Matching non-relativistic EFT and dispersion relations for $\eta \to 3\pi$ decays

Group Seminar

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9th Apr 2019

What can we learn from $\eta \to 3\pi$ decays?

- hadron spectrum has approximate symmetry under exchange of up and down quarks $\to$ isospin group $SU(2)$
  - proton and neutron form doublet
  - charged and neutral pions form triplet
  - η is a singlet
- $\eta \to 3\pi$ decays forbidden by isospin symmetry
  $G = C \times (-1)^I$, η: $I^G = 0^+$, π: $I^G = 1^- \to G$ not conserved
- 2 sources of isospin breaking in the Standard Model:
  - electromagnetism
  - strong isospin breaking due to $m_u \neq m_d$
- electromagnetic contributions largely suppressed   Sutherland 1966

$\to$ clean source of information on light quark mass difference

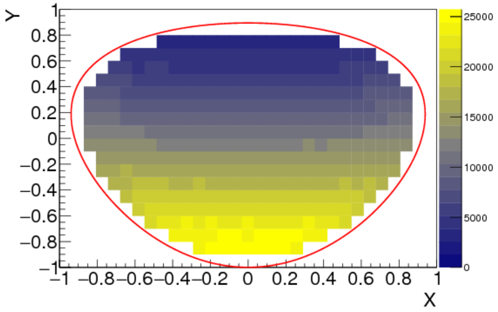
$\to$ need best possible theoretical description of amplitude with quark masses as free parameters to compare to experiment

Contents

- Chiral perturbation theory
- Khuri–Treiman equations
- Modified non-relativistic effective field theory
- Matching the three amplitudes and results
- Conclusions

Dalitz plots for $\eta \to 3\pi$ decays

- describe momentum distribution by Mandelstam variables s_1, s_2, s_3
- only two of them independent $\to$ 2D density plot called Dalitz plot
- flat phase space inside allowed region



KLOE 2016

Dalitz plot parameters

$$X \propto s_1 - s_2$$
$$Y \propto s_0 - s_3$$
$$s_0 = (M_\eta - 3M_\pi)/3$$

- measured distributions can be described by low-order polynomials

Chiral perturbation theory

- ChPT is the low-energy effective field theory of QCD
- QCD Lagrangian:

$$\mathcal{L} = \sum_f \bar{q}_f (\mathrm{i}\not{D} - m_f) q_f - \frac{1}{4} \mathrm{Tr}(G_{\mu\nu} G^{\mu\nu})$$

- in limit $m_f \to 0$ chiral symmetry

$$\mathcal{L} = \mathrm{i} \sum_f (\bar{q}_f^{\mathrm{R}} \not{D} q_f^{\mathrm{R}} + \bar{q}_f^{\mathrm{L}} \not{D} q_f^{\mathrm{L}}) - \frac{1}{4} \mathrm{Tr}(G_{\mu\nu} G^{\mu\nu})$$

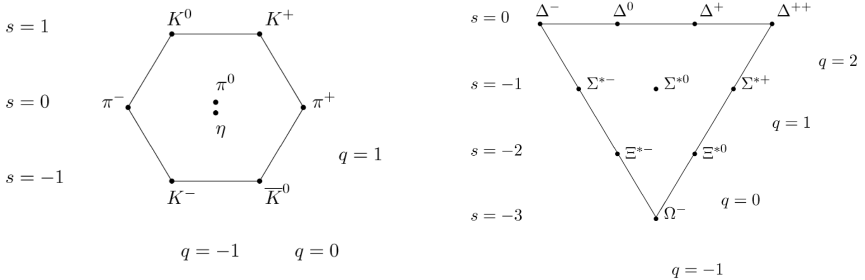
- reasonable approximation for u, d, s-quarks

$\rightarrow$ approximate symmetry of QCD

$$U(3)_{\mathrm{R}} \times U(3)_{\mathrm{L}} = U(1)_{\mathrm{V}} \times U(1)_{\mathrm{A}} \times SU(3)_{\mathrm{V}} \times SU(3)_{\mathrm{A}}$$

