The perturbative gradient flow

Robert Harlander

Institute for Theoretical Particle Physics and Cosmology | RWTH AACHEN UNIVERSITY

Particle Physics Seminar
Vienna, December 10, 2019

supported by:
DFG Deutsche Forschungsgemeinschaft

based on
RH, T. Neumann, JHEP 1606 (2016) 161 [arXiv:1606.03756]
RH, F. Lange, Y. Kluth, EPJ C78 (2018) 944 [arXiv:1808.09837]
J. Artz *et al.* [arXiv:1905.00882]

The QCD gradient flow

fundamental QCD: $$\mathscr{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu}$$

$$F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] \qquad D_\mu = \partial_\mu - iT^a A^a_\mu(x)$$

The QCD gradient flow

fundamental QCD: $$\mathscr{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu}$$

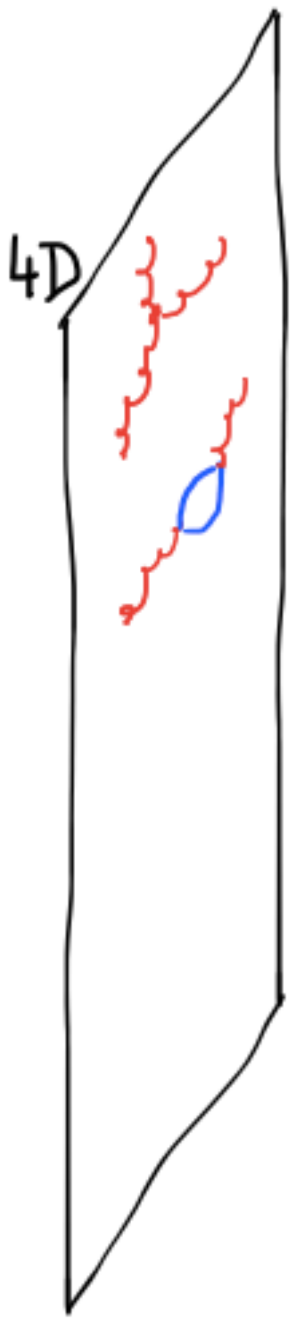
$$F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] \qquad D_\mu = \partial_\mu - iT^a A^a_\mu(x)$$

flowed gauge field:
$$\frac{\partial}{\partial t}B_\mu(t,x) = \mathscr{D}_\nu G_{\nu\mu}(t,x)$$
$$B_\mu(t=0,x) = A_\mu(x)$$

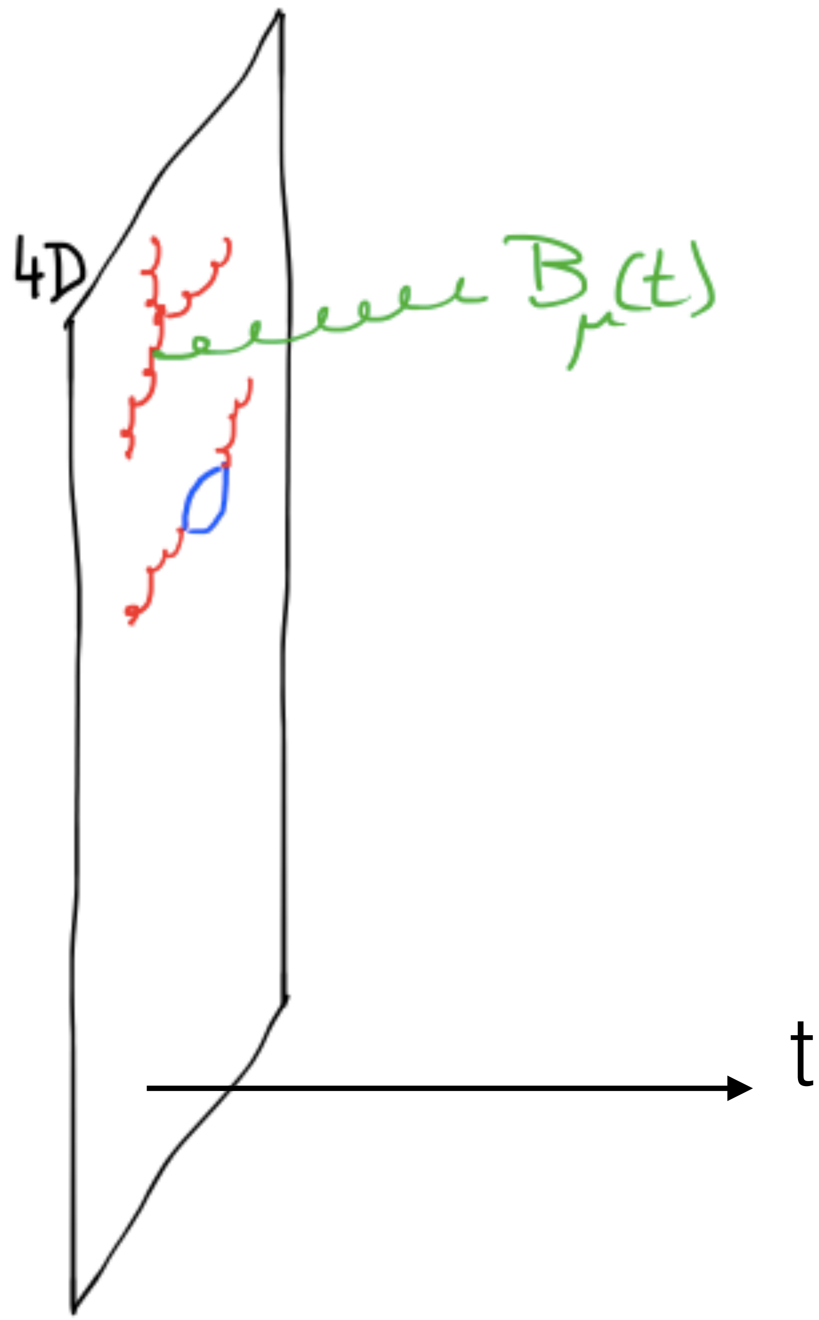
t: flow time

$$G_{\mu\nu} = \frac{i}{g}[\mathscr{D}_\mu, \mathscr{D}_\nu] \qquad \mathscr{D}_\mu = \partial_\mu - iT^a B^a_\mu(t,x)$$

$$\mathscr{L} = \mathscr{L}_{\rm QCD}$$



$$\mathscr{L} = \mathscr{L}_{\text{QCD}} + \mathscr{L}_B$$



The QCD gradient flow

fundamental QCD: $$\mathscr{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu}$$

$$F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] \qquad D_\mu = \partial_\mu - iT^a A^a_\mu(x)$$

flowed gauge field:

$$\frac{\partial}{\partial t}B_\mu(t,x) = \mathscr{D}_\nu G_{\nu\mu}(t,x)$$

$$B_\mu(t=0,x) = A_\mu(x)$$

t: flow time

$$G_{\mu\nu} = \frac{i}{g}[\mathscr{D}_\mu, \mathscr{D}_\nu] \qquad \mathscr{D}_\mu = \partial_\mu - iT^a B^a_\mu(t,x)$$

Schematically…

$$\frac{\partial}{\partial t} B_\mu(t,x) = \mathscr{D}_\nu G_{\nu\mu}(t,x)$$

$$\mathscr{D}_\mu = \partial_\mu - iT^a B^a_\mu(t,x)$$

Schematically…

$$\frac{\partial}{\partial t} B \qquad = \mathscr{D}\, G$$

$$\mathscr{D} \; = \partial \; - \qquad B$$

Schematically…

$$\frac{\partial}{\partial t}B = \mathscr{D}\, G$$

$$\mathscr{D} = \partial - B$$

$$G \sim [\mathscr{D}, \mathscr{D}] \sim \partial B + B^2$$

Schematically…

$$\frac{\partial}{\partial t} B \qquad = \mathscr{D}\, G$$

$$\mathscr{D} \;= \partial \;-\quad B$$

$$G \sim [\mathscr{D}, \mathscr{D}] \sim \partial B + B^2$$

flow equation: $\dot{B} \sim \partial^2 B + \partial B^2 + B^3$

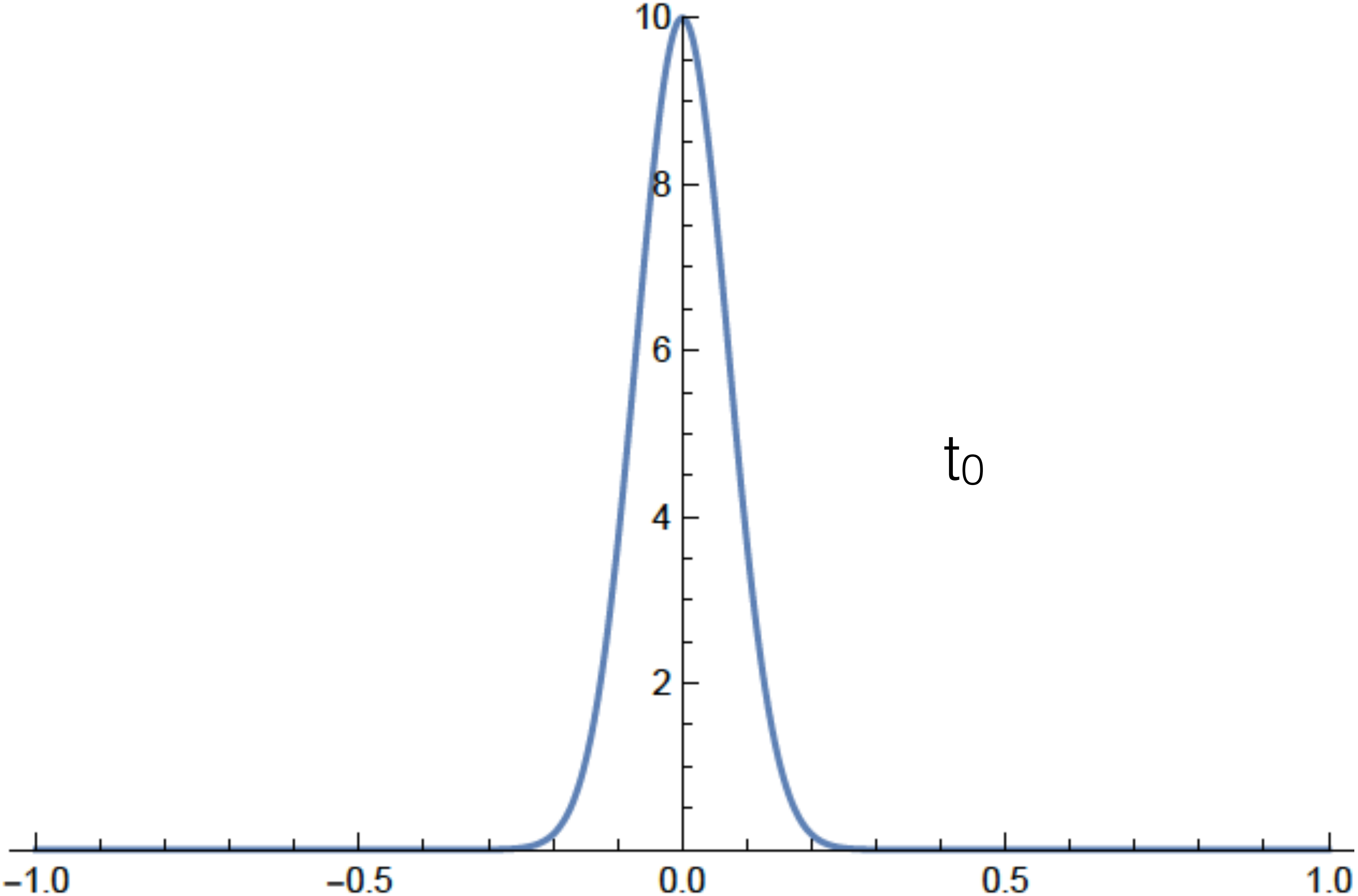
Perturbative solution

flow equation: $\dot{B} \sim \partial^2 B + \partial B^2 + B^3$

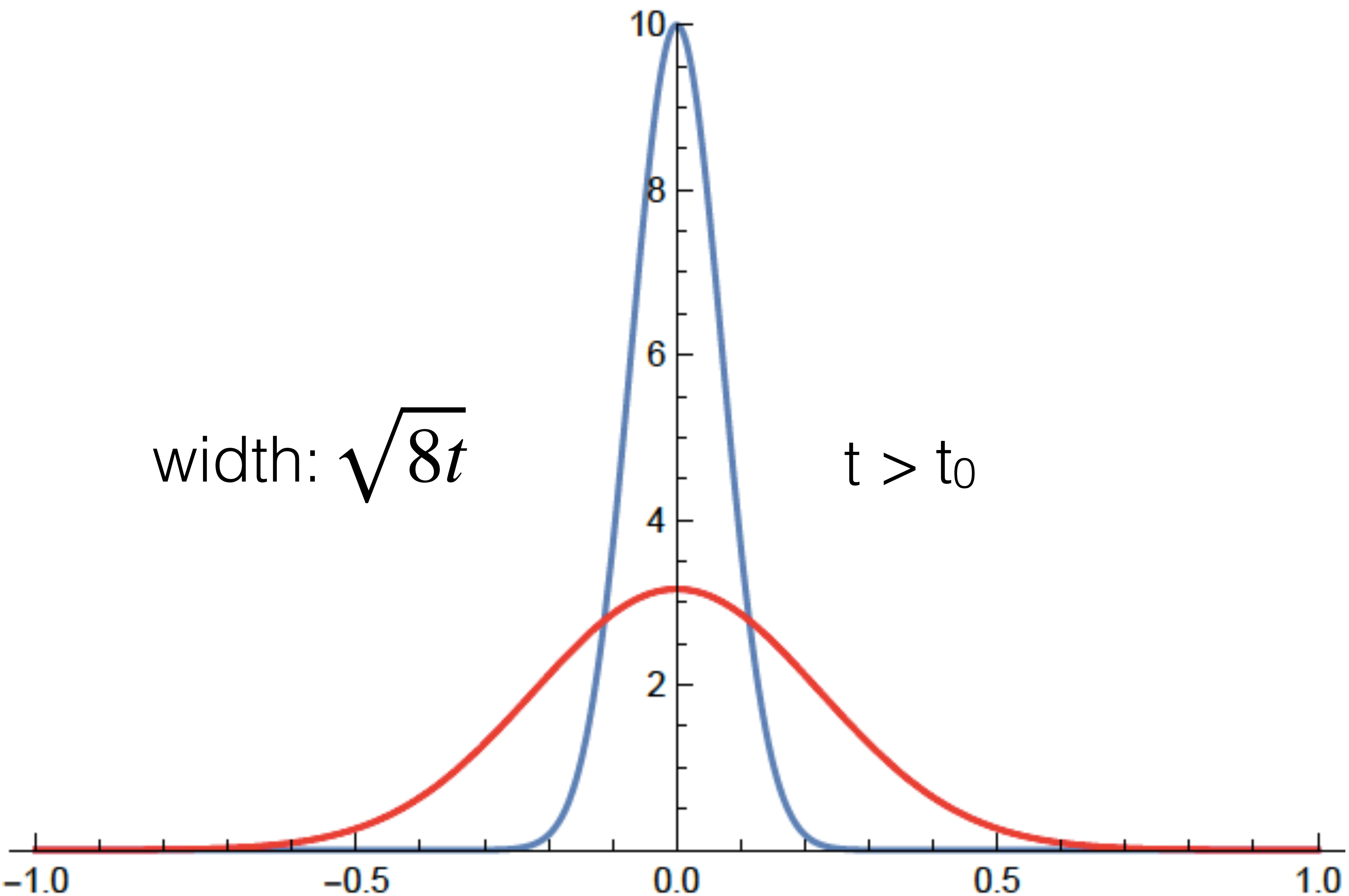
perturbative ansatz: $B = gB_1 + g^2 B_2 + \ldots$

