \( \neq \) Generalized Unitarity

**FOUR DIMENSIONAL UNSUBTRACTION (FDU)**
Introduce mappings of momenta between the virtual and real kinematics such that soft and collinear singularities are cancelled locally in \( d = 4 \) space-time dimensions. Real and virtual contributions generated simultaneously.

**LOCAL RENORMALISATION**
Suppress bad behavior at very high energies such that UV singularities are cancelled locally in \( d = 4 \) space-time dimensions.
We shall show that any diagram with closed loops can be expressed in terms of sums (actually integrals) of tree diagrams. In each of these tree diagrams there is, in addition to the external particles of the original closed loop diagram, certain particles in the initial and in the final state of the tree diagram. These particles form, together with the external lines, a set of closed loops of particles in the tree diagram. It is the purpose of this paper to show how one can correspond these closed loops to the integrals of the closed loop diagrams and recover the divergent terms in the theory.
IT’S ALL ABOUT THE TINY $+\text{i}0$ FROM

MATH: the $+\text{i}0$ is a small quantity usually ignored, assuming that the \textbf{analytical continuation} to the physical kinematics is well defined.

PHYSICS: the $+\text{i}0$ encodes \textbf{CAUSALITY} | positive frequencies are propagated forward in time, and negative backward.

\[
G_F(q_i) = \frac{1}{q_i^2 - m_i^2 + \text{i}0}
\]
THE LOOP–TREE DUALITY (LTD)

Cauchy residue theorem in the loop energy complex plane

Feynman Propagator $+i0$:
positive frequencies are propagated forward in time, and negative backward

$$G_F(q_i) = \frac{1}{q_i^2 - m_i^2 + i0}$$

selects residues with definite **positive energy and negative imaginary part**

in arbitrary coordinate systems: reduce the dimension of the integration domain by one unit

[Catani et al. JHEP 0809, 065]
THE LOOP–TREE DUALITY (LTD)

One-loop amplitudes in any relativistic, local and unitary QFT represented as a linear combination of \( N \) single-cut phase-space/dual amplitudes | non-disjoint trees (at higher orders: number of cuts equal to the number of loops)

\[
\int_{\ell_1} N(\ell_1) \prod G_F(q_i) = -\int_{\ell_1} N(\ell_1) \otimes \sum \tilde{\delta}(q_i) \prod_{i \neq j} G_D(q_i; q_j)
\]

- \( \tilde{\delta}(q_i) = i2\pi \theta(q_{i,0}) \delta(q_i^2 - m_i^2) \) sets internal line on-shell, positive energy mode
- \( G_D(q_i; q_j) = \frac{1}{q_i^2 - m_i^2 - i0 \eta_{k_{ji}}} \) dual propagator \( k_{ji} = q_j - q_i \) \( q_{i,0}^{(+)} = \sqrt{q_i^2 + m_i^2 - i0} \)

LTD realised by modifying the customary \( +i0 \) prescription of the Feynman propagators (only the sign matters), it encodes in a compact way the effect of multiple-cut contributions that appear in the Feynman’s Tree Theorem

- Lorentz invariant, best choice \( \eta^\mu = (1,0) \) : energy component integrated out, remaining integration in Euclidean space
\[ \lambda_{ij}^{\pm} = \pm q_{i,0}^{(+)} \pm q_{j,0}^{(+)} + k_{ji,0} \rightarrow 0 \]

where

\[ q_{r,0}^{(+)} = \sqrt{q_r^2 + m_r^2 - i0} \quad k_{ji} = q_j - q_i \]
THE DUAL FOREST | CAUSALITY

\[
S_{ij}^{(1)} = (2\pi i)^{-1} G_D(q_i; q_j) \tilde{\delta}(q_i) + (i \leftrightarrow j)
\]

- **Time-like distance (causally connected):** generates physical threshold singularities: always +i0

\begin{align*}
S_{ij}^{(1)} &= \frac{\theta(-k_{ji,0}) \theta(k_{ji}^2 - (m_i + m_j)^2)}{x_{ij} \left(-\lambda_{ij}^{++} - i0 k_{ji,0}\right)} + \mathcal{O}\left((\lambda_{ij}^{++})^0\right) \\
x_{ij} &= 4 q_{i,0}^{(+)} q_{j,0}^{(+)} + i0
\end{align*}