Chiral perturbation theory

- only $SU(3)$ seems to be (approximate) symmetry of hadron spectrum



$3 \times \bar{3} = 8 + 1$ $\qquad$ $3 \times 3 \times 3 = 10 + 8 + 8 + 1$

- $SU(3)_A$ symmetry spontaneously broken $\rightarrow$ 8 Goldstone bosons
- GB masses not exactly vanishing due to finite quark masses

Chiral perturbation theory for mesons

- construct most general Lagrangian for Goldstone bosons allowed by symmetries
- order terms by powers of p/Λ_χ (p: external momentum, meson mass)
- leading Lagrangian

$$\mathcal{L}_2 = \frac{{F_\pi}^2}{4}\,\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger + 2BM(U + U^\dagger))$$

$$U = \exp\left(\frac{\mathrm{i}\phi}{F_\pi}\right) \qquad \phi = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}$$

$$M = \mathrm{diag}(m_\mathrm{u}, m_\mathrm{d}, m_\mathrm{s})$$

- contains interactions of arbitrary even numbers of mesons

The quark mass ratio Q

- meson masses related to quark masses (and B) at LO
- B drops out in ratios $\rightarrow$ ratios of quark masses depend only on meson masses
- at NLO additional LECs enter
- only one ratio is independent of them: Leutwyler 1996

$$Q^2 = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} = \frac{M_K^2(M_K^2 - M_\pi^2)}{M_\pi^2(M_{K^0}^2 - M_{K^+}^2)} \left(1 + \mathcal{O}(m_q^2, \delta, e^2)\right)$$

- QED corrections important for meson masses
- Dashen's theorem: $(M_{K^+}^2 - M_{K^0}^2)_{\text{QED}} = (M_{\pi^+}^2 - M_{\pi^0}^2)_{\text{QED}} + \mathcal{O}(e^2 m_q)$ Dashen 1969
- result assuming Dashen's theorem: $Q = 24.3$
- **Can Dashen's theorem be trusted?**

$\eta \to 3\pi$ decays in chiral perturbation theory

- at leading order in isospin breaking amplitude proportional to

$$N = -\frac{1}{Q^2}\frac{M_K^2(M_K^2 - M_\pi^2)}{3\sqrt{3}M_\pi^2 F_\pi^2}\,, \quad Q = \sqrt{\frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}}$$

- $\eta \to 3\pi$ amplitude at leading order in p/Λ_χ given by tree diagram

$$\mathcal{M}_{\mathrm{c}}(s_1, s_2, s_3) = -N\frac{3s_3 - 4M_{\pi^0}^2}{M_\eta^2 - M_{\pi^0}^2}\,, \quad \mathcal{M}_{\mathrm{n}}(s_1, s_2, s_3) = 3N$$

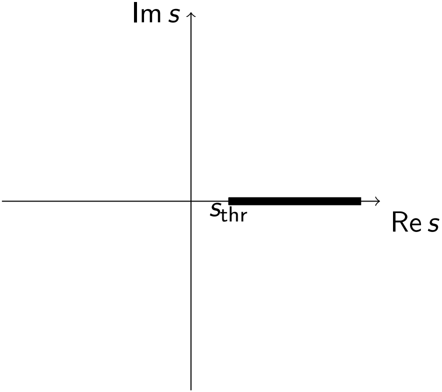
$$s_3 = (P_\eta - P_{\pi^0})^2$$

- bad description of measured momentum dependence (even at NLO)

Can we do better?

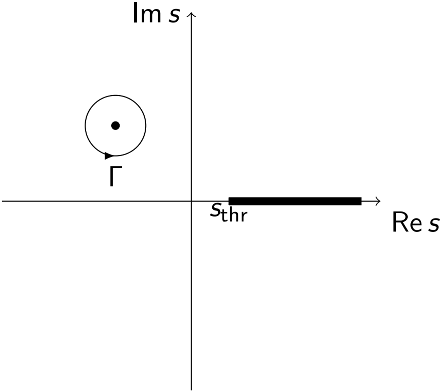
- ChPT does not converge fast enough
- need framework to include rescattering to all orders:
  dispersion relations
- rely on basic properties of amplitudes:
  - analyticity related to causality
  - unitarity related to probability conservation
  - crossing symmetry

Introduction to dispersion relations



- Amplitudes analytic apart from branch cut on real axis above threshold

Introduction to dispersion relations



- Amplitudes analytic apart from branch cut on real axis above threshold
- Cauchy's Theorem: $F(s) = \frac{1}{2\pi i} \oint_\Gamma d\, s' \frac{F(s')}{s'-s}$

