Variants and parametrization of self-interacting dark matter

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(Particle physics group seminar)

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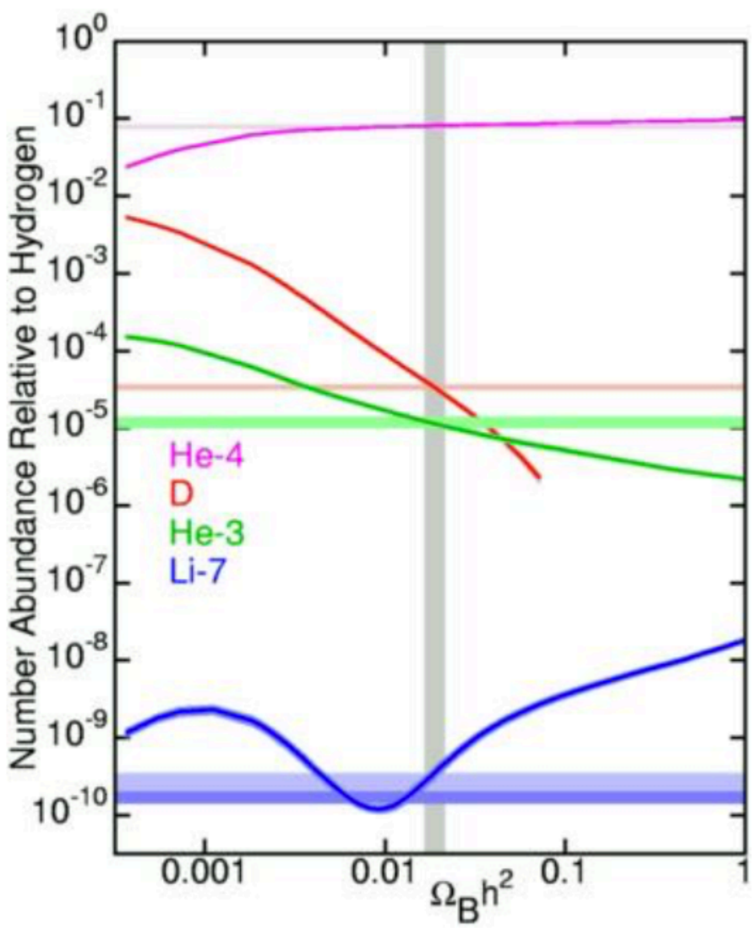
(1810.04709, 1908.06067)

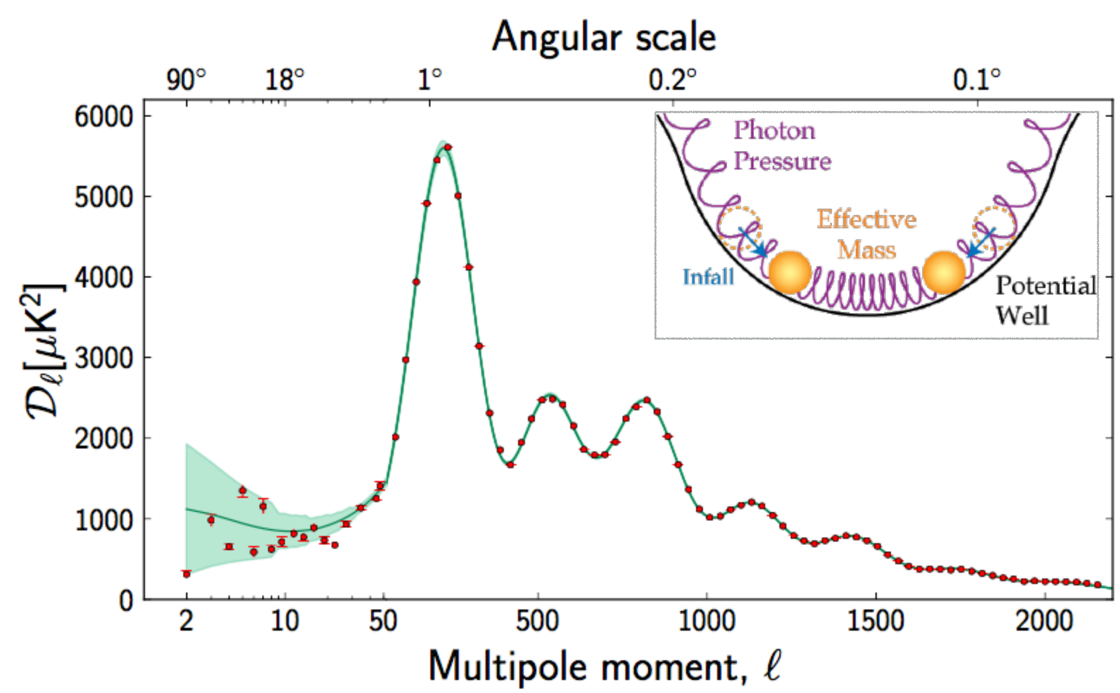
I. Introduction

Standard Cosmology is well established.

$\Omega_B h^2 = 0.021 \pm 0.03$[1] | $\Omega_B h^2 = 0.02225 \pm 0.00016$ | small perturbations

Reheating $\xrightarrow{\text{B-violation}}$ Nucleosynthesis $\xrightarrow{\text{R-M equality}}$ CMB radiation





[1] Lithium-7 deficiency at 4-5σ, e.g. [B.D.Fields 2012, R. Cyburt et al. 2015, A. Goudelis et al. 2016, ...]

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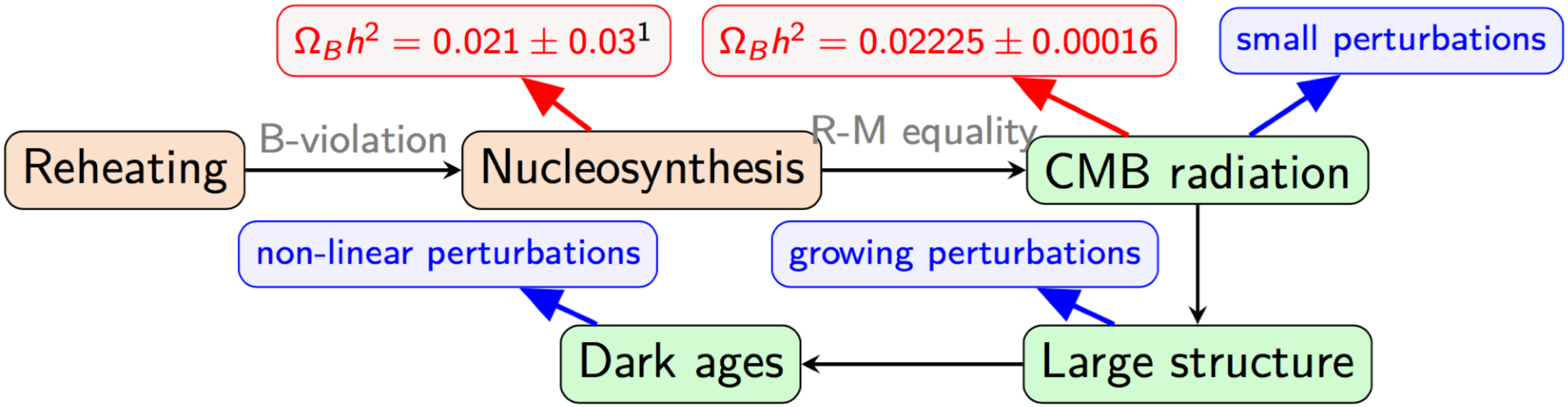
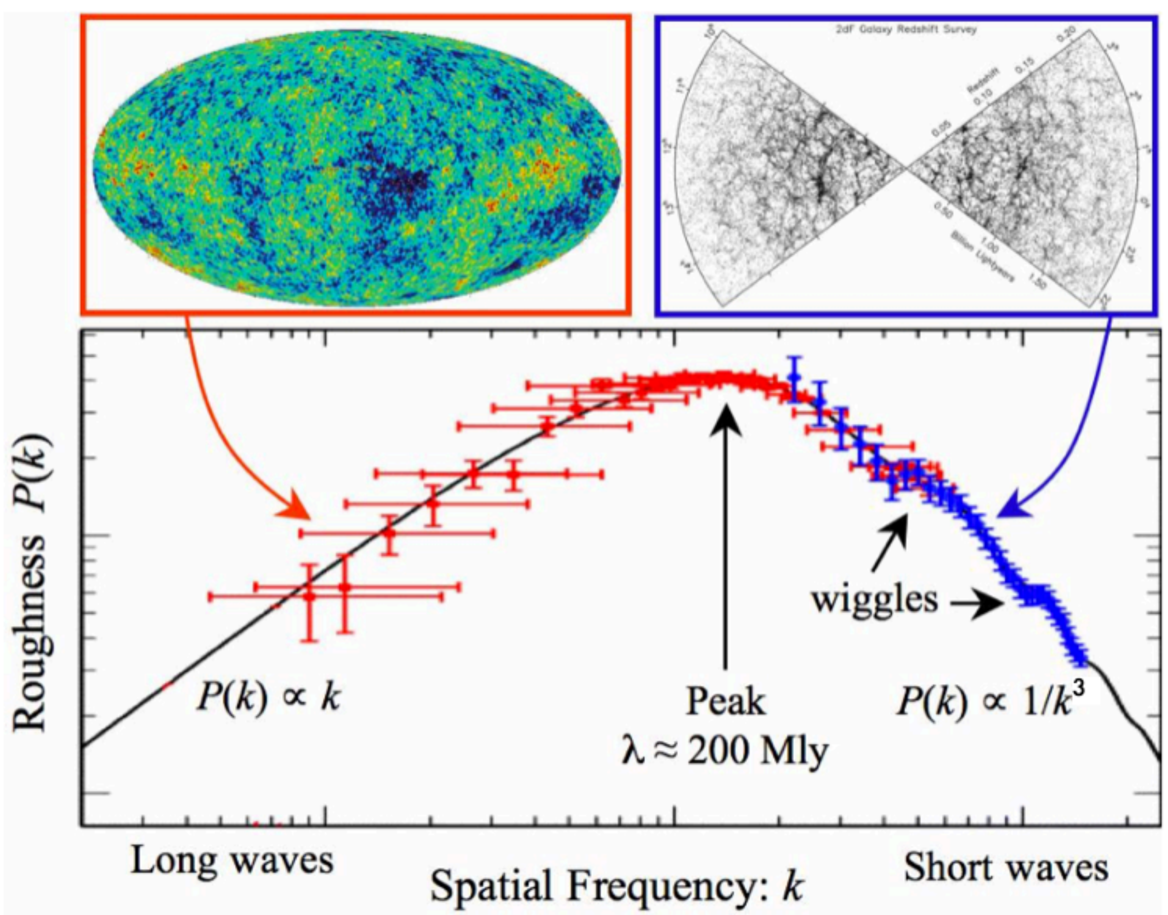
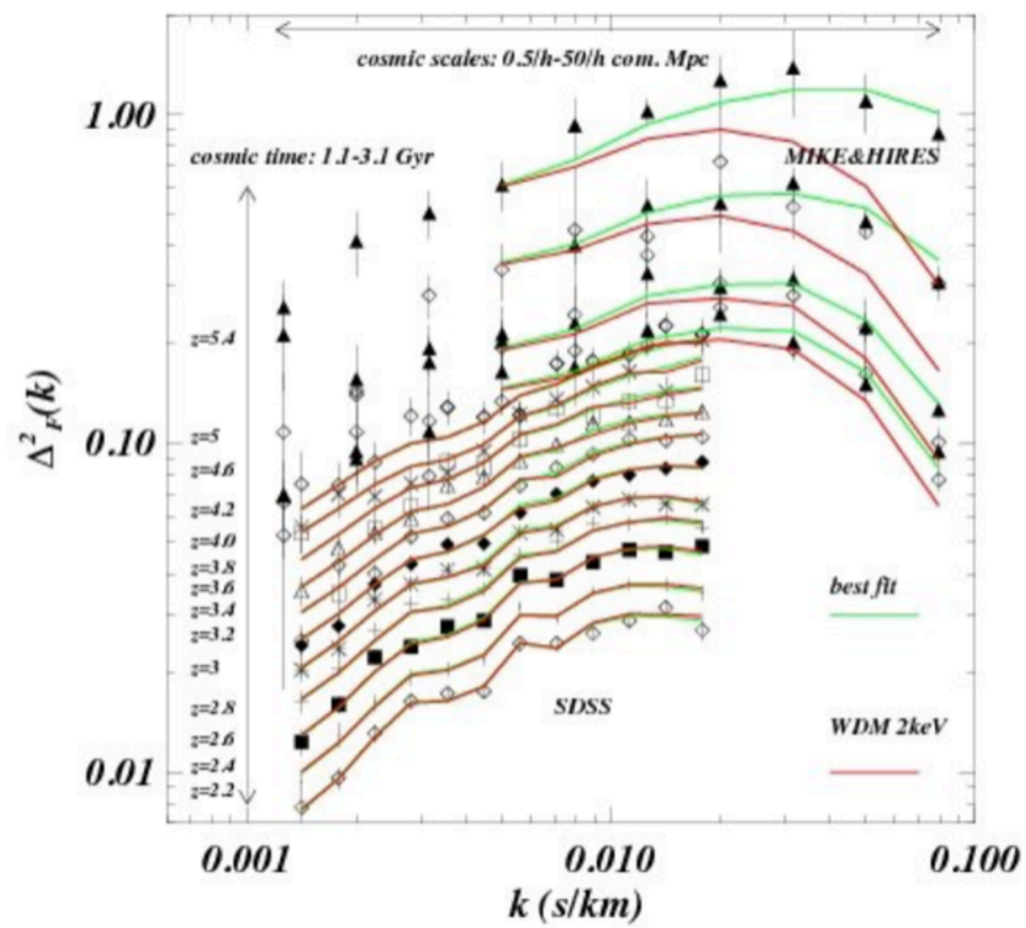
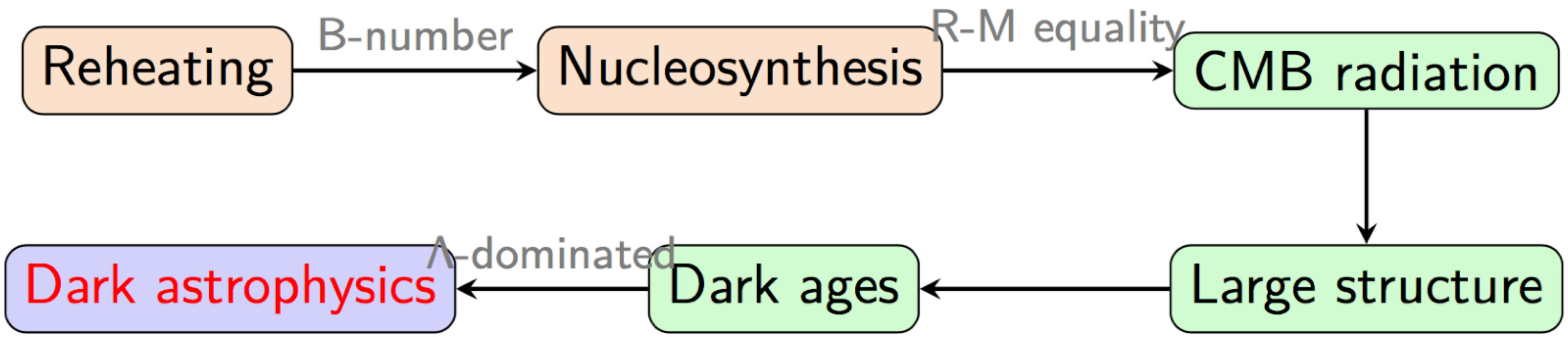
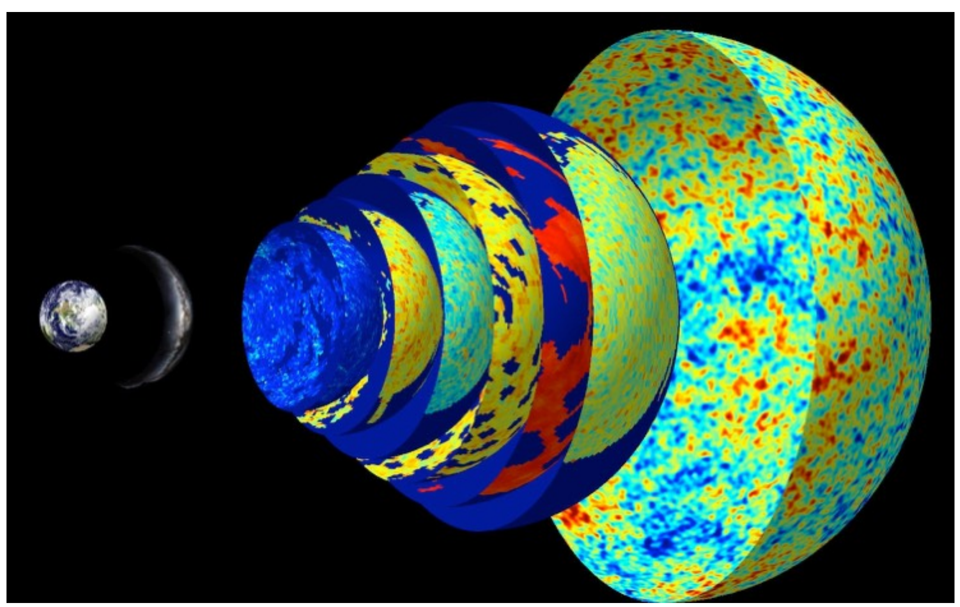






We know dark matter (DM) from the sky.





Decoupled from SM plasma

Non-relativistic before CMB

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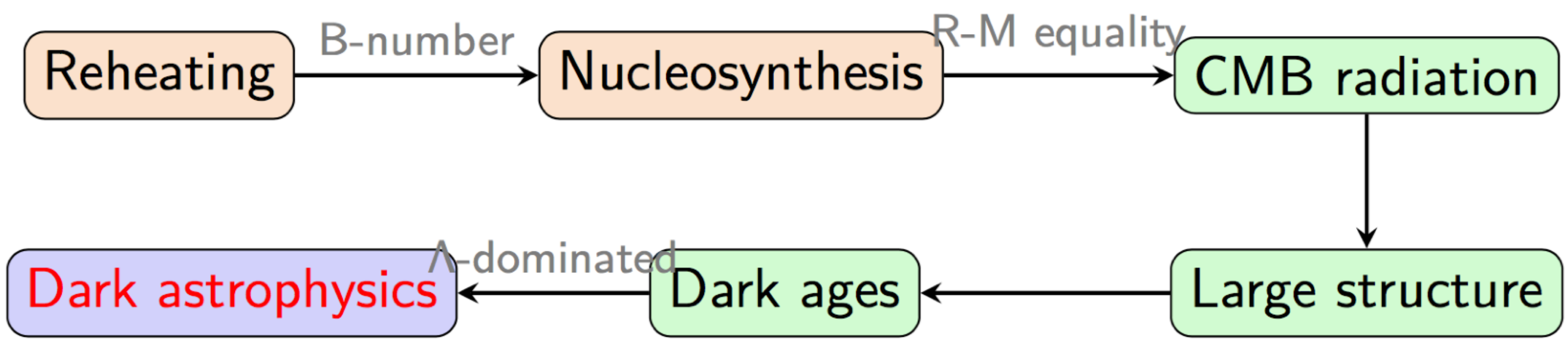
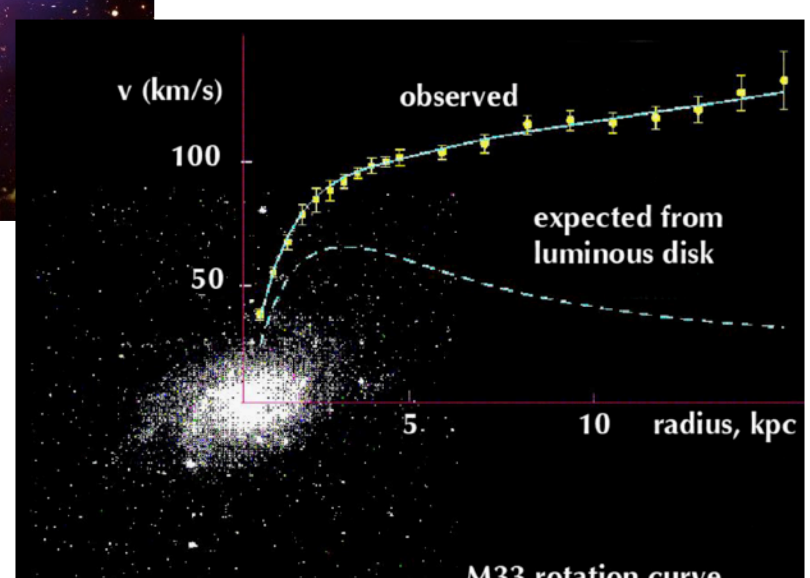




<u>Decoupled from SM plasma</u>

<u>Non-relativistic before CMB</u>

<u>**Self-interaction below ~ barn/GeV**</u>

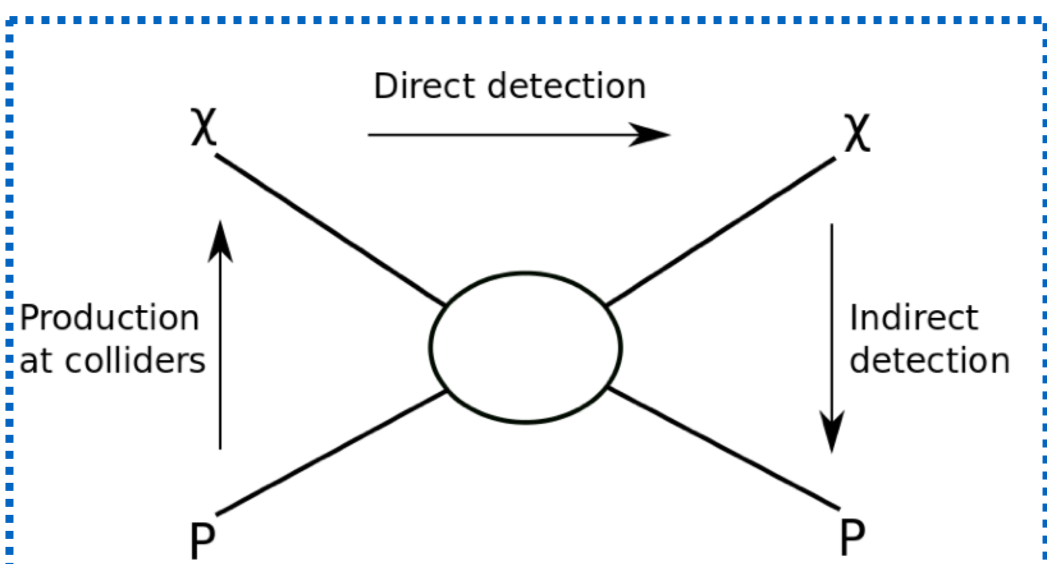
<u>**Diffuse halo (no efficient cooling)**</u>

We know dark matter (DM) from the sky.