- **Space-like distance:** there is a perfect cancellation of singularities, due to the dual +i0 prescription

\[
\lim_{\lambda_{ij}^{+-} \to 0} S_{ij}^{(1)} = \mathcal{O}\left((\lambda_{ij}^{+-})^0\right) \quad k_{ji}^2 - (m_j - m_i)^2 \leq 0
\]

- **Light-like distance:** both singular configurations, partial cancellation, IR singularities remain in a compact region

IR and threshold singularities are restricted to a **compact region** of the loop three-momentum.
THE FEYNMAN’S FOREST | CAUSALITY

FTT:

\[ \mathcal{F}^{(1)}_{ij} = (2\pi i)^{-1} G_F(q_j) \tilde{\delta}(q_i) + (i \leftrightarrow j) \]

- **Time-like distance (causally connected):** physics does not depend on the FTT or LTD representation

\[
\lim_{\lambda_{ij+}-0} \mathcal{F}^{(1)}_{ij} = \frac{\theta(-k_{ji,0}) \theta(k_{ji}^2 - (m_i + m_j)^2)}{x_{ij}(-\lambda_{ij+} + i0)} + \mathcal{O}\left((\lambda_{ij+})^0\right)
\]

- **Space-like distance:** there is mismatch in the +i0 prescription

\[
\lim_{\lambda_{ij-}-0} \mathcal{F}^{(1)}_{ij} \sim \frac{1}{-\lambda_{ij-} + i0} + \frac{1}{\lambda_{ij-} + i0} + \mathcal{O}\left((\lambda_{ij-})^0\right)
\]

- needs to be compensated by the contribution from multiple cuts
non-causal singularities (forward-forward in blue): undergo dual cancellations among dual pairs

causal singularities (forward-backward in orange): bounded to a compact region, which is of the size of the hard scale, collapse to a finite segment for infrared singularities (→ FDU)

Numerical integration in the Euclidean space of the loop three-momenta, CPU/GPU time do not scale significantly with the number of legs
NUMERICAL INTEGRATION

<table>
<thead>
<tr>
<th>Rank</th>
<th>Tensor</th>
<th>Hexagon</th>
<th>Real Part</th>
<th>Imaginary Part</th>
<th>Time [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>P20</td>
<td>1</td>
<td>SecDec</td>
<td>$-1.21585(12) \times 10^{-15}$</td>
<td></td>
<td>36</td>
</tr>
<tr>
<td></td>
<td></td>
<td>LTD</td>
<td>$-1.21552(354) \times 10^{-15}$</td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>P21</td>
<td>3</td>
<td>SecDec</td>
<td>$4.46117(37) \times 10^{-9}$</td>
<td></td>
<td>5498</td>
</tr>
<tr>
<td></td>
<td></td>
<td>LTD</td>
<td>$4.461369(3) \times 10^{-9}$</td>
<td></td>
<td>11</td>
</tr>
<tr>
<td>P22</td>
<td>1</td>
<td>SecDec</td>
<td>$1.01359(23) \times 10^{-15}$</td>
<td>$+i 2.68657(26) \times 10^{-15}$</td>
<td>33</td>
</tr>
<tr>
<td></td>
<td></td>
<td>LTD</td>
<td>$1.01345(130) \times 10^{-15}$</td>
<td>$+i 2.68633(130) \times 10^{-15}$</td>
<td>72</td>
</tr>
<tr>
<td>P23</td>
<td>2</td>
<td>SecDec</td>
<td>$2.45315(24) \times 10^{-12}$</td>
<td>$-i 2.06087(20) \times 10^{-12}$</td>
<td>337</td>
</tr>
<tr>
<td></td>
<td></td>
<td>LTD</td>
<td>$2.45273(727) \times 10^{-12}$</td>
<td>$-i 2.06202(727) \times 10^{-12}$</td>
<td>75</td>
</tr>
<tr>
<td>P24</td>
<td>3</td>
<td>SecDec</td>
<td>$-2.07531(19) \times 10^{-6}$</td>
<td>$+i 6.97158(56) \times 10^{-7}$</td>
<td>14280</td>
</tr>
<tr>
<td></td>
<td></td>
<td>LTD</td>
<td>$-2.07526(8) \times 10^{-6}$</td>
<td>$+i 6.97192(8) \times 10^{-7}$</td>
<td>85</td>
</tr>
</tbody>
</table>