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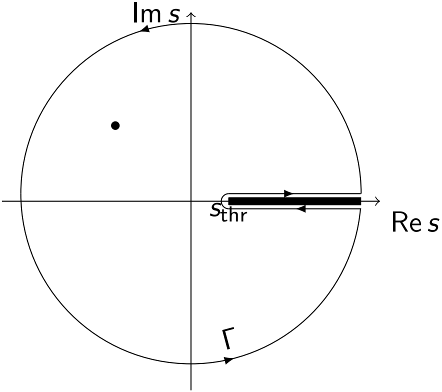

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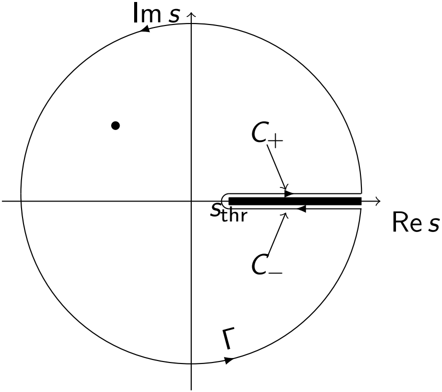

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  $F(s) = \frac{1}{2\pi i} \oint_\Gamma \mathrm{d}\, s' \frac{F(s')}{s'-s}$
- deform integration contour
- If F falls off sufficiently fast, only C_+ and C_- contribute
  $F(s) = \frac{1}{2\pi i} \int_{s_{\text{thr}}}^{\infty} \mathrm{d}\, s' \frac{\operatorname{disc} F(s')}{s'-s}$
  $\operatorname{disc} F(s) = F(s + i\epsilon) - F(s - i\epsilon)$

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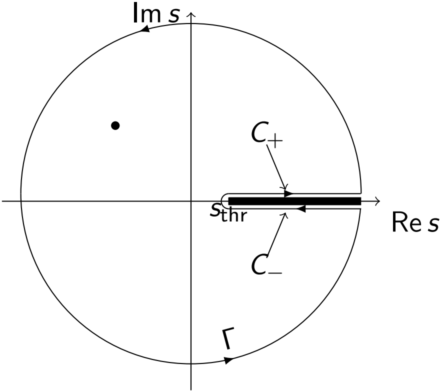

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- deform integration contour
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  $\mathrm{disc}\,F(s) = F(s+i\epsilon) - F(s-i\epsilon)$
- improve convergence by subtractions
  $F(s) = P_{n-1}(s) + \frac{1}{2\pi i} \int_{s_{thr}}^{\infty} \frac{d\,s'}{s^n} \frac{\mathrm{disc}\,F(s')}{s'-s}$

Sample application: pion vector form factor

- interaction of charged pion with virtual photon

$$\langle \pi^+(p_1) | J^\mu | \pi^+(p_2) \rangle = e(p_1 + p_2)^\mu F_V((p_1 + p_2)^2)$$

- Watson's final state theorem: $F_V(s)$ and pion scattering amplitude have same phase (in elastic regime)

$$F_V(s) = |F_V(s)| e^{i\delta_1^1(s)} \Rightarrow \operatorname{Im} F_V(s) = F_V(s) \sin \delta_1^1(s) e^{-i\delta_1^1(s)}$$

(δ_1^1 pion scattering phase shift with $I = L = 1$)

- special solution for F_V: Omnès function has $\Omega_1^1(s) \neq 0$ and $\Omega_1^1(0) = 1$

$$\operatorname{disc} \ln \Omega_1^1(s) = 2i\delta_1^1(s)$$

- write once-subtracted dispersion relation for $\ln \Omega_1^1(s)$ Omnès 1958

$$\Omega_1^1(s) = \exp\left(\frac{s}{\pi} \int_{4M_\pi^2}^{\infty} \frac{d s'}{s} \frac{\delta_1^1(s')}{s' - s - i\epsilon} \right)$$

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$$\operatorname{disc} \ln \Omega_1^1(s) = 2i\delta_1^1(s)$$

- general solution given by ($P(s)$ real polynomial with $P(0) = 1$)