First conclusion:

"*Cold-and-dark*" particle/field/fluid seems both inevitable and enough for high-redshift/large-scale data.[S. Dodelson 2011, G. Starkman 2012, J. Khoury 2014, ...]

Assuming DM is **particle-like**



<u>Decoupled from SM plasma</u>

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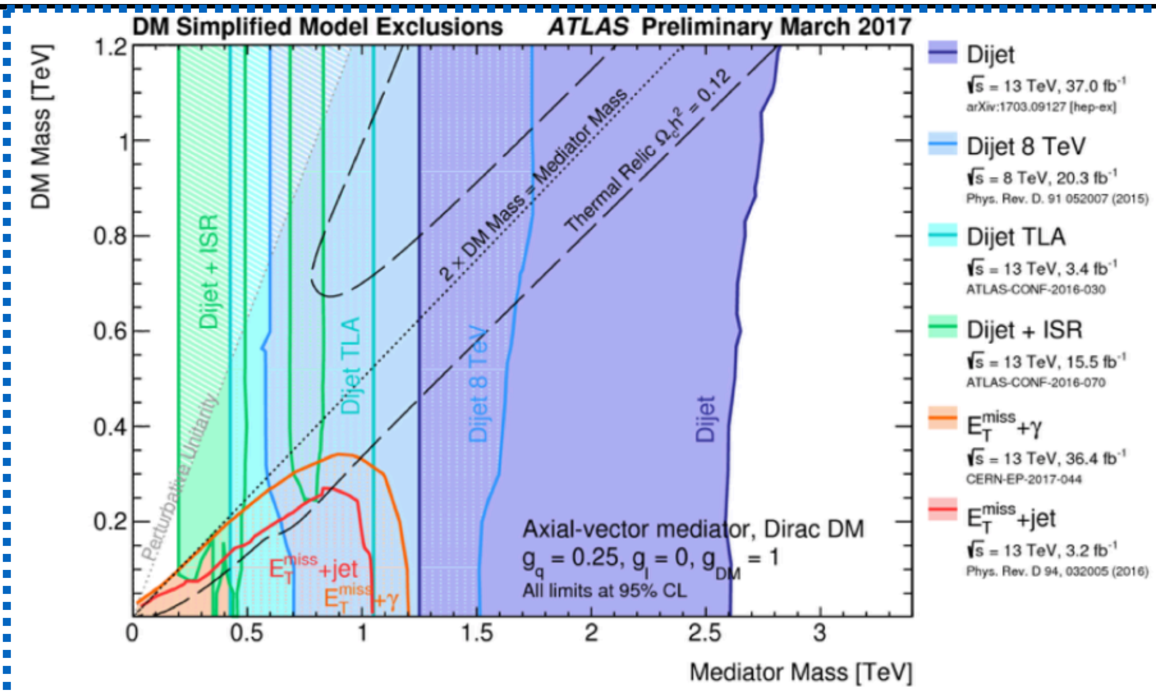
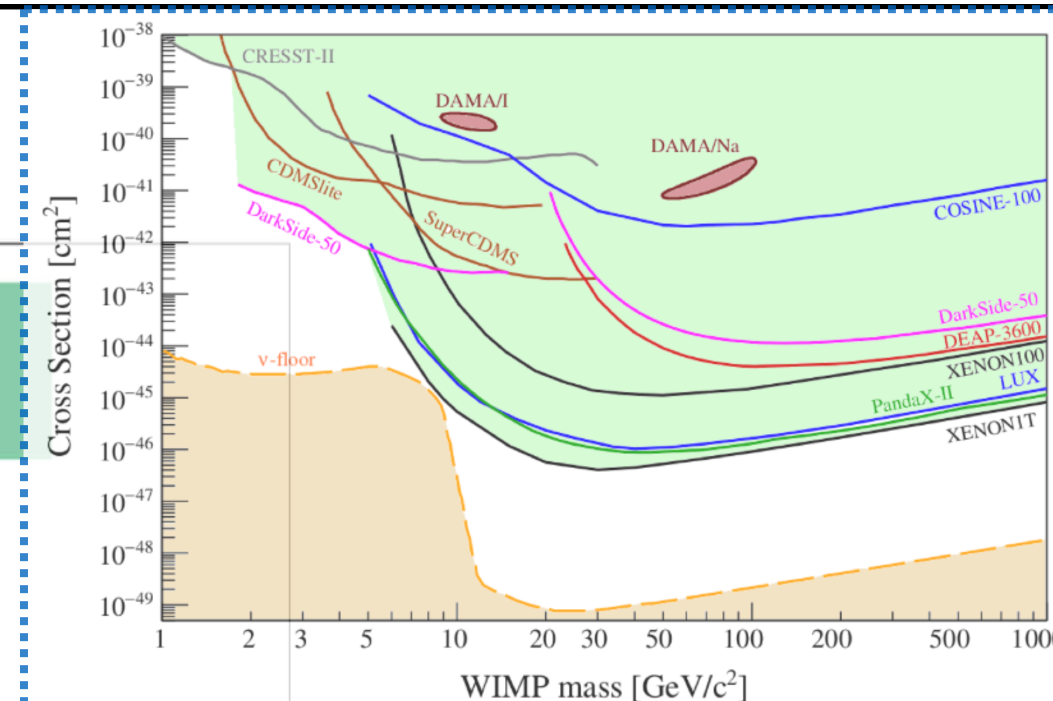
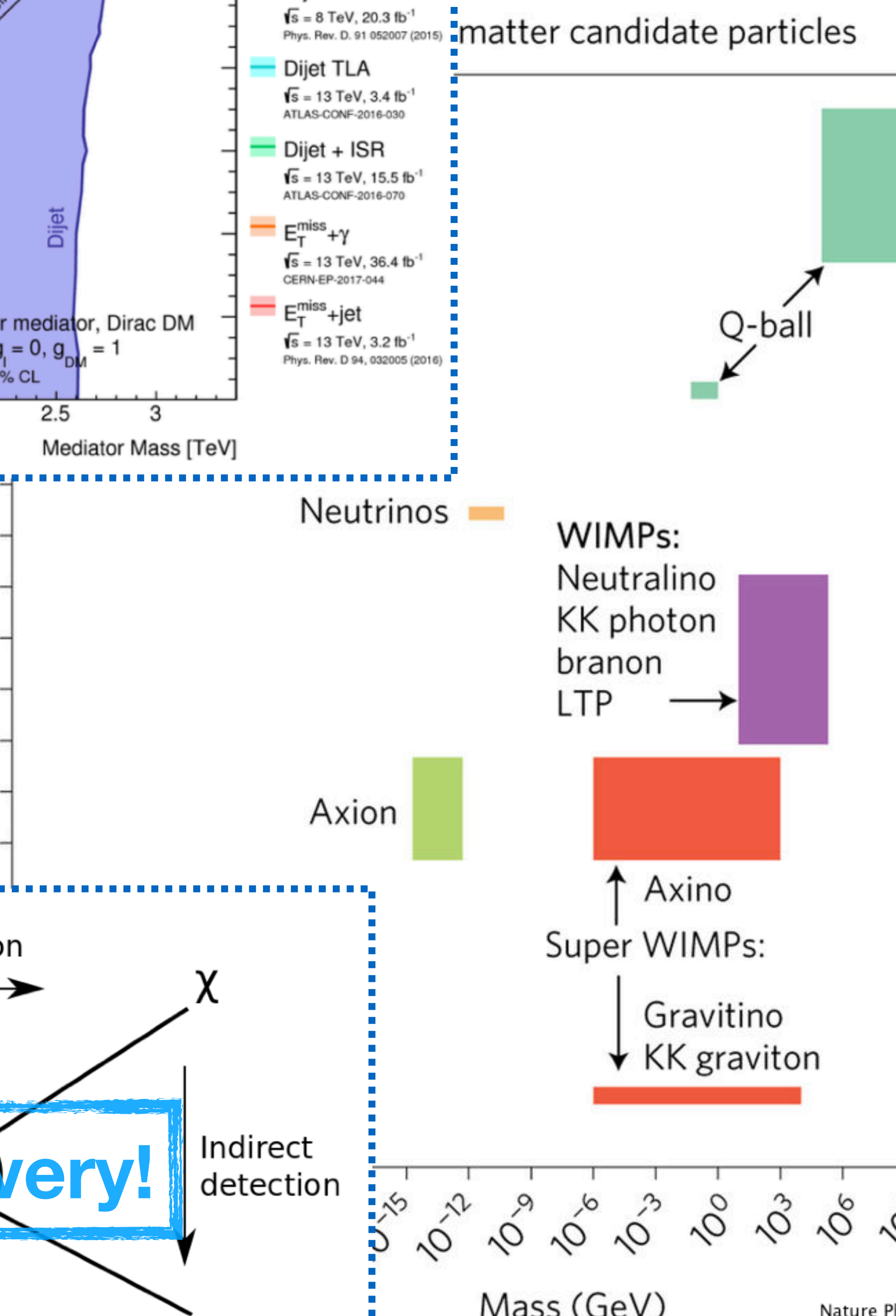
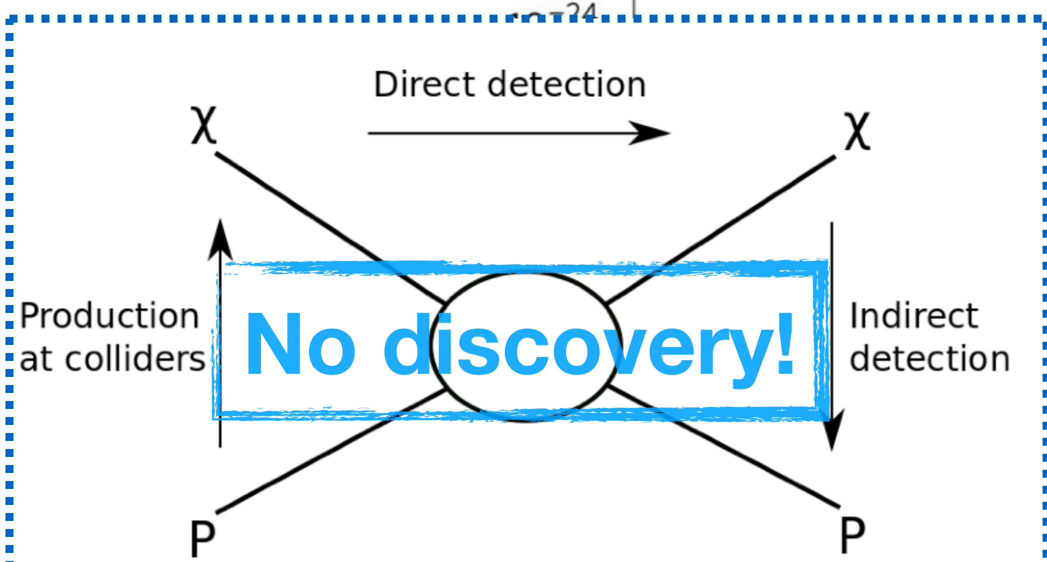
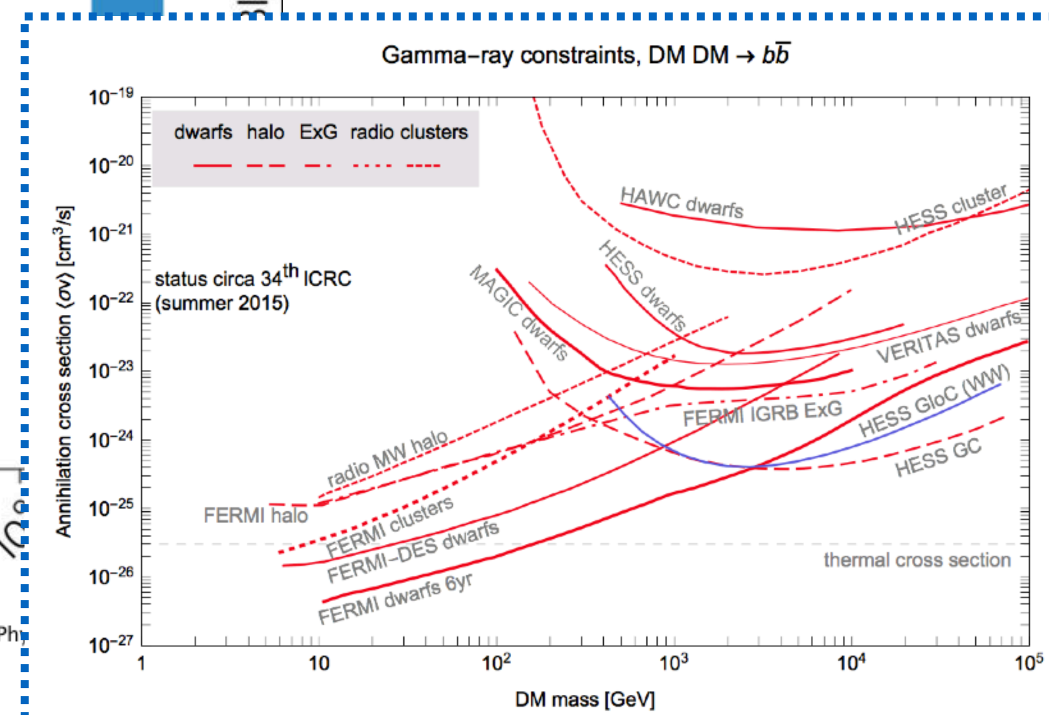
We only see dark matter from the sky.











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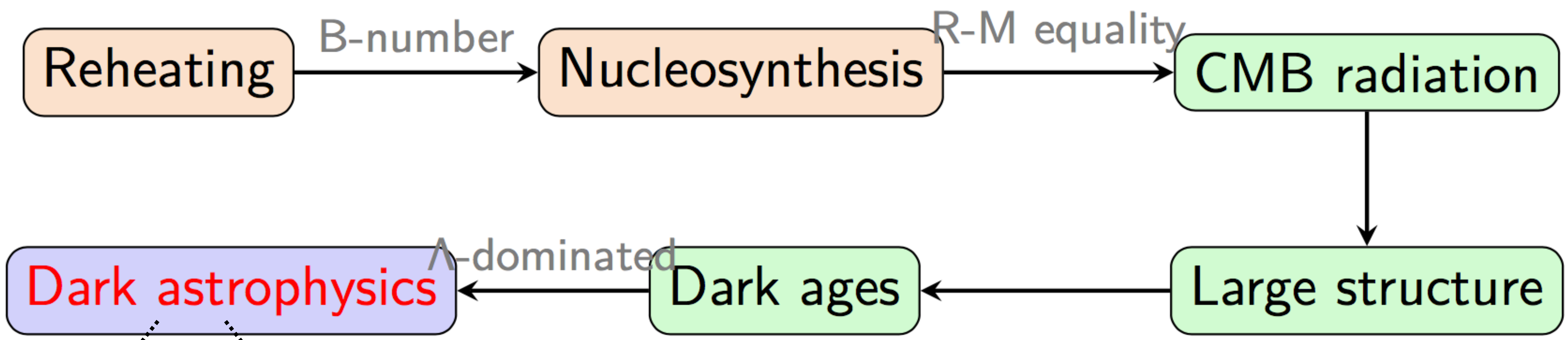


kpc - Mpc

What else can we learn from the sky?

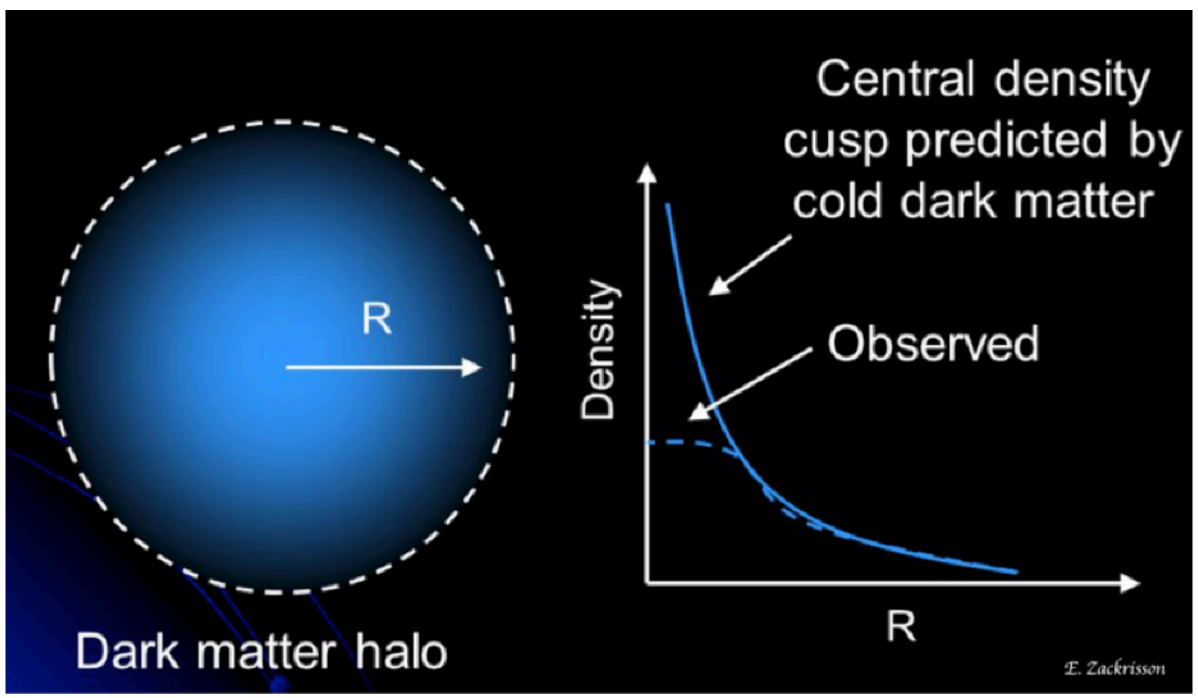
There are astrophysical **puzzles** at scales below cluster sizes. One example is:

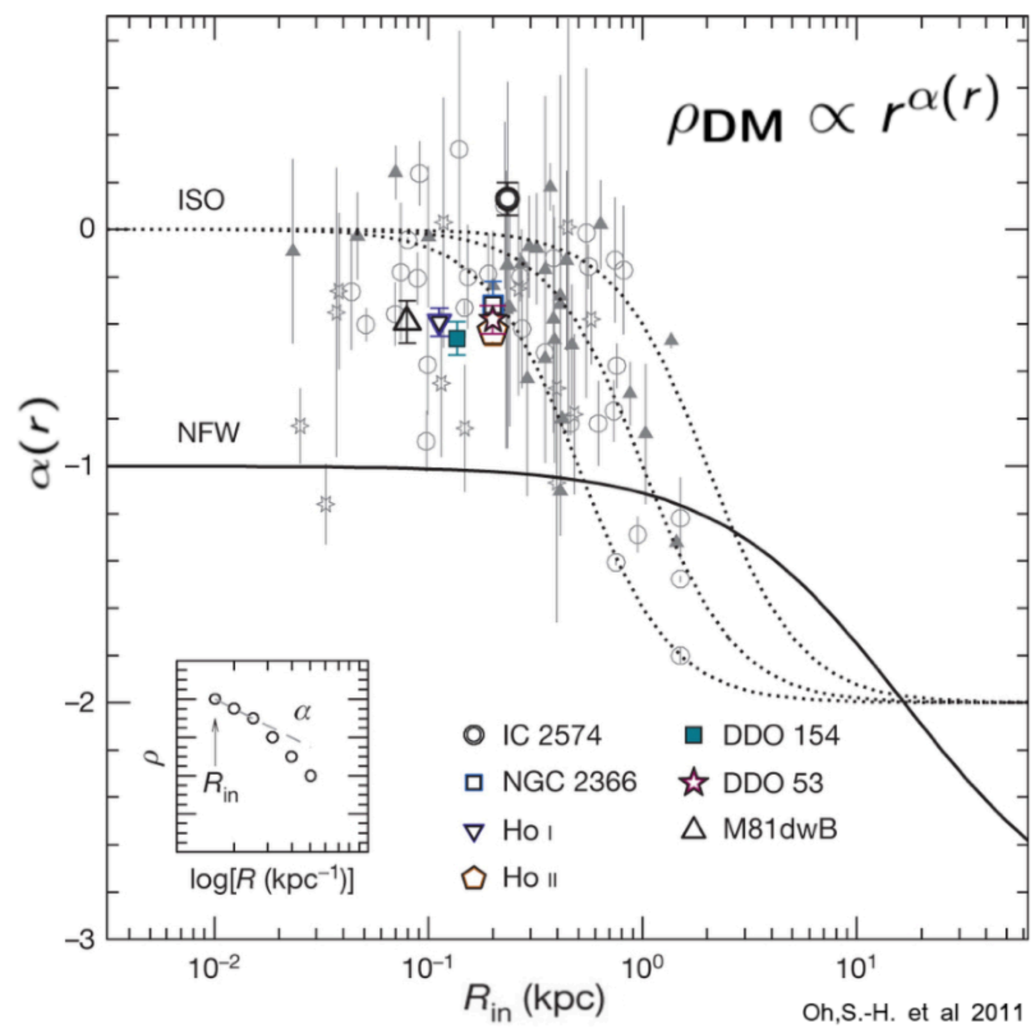
Mass deficit in small dark matter halos.