Table 5: Tensor hexagons involving numerators of rank one to three.

<table>
<thead>
<tr>
<th>Propagator</th>
<th>Real Part</th>
<th>Imaginary Part</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2.530(4) \times 10^{-14}$</td>
<td>$+i 8.514(1) \times 10^{-14}$</td>
</tr>
<tr>
<td>$\ell.p_3 \times \ell.p_5$</td>
<td>$8.08(4) \times 10^{-15}$</td>
<td>$+i 6.144(5) \times 10^{-13}$</td>
</tr>
</tbody>
</table>

Table 1: Scalar and tensor decagon with all internal masses different.
energy of the on-shell propagator smaller than the energy of the emitted particles

Threshold singularities occur when a second propagator gets on-shell: consistent with Cutkosky

It becomes collinear (soft) when a single massless particle is emitted

Causally connected

Virtual particle emitted and absorbed on-shell

Potential threshold and IR singularities cancel in the sum of single-cut trees: non-causal

Non-singular configurations at very large energies (UV) expected to be suppressed. If not sufficiently suppressed, renormalise
AT TWO LOOPS

UNITARITY THRESHOLD / TRIPLE COLLINEAR

\[ q_{i,0}^{(+)} + q_{j,0}^{(+)} + q_{k,0}^{(+)} - k_{ijk,0} = 0 \]

IR and threshold singularities are restricted to a **compact region** of the loop three-momenta

Essential feature for FDU at NNLO
LOCAL UV RENORMALISATION

- Expand propagators and numerators around a UV propagator [Reuschle et al., similar to Pittau’s FDR in the UV]

\[ G_F(q_{UV}) = \frac{1}{q_{UV}^2 - \mu_{UV}^2 + i0} \quad \{ \ell_j^2 \mid \ell_j \cdot k_i \} \rightarrow \{ \lambda^2 q_{UV}^2 + (1 - \lambda^2) \mu_{UV}^2 \mid \lambda q_{UV} \cdot k_i \} \]

- and adjust subleading terms, \( c_{UV} \), to subtract only the pole (\( \overline{MS} \) scheme), or to define any other renormalisation scheme. For the scalar two point function

\[ I^{\text{cnt}}_{UV} = \int \frac{1}{\ell_j \left( q_{UV}^2 - \mu_{UV}^2 + i0 \right)^2} \left( 1 + c_{UV} \frac{\mu_{UV}^2}{q_{UV}^2 - \mu_{UV}^2 + i0} \right) \]

- dual representation needs to deal with multiple poles [Bierenbaum et al.]

\[ I^{\text{cnt}}_{UV} = \int \frac{\tilde{\delta}(q_{UV})}{2 q_{UV,0}^{(+)^2}} \left( 1 - \frac{3 c_{UV} \mu_{UV}^2}{4 \left( q_{UV,0}^{(+)} \right)^2} \right) \quad q_{UV,0}^{(+)} = \sqrt{q_{UV}^2 + \mu_{UV}^2 - i0} \]

\[ \text{Hernández-Pinto, Sborlini, GR, JHEP 1602, 044} \]

- Integration on the UV on-shell hyperboloid: loop three-momentum unconstrained, but loop contributions suppressed for loop energies larger than \( \mu_{UV} \)
Finite helicity amplitudes, but unintegrated amplitudes locally singular.

\[ \mathcal{A}_R^{(L)} = \mathcal{A}^{(L)} - \mathcal{A}^{(L)}_{UV} \bigg|_{d=4} \quad \mathcal{A}^{(L)}_{UV} \bigg|_{d} = 0 \]

Subtract not only logarithmic UV singularities, but also linear and quadratic.