Omnès 1958

$$F_V(s) = P(s)\Omega_1^1(s)$$

Application to $\eta \to 3\pi$ decays: Khuri–Treiman equations

- $\eta \to 3\pi$ more difficult because two independent kinematic variables
- reconstruction theorem assuming discontinuity only in leading partial wave in each isospin channel:

$$\mathcal{M}_c(s_1, s_2, s_3) = \mathcal{M}_0(s_3) + (s_3 - s_2)\mathcal{M}_1(s_1) + (s_3 - s_1)\mathcal{M}_1(s_2) + \mathcal{M}_2(s_1) + \mathcal{M}_2(s_2) - \frac{2}{3}\mathcal{M}_2(s_3)$$

- decomposition ambiguous, parametrized by 5 complex parameters
- dispersion relations for isospin functions $\mathcal{M}_I$ Khuri & Treiman 1960

$$\mathcal{M}_I(s) = \Omega_I(s)\left\{P_I(s) + \frac{s^{n_I}}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\mathrm{d}\,s'}{s'^{n_I}}\frac{\sin\delta_I(s')\hat{\mathcal{M}}_I(s')}{|\Omega_I(s')|(s' - s - i\varepsilon)}\right\}$$

- $\hat{\mathcal{M}}_I(s)$ accounts for crossed channel rescattering and includes integrals over all $\mathcal{M}_I \to$ coupled equations

Application to $\eta \to 3\pi$ decays: Khuri–Treiman equations

- subtraction constants to be determined from ChPT and/or data
  Colangelo et al. 2018
- amplitude relativistically covariant $\to$ can also be used outside physical decay region (at not too high energies)
- need to match to ChPT to obtain amplitude including quark masses
  $\to$ ChPT converges fastest at $s = 0$
- but: assumes isospin limit (for final state interactions)

Higher-order isospin breaking

- included only first order isospin breaking until now
- but: excess energy $M_\eta - 3M_\pi$ differs by $7\,\%$ between channels

$\rightarrow$ kinematically allowed region in Dalitz plot substantially distorted

$\rightarrow$ need to correct for pion mass difference (at least)

$\rightarrow$ not (directly) possible in Khuri–Treiman formalism

$\rightarrow$ Modified non-relativistic EFT perfectly suited

Modified non-relativistic effective field theory (NREFT)

e.g. Gasser et al. 2011

- relativistic propagator can be split up

$$\frac{\mathrm{i}}{p^2 - M^2} = \frac{1}{\mathrm{i}}\left(\frac{1}{2w(\vec{p})\big(w(\vec{p}) - p^0\big)} + \frac{1}{2w(\vec{p})\big(w(\vec{p}) + p^0\big)}\right)$$

  with $w(\vec{p}) = \sqrt{M^2 + \vec{p}^2}$

- in non-relativistic regime: $p^0 \approx w(\vec{p}) \approx M$

$\rightarrow$ anti-particle propagator can be approximated by polynomial and included in couplings

$\rightarrow$ integrating out anti-particles

- but retain relativistic energy momentum relation and normalization of states!

Advantages of NREFT

- analytic structure of corresponding relativistic diagrams reproduced (in range of validity)
- scattering lengths, effective ranges, etc. matched to phenomenological values, only approached perturbatively in ChPT

$\rightarrow$ NREFT converges faster

- number of diagrams drastically reduced compared to ChPT (no antiparticles!)

Sample application: $\pi\pi$ scattering

- construct most general Lagrangian $\mathcal{L}_{\pi\pi}$ up to $\mathcal{O}(\epsilon^6)$ in non-rel. momenta $\epsilon = |\vec{p}_i|/M_\pi$
- all loop diagrams are products of one elementary loop diagram:

$$= J(s) = \frac{i}{16\pi}\sqrt{1 - \frac{4M_\pi^2}{s}}$$

- coincides with imaginary part of same diagram in relativistic theory
- diagrams can be resummed to obtain exact result (loop expansion)

$$\mathcal{M} = C\sum_{i=0}^{\infty}(CJ(s))^i = \frac{C}{1 - CJ(s)} \qquad C\text{: coupling constant}$$

- more complicated in reality (different channels and higher-order couplings), but still doable
- match coupling constants to phenomenological phase shifts