Dark matter (DM) halo **mass deficit?**

1. **DM cores** are preferred by observed circular velocities in dwarf/low-surface-brightness (LSB) galaxies [Moore 1994; Burkert 1995, …], and probably at cluster scales [Newman et al 2013, …] .

(core/cusp problem)

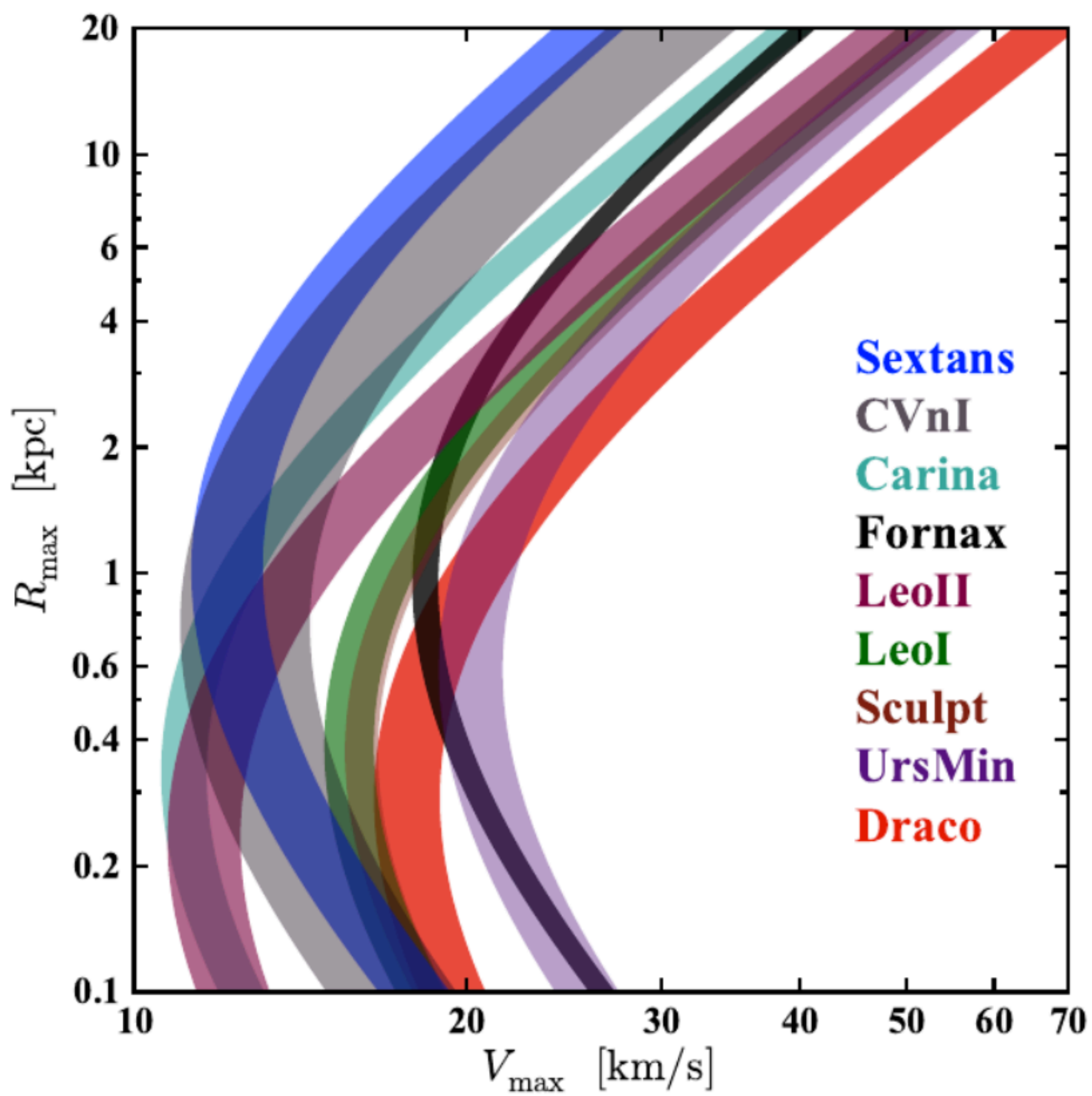


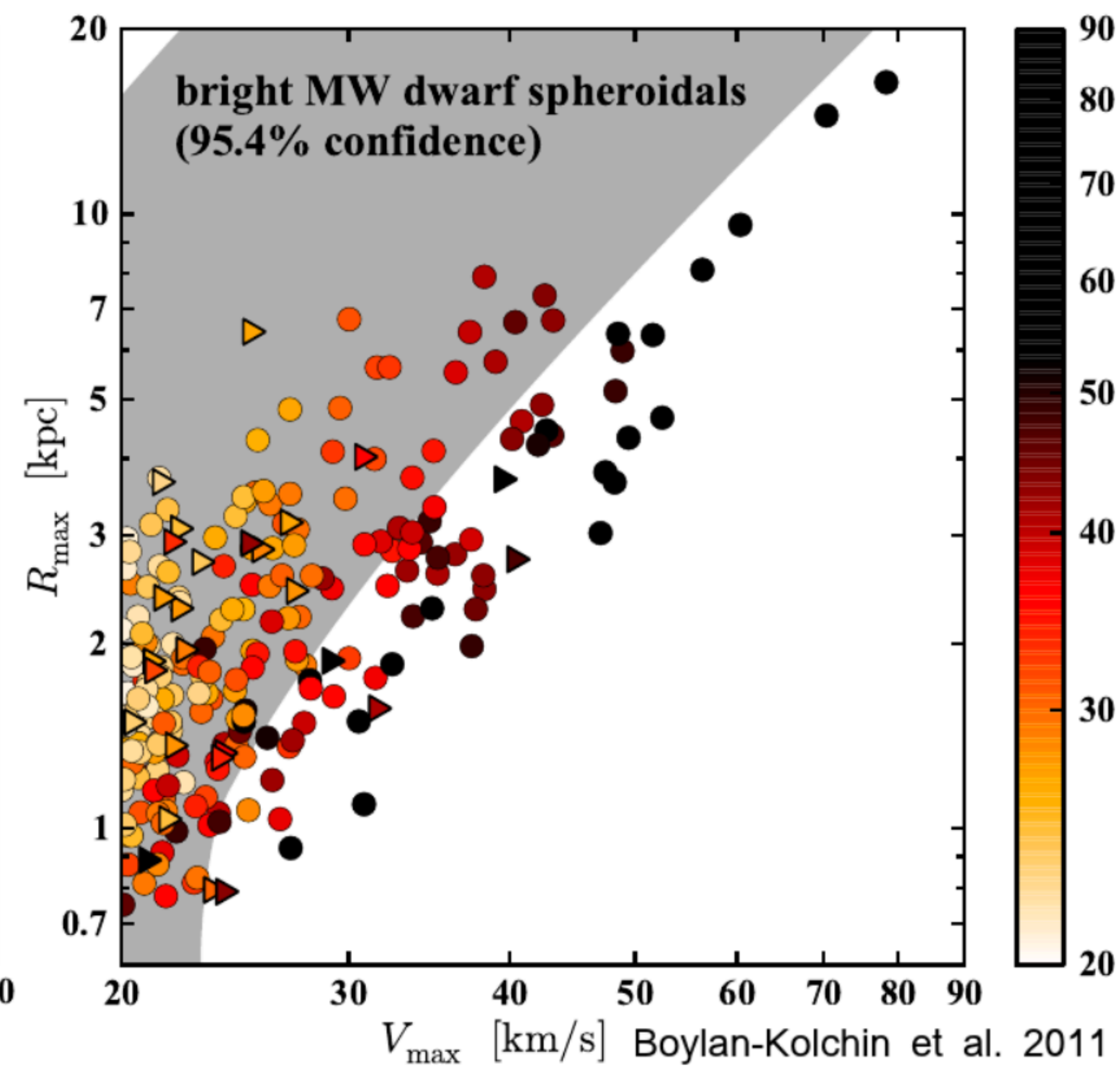


Dark matter (DM) halo mass deficit?

2. **Non-observation of massive sub-halos** which should host brightest dwarfs predicted by simulations in Milky Way [M.Boylan-Kolchin et al. 2011, 2012] and in field [Ferrero et al. 2011].

(too-big-to-fail problem)



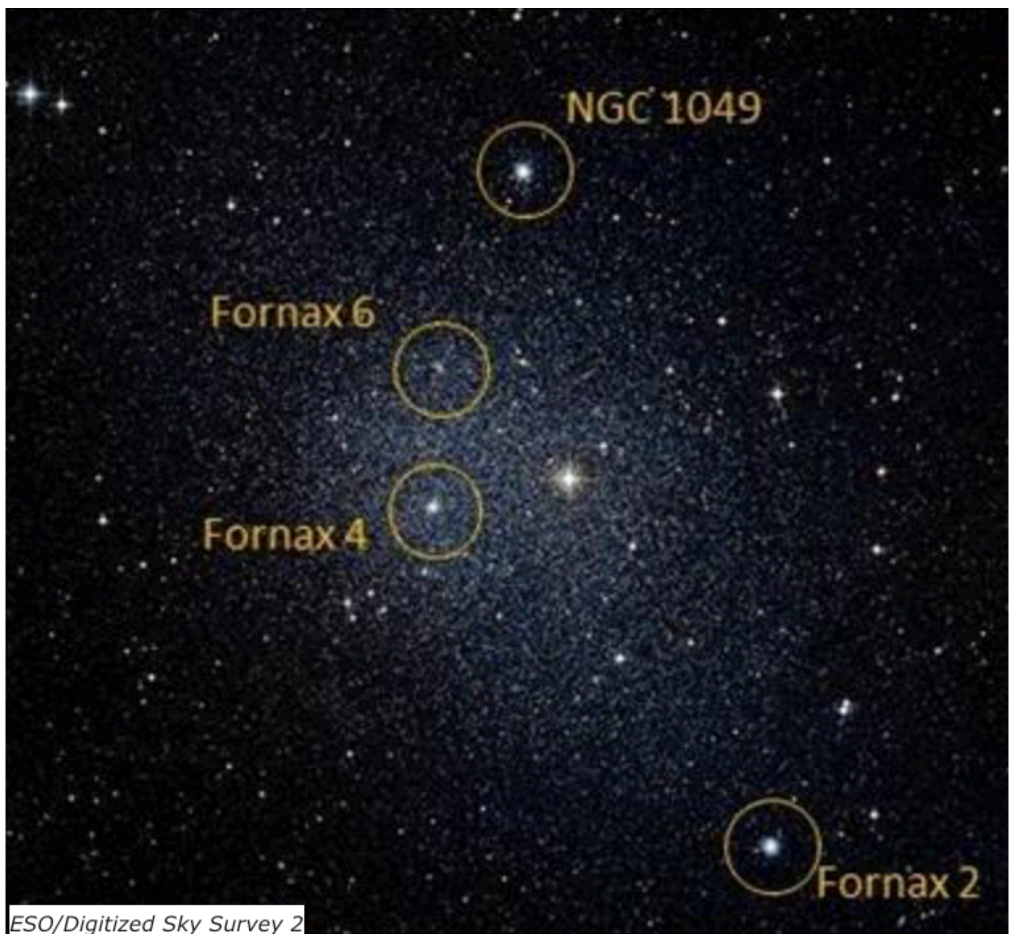


Boylan-Kolchin et al. 2011

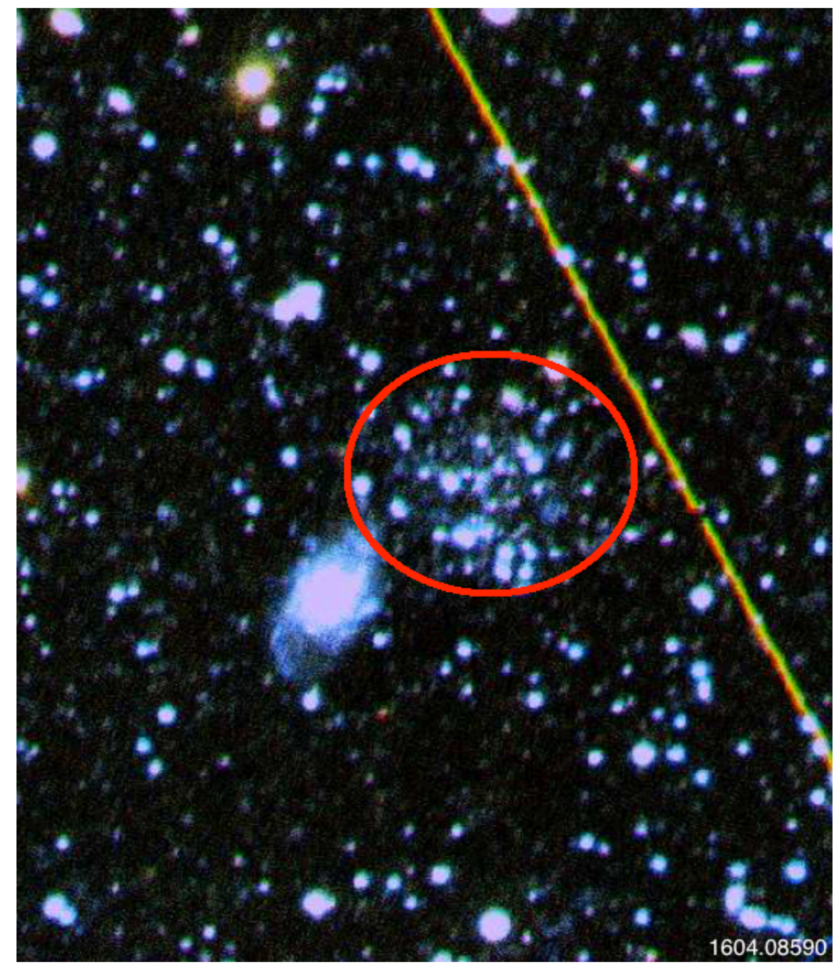
Dark matter (DM) halo **mass deficit?**

3. Given the long lifetime of dwarf galaxies, some globular/star clusters are expected to be destroyed, or sink to the center **if dwarf-sized halos are cuspy.** [J. Binney & S.Tremaine 2008, F. Contenta et al. 2017, P. Boldrini et al. 2018, …]

(timing problem)



Fornax



Eridanus II

Possible explanations:

Systematic uncertainties?

- **HI gas and star motions** may not be faithful tracers of gravity, e.g. observation bias [A. M. Brooks et al. 2017, R. Verbeke et al. 2017, …], pressure/non-circular motions [J. C. B. Pineda et al 2017, K. A. Oman et al. 2017, …], inclination [A. Schneider et a. 2016, Read et al. 2016, …] and so on.
- A common **preference of** mass deficit in the halo center?

Possible explanations:

Systematic uncertainties?

Baryonic effects (by bursty star formation)?

$\boxed{10^{53} - 10^{55}\ \mathrm{erg}}$

- **Heated by supernova** / in-falling clumps, etc.

  Each supernova deposits $\sim 10^{51}$erg in interstellar medium *[e.g. Madau, Shen, Governato 2014, …]*

- Need **star formation threshold > 10/cm³** [A. Pontzen et al. 2012,…], but 0.1/cm³ used to produce galaxies distributions, and no cores [S. Bose et al. 2018].

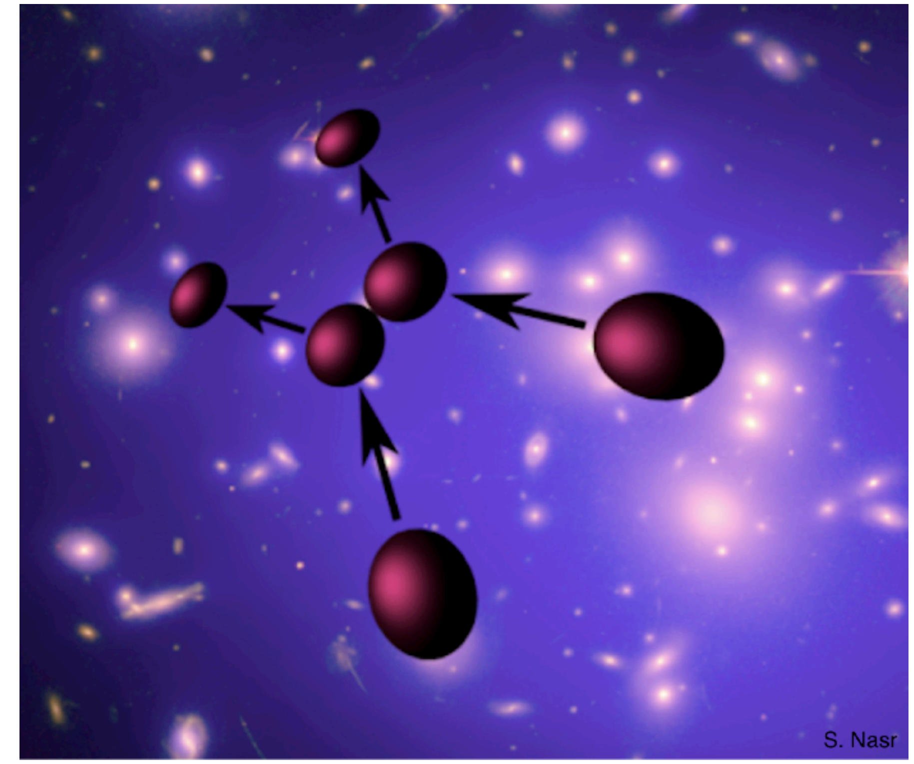
- **Core-cusp may be related to when star formation stops** [J.I Read et al. 2018]. But may be due to tidal effects [O. Semeie et al. 2019, …].

Self-interacting dark matter?

Observational evidence for self-interacting cold dark matter

D.N. Spergel and P J. Steinhardt [astro-ph/9909386]

Infalling dark matter is scattered before reaching the center of the galaxy so that the orbit distribution is isotropic rather than radial. These collisions increase the entropy of the dark matter phase space distribution and lead to a dark matter halo profile with a shallower density profile.



Strong DM self-scattering for dwarf ➡ inner halo **DM self-thermalization**
(heating up the halo center)

$$\frac{\sigma_{\rm SI}}{m_{\rm DM}} \sim 0.5 - 5{\rm cm}^2/{\rm g}$$

O(1) scatters per DM particle

II. Self-interacting dark matter model building

What we need in self-interacting dark matter?

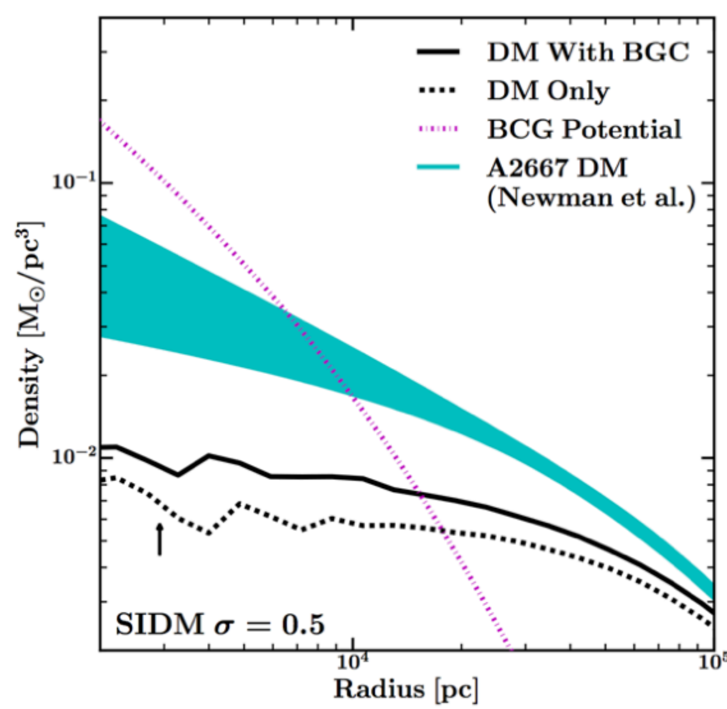
- Stronger self-scattering **needed** for (dwarf-sized) halos

$$\frac{\sigma_{\rm SI}}{m_{\rm DM}} \sim 0.5 - 10 {\rm cm}^2/{\rm g}$$ at dwarf scales of DM velocity ~ 10 km/s

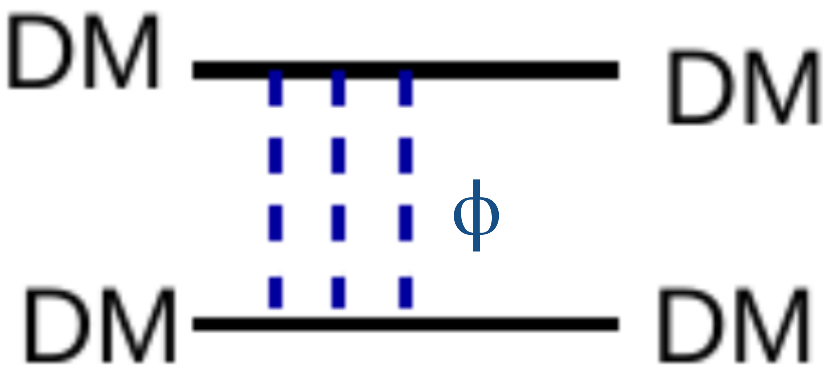
- Weaker self-scattering **favored** by cluster merging/halo profiles etc.