Example: $B(t=0,x) = c\,\delta(x)$

$$\mathcal{O}(g): \qquad B(t,x) = \frac{c}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

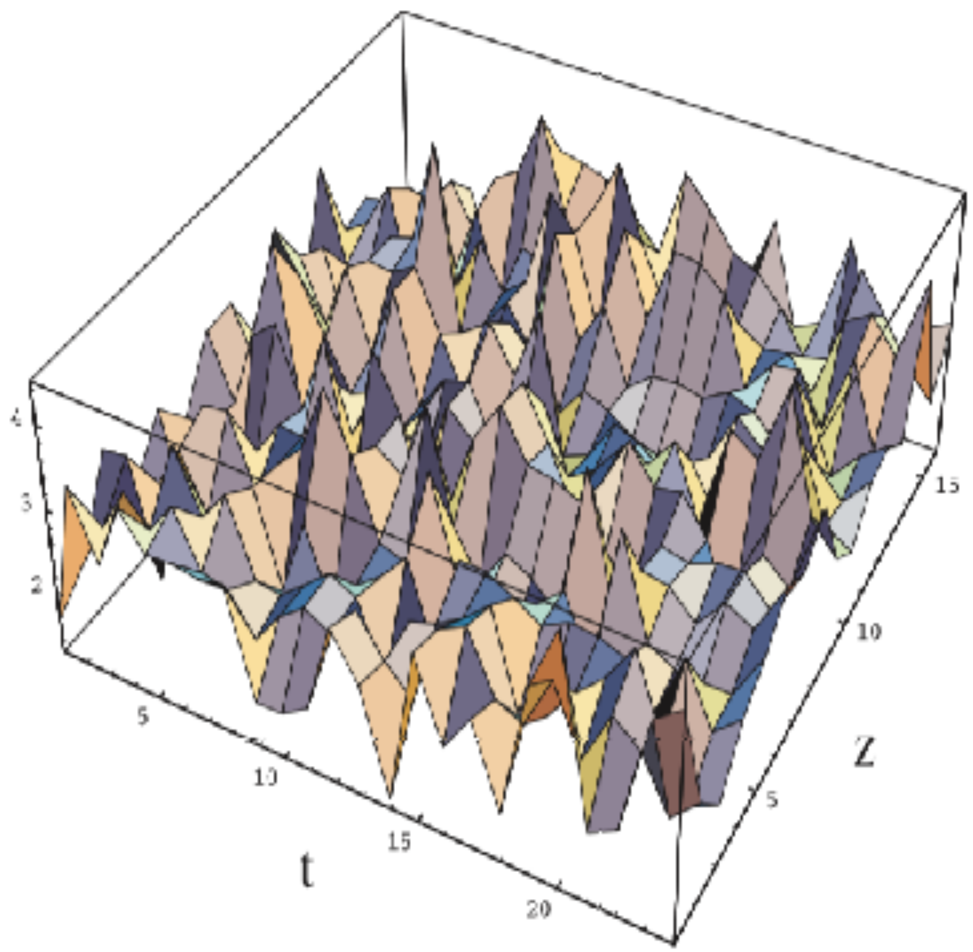


t_0


10
8
6
4
2
−1.0
−0.5
0.0
0.5
1.0
width: $\sqrt{8t}$
$t > t_0$

Lattice QCD

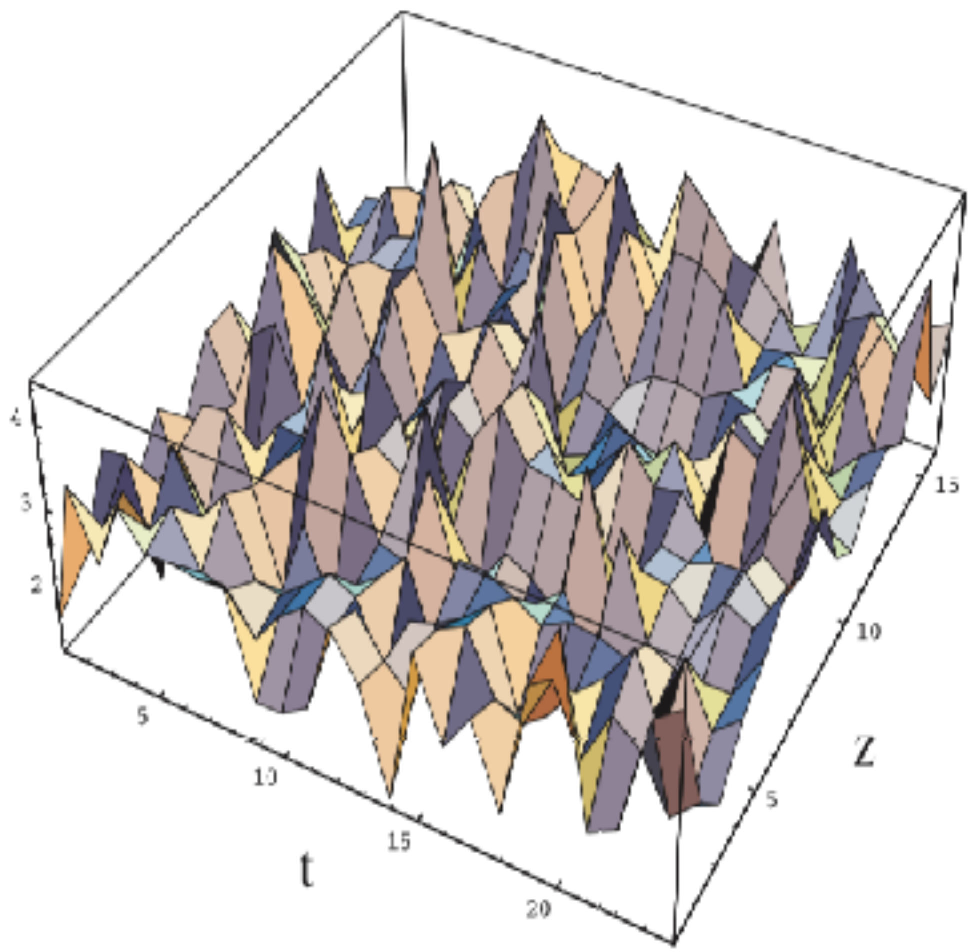
quantum fluctuations:



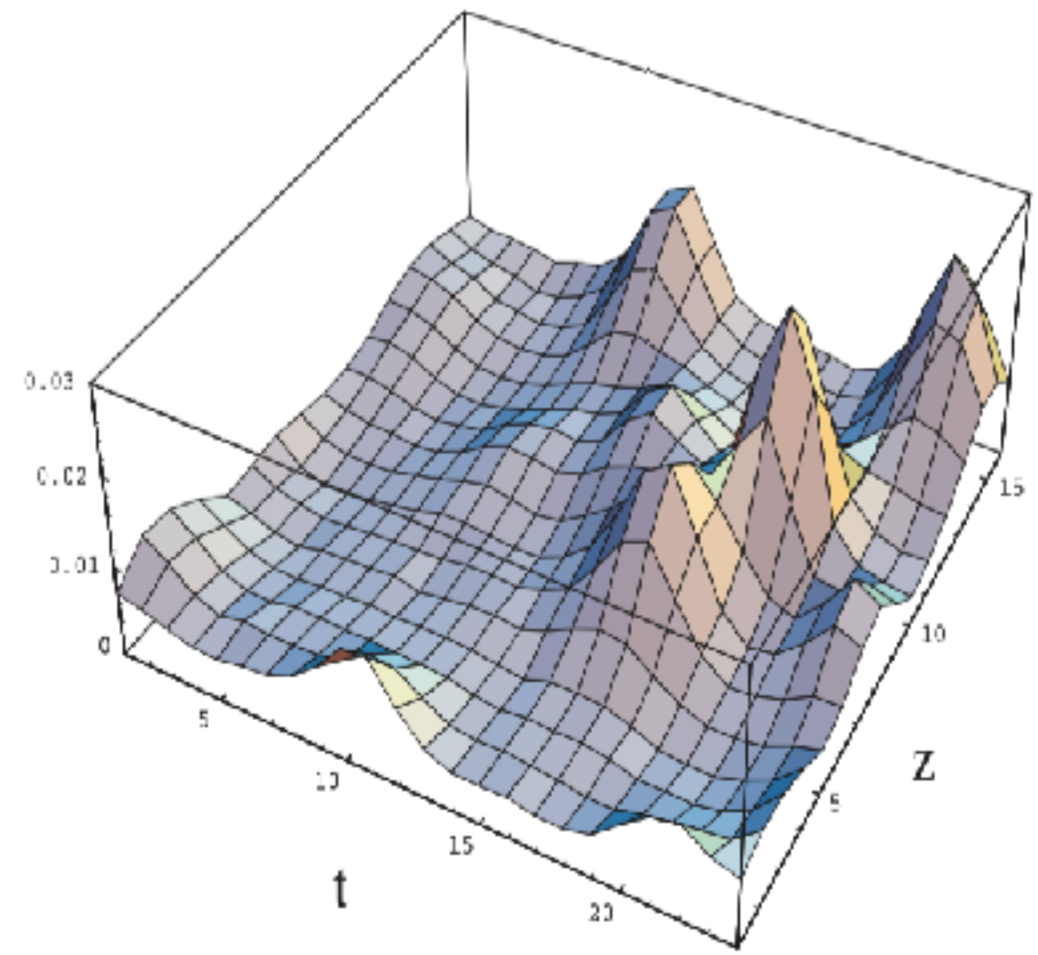
Engel 2009

Lattice QCD

quantum fluctuations:



"smearing":



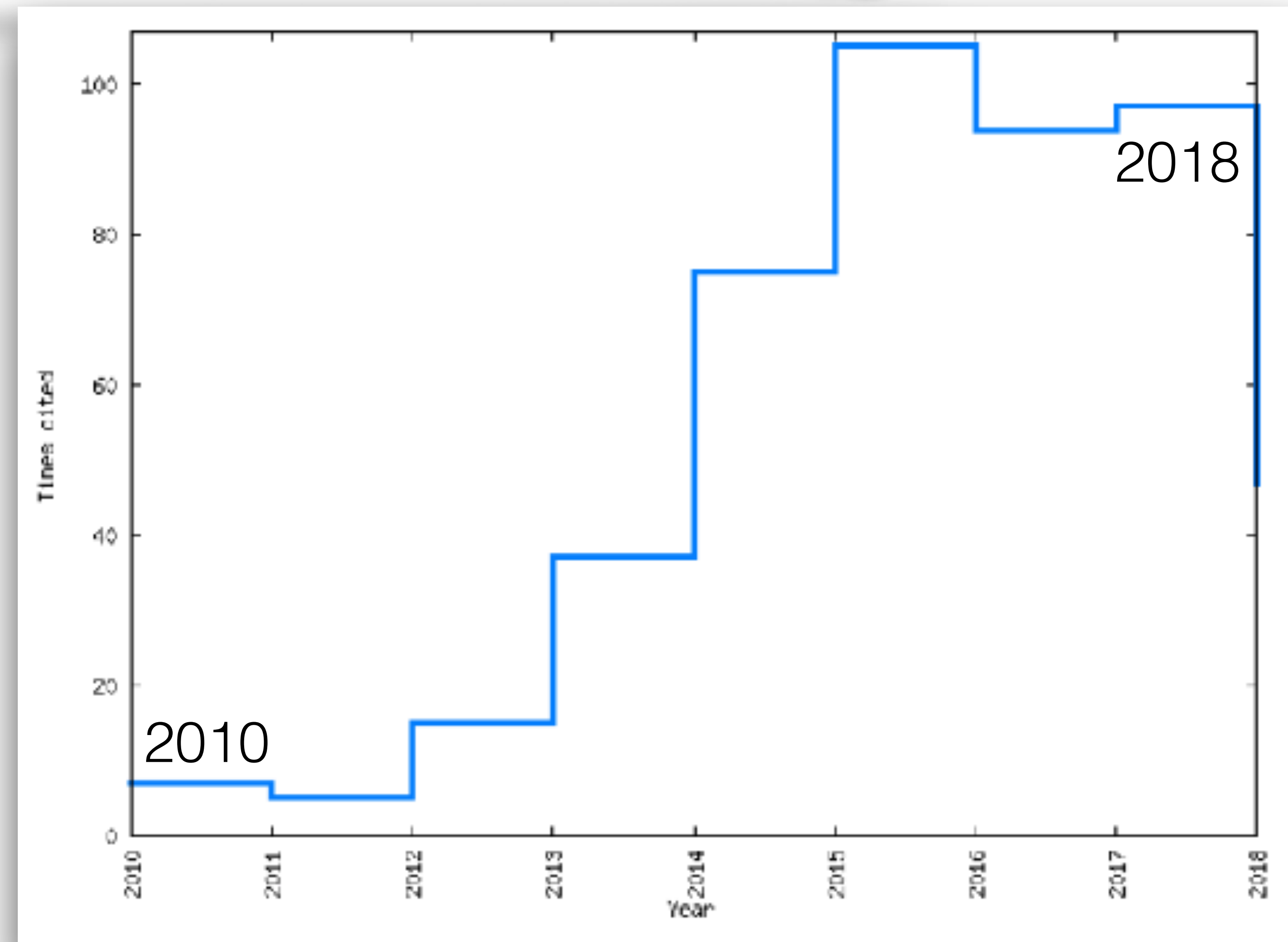
Engel 2009

Properties and uses of the Wilson flow in lattice QCD

Martin Lüscher (CERN & Geneva U.)