Colangelo et al. 2001 & Garcia-Martin et al. 2011

$\eta \to 3\pi$ in NREFT

Schneider et al. 2011

- construct most general Lagrangian up to $\mathcal{O}(\epsilon^6)$ in non-rel. momenta $\epsilon = |\vec{p}_i|/M_\pi$ of the form

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\eta\to 3\pi} + \mathcal{L}_{\pi\pi}$$

- use $\mathcal{L}_{\pi\pi}$ coupling constants obtained before (phen. phase shifts)
- tree level amplitudes

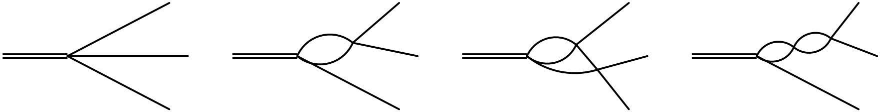
$$\begin{aligned}\mathcal{M}_c^{\text{tree}} =& L_0 + L_1(p_3^0 - M_{\pi^0}) + L_2(p_3^0 - M_{\pi^0})^2 + L_3(p_1^0 - p_2^0)^2 \\ & + L_4(p_3^0 - M_{\pi^0})^3 + L_5(p_3^0 - M_{\pi^0})(p_1^0 - p_2^0)^2 + \mathcal{O}(\epsilon^8) \\ \mathcal{M}_n^{\text{tree}} =& K_0 + K_1[(p_1^0 - M_{\pi^0})^2 + (p_2^0 - M_{\pi^0})^2 + (p_3^0 - M_{\pi^0})^2] \\ & + K_2[(p_1^0 - M_{\pi^0})^3 + (p_2^0 - M_{\pi^0})^3 + (p_3^0 - M_{\pi^0})^3] + \mathcal{O}(\epsilon^8)\end{aligned}$$

- K_i are functions of L_i in isospin limit $\to$ 6 unknown coupling constants

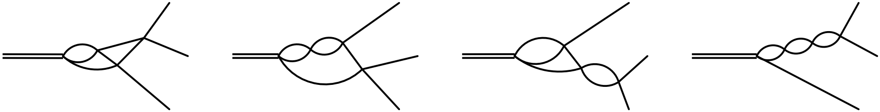
$\to$ need to fit to measured Dalitz plot(s)

Diagrams

- tree-level, one and two loop diagrams have been calculated before
  Schneider et al. 2011



- outer vertex unitarized
- 4 three-loop diagrams exist



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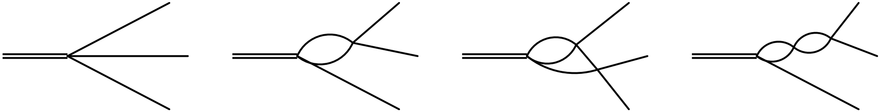

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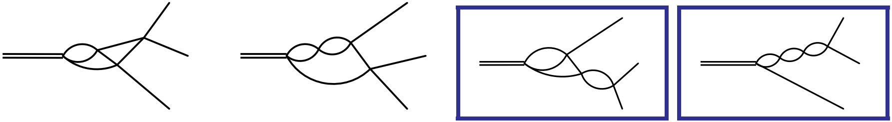

- included by unitarization

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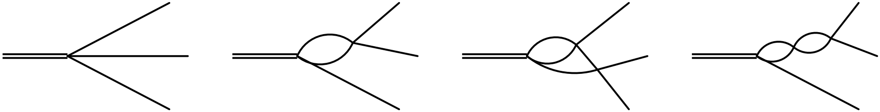

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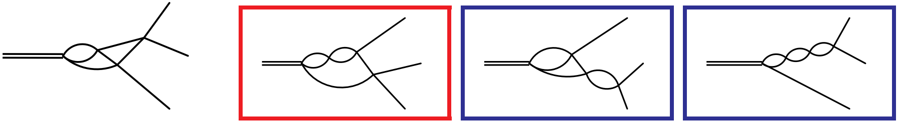
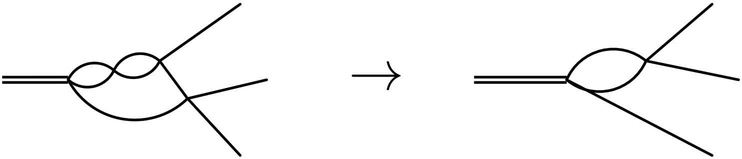