[O. D. Elbert et al. 2016, K. Bondarenko 2016,….]

$$\frac{\sigma_{\rm SI}}{m_{\rm DM}} \leq 0.2 - 1 {\rm cm}^2/{\rm g}$$ at cluster scales of DM velocity ~ 1000km/s



SIDM via a light mediator [D.N. Spergel & P. J. Steinhardt 1999, J. Feng, M. Kaplinghat & H.-B. Yu 2009, …]

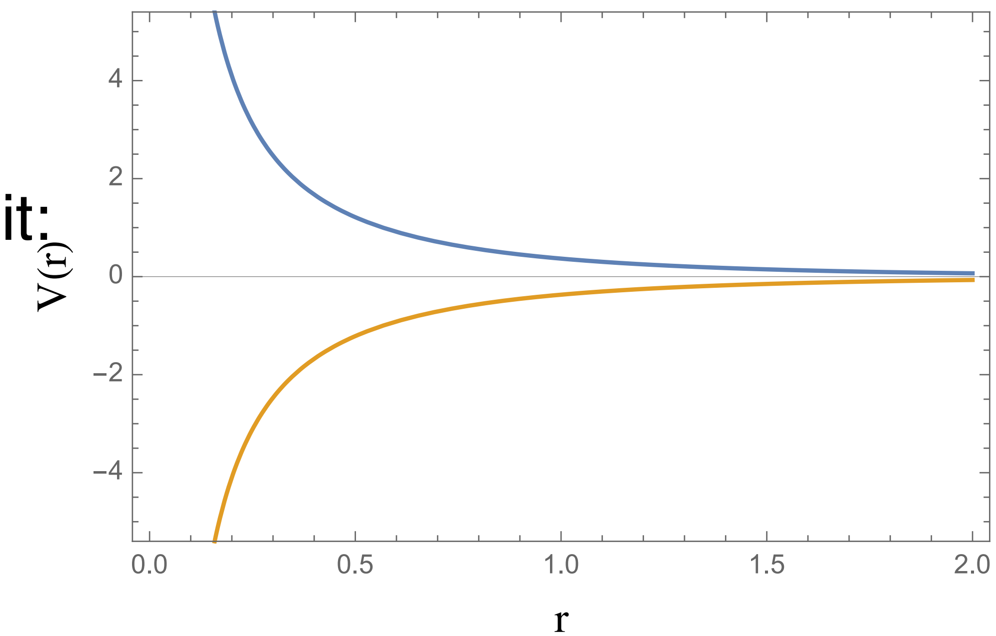


t-channel elastic scattering

For **a light ϕ**: enhanced cross section at low velocity

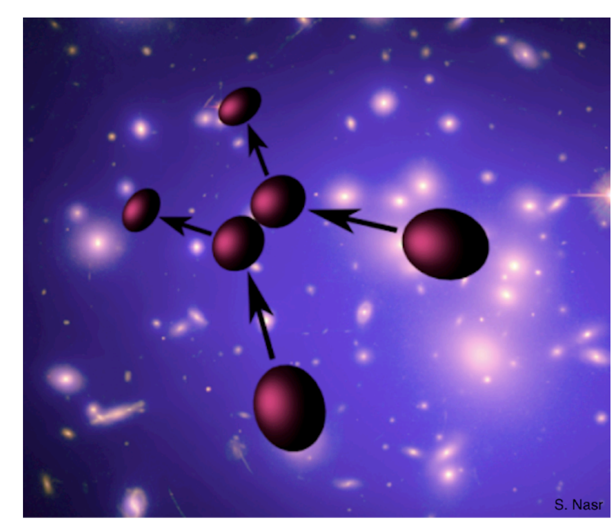
Described by a **Yukawa** potential at non-relativistic limit:

$$V(r) = \pm \frac{\alpha_X}{r} e^{-m_\phi r}$$



The impact of **(soft) scattering** can be characterised by

$$\sigma_T = \int d\Omega \, (1 - \cos\theta) \, \frac{d\sigma}{d\Omega}$$



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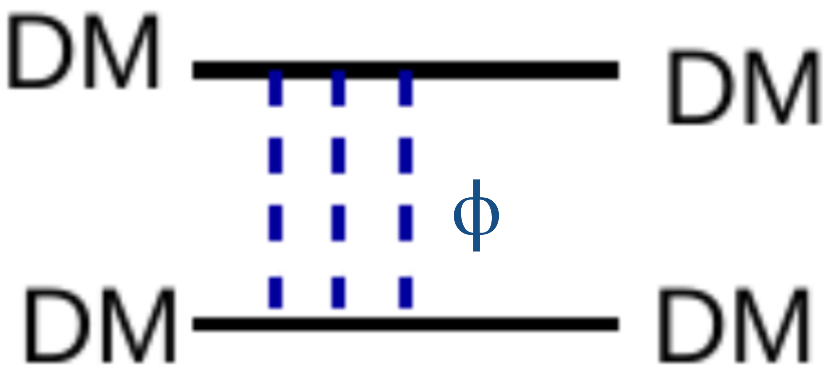

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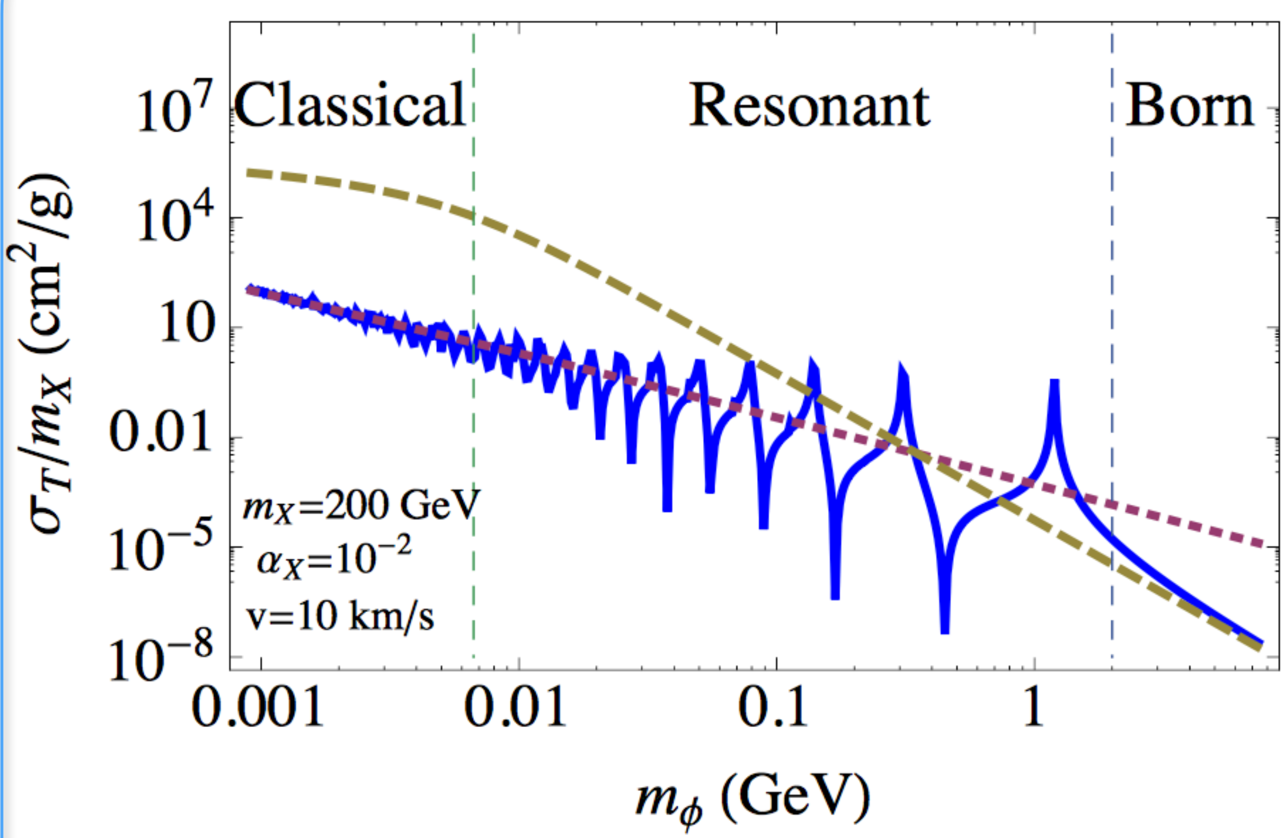


[S. Tulin, H.-B. Yu, & K. M. Zurek 2013]

Non-trivial calculation needed:

1. **Born regime**:

   $m_\phi \gg \alpha_\chi m_\chi$

2. **Resonant regime** (for attractive force):

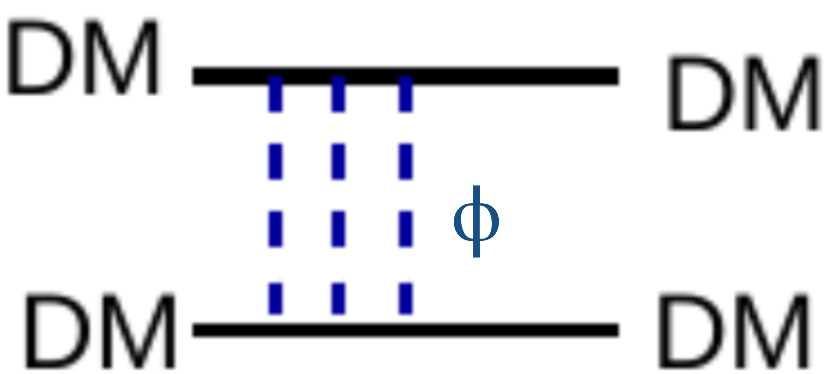
   $m_\phi \leq \alpha_\chi m_\chi$ (bound state formation)

3. **Classical** (non-perturbative) regime:

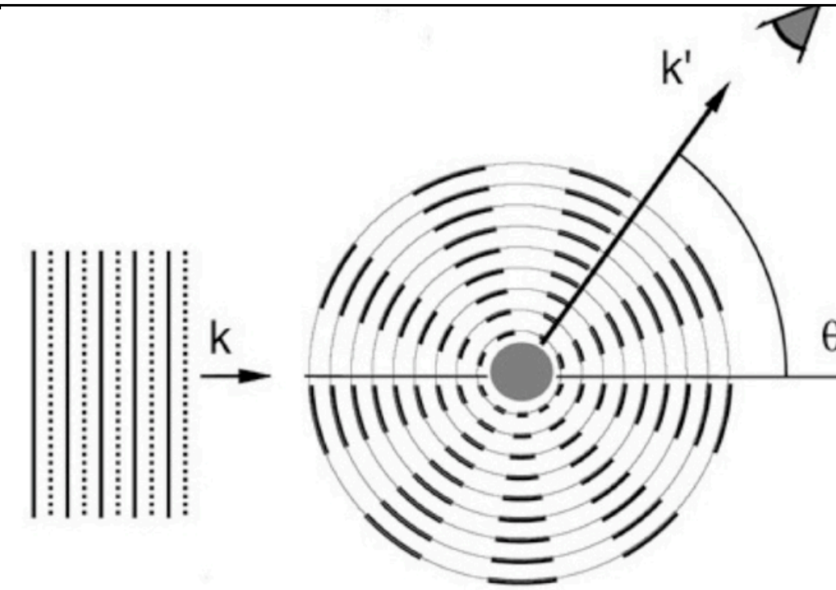
   $m_\phi \leq \alpha_\chi m_\chi \;\&\; m_\phi \leq m_\chi v$

DM de Brogile wavelength larger than potential range

SIDM via a light mediator [D.N. Spergel & P. J. Steinhardt 1999, J. Feng, M. Kaplinghat & H.-B. Yu 2009, …]



Quantum scattering



- Solve the Schrödinger equation in such **a potential at rest**

$$V(r) = \pm \frac{\alpha_X}{r} e^{-m_\phi r}$$

$$\psi(r) \simeq e^{i\mathbf{kr}} + f(k,\theta)\frac{e^{ikr}}{r}$$

leading to differential **cross section**:

$$\frac{d\sigma}{d\Omega} = |f(k,\theta)|^2$$

- Or in terms of **phase shift** in asymptotic form of radial wave-function

$$R_{\ell,k}(r) \propto \cos\delta_\ell\, j_\ell(kr) - \sin\delta_\ell n_\ell(kr) \approx \frac{1}{r}\sin\left(kr - \frac{l\pi}{2} + \delta_\ell\right)$$

$$f(k,\theta) = \sum_{\ell=0}^{\infty}(2l+1) f_\ell(k) P_\ell(\cos\theta)\ ,$$

$$\text{with}\quad f_\ell(k) \equiv \frac{e^{2i\delta_\ell(k)}-1}{2ik} = \frac{1}{k\left(\cot\delta_\ell(k) - i\right)}$$

SIDM via a light mediator [D.N. Spergel & P. J. Steinhardt 1999, J. Feng, M. Kaplinghat & H.-B. Yu 2009, …]

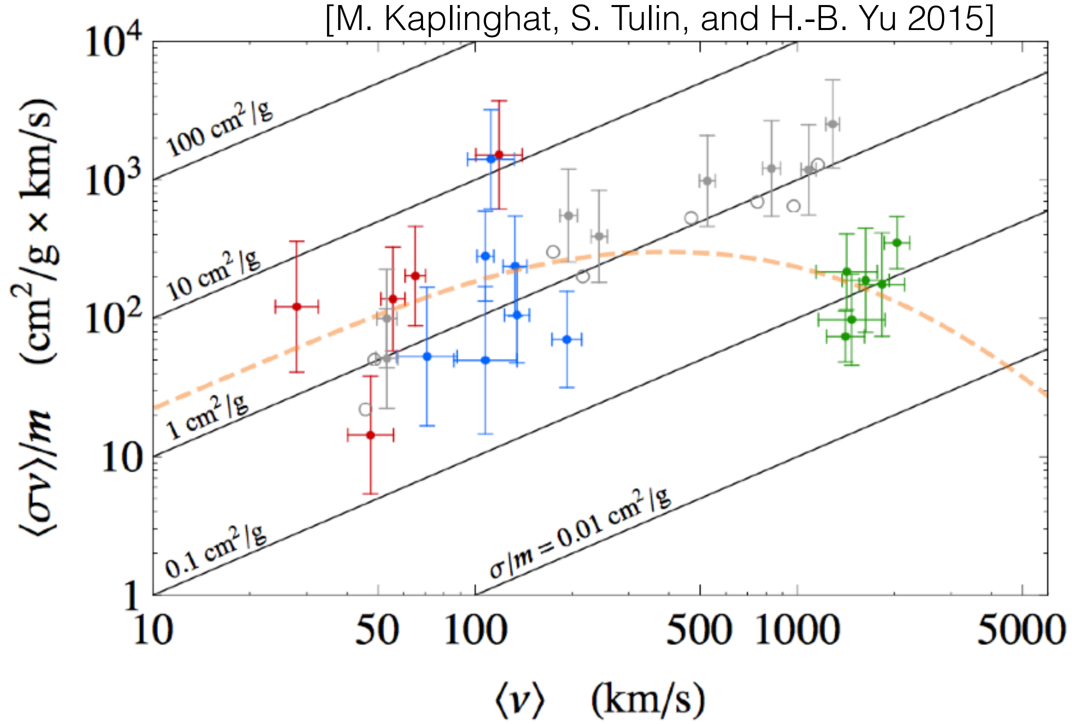
One example：

DM mass: tens of GeV

mediator mass: tens of MeV

dark coupling ~ 0.01 - 0.001

[M. Kaplinghat, S. Tulin, and H.-B. Yu 2015]



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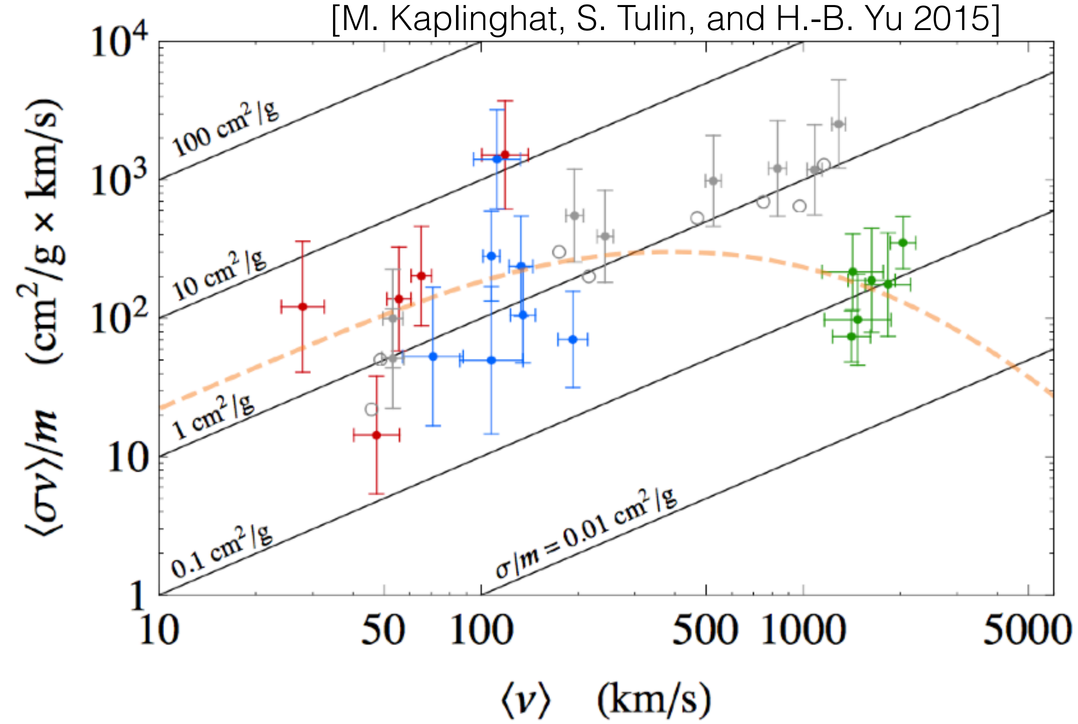


One example：

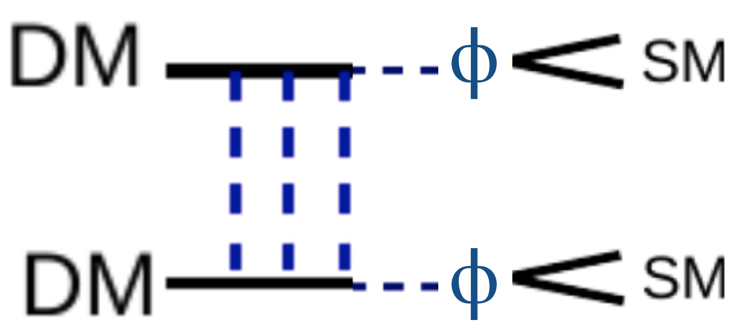
DM mass: tens of GeV

mediator mass: tens of MeV

dark coupling ~ 0.01 - 0.001

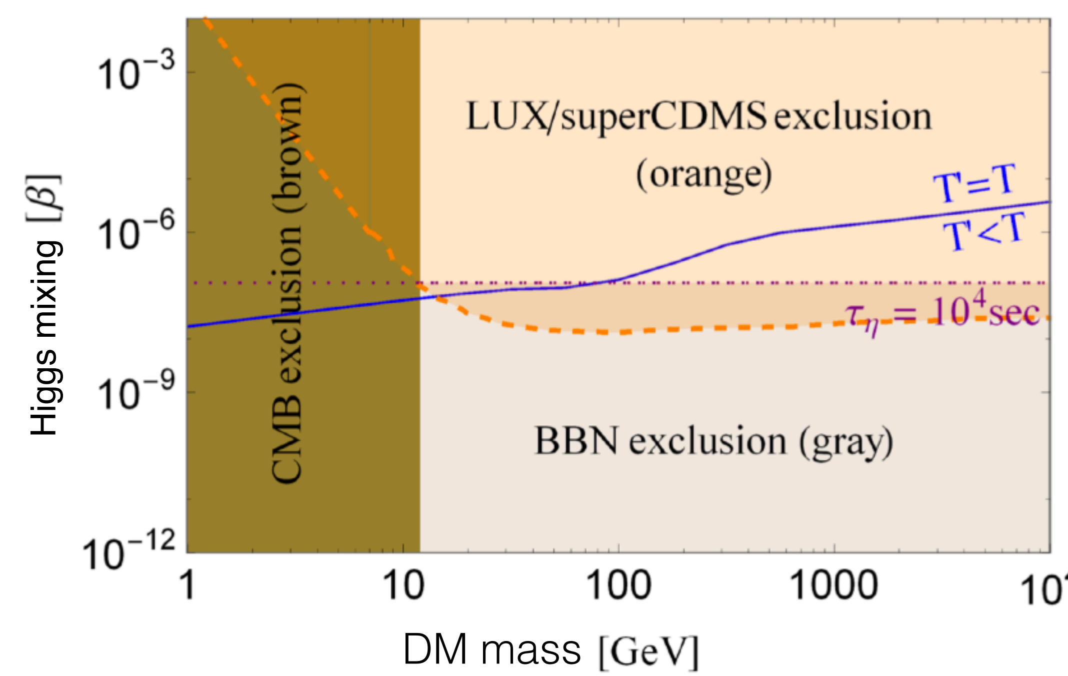
Same light mediator also enhances DM annihilation and scattering with SM.

Very strongly constrained if coupled to visible particles!