Disentangle the UV from the IR behaviour in scaleless integrals (e.g. self-energies).
**LOCAL UV RENORMALISATION: MULTILOOP**

\[
\begin{cases}
|\ell_1| \to \infty \\
|\ell_2| \text{ fixed}
\end{cases}, \quad \begin{cases}
|\ell_1| \text{ fixed} \\
|\ell_2| \to \infty
\end{cases}, \quad \begin{cases}
|\ell_1| \to \infty \\
|\ell_2| \to \infty
\end{cases}
\]

- Multiple UV limit

\[
\text{UV}^2 : \{ \ell_j^2 \mid \ell_j \cdot \ell_k \mid \ell_j \cdot k_i \}
\to \{ \lambda^2 q_{j,UV}^2 + (1 - \lambda^2) \mu_{UV}^2 \mid \lambda^2 q_{j,UV} \cdot q_{k,UV} + (1 - \lambda^2) \mu_{UV}^2/2 \mid \lambda \cdot q_{j,UV} \cdot k_i \}
\]

- Most subtle steep the adjustment of the subleading terms, \(d_{UV2}\), to be in agreement with e.g. the \(\overline{MS}\) scheme

\[
\left( \mathcal{A}^{(L)} - \mathcal{A}_{1,UV}^{(L)} - \mathcal{A}_{2,UV}^{(L)} \right)_{UV^2} - d_{UV2} \mu_{UV}^4 \int \ell_1 \ell_2 \left( G_F(q_{1,UV}) \right)^3 \left( G_F(q_{2,UV}) \right)^3
\]

**\(H \to \gamma\gamma \text{ at two-loops}\)**

[Analytic expressions from Aglietti, Bonciani, Degrassi, Vicini, JHEP 0701 (2007) 021]
**FOUR-DIMENSIONAL UNSUBTRACTION (FDU) @ NLO**

- **The LTD representation** of the renormalised loop cross-section: one single integral in the loop three-momentum

\[
\int_N d\sigma^{(1,R)}_V = \int_N \int_{\ell_1} 2 \Re \langle \mathcal{M}^{(0)}_N | \left( \sum_i \mathcal{M}^{(1)}_N (\tilde{\delta}(q_i)) \right) - \mathcal{M}^{(1)}_{\text{UV}} (\tilde{\delta}(q_{\text{UV}})) \rangle
\]

- A **partition** of the real phase-space

\[
\sum_i \mathcal{R}_i (\{p'_j\}_{N+1}) = 1
\]

- The real contribution **mapped** to the **Born kinematics + loop three-momentum**

\[
\int_{N+1} d\sigma^{(1)}_R = \int_N \int_{\ell_1} \sum_i \mathcal{J}_i(q_i) \mathcal{R}_i (\{p'_j\}) |\mathcal{M}^{(0)}_{N+1} (\{p'_j\})|^2 \bigg|_{\{p'_j\}_{N+1} \rightarrow (q_i,\{p_k\}_N)}
\]
MOMENTUM MAPPING: MULTI-LEG

Motivated by the factorisation properties of QCD: assuming $q_i^\mu$ on-shell, and close to collinear with $p_i^\mu$, we define the momentum mapping

$$p_r^\mu = q_i^\mu,$$

$$p_i^\mu = p_i^\mu - q_i^\mu + \alpha_i p_j^\mu,$$

$$p_j^\mu = (1 - \alpha_i) p_j^\mu,$$

$$p_k^\mu = p_k^\mu, \quad k \neq i, j$$

All the primed momenta (real process) on-shell and momentum conservation: $p_i^\mu$ is the emitter, $p_j^\mu$ the spectator needed to absorb momentum recoil.