- included by unitarization
- calculated in my Master's thesis

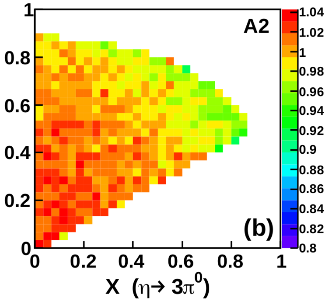
Three-loop caluclation



- recall:  $= J(s) = \frac{i}{16\pi}\sqrt{1 - \frac{4M_\pi^2}{s}}$
- diagram includes $J^2(s(\vec{l}))$, which is analytic in $w(\vec{l}) = \sqrt{M\pi^2 + \vec{l}^2}$
- Taylor expand this at $w(\vec{l}) = M_\pi$
- terms take form of (higher-order) couplings
- calculate effective one-loop diagram with series of couplings at $\eta \to 3\pi$ vertex
- result converges after only few terms

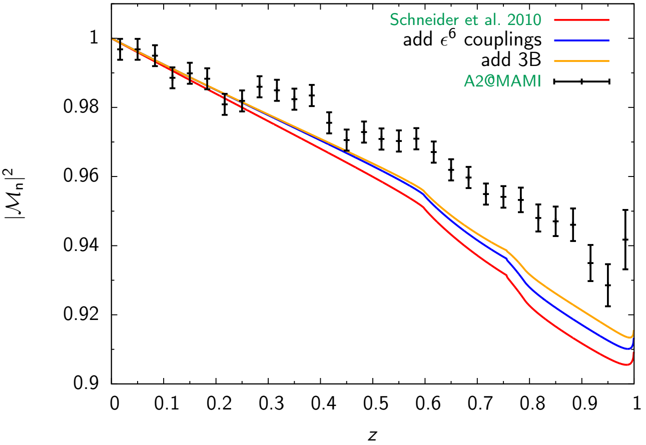
Fitting the charged channel Dalitz plot

- determine $\eta \to 3\pi$ coupling constants from fit only to $\eta \to \pi^+\pi^-\pi^0$ KLOE Dalitz plot measurement
- fit quality is excellent ($\chi^2 = 366$ for 371 bins and 6 fit parameters)
- amplitude completely fixed $\to$ can predict neutral Dalitz plot distribution
- measured distribution very flat with main variation in radial direction $\to$ show only radial distribution
- normalization not measured in Dalitz plot measurements $\to$ adjust to 1 at center



A2@MAMI 2018

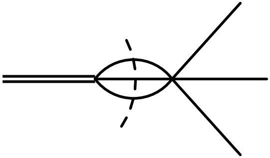
Prediction for the neutral decay channel



$$z = x^2 + y^2 \propto \sum_{i=1}^{3} (s_0 - s_i)^2 \qquad s_0 = (M_\eta - 3M_\pi)/3$$

Imaginary part at two loops

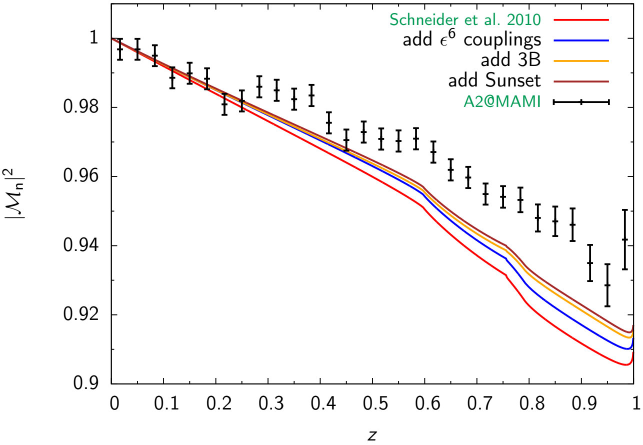
- in NREFT: tree diagram real, one-loop imaginary, two-loop real, ...
- exception: three-pion cut in the sunset diagram



- only present if $3\pi \to 3\pi$ couplings added to NREFT Lagrangian
- estimate these through tree-level ChPT at threshold
- absorb contribution by adding an imaginary part of $\sim 0.5\,\%$ to coupling constants of leading $\eta \to 3\pi$ vertices (L_0 and K_0)

$\to$ only rough estimation!