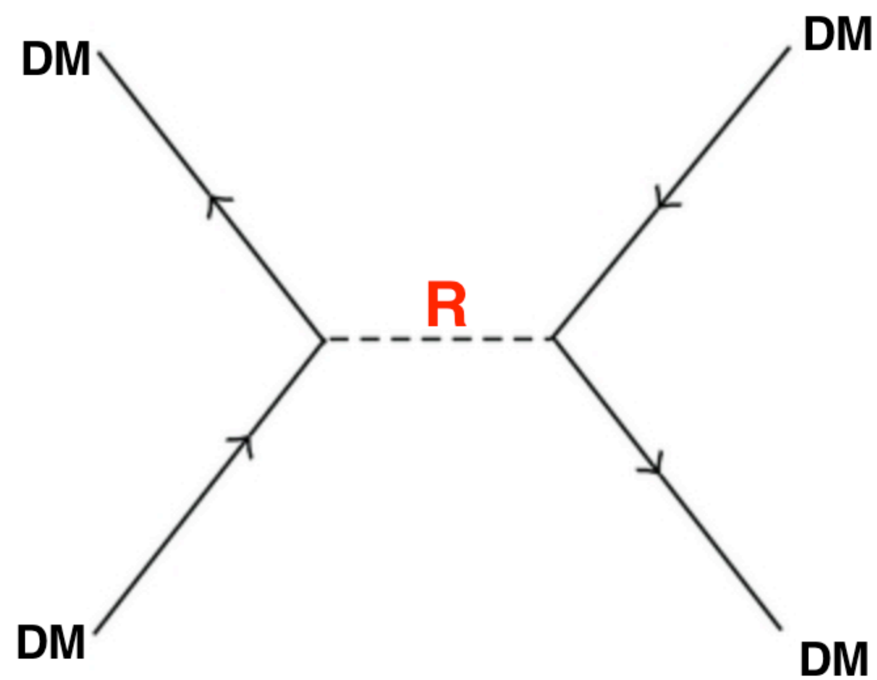
Sommerfeld enhancement

Mediator mass ∼ 30 MeV



[N. Bernal, XC, C.Garcia-Cely, T. Hambye, B. Zaldivar 2015]

SIDM via Breit-Wigner resonance [XC, C. Garcia-Cely, H. Murayama 2018]



$$\text{If } v_R^2 \equiv \frac{m_R - 2m_{\rm DM}}{m_{\rm DM}} \ll 1$$

Large enhancement
&
a velocity-dependence

$$\sigma \; = \; \sigma_0 + \frac{4\pi\, S}{mE(v)} \cdot \frac{\Gamma(v)^2/4}{(E(v) - E(v_R))^2 + \Gamma(v)^2/4}$$

t/u - channel

$$E(v) = \frac{1}{2}\frac{m}{2}v^2 \qquad \text{and} \qquad S = \frac{2J_R + 1}{(2J_{\rm DM} + 1)^2}$$

$$\Gamma(v) = m_R \gamma v^{2L+1}$$

L — partial wave

γ — couplings

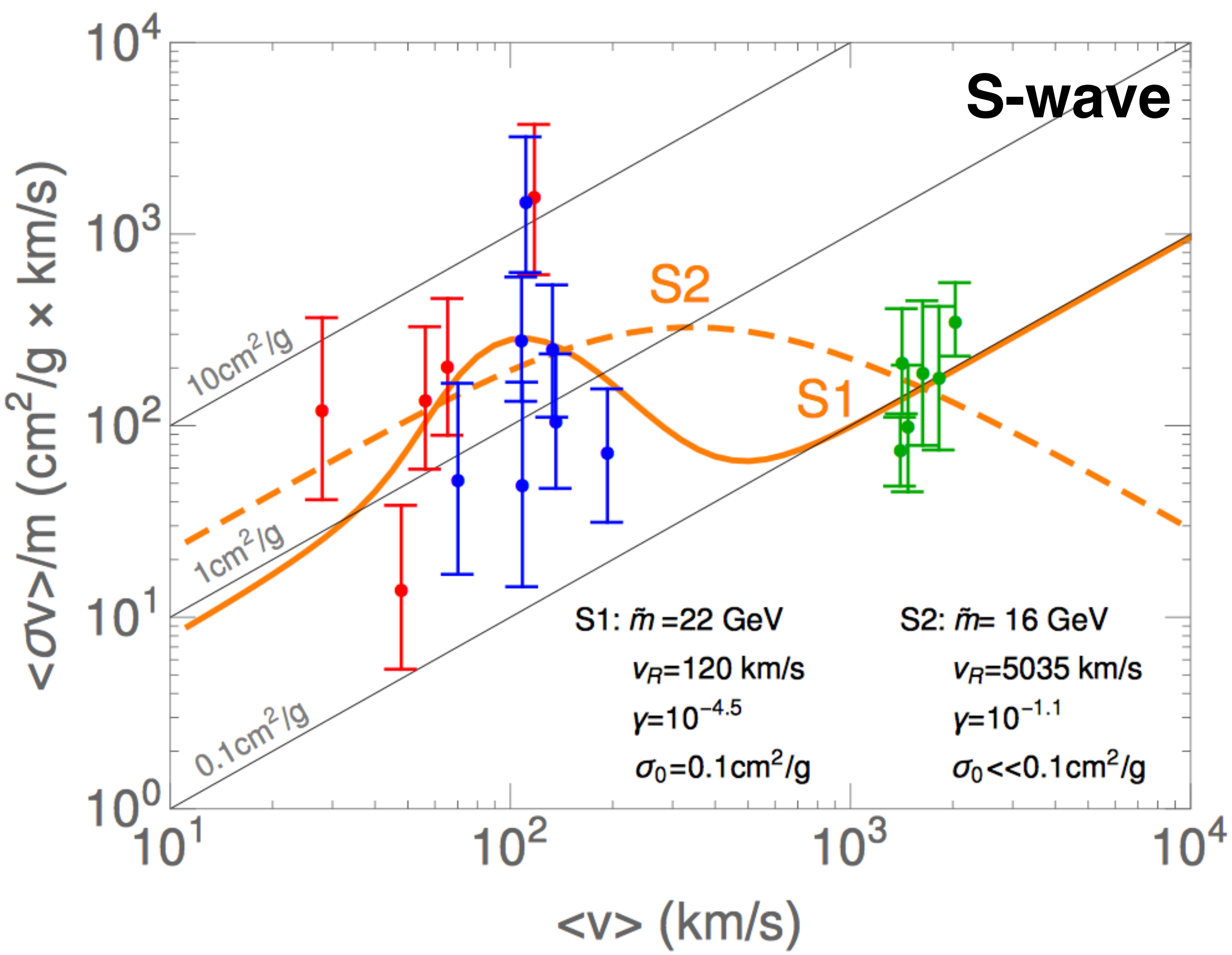
v_R — **near resonance**

SIDM via Breit-Wigner resonance [XC, C. Garcia-Cely, H. Murayama 2018]

Self-scattering via $\mathrm{DM} + \mathrm{DM} \to R^{(*)} \to \mathrm{DM} + \mathrm{DM}$

$$\sigma = \sigma_0 + \frac{4\pi\, S}{mE(v)} \cdot \frac{\Gamma(v)^2/4}{(E(v) - E(v_R))^2 + \Gamma(v)^2/4}$$

then averaged over velocity distribution



Data points from [M. Kaplinghat, S. Tulin, and H.-B. Yu 2015]

Fine-Tuning?

- S1: narrow-width:

$$1/10^6$$

- S2: broad-width:

$$1/10^2$$

SIDM via Breit-Wigner resonance [XC, C. Garcia-Cely, H. Murayama 2018]

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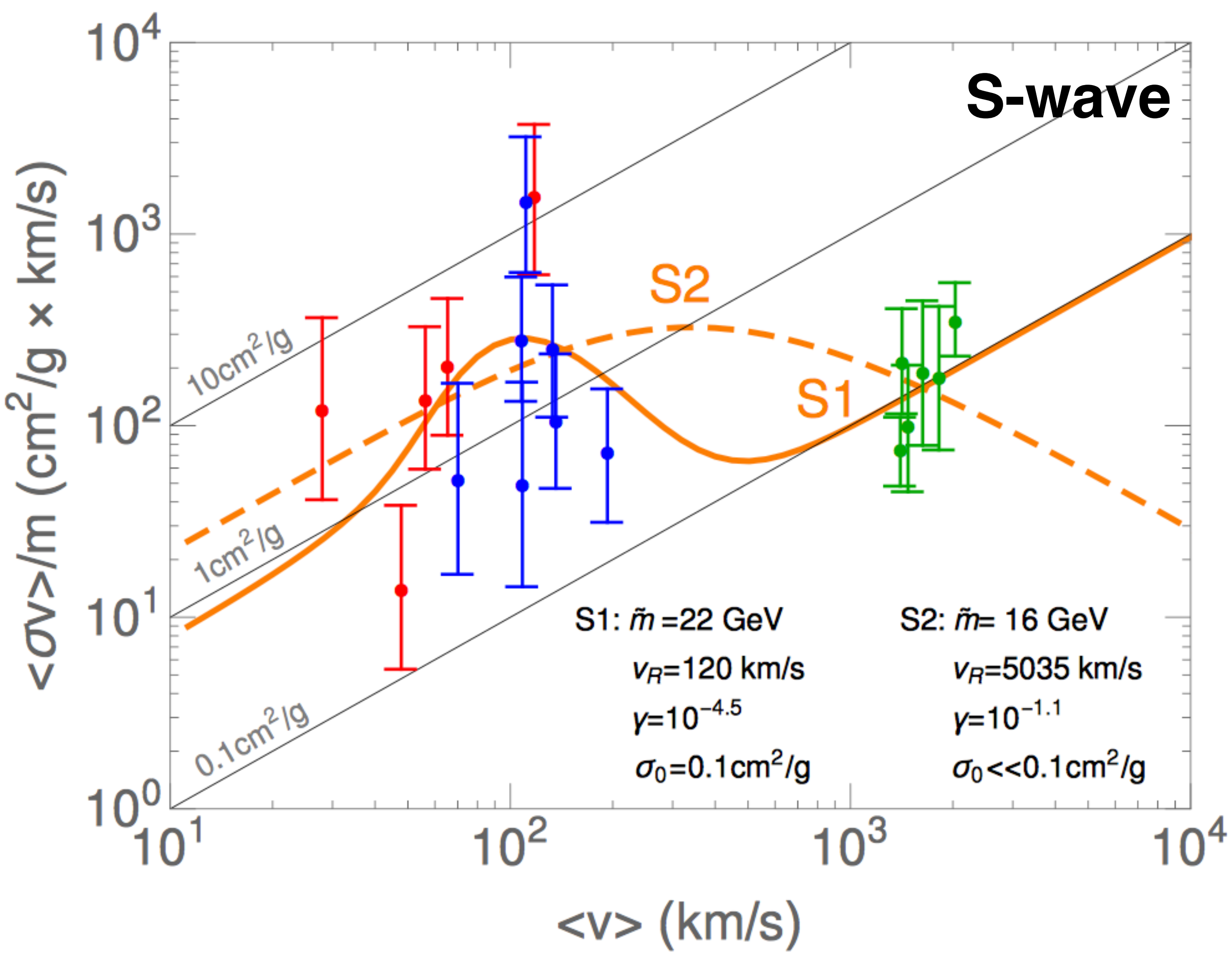
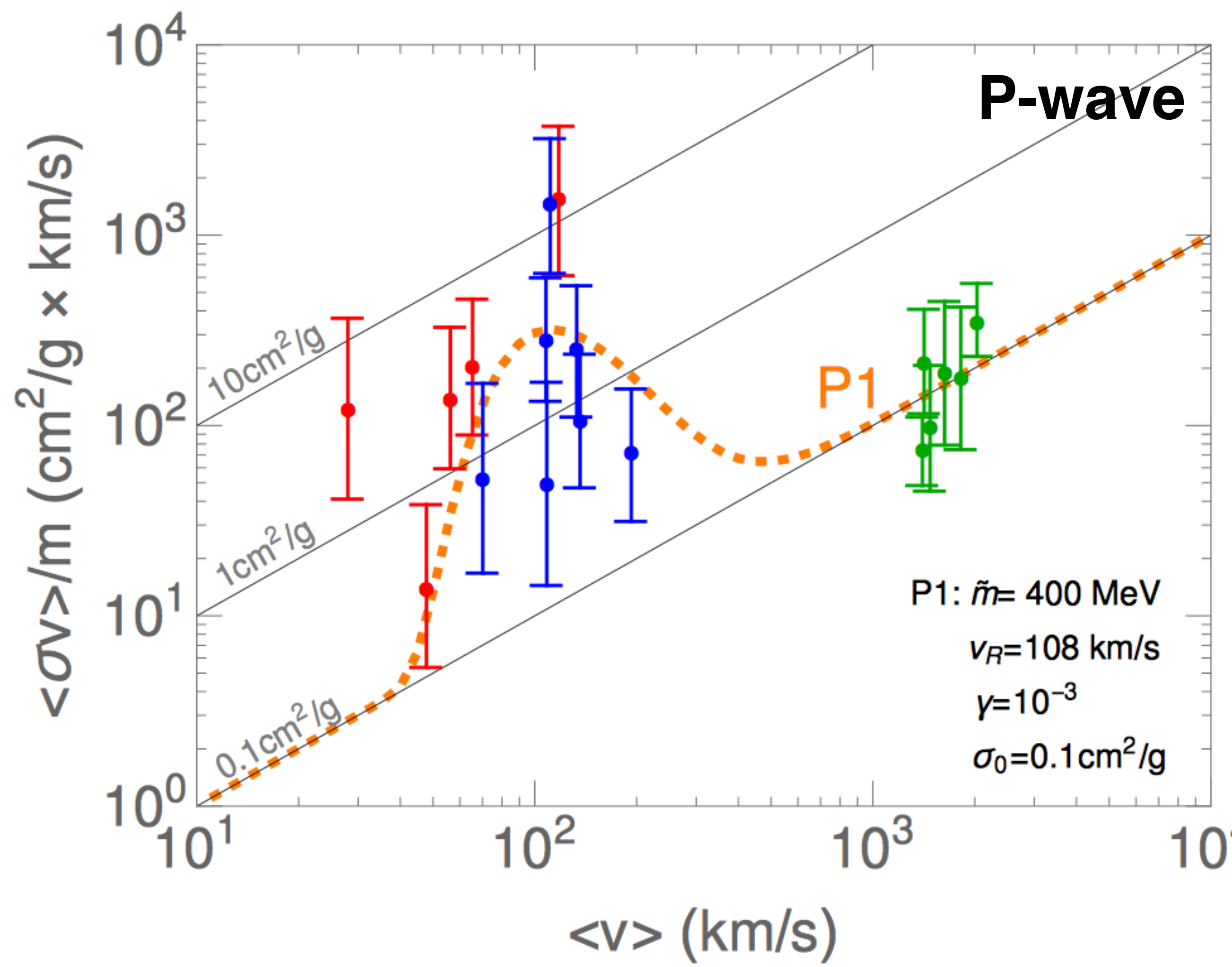


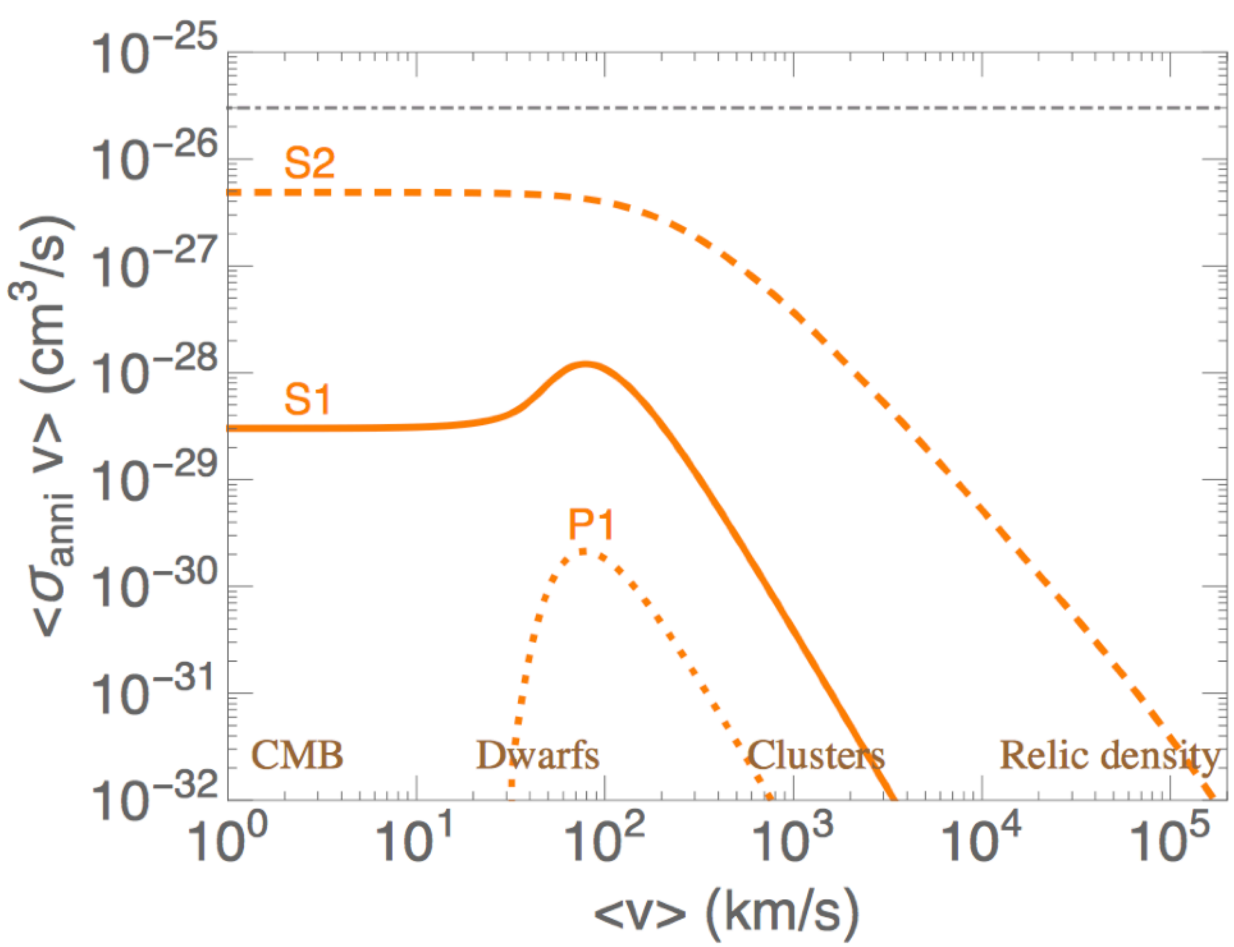


Data points from [M. Kaplinghat, S. Tulin, and H.-B. Yu 2015]

Resonant annihilation [XC, C. Garcia-Cely, H. Murayama 2018]

Possible annihilation via $\mathrm{DM} + \mathrm{DM} \to R^{(*)} \to \mathrm{SM} + \mathrm{SM}$

$$\sigma_{\rm anni} \simeq \frac{4\pi\, S}{mE(v)} \cdot \frac{\Gamma(v) \cdot \boxed{m_R \gamma_f/4}}{(E(v) - E(v_R))^2 + \Gamma(v)^2/4}$$



1. **Second fast annihilation** is possible;
2. For symmetric DM, **freeze-in** is needed.

III. A consistent parametrization

Effective-range theories (ERT)

Back to **quantum scattering theory** using **phase shift**:

$$\frac{d\sigma}{d\Omega} = |f(k,\theta)|^2 \qquad \text{with} \quad f(k,\theta) = \sum_{\ell=0}^{\infty}(2l+1)f_\ell(k)P_\ell(\cos\theta)\ ,$$

$$\text{with} \quad f_\ell(k) \equiv \frac{e^{2i\delta_\ell(k)}-1}{2ik} = \frac{1}{k\left(\cot\delta_\ell(k)-i\right)}$$

Boundary conditions of short-range potential suggest [Schwinger, Blatt & Jackson, Bethe, 1940s]

$$\lim_{k\to 0}\frac{\tan\delta_l}{k^{2l+1}} = const.$$

Effective-range theories (ERT)

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Boundary conditions of short-range potential suggest [Schwinger, Blatt & Jackson, Bethe, 1940s]

$$\lim_{k\to 0} \frac{\tan\delta_l}{k^{2l+1}} = const.$$

allowing an expansion of phase shift at low velocities of DM:

$$k^{2\ell+1}\cot\delta_\ell = -\frac{1}{a_\ell^{2\ell+1}} + \frac{1}{2\, r_{e,\ell}^{2\ell-1}} k^2 + \mathcal{O}(k^4)$$

scattering length   effective range

And in general, s-wave parameters (ℓ=0) dominate at low velocities.