Jun 23, 2010 - 21 pages

JHEP 1008 (2010) 071
Erratum: JHEP 1403 (2014) 092
(2010)
DOI: 10.1007/JHEP08(2010)071, 10.1007/JHEP03(2014)092
CERN-PH-TH-2010-143
e-Print: arXiv:1006.4518 [hep-lat] | PDF

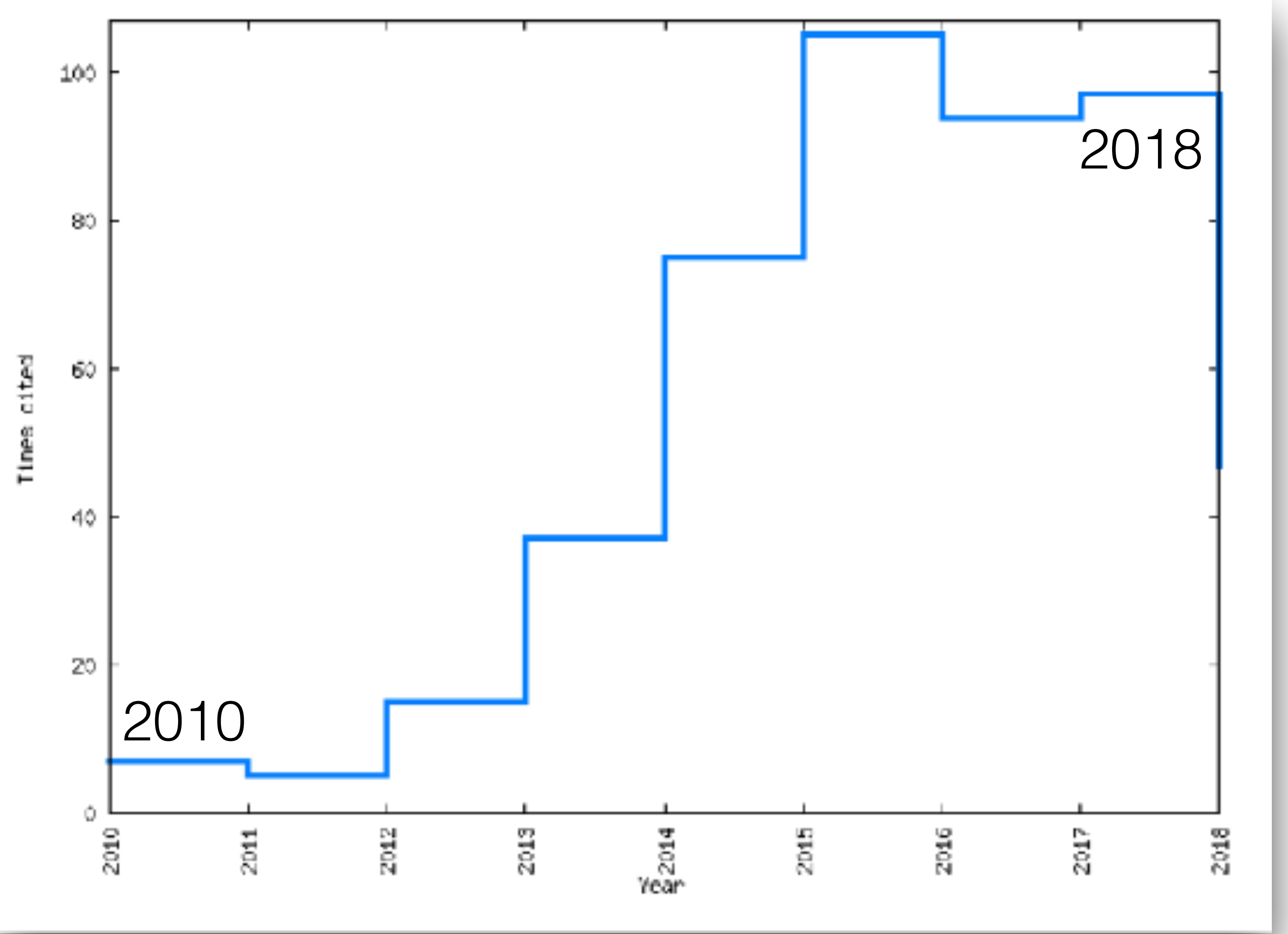


Properties and uses of the Wilson flow in lattice QCD

Martin Lüscher (CERN & Geneva U.)

Jun 23, 2010 - 21 pages

JHEP 1008 (2010) 071
Erratum: JHEP 1403 (2014) 092
(2010)
DOI: 10.1007/JHEP08(2010)071, 10.1007/JHEP03(2014)092
CERN-PH-TH-2010-143
e-Print: arXiv:1006.4518 [hep-lat] | PDF

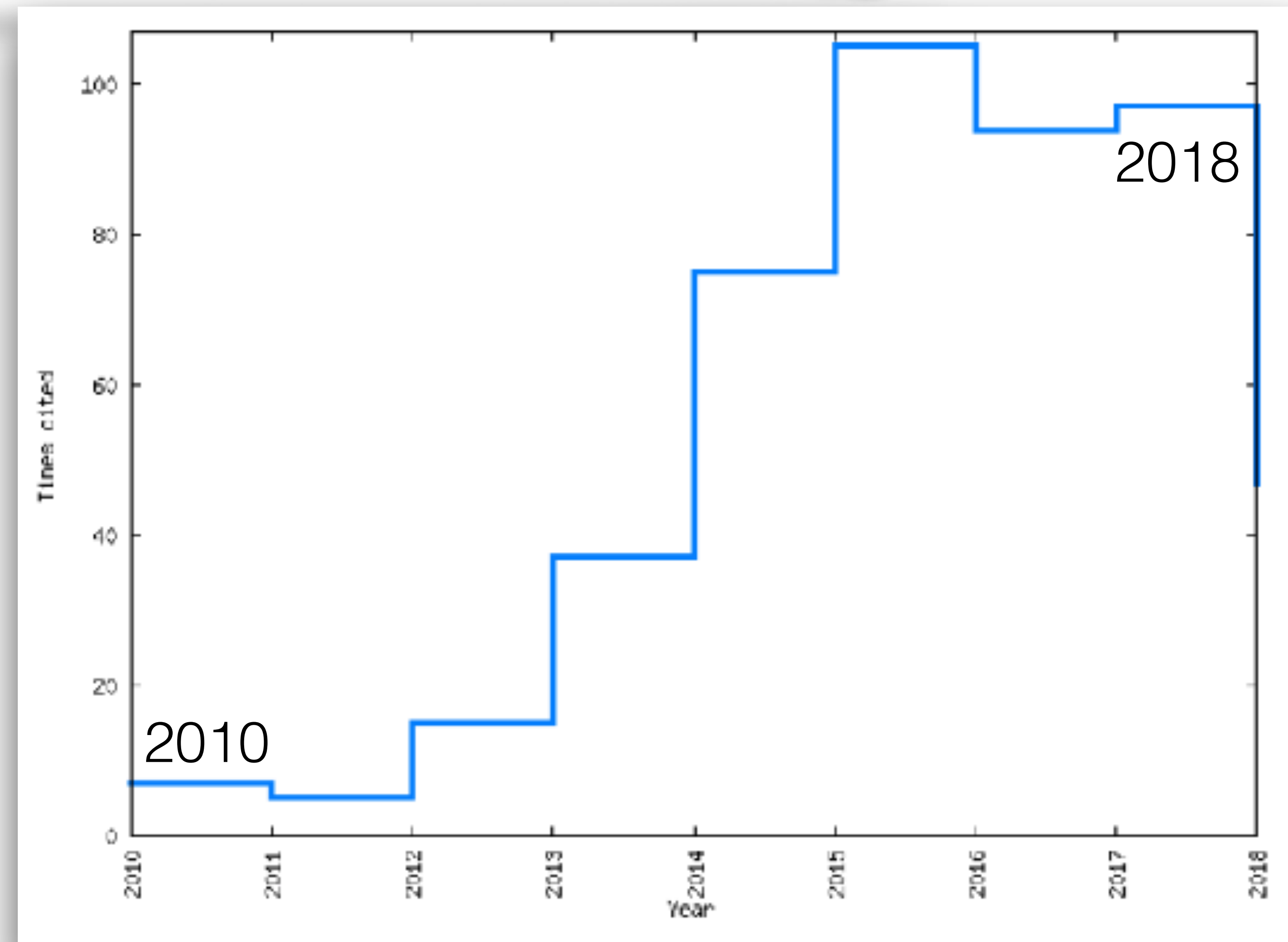


Properties and uses of the Wilson flow in lattice QCD

Martin Lüscher (CERN & Geneva U.)

Jun 23, 2010 - 21 pages

JHEP 1008 (2010) 071
Erratum: JHEP 1403 (2014) 092
(2010)
DOI: 10.1007/JHEP08(2010)071, 10.1007/JHEP03(2014)092
CERN-PH-TH-2010-143
e-Print: arXiv:1006.4518 [hep-lat] | PDF



see also
Narayanan, Neuberger '06

Higher orders

flow equation: $\dot{B} = \partial^2 B + \partial B^2 + B^3$

perturbative ansatz: $B = gB_1 + g^2 B_2 + \ldots$

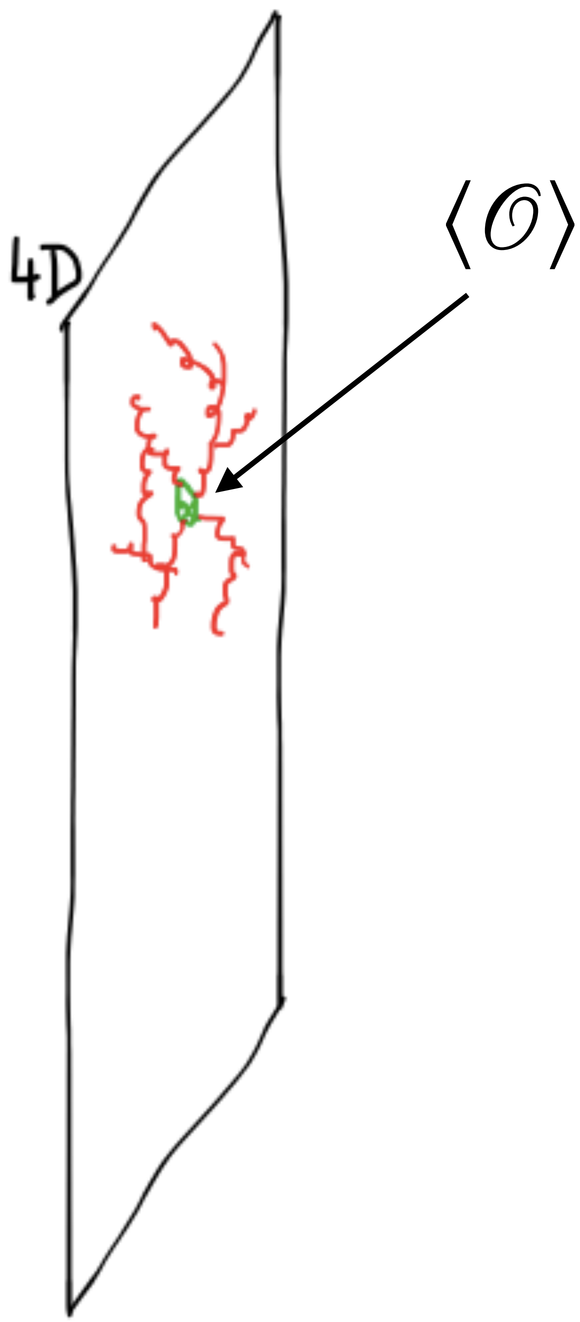
$$\Rightarrow \quad \tilde{B}_2(t,p) = \int_0^t ds \int d^4q\, K(t,s,p,q)\, A(p)\, A(p-q)$$

$$K(t,s,p,q) \sim \exp\left[-tp^2 - 2sq(q-p)\right]$$

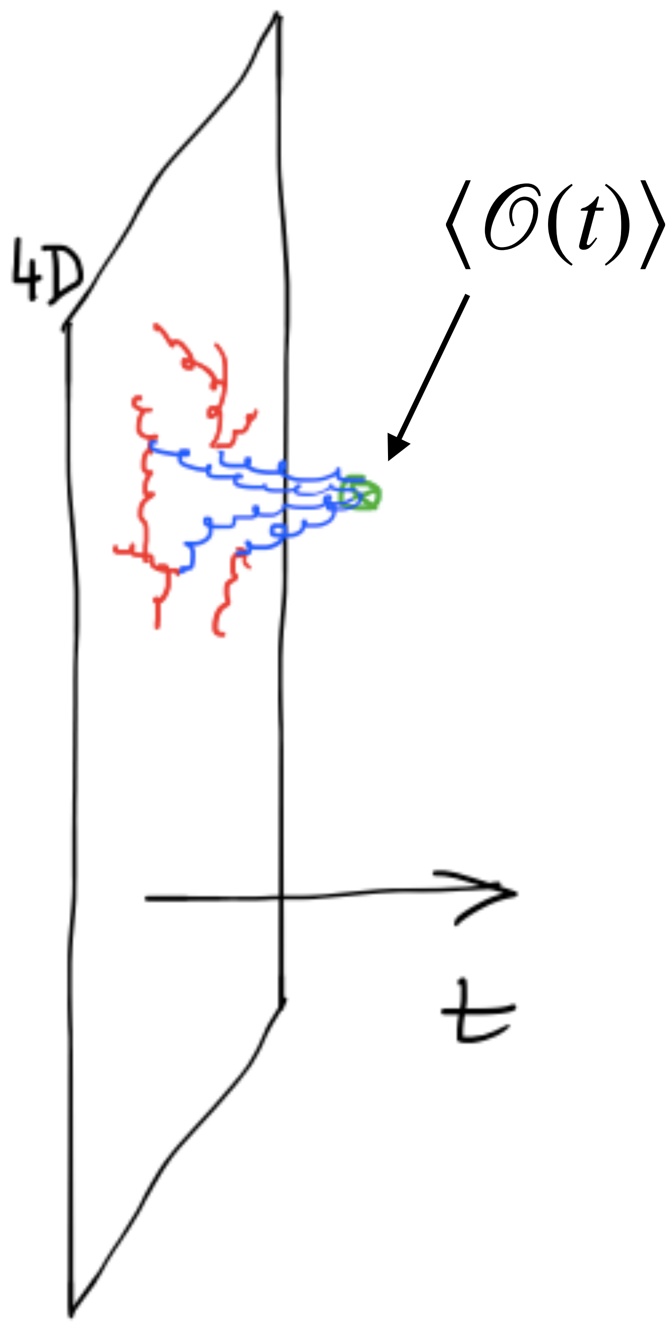
etc.