Quasi-collinear configurations can also be conveniently mapped such that the massless limit is smooth [Sborlini, Driencourt-Mangin, GR, JHEP 1610, 162].
The bulk of the physics is in the "low" energy region of the loop momentum.
BENCHMARK APPLICATION: $A^* \rightarrow q\bar{q}(g)$

- Excellent agreement with analytic DREG
- Efficient numerical implementation
- Smooth massless limit
Expansion of dual propagators

\[ G_D(q_i; q_j) = \frac{1}{2q_i \cdot k_{ji} + \Gamma_{ij} + \Delta_{ij} - i0\eta \cdot k_{ji}} \]

\[ \Delta_{ij} \text{ small} \]

\[ = \sum_{n=0} \frac{(-\Delta_{ij})^n}{(2q_i \cdot k_{ji} + \Gamma_{ij} - i0\eta \cdot k_{ji})^{n+1}} \]

wrt Expansion by Regions [Smirnov, Beneke]: it does not mix UV with IR. Only the first terms might need local renormalisation

Expansion by dual regions in the loop three-momentum

<table>
<thead>
<tr>
<th>Dual propagator components</th>
<th>( M^2 \gg {m^2, p^2} )</th>
<th>( p^2 \gg {m^2, M^2} )</th>
<th>( p^2 = (m + M)^2(1 - \beta) ), ( \beta \to 0^\pm )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_D(q_i; q_j) )</td>
<td>( \Gamma_{12} = M^2 + p^2 )</td>
<td>( p^2 + M^2 )</td>
<td>( 2Mp \cosh \left( \sqrt{-\frac{m^2}{M^2}} - i0 \right) )</td>
</tr>
<tr>
<td>( \Delta_{12} )</td>
<td>( -m^2 )</td>
<td>( -m^2 )</td>
<td>( p^2 + M^2 - m^2 - \Gamma_{12} )</td>
</tr>
<tr>
<td>( r_{12} )</td>
<td>( \frac{\not{p}}{M} )</td>
<td>( \frac{M}{\sqrt{p^2}} )</td>
<td>( \exp \left( \sqrt{-\frac{m^2}{M^2}} - i0 \right) )</td>
</tr>
<tr>
<td>( Q_i^2 )</td>
<td>( M^2 )</td>
<td>( p^2 )</td>
<td>( M p \exp \left( -\sqrt{-\frac{m^2}{M^2}} - i0 \right) )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Dual propagator components</th>
<th>( G_D(\ell; q_i) )</th>
<th>( \Gamma_{12} = p^2 + m^2 + \frac{m^2 M^2}{p^2} )</th>
<th>( p^2 + m^2 )</th>
<th>( 2m p \cosh \left( \sqrt{-\frac{M^2\beta}{m}} + i0 \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta_{12} )</td>
<td>( -M^2 - \frac{m^2 p^2}{M^2} )</td>
<td>( -M^2 )</td>
<td>( p^2 + m^2 - M^2 - \Gamma_{21} )</td>
<td></td>
</tr>
<tr>
<td>( r_{12} )</td>
<td>( \frac{m\not{p}}{M} )</td>
<td>( -\frac{m}{\sqrt{p^2}} + i0 )</td>
<td>( -\exp \left( -\sqrt{\frac{M^2\beta}{m}} + i0 \right) )</td>
<td></td>
</tr>
<tr>
<td>( Q_i^2 )</td>
<td>( -M^2 )</td>
<td>( p^2 )</td>
<td>( m p \exp \left( -\sqrt{\frac{M^2\beta}{m}} + i0 \right) )</td>
<td></td>
</tr>
</tbody>
</table>
After the first LTD round the position of the poles in the complex plane is momentum dependent.