Prediction for the neutral decay channel



$$z = x^2 + y^2 \propto \sum_{i=1}^{3} (s_0 - s_i)^2 \qquad s_0 = (M_\eta - 3M_\pi)/3$$

Alternative fit strategy

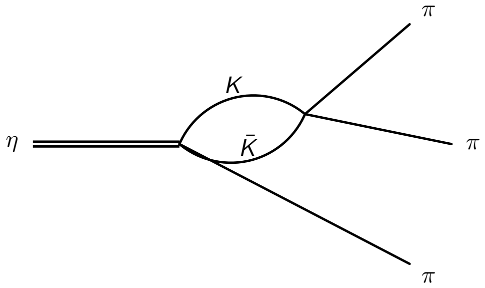
- can both Dalitz plots be described simultaneously?
- include charged and neutral data in fit
- take different normalizations of measured Dalitz plots into account
  $\rightarrow$ 7 fit parameters
- χ^2/ndof decreases from 1.3 to 1.1 for the different amplitudes
- corresponds to p-values of $4 \times 10^{-6}\,\%$ and $1.5\,\%$, respectively

$\rightarrow$ added contributions are important, but do not suffice

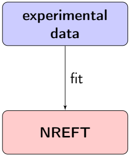
- use best of these fits for further analysis

What is missing?

- included higher-order diagrams and couplings improve consistency with data
- if nice convergence assumed: something else missing
- possibility: other effects of higher-order in isospin breaking:
  - virtual photon contributions feasible in NREFT but probably negligible (universal radiative corrections already applied to data)
  - kaon mass difference in ChPT loops leads to corrections to isospin relation between charged and neutral channel

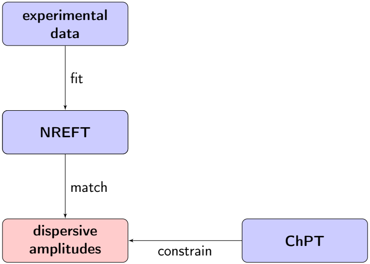


Matching procedure



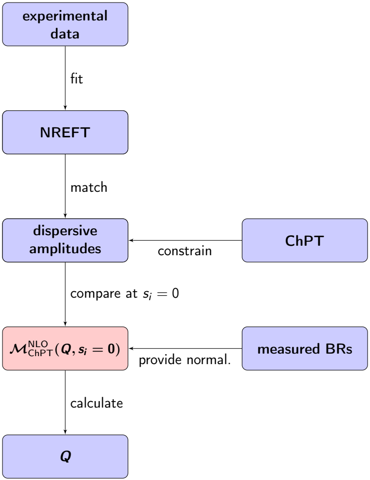
- NREFT includes higher-order isospin breaking corrections and converges faster than ChPT $\rightarrow$ want to describe momentum dependence by NREFT
- NREFT coupling constants determined from fit to data
- but: NREFT cannot be used at $s = 0$ where ChPT converges fastest

Matching procedure



- match to dispersive formalism
- need to determine 6 subtraction constants from 6 NREFT couplings
- set $M_{\pi^+} = M_{\pi^0} = M_\pi$ in NREFT amplitude after fitting
- decompose NREFT amplitude into isospin amplitudes
- calculate isospin limit of NREFT coupling constants by matching to tree-level ChPT
- determine subtraction constants (and ambiguity parameters of reconstruction theorem) from unweighted fit
- use constraints from ChPT

Matching procedure



- match dispersive amplitude to NLO ChPT amplitude at $s_i = 0$
- use normalization provided by experimental branching ratios
- calculate Q

Quark mass ratio Q

- result: $Q = \begin{cases} 21.86 & \eta \to \pi^+\pi^-\pi^0 \\ 22.01 & \eta \to 3\pi^0 \end{cases}$ (error estimation in progress)
- compare to
  - dispersive + ChPT analysis: $Q = 22.1 \pm 0.7$ Colangelo et al. 2018
  - lattice: $Q = 24.0 \pm 0.8$ FLAG 2019
  - NLO ChPT using Dashen's theorem ($\Delta M_\pi^{em} = \Delta M_K^{em}$): $Q = 24.3$