Exponential damping in momentum integrals!

$$\mathscr{L} = \mathscr{L}_{\text{QCD}}$$



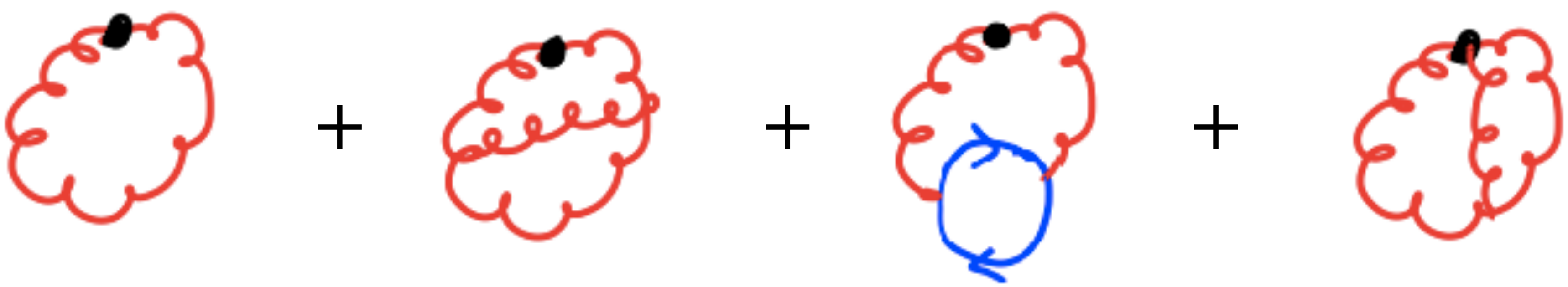
$$\mathscr{L} = \mathscr{L}_{\rm QCD} + \mathscr{L}_B$$



Let’s calculate

$$E(t,x) \equiv \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} \qquad \langle E \rangle \sim \langle \partial B\, \partial B \rangle + \langle B^2 \partial B \rangle + \langle B^4 \rangle$$

at t=0 (i.e. fundamental QCD):



+ ... = 0

in dim. reg. (m_q=0)

with gradient flow: $\tilde{B}_1(t,p) = e^{-tp^2} \tilde{A}(p)$

t has dimension mass^{-2} !

Let's calculate

$$E(t,x) \equiv \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu}$$

$$\langle E \rangle \sim \langle \partial B \, \partial B \rangle + \langle B^2 \partial B \rangle + \langle B^4 \rangle$$

$$\tilde{B}_1(t,p) = e^{-tp^2} \tilde{A}(p)$$

LO:  $\sim \int d^D p \, e^{-2tp^2} \sim t^{-D/2} \to t^{-2} \quad \neq 0$

Let's calculate

$$E(t,x) \equiv \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu}$$

$$\langle E \rangle \sim \langle \partial B \, \partial B \rangle + \langle B^2 \partial B \rangle + \langle B^4 \rangle$$

$$\tilde{B}_1(t,p) = e^{-tp^2} \tilde{A}(p)$$

LO:  $\sim \int d^D p \, e^{-2tp^2} \sim t^{-D/2} \to t^{-2} \quad \neq 0$

explicitly: $\langle E(t) \rangle = \dfrac{3\alpha_s}{4\pi t^2} \quad + \quad \mathcal{O}(\alpha_s^2)$

Let's calculate

$$E(t,x) \equiv \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu}$$

$$\langle E \rangle \sim \langle \partial B \, \partial B \rangle + \langle B^2 \partial B \rangle + \langle B^4 \rangle$$

$$\tilde{B}_1(t,p) = e^{-tp^2} \tilde{A}(p)$$

LO:  $\sim \int d^D p \, e^{-2tp^2} \sim t^{-D/2} \to t^{-2} \quad \neq 0$

explicitly: $\langle E(t) \rangle = \frac{3\alpha_s}{4\pi t^2} \quad + \quad \mathcal{O}(\alpha_s^2)$

→ measure α_s on the lattice?

Let’s calculate

$$E(t,x) \equiv \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu}$$

$$\langle E \rangle \sim \langle \partial B\, \partial B \rangle + \langle B^2 \partial B \rangle + \langle B^4 \rangle$$

$$\tilde{B}_1(t,p) = e^{-tp^2} \tilde{A}(p)$$

LO:  $\sim \int d^D p\, e^{-2tp^2} \sim t^{-D/2} \to t^{-2} \quad \neq 0$

explicitly: $$\langle E(t) \rangle = \frac{3\alpha_s}{4\pi t^2} \quad + \quad \mathcal{O}(\alpha_s^2)$$

→ measure α_s on the lattice?

actually: $$\langle E(t) \rangle = \frac{3\alpha_s(\mu)}{4\pi t^2}$$

Let's calculate

$$E(t,x) \equiv \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu}$$

$$\langle E \rangle \sim \langle \partial B \, \partial B \rangle + \langle B^2 \partial B \rangle + \langle B^4 \rangle$$

$$\tilde{B}_1(t,p) = e^{-tp^2} \tilde{A}(p)$$

LO:  $\sim \int d^D p \, e^{-2tp^2} \sim t^{-D/2} \to t^{-2} \quad \neq 0$

explicitly: $\langle E(t) \rangle = \dfrac{3\alpha_s}{4\pi t^2} \quad + \quad \mathcal{O}(\alpha_s^2)$

→ measure α_s on the lattice?

actually: $\langle E(t) \rangle = \dfrac{3\alpha_s(\mu)}{4\pi t^2} \qquad \mu = \dfrac{1}{\sqrt{8t}}$?

Higher orders

$$E(t,x) \equiv \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} \qquad \langle E \rangle \sim \langle \partial B\, \partial B \rangle + \langle B^2 \partial B \rangle + \langle B^4 \rangle$$

$$\tilde{B}_1(t,p) = e^{-tp^2} \tilde{A}(p)$$

$$\tilde{B}_2(t,p) = \int_0^t ds \int d^4q\, K(t,s,p,q)\, A(p)\, A(p-q)$$

$+$

$+ \dots$

Higher orders

$$E(t,x) \equiv \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} \qquad \langle E \rangle \sim \langle \partial B\, \partial B \rangle + \langle B^2 \partial B \rangle + \langle B^4 \rangle$$

$$\tilde{B}_1(t,p) = e^{-tp^2} \tilde{A}(p)$$

$$\tilde{B}_2(t,p) = \int_0^t ds \int d^4q\, K(t,s,p,q)\, A(p)\, A(p-q)$$

 +  + …

- one more momentum integration

Higher orders

$$E(t,x) \equiv \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} \qquad \langle E \rangle \sim \langle \partial B\, \partial B \rangle + \langle B^2 \partial B \rangle + \langle B^4 \rangle$$

$$\tilde{B}_1(t,p) = e^{-tp^2} \tilde{A}(p)$$

$$\tilde{B}_2(t,p) = \int_0^t ds \int d^4q\, K(t,s,p,q)\, A(p)\, A(p-q)$$

- one more momentum integration
- additional integration over flow-time parameter *s*

Higher orders

$$E(t,x) \equiv \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} \qquad \langle E \rangle \sim \langle \partial B\, \partial B \rangle + \langle B^2 \partial B \rangle + \langle B^4 \rangle$$

$$\tilde{B}_1(t,p) = e^{-tp^2} \tilde{A}(p)$$

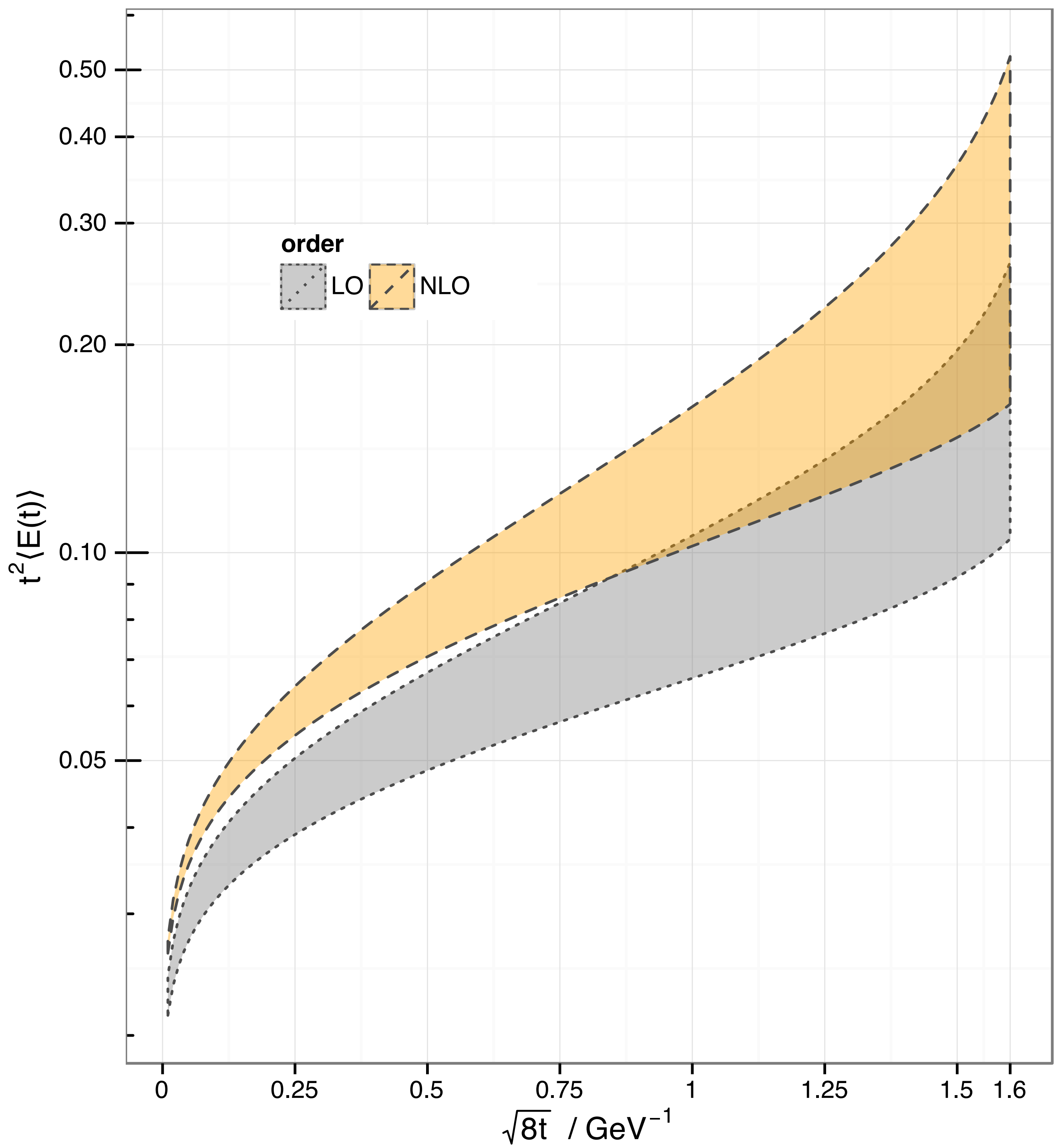
$$\tilde{B}_2(t,p) = \int_0^t ds \int d^4q\, K(t,s,p,q)\, A(p)\, A(p-q)$$

 +  + …

- one more momentum integration
- additional integration over flow-time parameter *s*
- renormalization: same as fundamental QCD!

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu)\right]$$

Lüscher '10



$$k_1 = \left(\frac{52}{9} + \frac{22}{3}\ln 2 - 3\ln 3 - \frac{11}{3}L_{t\mu}\right)C_A - \frac{8}{9}n_f T_R$$

$$L_{t\mu} = \ln 2\mu^2 t + \gamma_E$$

$$\mu = \frac{1}{\sqrt{8t}}\,?$$

resulting perturbative accuracy on α_s: ± 3-5%

PDG: ± 1%

5d QFT Lüscher, Weisz 2011

$$\mathscr{L} = \mathscr{L}_{\mathrm{QCD}} + \mathscr{L}_B$$

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu}\right)$$

L_μ Lagrange multiplier field

5d QFT Lüscher, Weisz 2011

$$\mathscr{L} = \mathscr{L}_{\mathrm{QCD}} + \mathscr{L}_B$$

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu}\right)$$

L_μ Lagrange multiplier field

analogously for quarks: Lüscher 2013

$$\mathscr{L}_\chi \sim \int_0^\infty dt\, \bar{\lambda} \left(\partial_t - \Delta\right) \chi + \mathrm{h.c.}$$

5d QFT Lüscher, Weisz 2011

$$\mathscr{L} = \mathscr{L}_{\text{QCD}} + \mathscr{L}_B + \mathscr{L}_\chi$$

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu}\right)$$

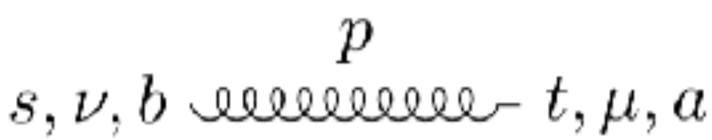
L_μ Lagrange multiplier field

analogously for quarks: Lüscher 2013

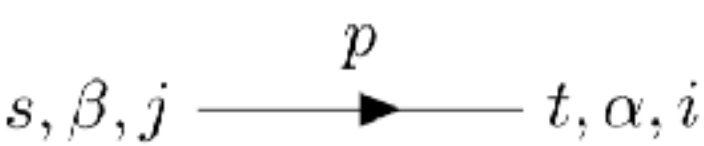
$$\mathscr{L}_\chi \sim \int_0^\infty dt\, \bar{\lambda} \left(\partial_t - \Delta\right) \chi + \text{h.c.}$$

Feynman rules

Flowed propagators:

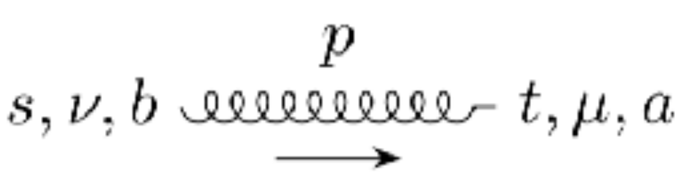


$$\delta^{ab}\delta_{\mu\nu}\frac{1}{p^2}\,e^{-(t+s)p^2}$$

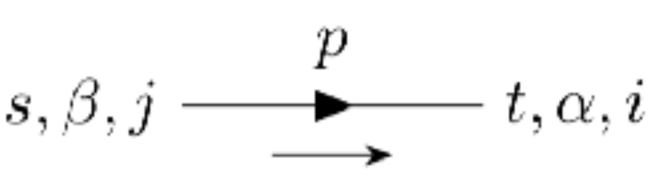


$$\delta_{ij}\frac{(-i\not{p}+m)_{\alpha\beta}}{p^2+m^2}e^{-(t+s)p^2}$$

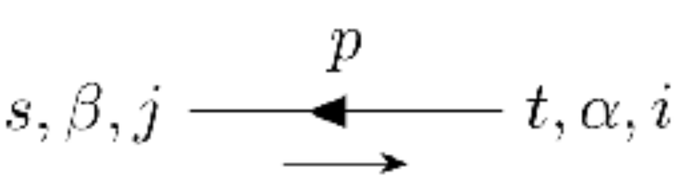
Flow lines:



$$\delta_{ab}\delta_{\mu\nu}\,\theta(t-s)\,e^{-(t-s)p^2}$$



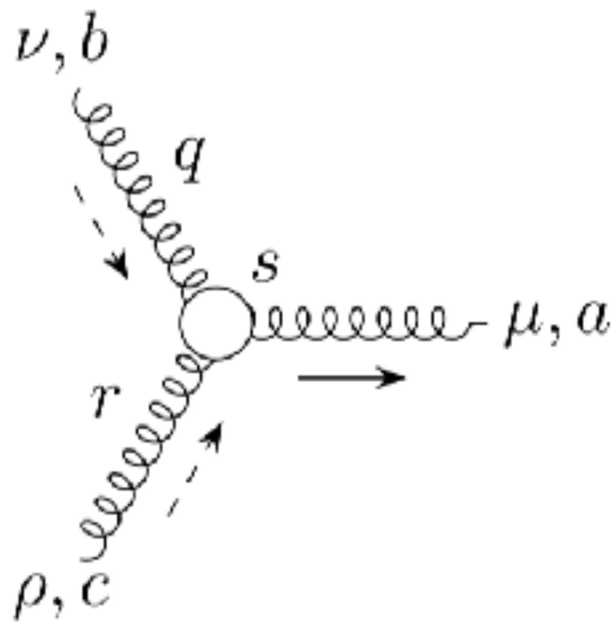
$$\delta_{ij}\delta_{\alpha\beta}\,\theta(t-s)\,e^{-(t-s)p^2}$$



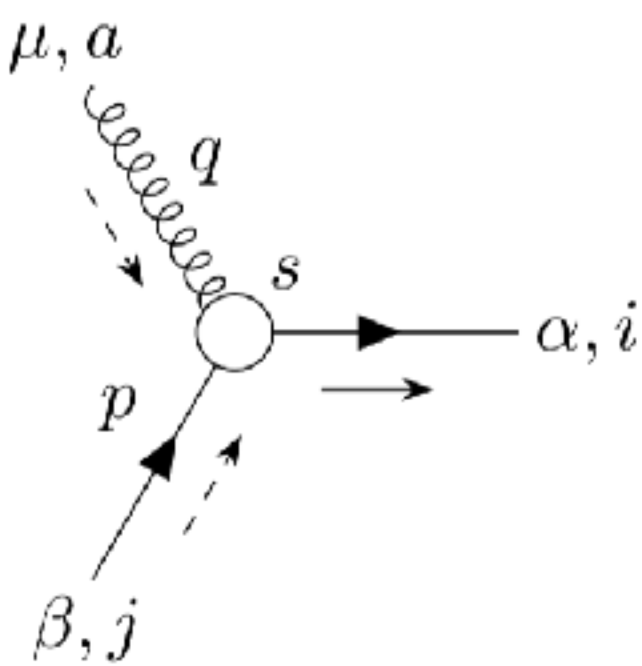
$$\delta_{ij}\delta_{\alpha\beta}\,\theta(t-s)\,e^{-(t-s)p^2}$$

Feynman rules

Flow vertices:



$$-igf^{abc}\int_0^\infty ds\left(\delta_{\nu\rho}(r-q)_\mu + 2\delta_{\mu\nu}q_\rho - 2\delta_{\mu\rho}r_\nu\right)$$



$$2ig\delta_{\alpha\beta}(T^a)_{ij}\int_0^\infty ds\, p_\mu$$

etc.

Three-loop calculation

The usual problems:

- many diagrams (NLO: 20; NNLO: 3651)
- many integrals
- complicated integrals

Three-loop calculation

The usual problems:

- many diagrams (NLO: 20; NNLO: 3651)
- many integrals
- complicated integrals

$$I(1,1,1,1,1) = \int_k \int_l \frac{1}{l^2\, k^2\, (k-q)^2\, (l-q)^2\, (l-k)^2}$$

Three-loop calculation

The usual problems:

- many diagrams (NLO: 20; NNLO: 3651)
- many integrals
- complicated integrals

$$I(1,1,1,1,1) = \int_0^t du \int_0^u dv \int_k \int_l \frac{e^{-tl^2-u(l-k)^2-v(l-q)^2}}{l^2\, k^2\, (k-q)^2\, (l-q)^2\, (l-k)^2}$$

Institute for Theoretical Particle Physics and Cosmology | RWTH AACHEN UNIVERSITY

Three-loop calculation

The usual problems:

- many diagrams (NLO: 20; NNLO: 3651)
- many integrals
- complicated integrals

$$I(1,1,1,1,1) = \int_0^t du \int_0^u dv \int_k \int_l \frac{e^{-tl^2-u(l-k)^2-v(l-q)^2}}{l^2\, k^2\, (k-q)^2\, (l-q)^2\, (l-k)^2}$$

The usual solutions:

- use automatic diagram generation `qgraf` Nogueira 1993
- reduce to master integrals
- evaluate master integrals

Reduction to masters

general form: $$I_n = \sum_{i=1}^{100} \frac{b_{n,i}(D)}{c_{n,i}(D)} I_i^{\text{master}}$$

Reduction to masters

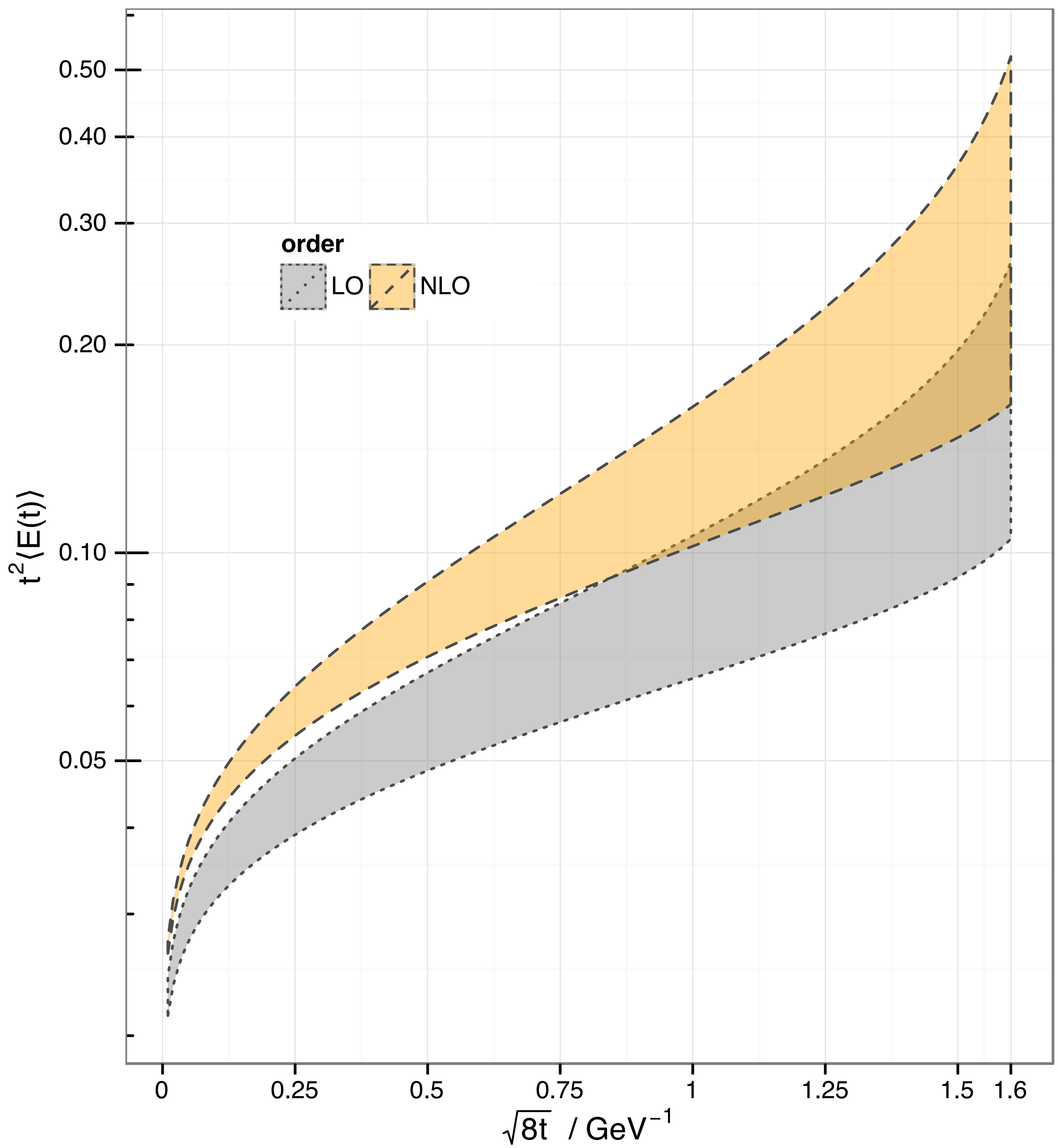
general form: $$I_n = \sum_{i=1}^{100} \frac{b_{n,i}(D)}{c_{n,i}(D)} I_i^{\text{master}}$$

Reconstruct rationals from Finite Fields:

FireFly Klappert, Lange 2019

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu)\right]$$

Lüscher '10



$$k_1 = \left(\frac{52}{9} + \frac{22}{3}\ln 2 - 3\ln 3 - \frac{11}{3}L_{t\mu}\right)C_A - \frac{8}{9}n_f T_R$$