1. **Use a general identity** to transform into Feynman propagators the dual propagators that enter the successive LTD rounds [Bierenbaum et al., 2010]
   - First **full two-loop** calculation ($H \rightarrow \gamma\gamma$) with local UV renormalization [Driencourt et al., 2019]
   - **Analytic proof** of the dual cancellation of unphysical (non-causal) singularities, causal and anomalous thresholds as well as infrared in a compact region ($\rightarrow$ FDU) [Aguilera et al., 2019]

2. **Average** over all possible momentum flows [Runkel et al., 2019]: cumbersome symmetry factors

3. **Keep track of the position of the poles and close the Cauchy contour either from above or from below** to cancel that dependence [Capatti et al., 2019]
   - **Numerical test** of dual cancellations
OPENING TO NON-DISJOINT TREES + CAUSALITY

G. Rodrigo, Wien seminar

LTD TO ALL ORDERS AND POWERS

- Multi-loop scattering amplitude: $n$ sets of momenta that depend on $L$ loop momenta or a linear combination

$$\mathcal{A}^{(L)}_N(1,\ldots,n) = \int_{\ell_1\ldots\ell_L} \mathcal{N}(|\ell_i⟩_L, \{p_j⟩_N) G_F(1,\ldots,n)$$

$$= \prod_{i\in 1\cup…\cup n} (G_F(q_i))^{a_i}$$

- The dual function involving two sets that depend on the same loop momentum: momenta in the set $t$ remain off-shell

$$G_D(s; t) = -2\pi i \sum_{i_s \in s} \text{Res} \left( G_F(s, t), \text{Im}(\eta q_{i_s}) < 0 \right)$$

- Cauchy contour always from below the real axis

- valid for arbitrary powers and Lorentz invariant [Catani et al. JHEP 0809, 065]

- reverse momenta, if necessary, to keep a coherent momentum flow

  $$t \rightarrow \bar{t} \quad (q_{i_t} \rightarrow - q_{i_t})$$

- The nested residue involving several sets

$$G_D(1,\ldots,r; n) = -2\pi i \sum_{i_r \in r} \text{Res} \left( G_D(1,\ldots,r-1; r, n), \text{Im}(\eta q_{i_r}) < 0 \right)$$
MULTILOOP TOPOLOGIES

**MLT**
Maximal Loop Topology
unique topology at two loops

**NMLT**
Next-to-maximal Loop Topology

**N2MLT**
Next-to-next-to-maximal Loop Topology
both topologies starting at three loops

arbitrary number of external legs attached to each line
\[ q_{is} = \ell_s + k_{is}, \quad q_{i12} = -\ell_1 - \ell_2 + k_{i12} \]
MAXIMAL LOOP TOPOLOGY

- **extremely simple and symmetric** LTD representation, proven by induction and directly *independent of the position of the poles* in the complex plane

\[
\mathcal{A}^{(L)}_{\text{MLT}}(1, \ldots, n) = \int_{\ell_1 \cdots \ell_L} \sum_{i=1}^{n} \mathcal{A}^{(L)}_D(1, \ldots, i-1, i+1, \ldots, \bar{n}; i)
\]

- **causal singularities** when on-shell momenta get aligned [Aguilera et al. JHEP 1912, 163]

\[
\mathcal{A}^{(L)}_D(1, \ldots, n-1; n) \big|_{\text{onshell}} \rightarrow \mathcal{A}^{(L)}_D(1, 2, \ldots, n) \quad \big| \quad \mathcal{A}^{(L)}_D(\bar{2}, \ldots, \bar{n}; 1) \big|_{\text{onshell}} \rightarrow \mathcal{A}^{(L)}_D(\bar{1}, 2, \ldots, \bar{n})
\]

- **non-causal singularities** (unphysical) entangled among dual pairs, they cancel

\[
\mathcal{A}^{(L)}_D(\bar{2}, \bar{3}, \ldots, \bar{n}; 1) + \mathcal{A}^{(L)}_D(1, \bar{3}, \ldots, \bar{n}; 2) \rightarrow \mathcal{A}^{(L)}_D(1, \bar{2}, \bar{3}, \ldots, \bar{n}) - \mathcal{A}^{(L)}_D(1, \bar{2}, \bar{3}, \ldots, \bar{n})
\]

- **What if we reorder the loop lines? Do we get a different representation?**
OPENING TO NON-DISJOINT TREES + CAUSALITY