A useful parametrization of SIDM [XC, C. Garcia-Cely, H. Murayama 2019]

A s-wave (ℓ=0) dominated cross section is parametrized by:

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 \approx \frac{4\pi a^2}{1 + k^2\left(a^2 - a r_e\right) + \frac{1}{4}a^2 r_e^2 k^4}$$

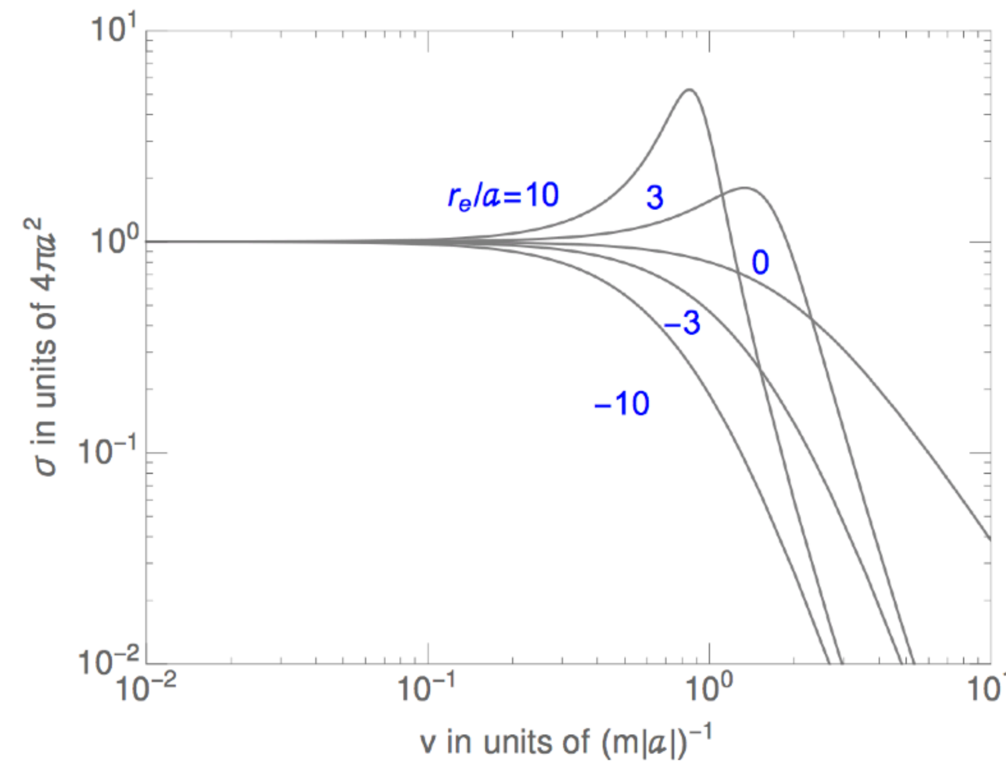
- For hard scattering $(1/k \gg |a| \gg |r_e|)$:

$$\sigma_0 \to 4\pi a^2$$ (contact interaction)

- For very large scattering $(a \to \infty)$:

$$\sigma_0 \to \frac{4\pi}{k^2}$$ (unitarity boundary)

- Generally, larger σ at low velocities



Connect DM models to s-wave EFT

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 \approx \frac{4\pi a^2}{1 + k^2\left(a^2 - a r_e\right) + \frac{1}{4}a^2 r_e^2 k^4}$$

DM Models:

1. **Contact interaction** (i.e. effective operator, quartic interaction).

2. **Born regime** of Yukawa potential:

$$V(r) = \pm\frac{\alpha_X}{r}e^{-m_\phi r}$$

$$a = -\frac{m\alpha}{m_\phi^2}\,, \quad \text{and} \quad r_e = \frac{4}{m\alpha}$$

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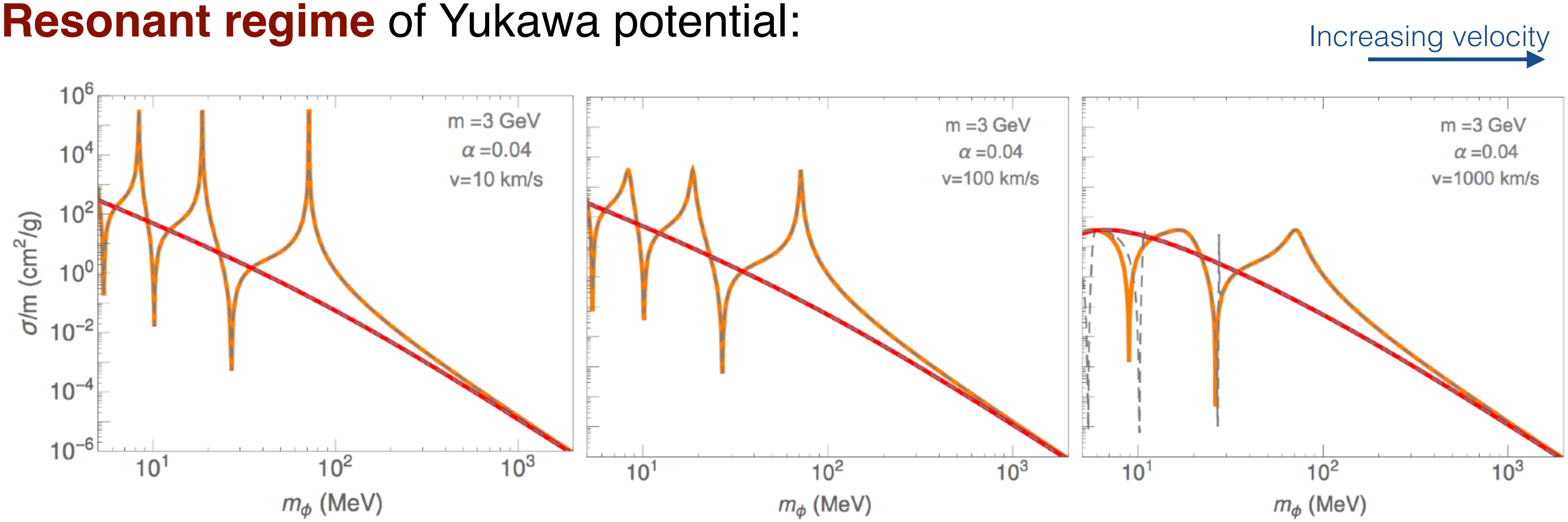
3. **Resonant regime** of Yukawa potential:



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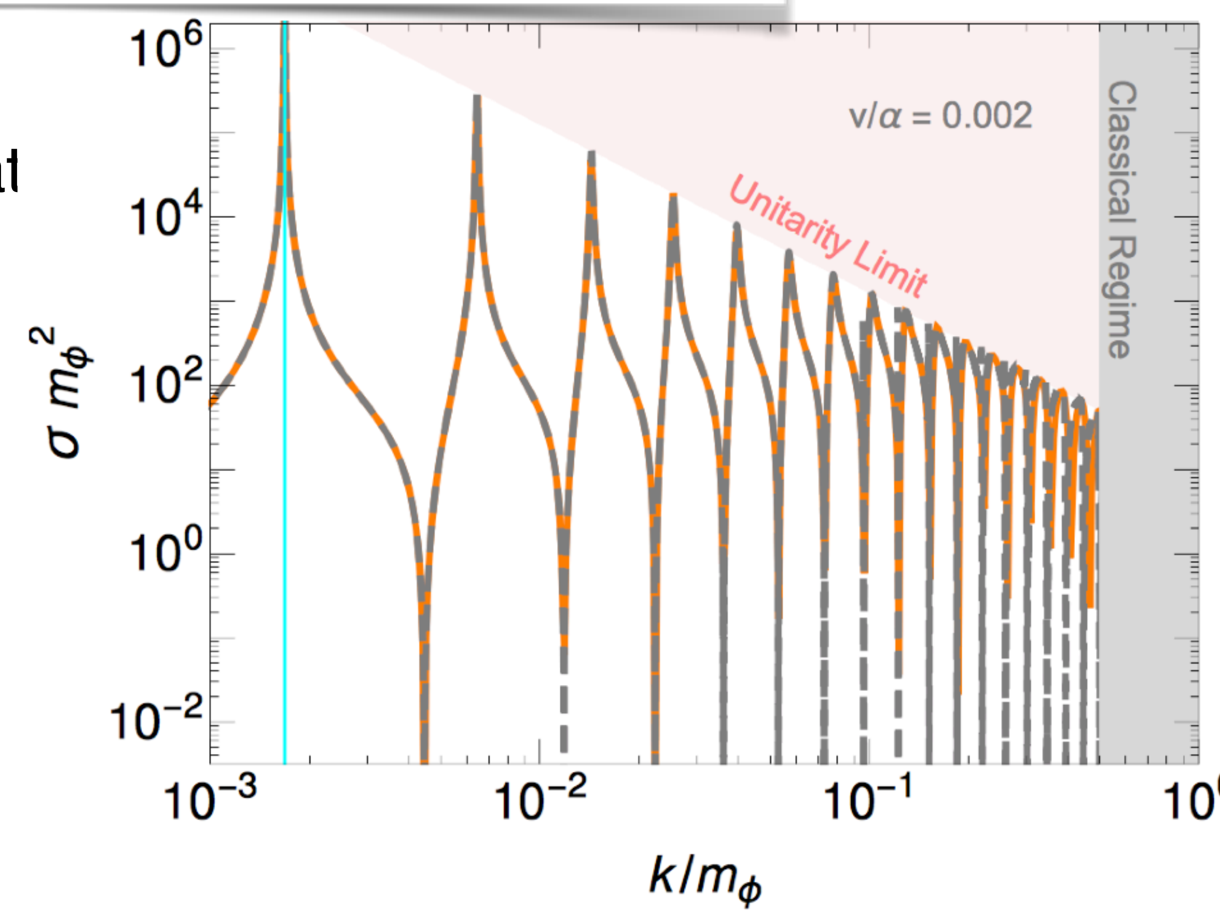


4. **Classical regime** of Yukawa potential: $m_\phi \leq \alpha_\chi m_\chi$ & $m_\phi \leq m_\chi v$

Not work, long-range interaction and not s-wave dominated (QED-like).

Connect DM models to s-wave EFT

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 \approx \frac{4\pi a^2}{1 + k^2\left(a^2 - a r_e\right) + \frac{1}{4}a^2 r_e^2 k^4}$$

DM Models:

1. **Contact interaction** (i.e. effective operator, quartic interaction).
2. **Born regime** of Yukawa potential:
3. **Resonant regime** of Yukawa potential:
4. **Classical regime** of Yukawa potential:
5. **S-wave resonance** self-scattering**:**

Equality for each partial wave

$$a_\ell = -\frac{\Gamma(E_R)^{\frac{1}{2\ell+1}}}{2^{\frac{1}{2\ell+1}} E_R^{\frac{2\ell+3}{4\ell+2}} m^{\frac{1}{2}}}\,, \quad r_{e,\ell} = -\frac{2^{\frac{2}{-2\ell+1}} E_R^{\frac{2\ell+1}{-4\ell+2}}}{\Gamma^{\frac{1}{-2\ell+1}} m^{\frac{1}{2}}}$$

$$\frac{4\pi\, S}{mE(v)} \cdot \frac{\Gamma(v)^2/4}{\left(E(v) - E(v_R)\right)^2 + \Gamma(v)^2/4}$$

$$E(v) = \frac{1}{2}\frac{m}{2}v^2 \quad \text{and} \quad S = \frac{2J_R + 1}{(2J_{\rm DM}+1)^2}$$

A lesson from nuclear physics

If we have measured **velocity dependence**, what it tells about the **particle properties of DM**?

- **For weak potential** (Born approximation applies)

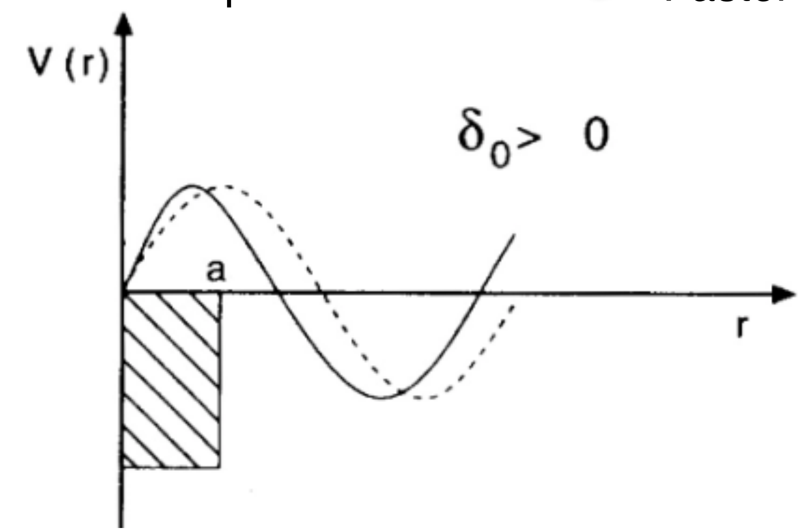
$$a = -\lim_{k\to 0}\frac{\tan\delta_0(k)}{k}$$

Attractive potential: $a < 0, \quad r_e > 0$

Repulsive potential: $a > 0, \quad r_e < 0$

$k = \sqrt{2m(E-V)}$ ⬆

Inside attractive potential → Faster oscillation



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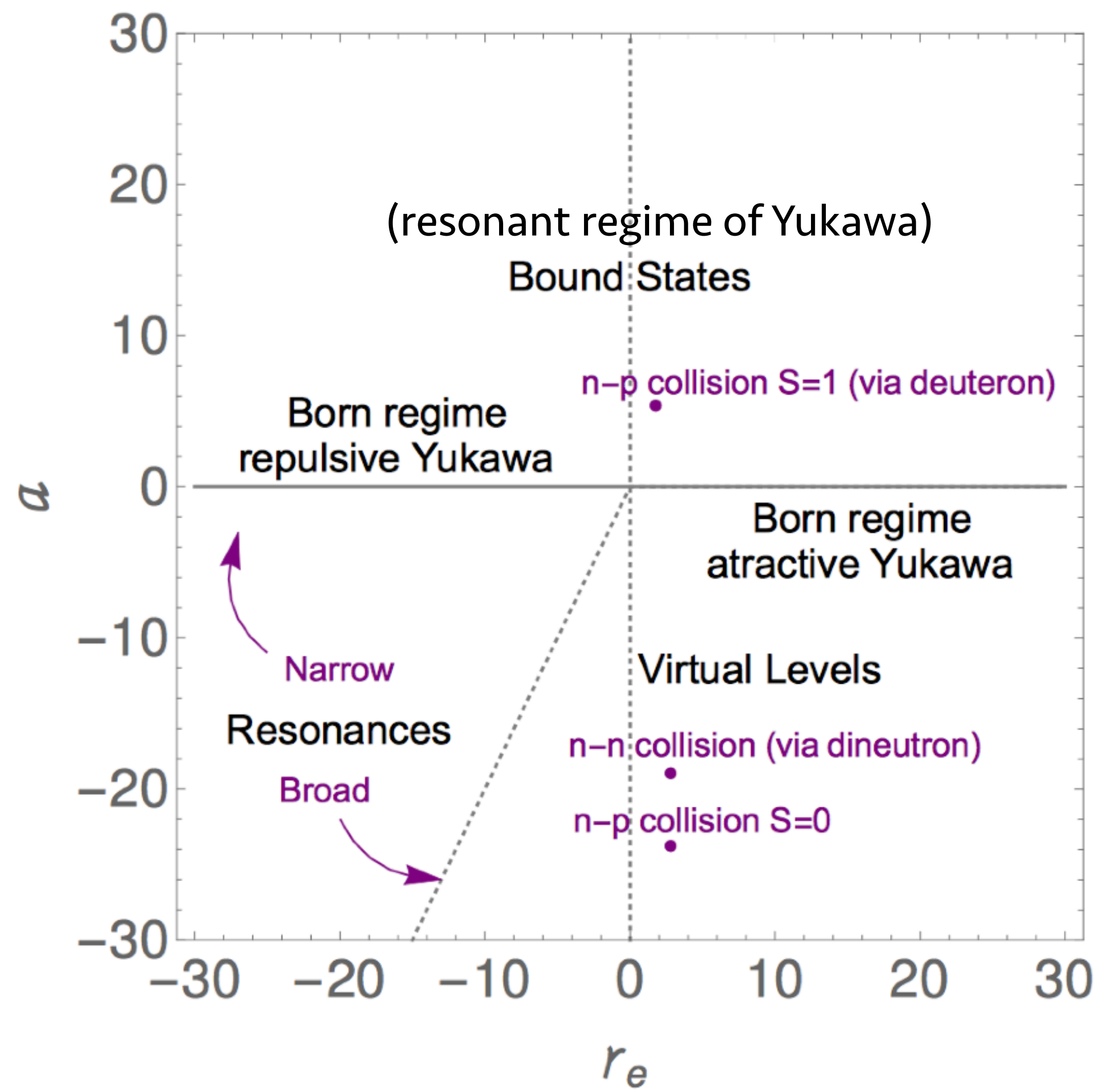
$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 \approx \frac{4\pi a^2}{1 + k^2\left(a^2 - a r_e\right) + \frac{1}{4}a^2 r_e^2 k^4}$$

The closer pole to real k-axis $\quad k^0_{\rm min} = \dfrac{2i}{a(1+\sqrt{1-2r_e/a})}$



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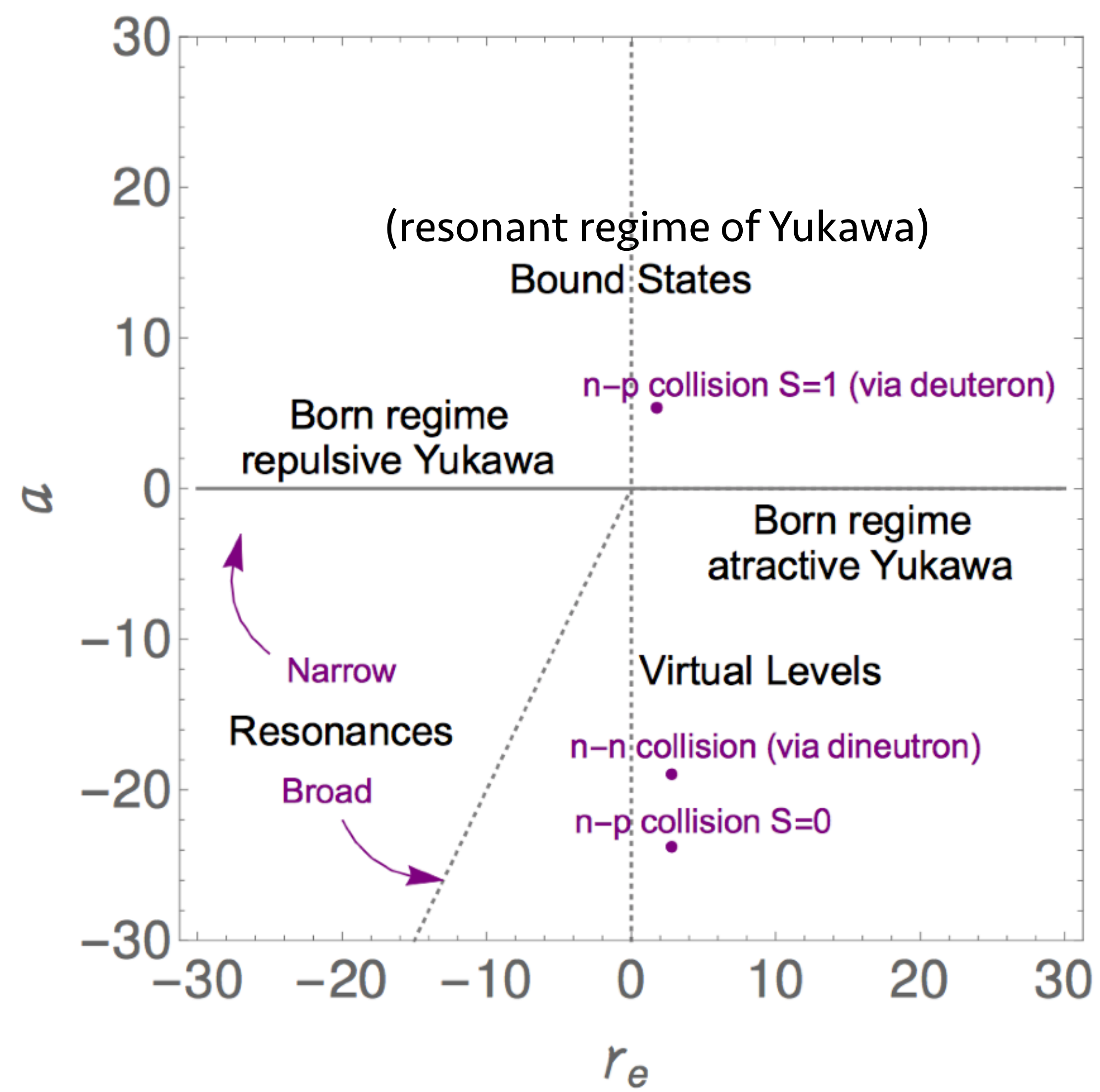