$\rightarrow$ further investigations needed to resolve discrepancy between lattice and phenomenology

Conclusions

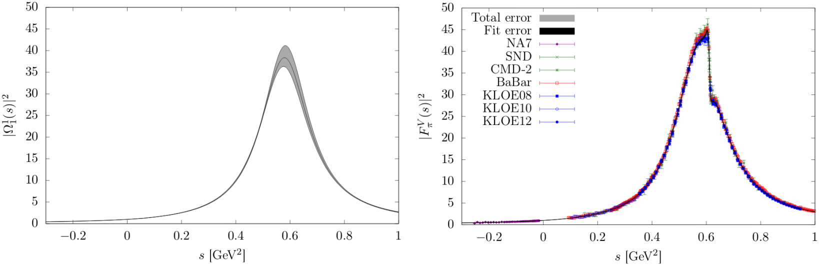
- $\eta \to 3\pi$ decays are isospin violating and provide clean information on $m_d - m_u$
- NREFT allows for consistent description of important higher-order isospin breaking effects
- ongoing work to further improve compatibility with Dalitz plot measurements (kaon loop, ...)
- NREFT cannot be used to extrapolate to unphysical points $\to$ match to dispersive formalism
- use measured rate and normalization of ChPT amplitude at $s = 0$ to determine Q
- result agrees well with other dispersive analyses, but differs from lattice determinations
- **further investigations needed!**

Sample application: pion vector form factor

- general solution given by ($P(s)$ real polynomial with $P(0) = 1$)

$$F_V(s) = P(s)\Omega_1^1(s)$$

- pion scattering phase shifts well-known from Roy-equation analyses
  Colangelo et al. 2000, Garcìa-Martìn et al. 2011
- $P(s)$ has to be determined from somewhere else



Colangelo et al. 2019

Higher-order isospin breaking: cusp

- $\eta \to 3\pi^0$: can have charged pion intermediate state
- amplitude has structure

$$\mathcal{M}_\mathrm{n}(s_1, s_2, s_3) = a(s_1, s_2, s_3) + b(s_1, s_2, s_3)\left(\sqrt{1 - \frac{s_1}{4M_{\pi^+}^2}} + \sqrt{1 - \frac{s_2}{4M_{\pi^+}^2}} + \sqrt{1 - \frac{s_3}{4M_{\pi^+}^2}}\right) + c(s_1, s_2, s_3)\left(\sqrt{1 - \frac{s_1}{4M_{\pi^0}^2}} + \sqrt{1 - \frac{s_2}{4M_{\pi^0}^2}} + \sqrt{1 - \frac{s_3}{4M_{\pi^0}^2}}\right) + \ldots$$

- a, b and c are real polynomials (inside physical region)
- interference between a and b terms leads to cusp inside Dalitz plot due to $M_{\pi^+} > M_{\pi^0}$
- cusp started to be seen in current data
- cusp much stronger in $K \to 3\pi$, was used to gain information on $\pi\pi$ scattering lengths   Gasser et al. 2011

Treatment of higher-order isospin breaking in Colangelo et al. 2018

- mapping from isospin symmetric Dalitz plot to physical ones preserving boundary $(s_1^{\mathrm{iso}}, s_2^{\mathrm{iso}}, s_3^{\mathrm{iso}}) \mapsto (s_{\mathrm{c/n}}^1, s_{\mathrm{c/n}}^2, s_{\mathrm{c/n}}^3)$
- correct mapped dispersive amplitudes by factor

$$K_{\mathrm{c/n}}(s_{\mathrm{c/n}}^1, s_{\mathrm{c/n}}^2, s_{\mathrm{c/n}}^3) = \frac{\mathcal{M}_{\mathrm{c/n}}(s_{\mathrm{c/n}}^1, s_{\mathrm{c/n}}^2, s_{\mathrm{c/n}}^3)}{\mathcal{M}_{\mathrm{c/n}}^{\mathrm{iso}}(s_1^{\mathrm{iso}}, s_2^{\mathrm{iso}}, s_3^{\mathrm{iso}})}$$

calculated at NLO in ChPT including em corrections
- assumes isospin breaking effects to factorize from ChPT amplitudes