NMLT AND N2MLT: CASCADE FACTORIZATION

\[ \mathcal{A}_{\text{NMLT}}^{(L)}(1, \ldots, n, 12) = \mathcal{A}_{\text{MLT}}^{(2)}(1,2,12) \otimes \mathcal{A}_{\text{MLT}}^{(L-2)}(3, \ldots, n) + \mathcal{A}_{\text{MLT}}^{(1)}(1,2) \otimes \mathcal{A}^{(0)}(12) \otimes \mathcal{A}_{\text{MLT}}^{(L-1)}(3, \ldots, \bar{n}) \]

- **causal singularities** determined by subtopologies

\[ \mathcal{A}_{\text{N2MLT}}^{(L)}(1, \ldots, n, 12, 23) = \mathcal{A}_{\text{NMLT}}^{(3)}(1,2,3,12,23) \otimes \mathcal{A}_{\text{MLT}}^{(L-3)}(4, \ldots, n) + \mathcal{A}_{\text{MLT}}^{(2)}(1 \cup 23,2,3 \cup 12) \otimes \mathcal{A}_{\text{MLT}}^{(L-2)}(4, \ldots, \bar{n}) \]

- **factorization** conjectured to hold to all potential topologies at higher orders with simpler topologies as building blocks
At **two loops** any scattering amplitude is opened as **MLT**

At **three loops** the master opening is **$N^2MLT$**

\[
N^2MLT \left[ \begin{array}{c}
1 \\
2 \\
3 \\
\vdots \\
n
\end{array} \right] = \sum_{i=1}^{n} \left[ \begin{array}{c}
i-1 \\
i \\
i+1 \\
\vdots \\
n
\end{array} \right]
\]

\[
N^2MLT \left[ \begin{array}{c}
1 \\
2 \\
3 \\
\vdots \\
n
\end{array} \right] = NMLT
\]

- **direct and efficient application to physical scattering processes**
What if we reorder the loop lines? Do we get a different representation?

\[ \mathcal{A}^{(L)}_{\text{MLT}}(1, \ldots, n) = \int_{\ell_1 \cdots \ell_L} \frac{1}{\prod 2q_{i,0}^{(+)}} \left( \frac{1}{\lambda_{1,n}^+} + \frac{1}{\lambda_{1,n}^-} \right), \quad \lambda_{1,n}^{\pm} = \sum q_{i,0}^{(+)} \pm k_{1n,0} \]

- Independent of the initial momentum flow assignments
- Free of non-causal singularities: conjectured for all topologies and internal configurations
CONCLUSIONS

- **Theory is already the limiting factor** in many LHC analysis.
- Current techniques insufficient to match the expected accuracy at future colliders (HL-LHC, FCC, HE-LHC, ILC/CLIC, CEPC-SPPC). New theoretical developments needed: numeric, semi-numeric or analytic.
- Back to the physical four space-time dimensions and fully local.
- **Loop-tree duality** powerful formalism to reveal intriguing properties of multi-loop scattering amplitudes: manifest causal structure. Reformulated to all orders in terms of the original Lorentz-invariant prescription.
- Still few phenomenological applications, more to come.
CANCELLATION OF IR SINGULARITIES & DREG

- Cancellation of IR singularities at **NLO** is satisfactorily solved: efficient algorithms applicable to any process for which matrix elements are known (**Slicing** [Giele, Glover, ...] vs **Subtraction**: dipole [Catani, Seymour], FKS [Frixione, Kunszt, Signer], NS [Nagy, Soper])

- At **NNLO** several working algorithms, successfully applied to specific processes with heavy computational costs
  - Antennae Subtraction [Gehrmann et al.]
  - ColourfulNNLO Subtraction [Del Duca et al.]
  - Geometric Subtraction [Herzog]
  - Leading Regions [Anastasiou, Sterman]
  - Nested Soft-Collinear Subtraction [Caola et al.]
  - N-Jettiness [Boughezal, Petriello et al., Gaunt et al.]
  - Projection to Born [Bonciani et al.]
  - $q_T$ Subtraction [Catani, Grazzini et al.]
  - Stripper [Czakon et al.]

**New strategy at $d=4$**

- Four-dimensional Regularization (FDR) [Pittau et al.]
- Four-dimensional Unsubtraction (FDU) [Sborlini et al.]