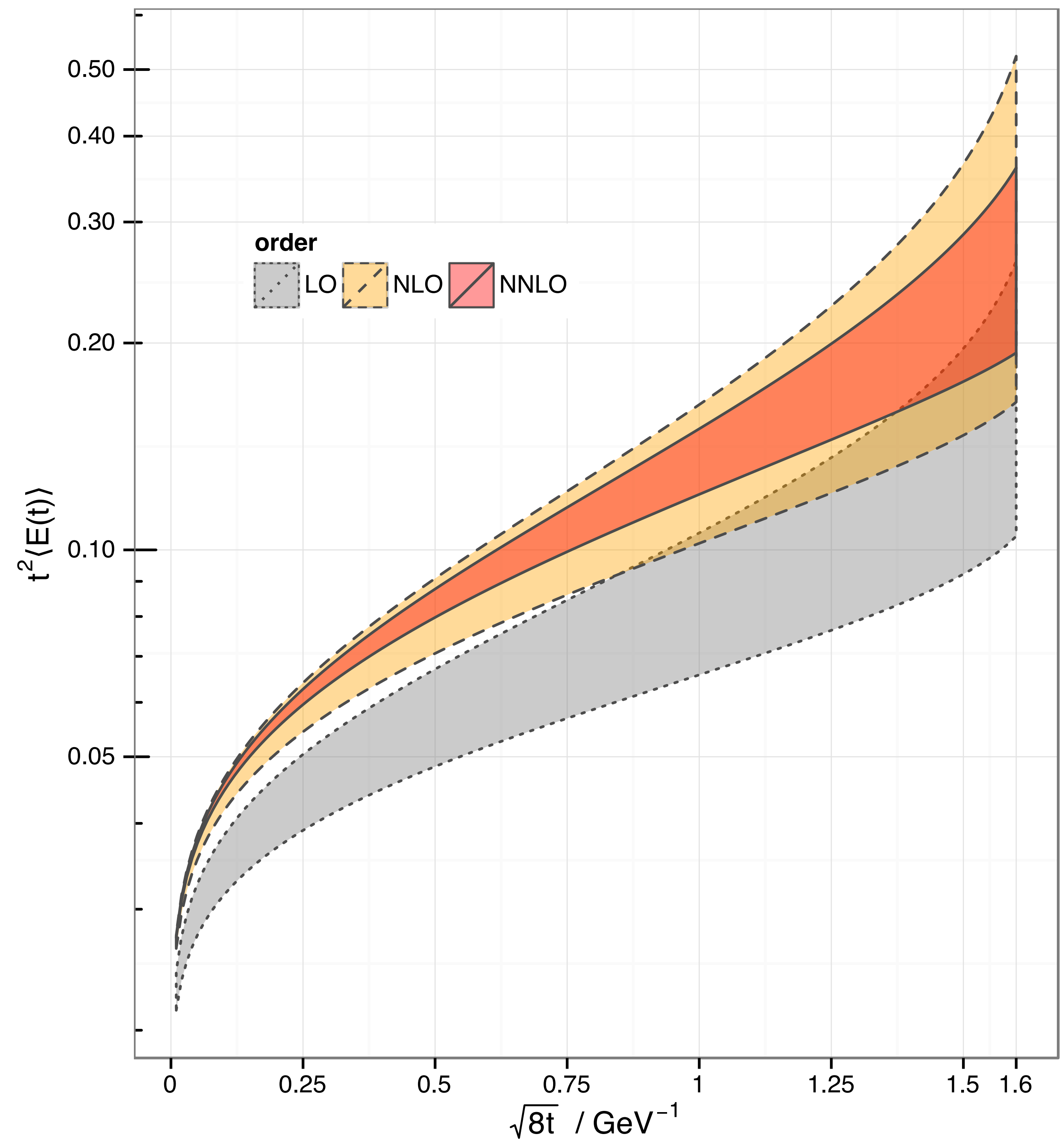
$$L_{t\mu} = \ln 2\mu^2 t + \gamma_E$$

resulting perturbative accuracy on α_S: ± 3-5%

PDG: ± 1%

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right]$$

RH, Neumann '16



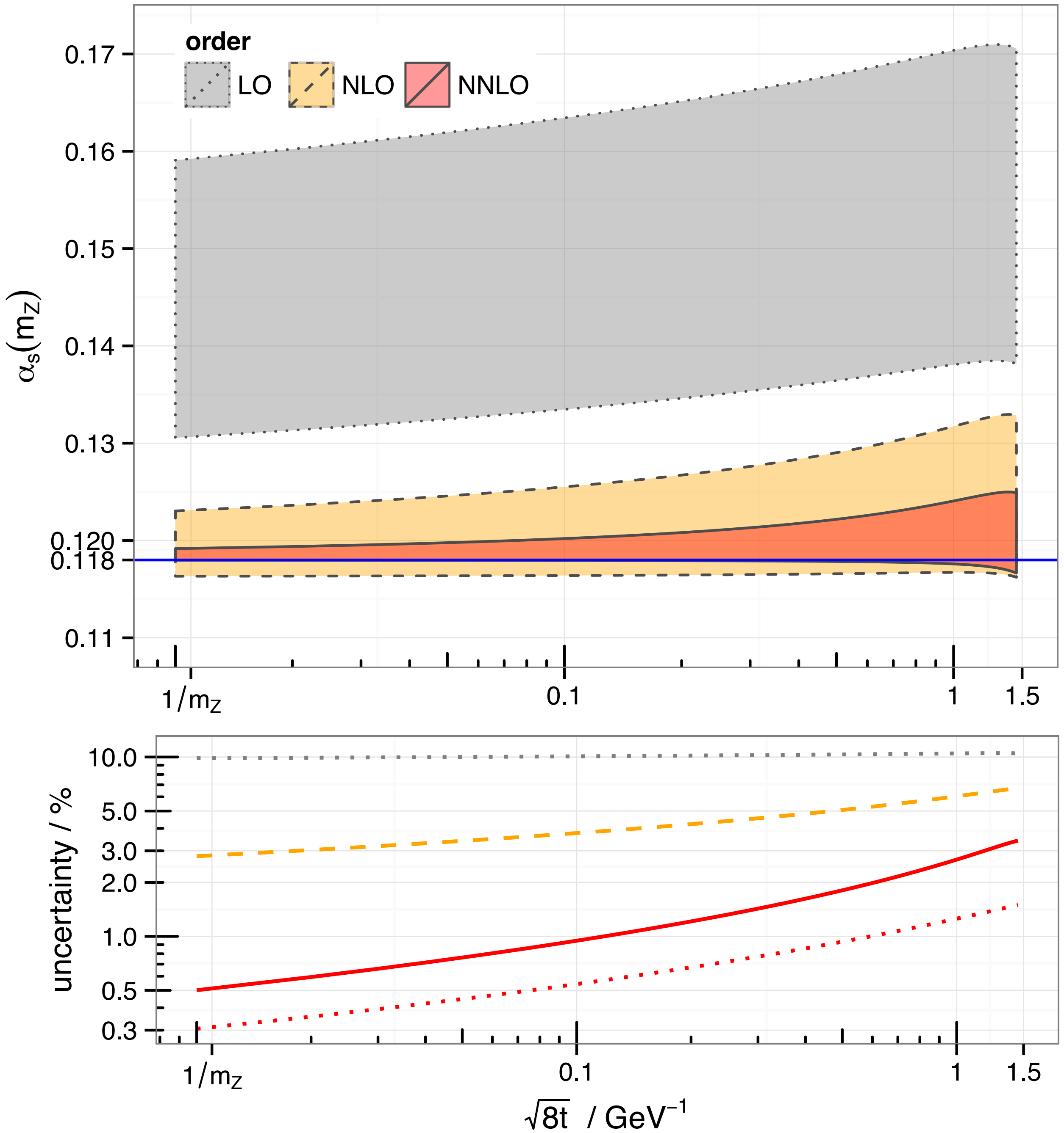
resulting perturbative accuracy on α_s: O(1%)

PDG: ± 1%

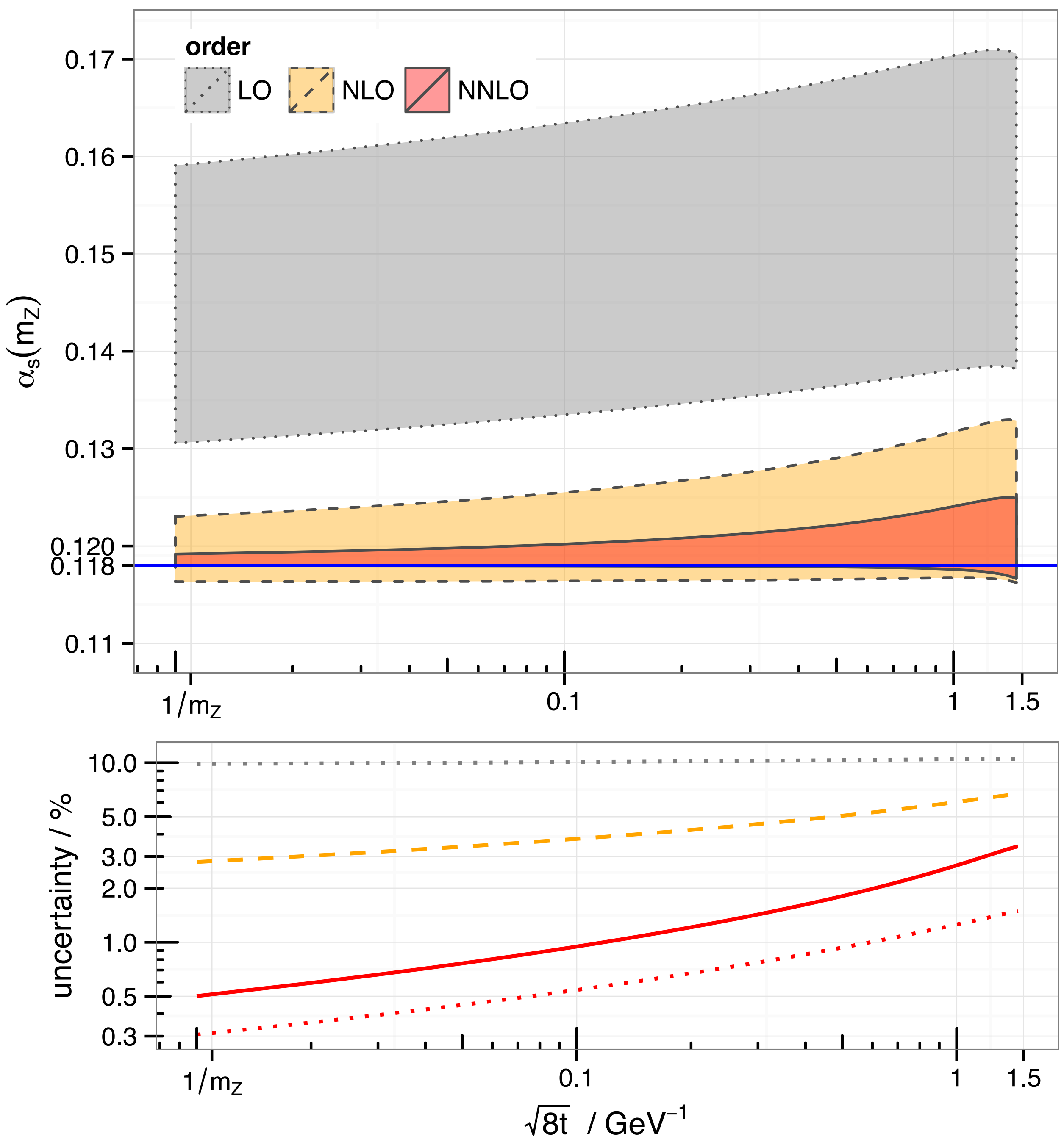
Derive $\alpha_s(m_Z)$

	$t^2\langle E(t)\rangle \cdot 10^4$							
q_8	2 GeV		10 GeV			m_Z		
$\alpha_s(m_Z)$	$n_f = 3$	$n_f = 4$	$n_f = 3$	$n_f = 4$	$n_f = 5$	$n_f = 3$	$n_f = 4$	$n_f = 5$
0.113	744	755	424	446	456	267	285	299
0.1135	753	764	426	449	459	268	286	301
0.114	762	773	429	452	462	269	287	302
0.1145	771	782	432	455	466	270	289	303
0.115	780	792	435	458	469	272	290	305
0.1155	789	802	438	461	472	273	291	306
0.116	798	811	440	465	476	274	292	308
0.1165	808	821	443	468	479	275	294	309
0.117	818	832	446	471	483	276	295	311
0.1175	827	842	449	474	486	277	296	312
0.118	837	852	452	478	490	278	298	314
0.1185	847	863	455	481	493	279	299	315
0.119	858	874	457	484	497	280	300	316
0.1195	868	885	460	488	500	281	301	318
0.12	879	896	463	491	504	282	303	319

Institute for Theoretical Particle Physics and Cosmology
RWTH AACHEN UNIVERSITY



RH, Neumann ‘16



RH, Neumann '16

However: Dalla Brida, Ramos '19

Composite Operators

$$E(t,x) \equiv \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu}$$

Composite Operators

$$E(t,x) \equiv \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu}$$

explicitly: $$\langle E(t) \rangle = \frac{3\alpha_s}{4\pi t^2} \quad + \quad \mathcal{O}(\alpha_s^2)$$

Composite Operators

$$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

e.g.: C_n in $D = 4 - 2\epsilon$ $\langle \mathcal{O}_n \rangle$ on lattice

Composite Operators

$$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

e.g.: C_n in $D = 4 - 2\epsilon$ $\langle \mathcal{O}_n \rangle$ on lattice

flowed operators: $\mathcal{O}_n \to \tilde{\mathcal{O}}_n(t)$ finite!

$$\tilde{\mathcal{O}}_n(t) = \sum_m \zeta_{nm}(t)\mathcal{O}_m + \ldots$$

Composite Operators

$$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

e.g.: C_n in $D = 4 - 2\epsilon$ $\langle \mathcal{O}_n \rangle$ on lattice

flowed operators: $\mathcal{O}_n \to \tilde{\mathcal{O}}_n(t)$ finite!

$$\tilde{\mathcal{O}}_n(t) = \sum_m \zeta_{nm}(t)\mathcal{O}_m + \ldots$$

$$\sum_n C_n \mathcal{O}_n = \sum_n \tilde{C}_n(t)\, \tilde{\mathcal{O}}_n(t)$$

Finite, and calculable in perturbation theory!

Energy-momentum tensor

in QCD:

$$T_{\mu\nu}(x) = \frac{1}{g_0^2}\left[\mathcal{O}_{1,\mu\nu}(x) - \frac{1}{4}\mathcal{O}_{2,\mu\nu}(x)\right] + \frac{1}{4}\mathcal{O}_{3,\mu\nu}(x)$$

$T_{\mu\nu}(x)$ is finite!

$$\mathcal{O}_{1,\mu\nu} = F^a_{\mu\rho}F^a_{\nu\rho}$$

$$\mathcal{O}_{2,\mu\nu} = \delta_{\mu\nu}F^a_{\rho\sigma}F^a_{\rho\sigma}$$

$$\mathcal{O}_{3,\mu\nu} = \bar{\psi}\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\psi$$

Energy-momentum tensor

in QCD:

$$T_{\mu\nu}(x) = \frac{1}{g_0^2}\left[\mathcal{O}_{1,\mu\nu}(x) - \frac{1}{4}\mathcal{O}_{2,\mu\nu}(x)\right] + \frac{1}{4}\mathcal{O}_{3,\mu\nu}(x)$$

$T_{\mu\nu}(x)$ is finite!

$$\mathcal{O}_{1,\mu\nu} = F^a_{\mu\rho}F^a_{\nu\rho}$$

$$\mathcal{O}_{2,\mu\nu} = \delta_{\mu\nu}F^a_{\rho\sigma}F^a_{\rho\sigma}$$

$$\mathcal{O}_{3,\mu\nu} = \bar{\psi}\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\psi$$

$$\mathcal{O}_{4,\mu\nu} = \delta_{\mu\nu}\bar{\psi}\overleftrightarrow{\not{D}}\psi$$

Energy-momentum tensor

in QCD:

$$T_{\mu\nu}(x) = \frac{1}{g_0^2}\left[\mathcal{O}_{1,\mu\nu}(x) - \frac{1}{4}\mathcal{O}_{2,\mu\nu}(x)\right] + \frac{1}{4}\mathcal{O}_{3,\mu\nu}(x)$$

$T_{\mu\nu}(x)$ is finite!

$$\mathcal{O}_{1,\mu\nu} = F^a_{\mu\rho}F^a_{\nu\rho}$$

$$\mathcal{O}_{2,\mu\nu} = \delta_{\mu\nu}F^a_{\rho\sigma}F^a_{\rho\sigma}$$

$$\mathcal{O}_{3,\mu\nu} = \bar{\psi}\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\psi$$

$$\mathcal{O}_{4,\mu\nu} = \delta_{\mu\nu}\bar{\psi}\overleftrightarrow{\not{D}}\psi$$

$$\mathcal{O}_n = \sum_m \zeta^{-1}_{nm}(t)\tilde{\mathcal{O}}_m(t) + \ldots$$

Energy-momentum tensor

in QCD:

$$T_{\mu\nu}(x) = \frac{1}{g_0^2}\left[\mathcal{O}_{1,\mu\nu}(x) - \frac{1}{4}\mathcal{O}_{2,\mu\nu}(x)\right] + \frac{1}{4}\mathcal{O}_{3,\mu\nu}(x)$$

$T_{\mu\nu}(x)$ is finite!

$$\mathcal{O}_{1,\mu\nu} = F^a_{\mu\rho}F^a_{\nu\rho}$$

$$\mathcal{O}_{2,\mu\nu} = \delta_{\mu\nu}F^a_{\rho\sigma}F^a_{\rho\sigma}$$

$$\mathcal{O}_{3,\mu\nu} = \bar{\psi}\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\psi$$

$$\mathcal{O}_{4,\mu\nu} = \delta_{\mu\nu}\bar{\psi}\overleftrightarrow{\not{D}}\psi$$

$$\mathcal{O}_n = \sum_m \zeta^{-1}_{nm}(t)\tilde{\mathcal{O}}_m(t) + \ldots$$

finite because of e^{-tp^2}

TTK Institute for Theoretical Particle Physics and Cosmology | RWTH AACHEN UNIVERSITY

Flowed operators

$$\mathcal{O}_n = \sum_m \zeta_{nm}^{-1}(t)\tilde{\mathcal{O}}_m(t) + \ldots$$

↑

finite because of e^{-tp^2}

Flowed operators

$$\mathcal{O}_n = \sum_m \zeta^{-1}_{nm}(t)\tilde{\mathcal{O}}_m(t) + \ldots \qquad \tilde{\mathcal{O}}_n(t) = \sum_m \zeta_{nm}(t)\mathcal{O}_m + \ldots$$