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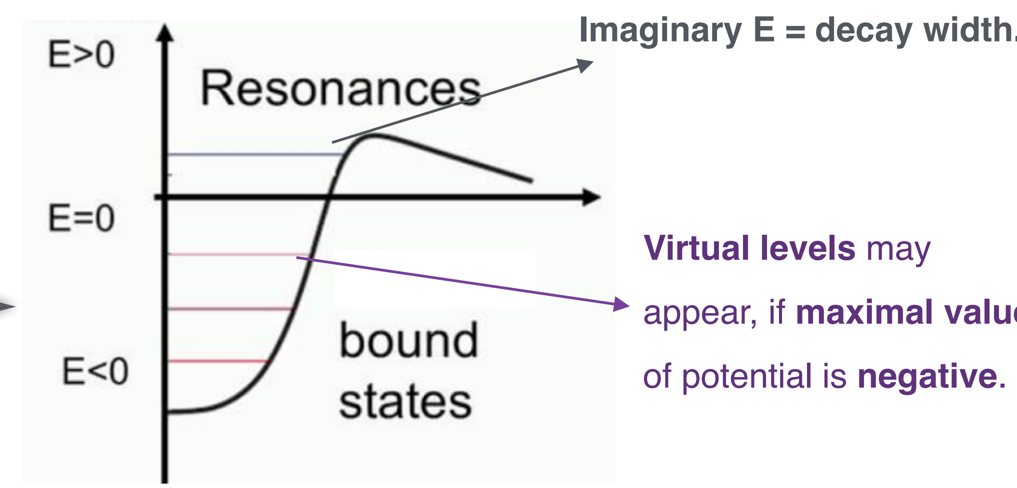
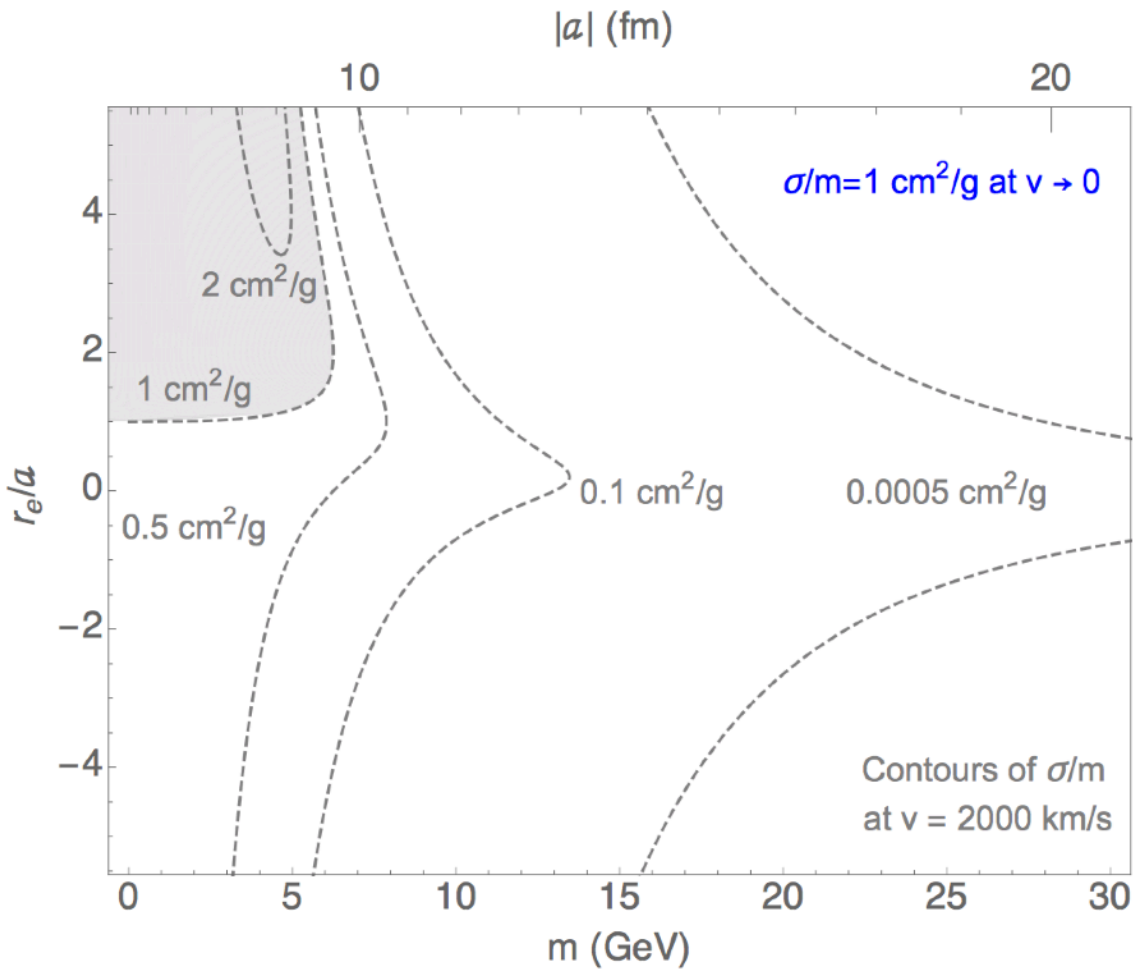
$$E^0 = \frac{k_0^2}{2m}$$

The closer pole to real k-axis $\quad k^0_{\rm min} = \dfrac{2i}{a(1+\sqrt{1-2r_e/a})}$

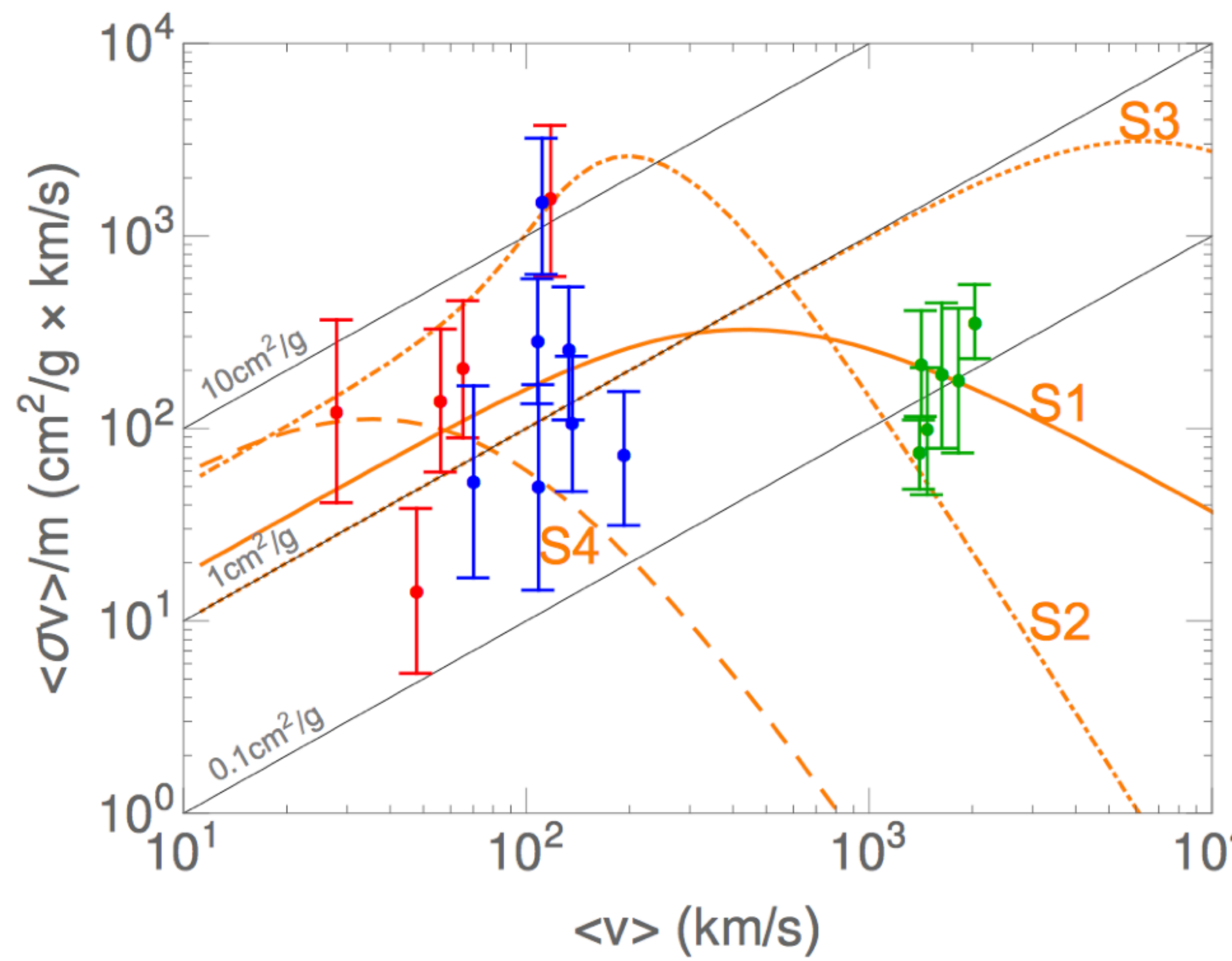


ERT Parametrization

Fix the self-scattering cross section to the preferred value at dwarf-scale:



Fitting the inferred values of self-scattering cross section at various velocities.

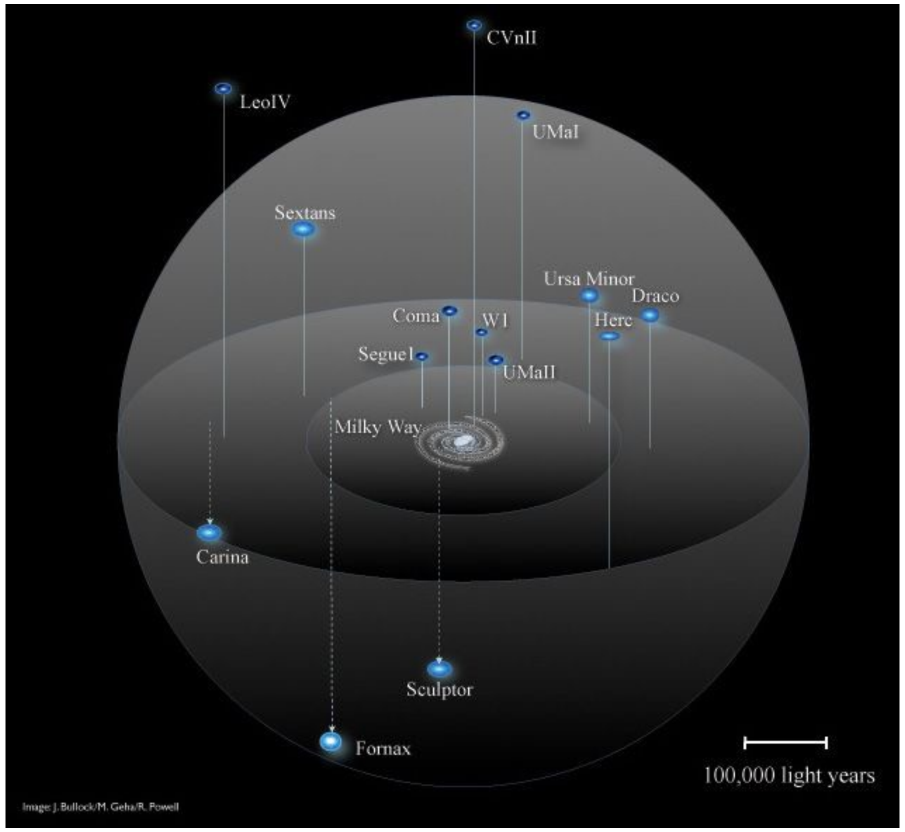


- So far constraints/hints from observations are **too weak**.
- Observation can never tell the total sign of **scattering length** and **effective range**.

ERT Parametrization

Mass deficit problems are mostly found in **satellite halos** (e.g. in Milky Way):

Cosmological simulations with a consistent description of such **cross section involving various scales of DM relative velocities** can be crucial, e.g. to understand evolution of satellite halos moving inside host large halo [A. Banerjee et al. 2019].



One nice way is **effective range theories** with **two parameters**, and **the DM mass**:

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 \approx \frac{4\pi a^2}{1 + k^2\left(a^2 - a r_e\right) + \frac{1}{4}a^2 r_e^2 k^4}$$

IV. Conclusions

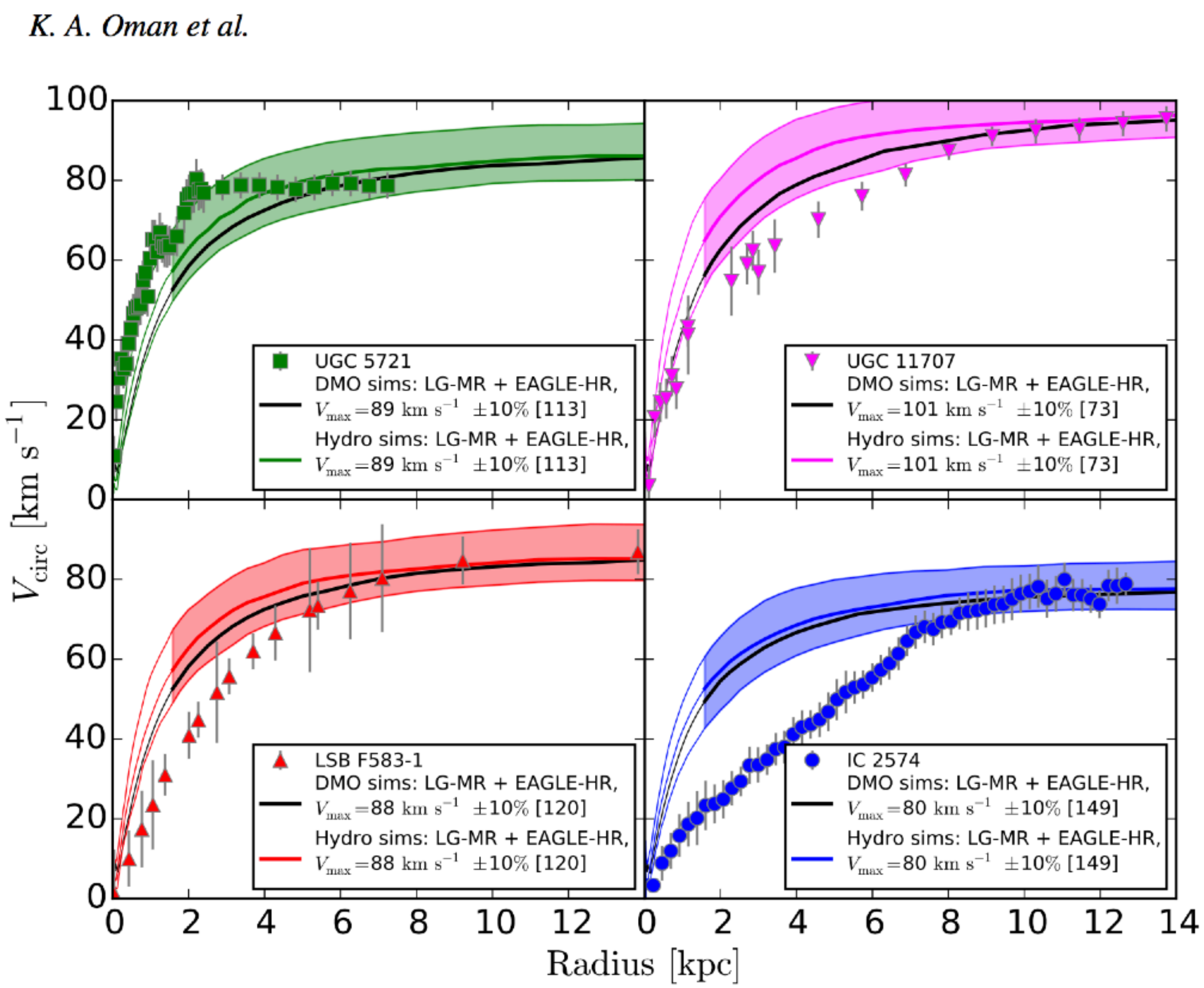
Conclusions

- Halo mass deficit may be **hint of non-conventional DM;**
- For the SIDM solution, **velocity-dependence** seems necessary;
    - Interaction induced by a light mediator or s-wave resonance.
- **Consistent parametrization** is possible for well-motivated models;
    - In general scattering is enhanced at smaller DM velocities,
    - Complex phase shift to describe both scattering and annihilation?
- Detailed **cosmological simulations are required** (precise baryon effects, halo evolution with time, tidal effects, …).
- Observing profiles of **small halos (with no star activities)** may be crucial.

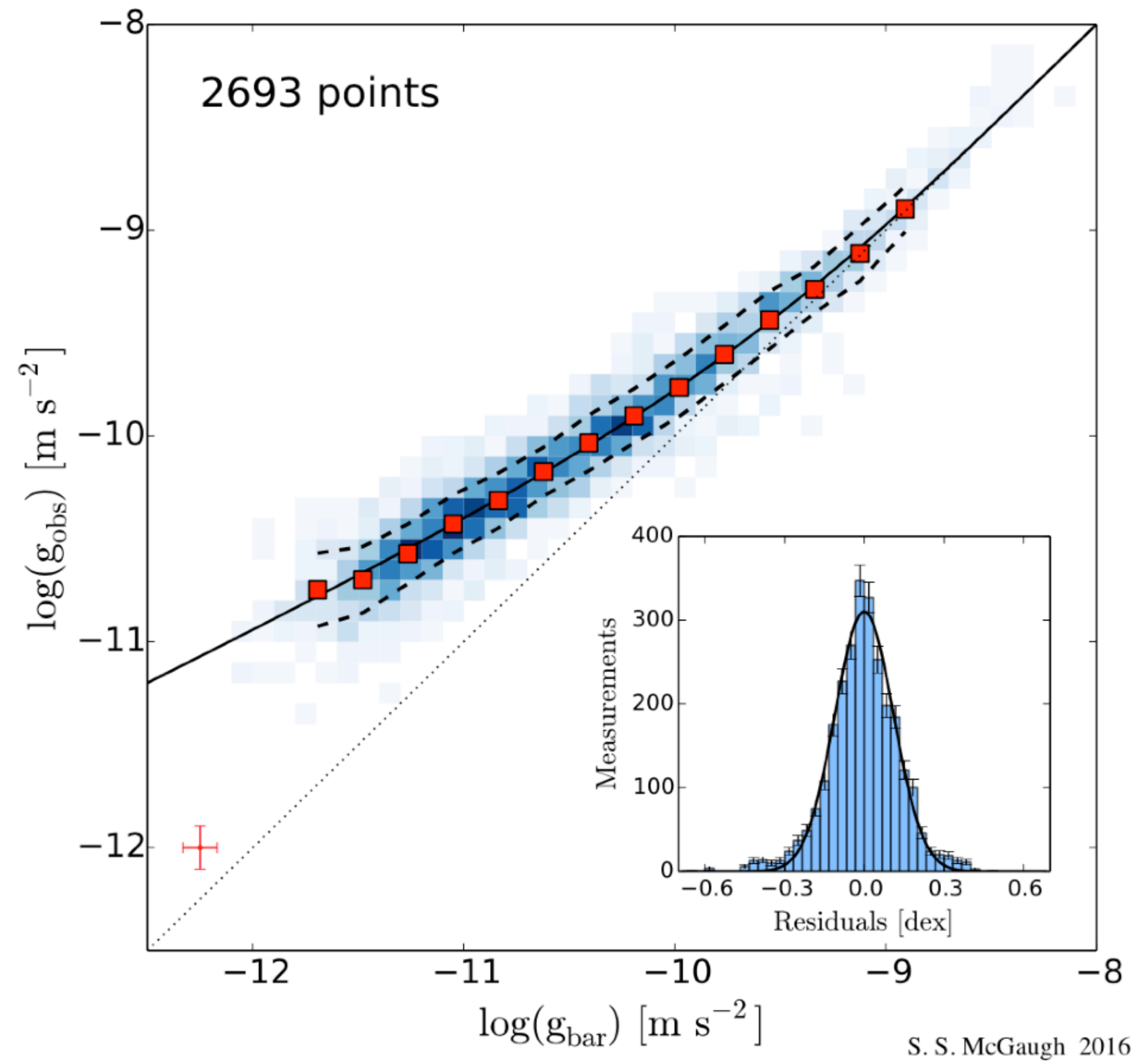
Thanks!

Diversity and MDAR in self-interacting DM (SIDM)

[1504.01437] Diversity of galactic rotational curves



[1609.05917] Matter-Discrepancy Acceleration Relation (MDAR)



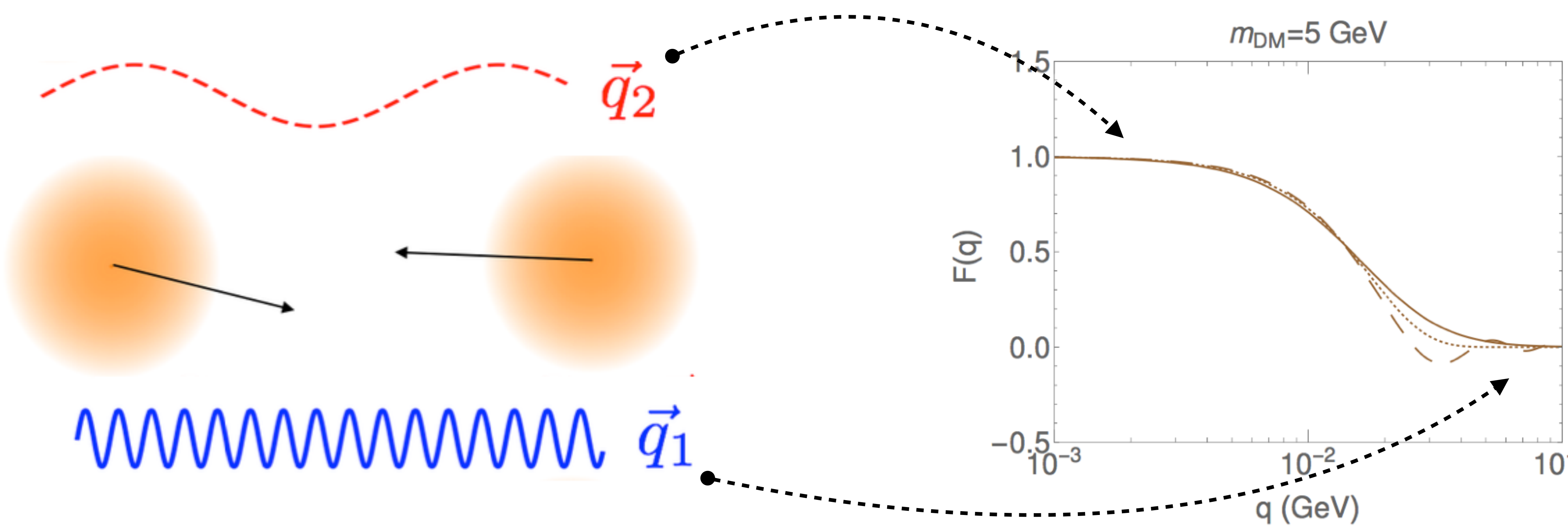
In SIDM, **thermalized DM particles** respond **efficiently and steadily** to baryonic contraction, leading to **larger diversity in rotational curves** and **smaller scatters in MDAR** [1611.02716, 1711.09096, 1808.05695,1811.02569, …].