↑

finite because of e^{-tp^2}

Flowed operators

$$\mathcal{O}_n = \sum_m \zeta^{-1}_{nm}(t)\tilde{\mathcal{O}}_m(t) + \ldots \qquad \tilde{\mathcal{O}}_n(t) = \sum_m \zeta_{nm}(t)\mathcal{O}_m + \ldots$$

finite because of e^{-tp^2}

$$P_n[\mathcal{O}_k] = \delta_{nk}$$

Flowed operators

$$\mathcal{O}_n = \sum_m \zeta^{-1}_{nm}(t)\tilde{\mathcal{O}}_m(t) + \ldots \qquad \tilde{\mathcal{O}}_n(t) = \sum_m \zeta_{nm}(t)\mathcal{O}_m + \ldots$$

finite because of e^{-tp^2}

$$P_n[\mathcal{O}_k] = \delta_{nk} \qquad \zeta_{nm}(t) = P_m[\tilde{\mathcal{O}}_n(t)]$$

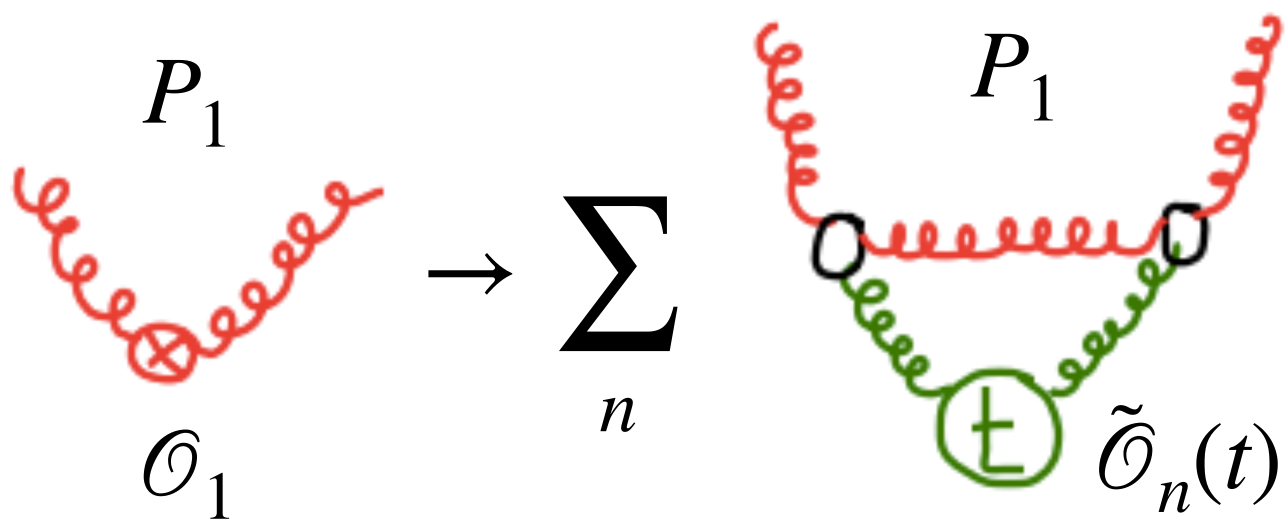
Flowed operators

$$\mathcal{O}_n = \sum_m \zeta^{-1}_{nm}(t)\tilde{\mathcal{O}}_m(t) + \ldots \qquad \tilde{\mathcal{O}}_n(t) = \sum_m \zeta_{nm}(t)\mathcal{O}_m + \ldots$$

↑ finite because of e^{-tp^2}

$$P_n[\mathcal{O}_k] = \delta_{nk} \qquad \zeta_{nm}(t) = P_m[\tilde{\mathcal{O}}_n(t)]$$

$$\mathcal{O}_{1,\mu\nu} = F^a_{\mu\rho}F^a_{\nu\rho}$$



Energy-momentum tensor

in QCD:

$$T_{\mu\nu}(x) = \frac{1}{g_0^2}\left[\mathcal{O}_{1,\mu\nu}(x) - \frac{1}{4}\mathcal{O}_{2,\mu\nu}(x)\right] + \frac{1}{4}\mathcal{O}_{3,\mu\nu}(x)$$

$$\mathcal{O}_{1,\mu\nu} = F^a_{\mu\rho}F^a_{\nu\rho}$$

$$\mathcal{O}_{2,\mu\nu} = \delta_{\mu\nu}F^a_{\rho\sigma}F^a_{\rho\sigma}$$

$$\mathcal{O}_{3,\mu\nu} = \bar{\psi}\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\psi$$

$$\mathcal{O}_{4,\mu\nu} = \delta_{\mu\nu}\bar{\psi}\overleftrightarrow{\not{D}}\psi$$

Energy-momentum tensor

in QCD:

$$T_{\mu\nu}(x) = \frac{1}{g_0^2}\left[\mathcal{O}_{1,\mu\nu}(x) - \frac{1}{4}\mathcal{O}_{2,\mu\nu}(x)\right] + \frac{1}{4}\mathcal{O}_{3,\mu\nu}(x)$$

$$\mathcal{O}_{1,\mu\nu} = F^a_{\mu\rho}F^a_{\nu\rho}$$

$$\mathcal{O}_{2,\mu\nu} = \delta_{\mu\nu}F^a_{\rho\sigma}F^a_{\rho\sigma}$$

$$\mathcal{O}_{3,\mu\nu} = \bar{\psi}\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\psi$$

$$\mathcal{O}_{4,\mu\nu} = \delta_{\mu\nu}\bar{\psi}\overleftrightarrow{\not{D}}\psi$$

$$\tilde{\mathcal{O}}_{1,\mu\nu}(t) = F^a_{\mu\rho}(t)F^a_{\nu\rho}(t)$$

$$\tilde{\mathcal{O}}_{2,\mu\nu}(t) = \delta_{\mu\nu}F^a_{\rho\sigma}(t)F^a_{\rho\sigma}(t)$$

$$\tilde{\mathcal{O}}_{3,\mu\nu}(t) = \bar{\psi}(t)\left(\gamma_\mu \overleftrightarrow{D}_\nu(t) + \gamma_\nu \overleftrightarrow{D}_\mu(t)\right)\psi(t)$$

$$\tilde{\mathcal{O}}_{4,\mu\nu}(t) = \delta_{\mu\nu}\bar{\psi}(t)\overleftrightarrow{\not{D}}(t)\,\psi(t)$$

Energy-momentum tensor

in QCD:

$$T_{\mu\nu}(x) = \frac{1}{g_0^2}\left[\mathcal{O}_{1,\mu\nu}(x) - \frac{1}{4}\mathcal{O}_{2,\mu\nu}(x)\right] + \frac{1}{4}\mathcal{O}_{3,\mu\nu}(x)$$

$$\mathcal{O}_{1,\mu\nu} = F^a_{\mu\rho}F^a_{\nu\rho}$$

$$\mathcal{O}_{2,\mu\nu} = \delta_{\mu\nu}F^a_{\rho\sigma}F^a_{\rho\sigma}$$

$$\mathcal{O}_{3,\mu\nu} = \bar{\psi}\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\psi$$

$$\mathcal{O}_{4,\mu\nu} = \delta_{\mu\nu}\bar{\psi}\overleftrightarrow{\not{D}}\psi$$

$$\tilde{\mathcal{O}}_{1,\mu\nu}(t) = F^a_{\mu\rho}(t)F^a_{\nu\rho}(t)$$

$$\tilde{\mathcal{O}}_{2,\mu\nu}(t) = \delta_{\mu\nu}F^a_{\rho\sigma}(t)F^a_{\rho\sigma}(t)$$

$$\tilde{\mathcal{O}}_{3,\mu\nu}(t) = \bar{\psi}(t)\left(\gamma_\mu \overleftrightarrow{D}_\nu(t) + \gamma_\nu \overleftrightarrow{D}_\mu(t)\right)\psi(t)$$

$$\tilde{\mathcal{O}}_{4,\mu\nu}(t) = \delta_{\mu\nu}\bar{\psi}(t)\overleftrightarrow{\not{D}}(t)\,\psi(t)$$

$$T_{\mu\nu}(x) = \sum_{n=1}^{4} c_n(t)\tilde{\mathcal{O}}_{n,\mu\nu}(t,x)$$

idea and NLO result: H. Suzuki '14

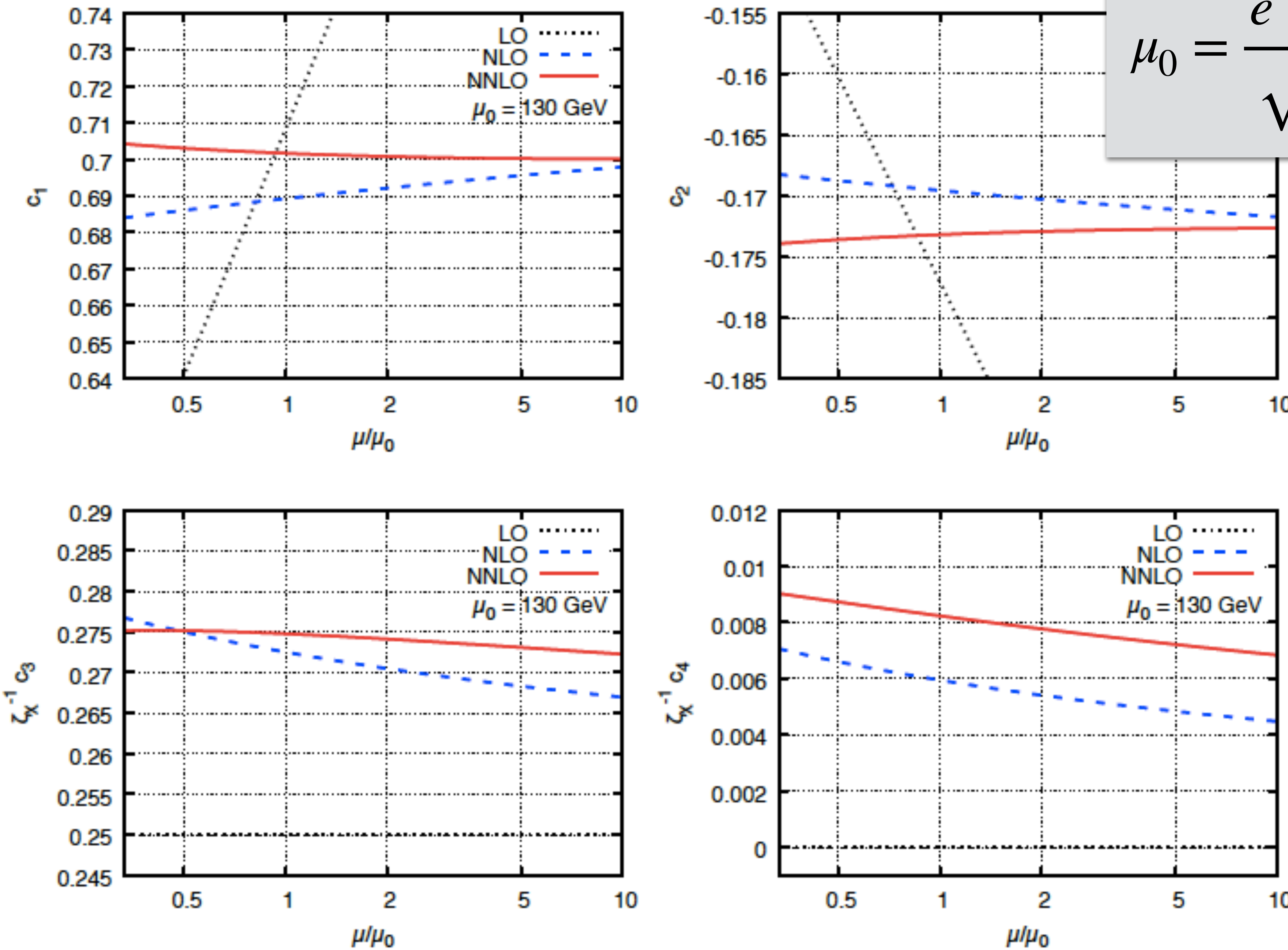
NNLO result

$$
\begin{aligned}
c_1(t) = \frac{1}{g^2}\Bigg\{ 1 &+ \frac{g^2}{(4\pi)^2}\left[-\frac{7}{3}C_A + \frac{3}{2}T_F - \beta_0\, L(\mu,t)\right] \\
&+ \frac{g^4}{(4\pi)^4}\Bigg[-\beta_1\, L(\mu,t) + C_A^2\left(-\frac{14482}{405} - \frac{16546}{135}\ln 2 + \frac{1187}{10}\ln 3\right) \\
&\quad + C_A T_F\left(\frac{59}{9}\mathrm{Li}_2\left(\frac{1}{4}\right) + \frac{10873}{810} + \frac{73}{54}\pi^2 - \frac{2773}{135}\ln 2 + \frac{302}{45}\ln 3\right) \\
&\quad + C_F T_F\left(-\frac{256}{9}\mathrm{Li}_2\left(\frac{1}{4}\right) + \frac{2587}{108} - \frac{7}{9}\pi^2 - \frac{106}{9}\ln 2 - \frac{161}{18}\ln 3\right)\Bigg] \\
&+ \mathcal{O}(g^6)\Bigg\}\,,
\end{aligned}
$$

$$
L(\mu,t) \equiv \ln\left(2\mu^2 t\right) + \gamma_{\mathrm{E}}
$$

RH, Kluth, Lange ‘18

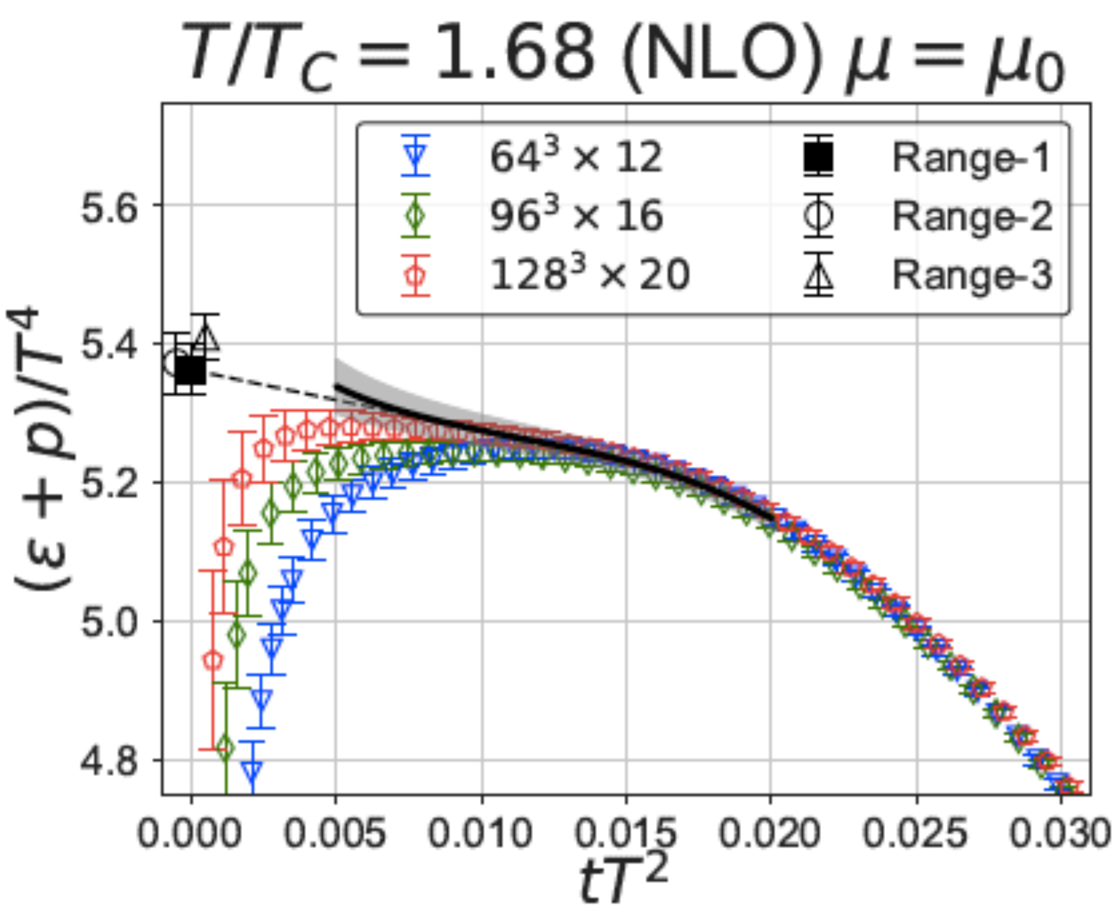
$$\mu_0 = \frac{e^{-\gamma_E/2}}{\sqrt{2t}}$$