SIDM via dark form factor [XC, C. Garcia-Cely, H. Murayama, 2019]

Dark matter with form factor has been considered for direct detections, and in various dark nucleon/nugget models.

$$F(\vec{\mathbf{q}}) = \int d^3\vec{\mathbf{r}}\, e^{i\vec{\mathbf{q}}\cdot\vec{\mathbf{r}}}\rho(r) \propto \begin{cases} 1/(M^2+\mathbf{q}^2) & \text{if } \rho(r) \propto e^{-Mr}/r\,, \\ 1/(M^2+\mathbf{q}^2)^2 & \text{if } \rho(r) \propto e^{-Mr}\,, \end{cases}$$

Velocity-dependence



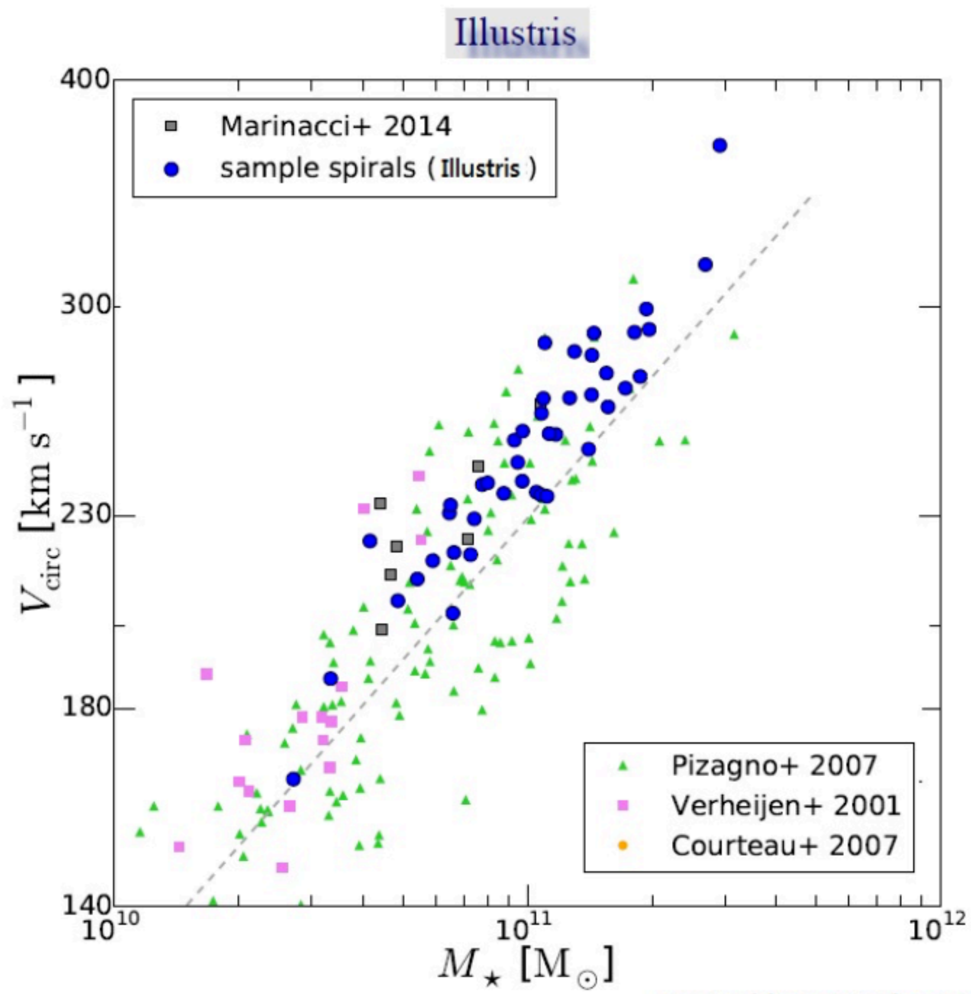
Other potential anomalies in Cold DM paradigm

· Tully-Fisher relation.-Typically luminous $M_b \propto V_{\rm rot}^4$ in spiral galaxies [McGauge 2012, ...]

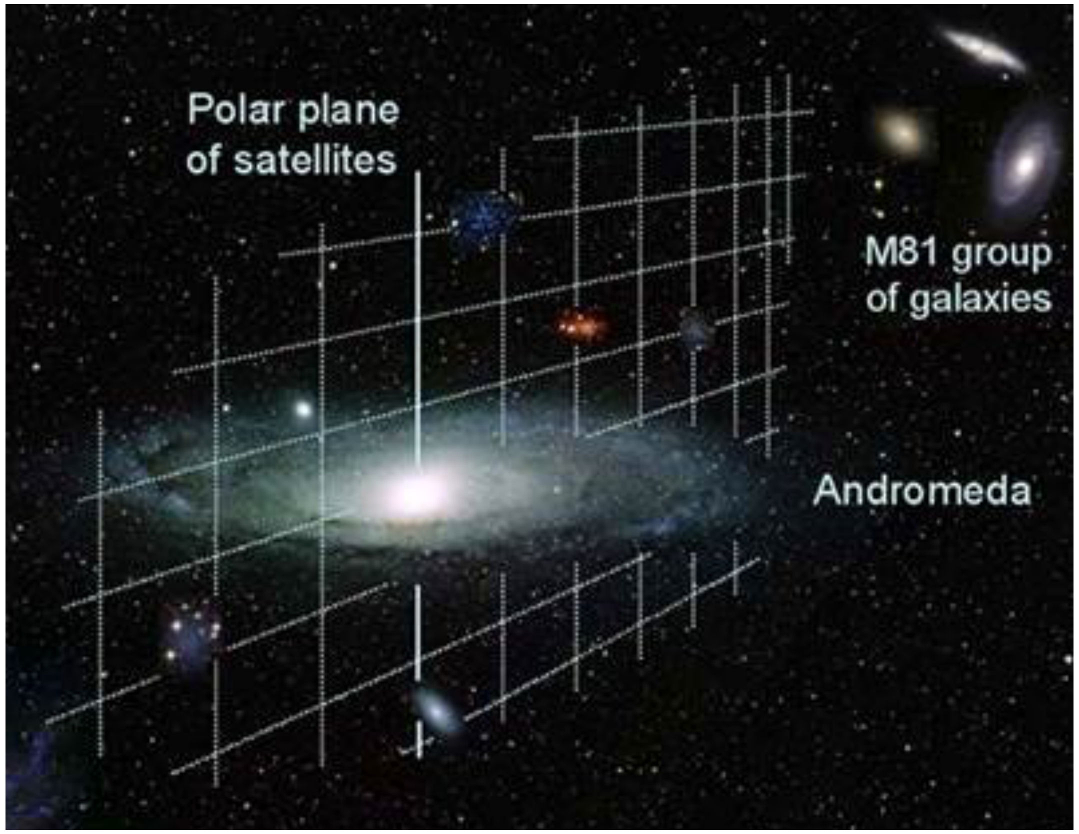
Maybe $M_b/M_{halo} \propto M_{halo}$ with (observed) large dispersions? [Guo et al. 2009, Desmond 2012, ...]

· Isothermal profile of total mass density.-in some elliptical galaxies.

· Fundamental plane,



M.Vogelsberger et al. 2014



Solutions required in Cold DM paradigm

- Lithium-7 problem at BBN
  - Stellar depletion? nuclear physics?
  - New particle, modifying history of the Universe?
- Empirical Tully-Fisher relations
  - Strong dynamical correlation between DM halo/galaxy evolutions?
  - Latest simulations suggest so, but not completely solved yet.
  - If not GR: baryon-DM interaction? exotic DM? DM+MOND?
- Dwarf galaxies observations
  - How to make inner mass deficient in halo mass profiles?
  - What is supporting tidal dwarfs, if they are self-bounded?

The Schrödinger equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR_{\ell,k}}{dr}\right)+\left(k^2-\frac{\ell(\ell+1)}{r^2}-mV(r)\right)R_{\ell,k}=0\,,$$

Scattering length

$$t_{\ell,k}(r)=\frac{j_\ell(kr)\left(\frac{R'_{\ell,k}(r)}{R_{\ell,k}(r)}-\frac{\ell}{r}\right)+k\,j_{\ell+1}(kr)}{n_\ell(kr)\left(\frac{R'_{\ell,k}(r)}{R_{\ell,k}(r)}-\frac{\ell}{r}\right)+k\,n_{\ell+1}(kr)}\,. \qquad \text{(A3)}$$

Simple algebra shows that

$$\frac{dt_{\ell,k}(r)}{dr}=-k\,m\,r^2V(r)\left(j_\ell(kr)-t_{\ell,k}(r)n_\ell(kr)\right)^2\,. \qquad \text{(A4)}$$

The fact that $R_{\ell,k}\propto r^\ell$ and Eq. (A2) fix the boundary conditions of this differential equation to

$$t_{\ell,k}(0)=0 \quad \text{and} \quad t_{\ell,k}(r)\to\tan\delta_\ell \quad \text{at} \quad r\to\infty\,. \qquad \text{(A5)}$$

Notice that $j_\ell(kr)\propto k^\ell$ and $n_\ell(kr)\propto k^{-(\ell+1)}$ in the limit $k\to 0$, which together with Eq. (A4) imply that $\tan\delta_\ell\propto k^{2\ell+1}$ for small momenta. The corresponding coefficient of proportionality defines scattering length a_ℓ. More precisely,

$$a_\ell^{2\ell+1}\equiv-\lim_{k\to0}\frac{\tan\delta_\ell}{k^{2\ell+1}}\,. \qquad \text{(A6)}$$

Effective range

$$u_k(r)\frac{du_0(r)}{dr}-u_0(r)\frac{du_k(r)}{dr}\Big|_0^r=k^2\int_0^r u_0(r')u_k(r')dr'\,. \qquad \text{(A14)}$$

Moreover, using the fact that $\psi_k(r)$ is the solution of the Schrödinger equation for $V(r)=0$, we find that

$$\psi_k(r')\frac{d\psi_0(r')}{dr}-\psi_0(r')\frac{d\psi_k(r')}{dr}\Big|_0^r=k^2\int_0^r \psi_0(r')\psi_k(r')dr'\,. \qquad \text{(A15)}$$

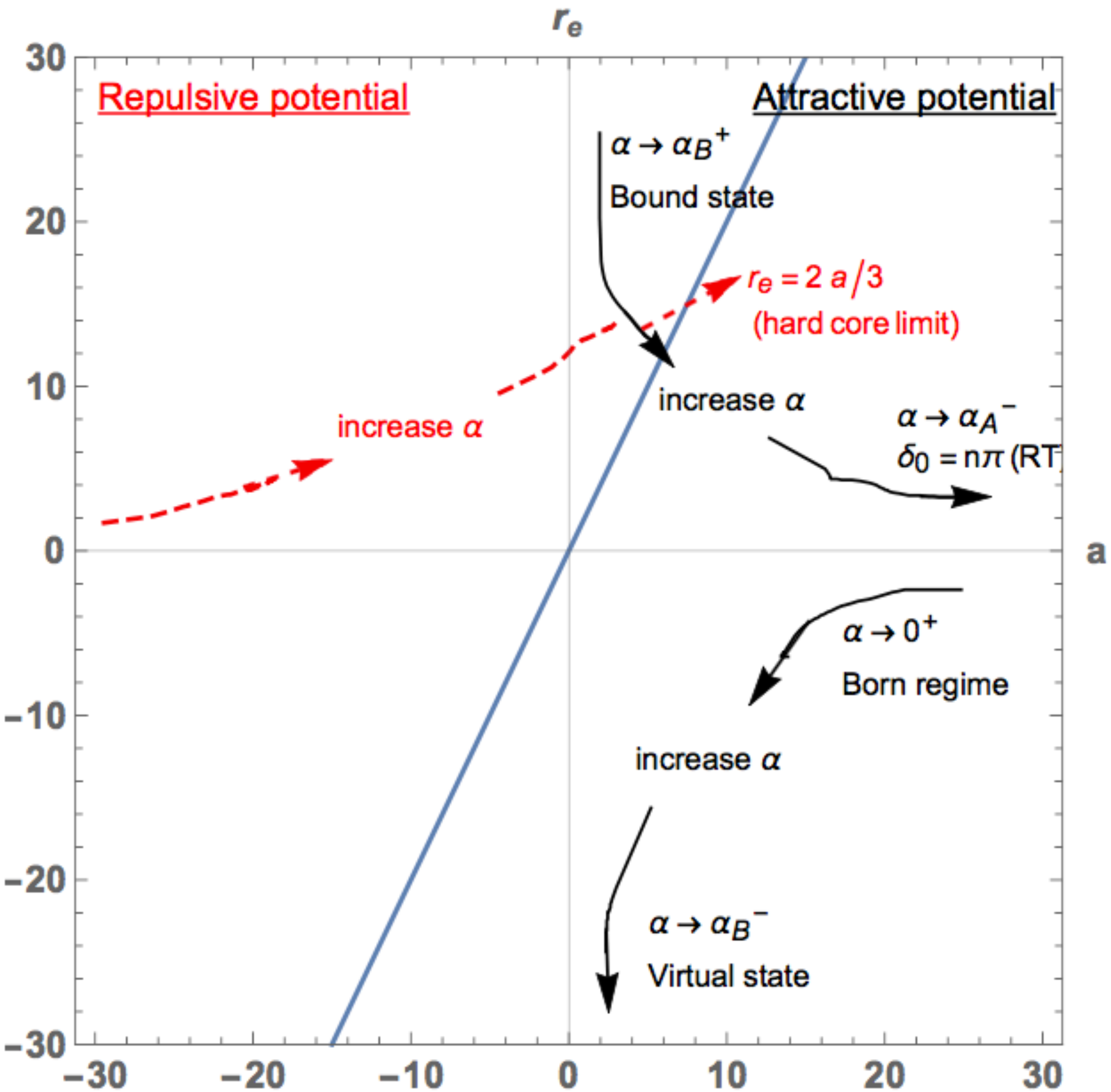
Notice that $\psi_0(r)=1-r/a_0$, where a_0 is the scattering length. Subtracting Eq. (A14) from Eq. (A15), taking $r\to\infty$ and using the fact that u_k and ψ_k approach to each other in that limit, we find that

$$\begin{aligned} k\cot\delta_0 &= -\frac{1}{a_0}+k^2\int_0^\infty(\psi_0\psi_k-u_0u_k)\,dr \\ &= -\frac{1}{a_0}+\frac{1}{2}r_{e,0}k^2+\mathcal{O}(k^4)\,. \end{aligned} \qquad \text{(A16)}$$

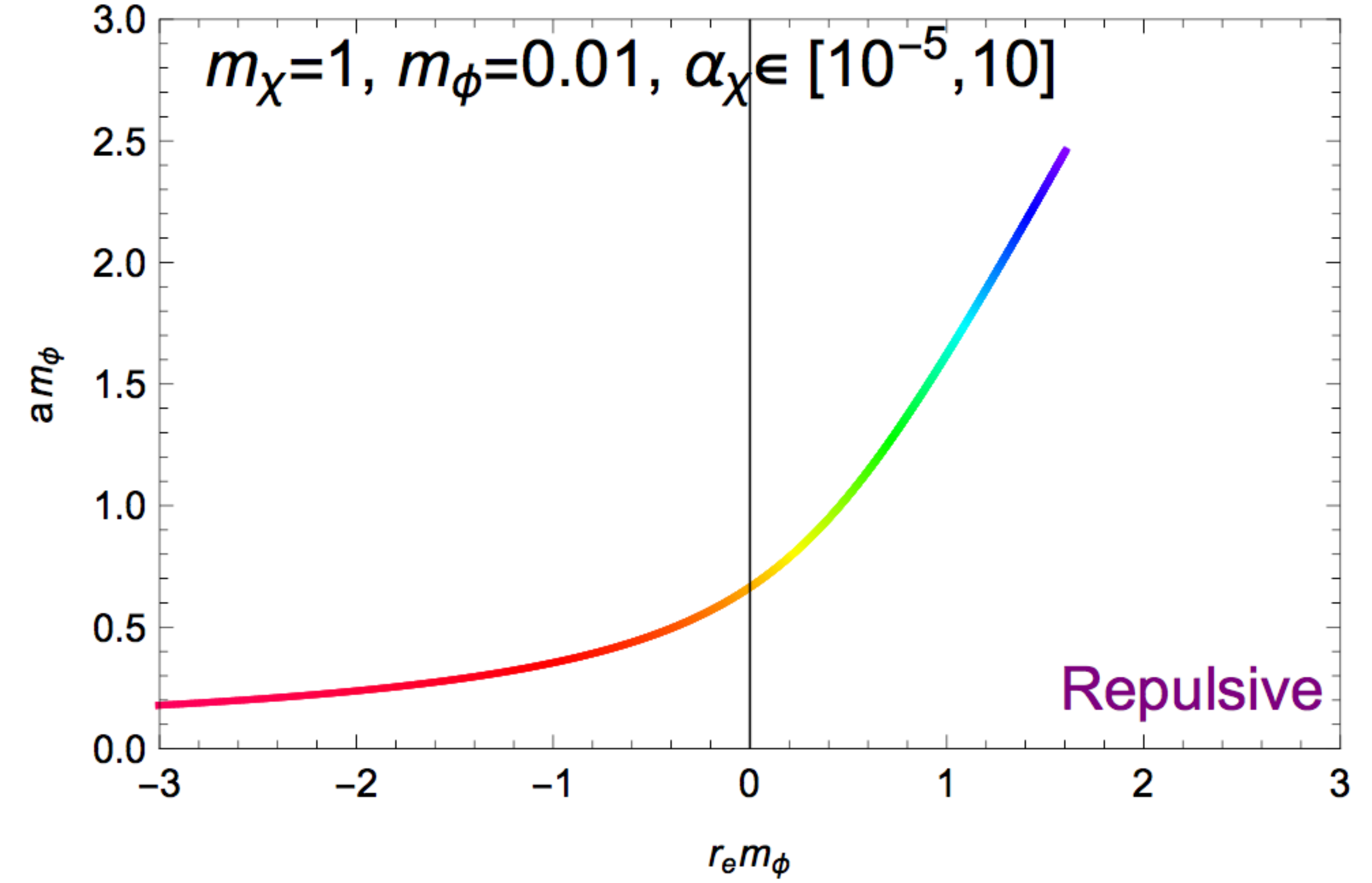
where

$$r_{e,0}=2\int_0^\infty\left(\psi_0^2-u_0^2\right)dr\,. \qquad \text{(A17)}$$

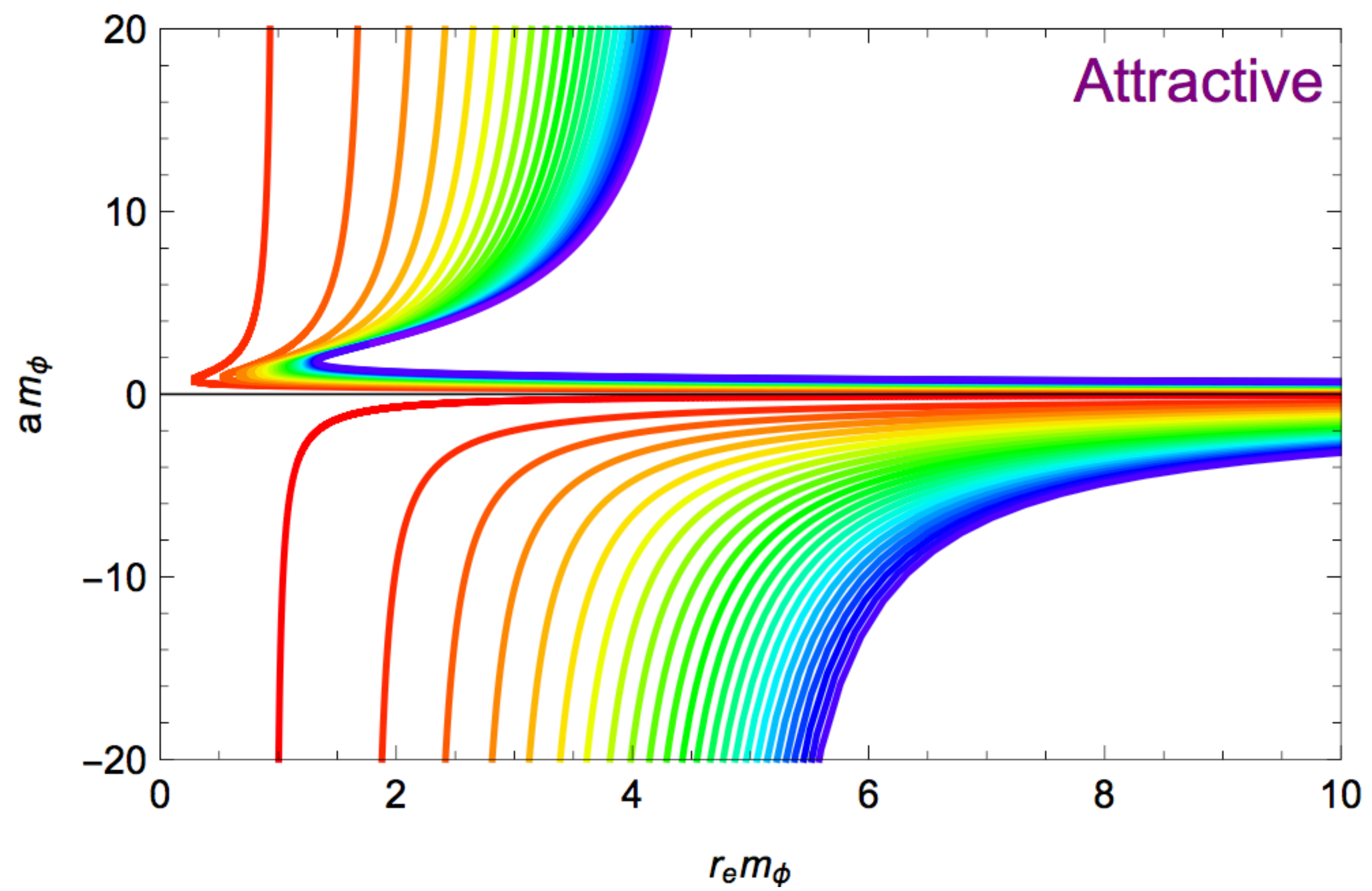
ArXiv: 1908.06067



Repulsive potential
Attractive potential
$\alpha \to \alpha_B{}^+$
Bound state
$r_e = 2\,a/3$
(hard core limit)
increase α
increase α
$\alpha \to \alpha_A{}^-$
$\delta_0 = n\pi$ (RT
$\alpha \to 0^+$
Born regime
increase α
$\alpha \to \alpha_B{}^-$
Virtual state



m_χ=1, m_ϕ=0.01, $\alpha_\chi \in [10^{-5},10]$
Repulsive



Attractive