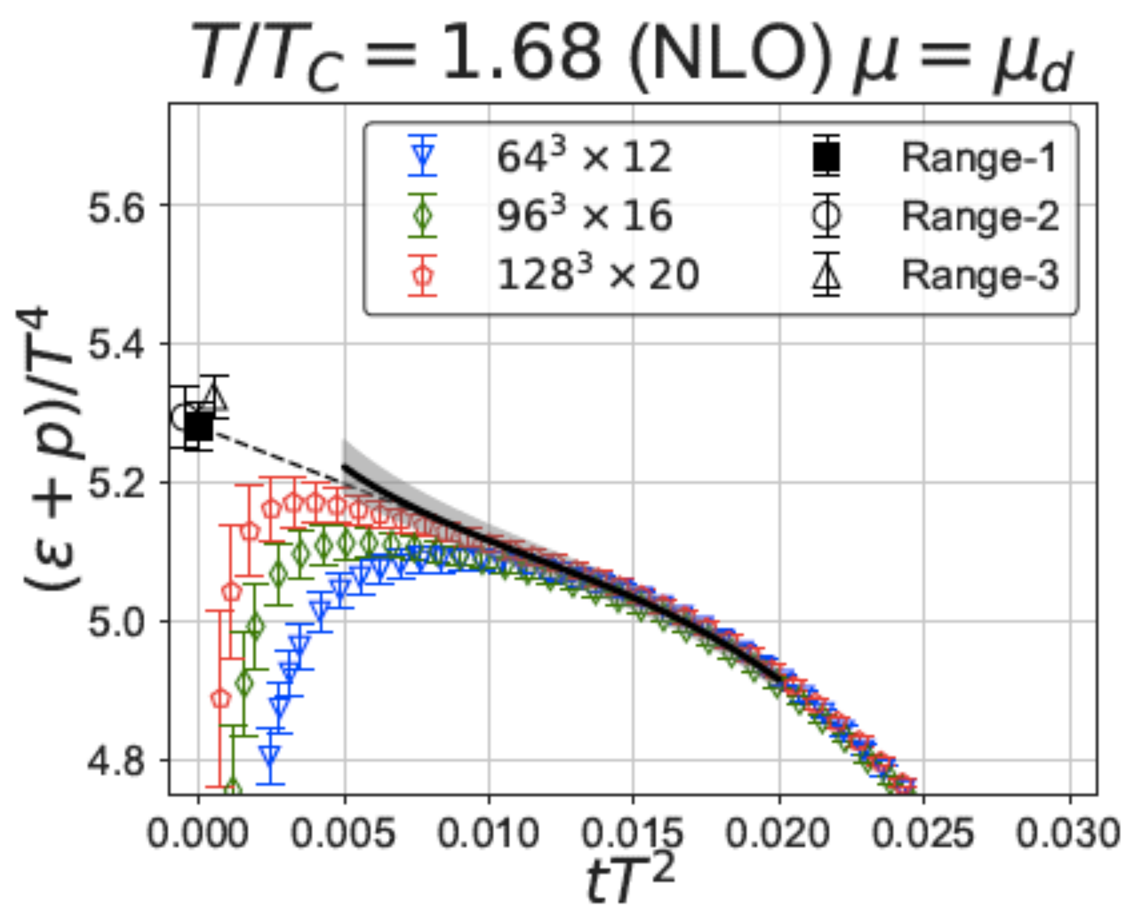


Application

$$\mu_0 = \frac{e^{-\gamma_E/2}}{\sqrt{2t}}$$

$$\mu_d = \frac{1}{\sqrt{8t}}$$

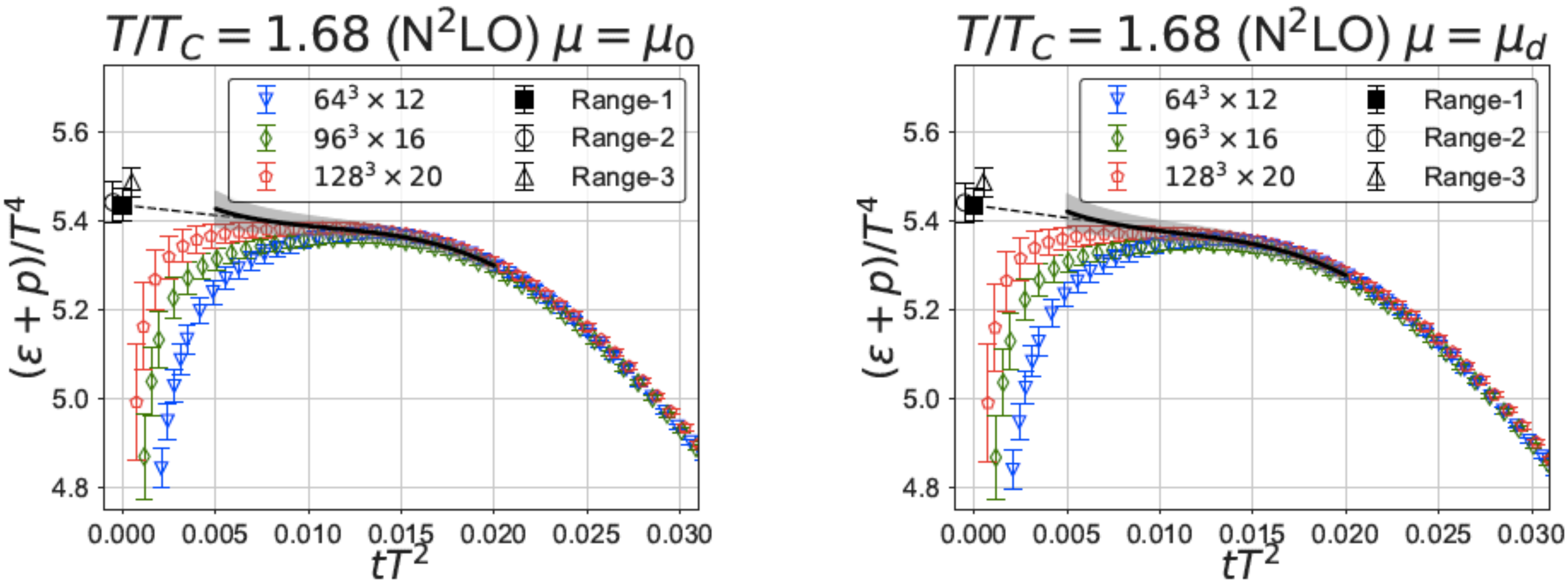




Iritani, Kitazawa, Suzuki, Takaura 2019

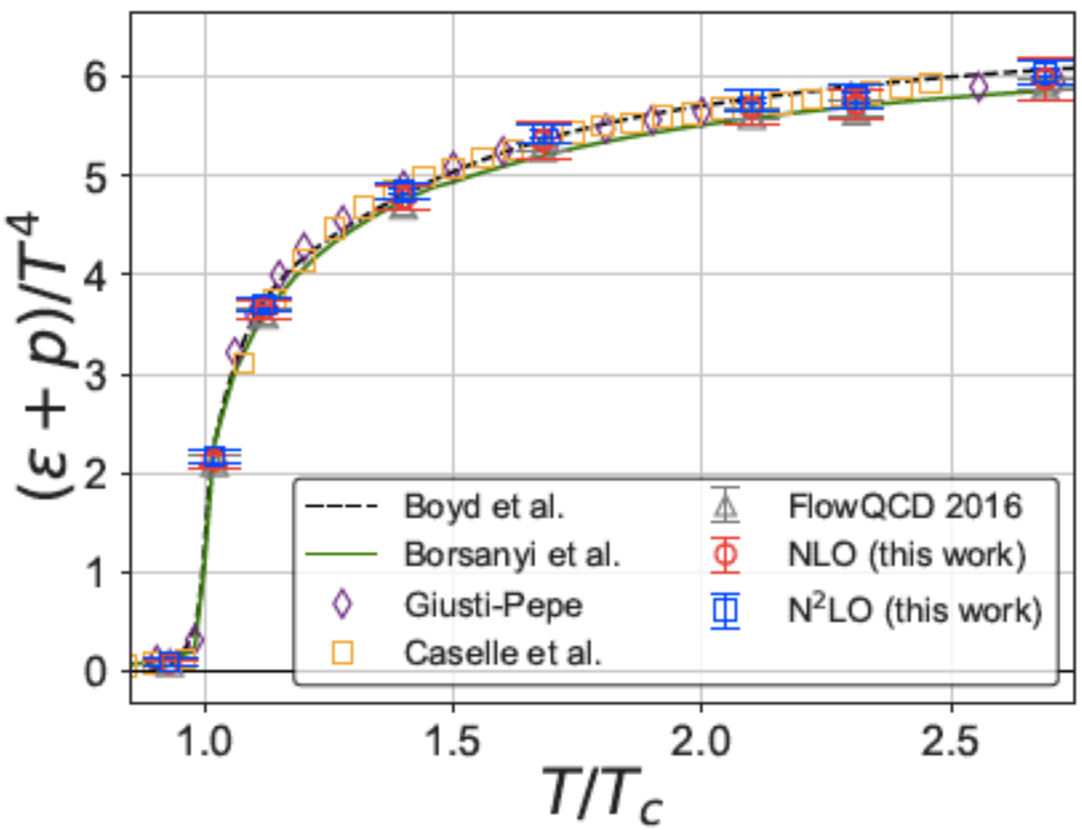
Application

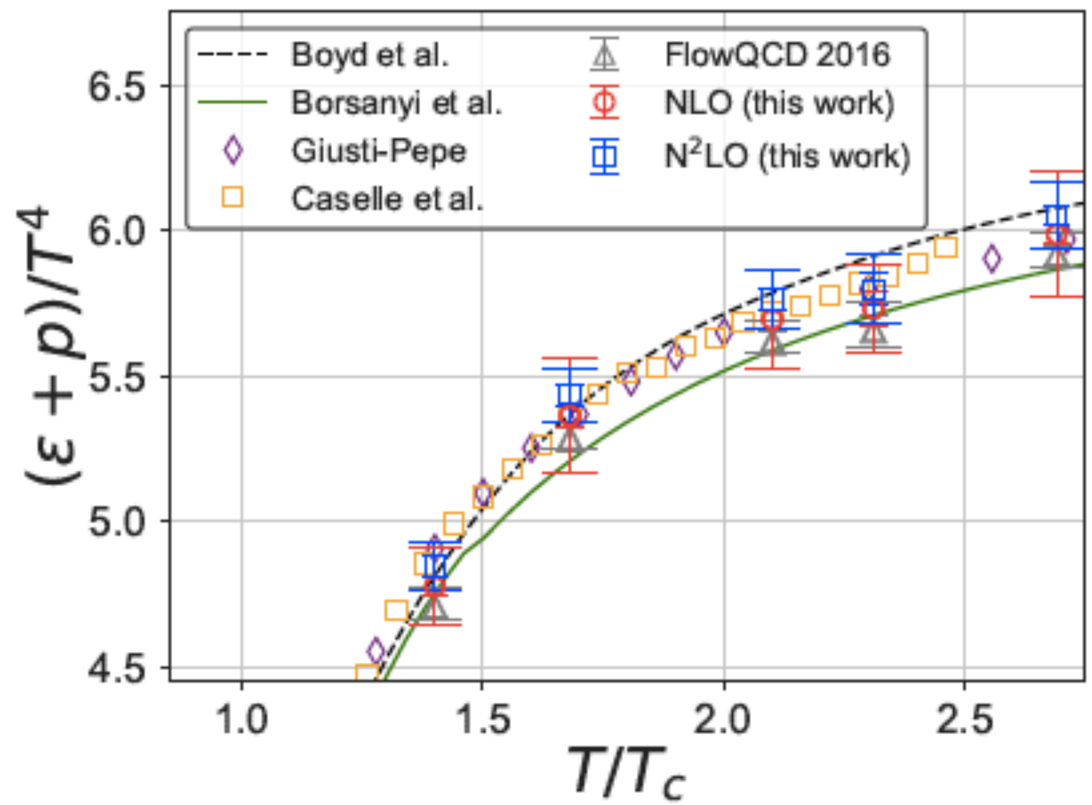
$$\mu_0 = \frac{e^{-\gamma_E/2}}{\sqrt{2t}} \qquad\qquad \mu_d = \frac{1}{\sqrt{8t}}$$



Iritani, Kitazawa, Suzuki, Takaura 2019

Application





Iritani, Kitazawa, Suzuki, Takaura 2019

Summary

- Gradient Flow is a (relatively) new tool
- Extremely successful in lattice QCD
- Perturbative approach not yet fully explored
- Here:
  - potential measurement of α_s on the lattice
  - lattice definition of energy-momentum tensor

Outlook:

- Applications in other fields? (Flavor?)
- Renormalization group vs. flow time?

TTK Institute for Theoretical Particle Physics and Cosmology | RWTH AACHEN UNIVERSITY