Next-to-soft QCD corrections to hadronic cross sections

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Outline

- Introduction to leading power soft approximation
  - Soft exponentiation
- Next-to-leading power threshold logarithms
  - Regions, factorization at NNLO
- Next-leading power at NLO
  - colour singlet, prompt photon
- LL resummation at NLP
  - Drell-Yan, prompt photon
A brief history of the eikonal approximation

- “Eikon” originally from Greek εἰκεναι [to resemble]
  - leading to εἰκον [icon, image]
- Predates quantum mechanics, and even Maxwell
  - also known in optics as “ray optics”
    - Rays are straight lines, perpendicular to wave fronts
Eikonal optics: rays

- Can describe formation of images/eikons
  - wavelength $\ll$ size of scatterer
- Cannot describe diffraction, polarization etc
  - these are wave phenomena
Eikonal quantum mechanics

- Eikonal approximation in QM scattering
  - R.J. Glauber [1959 lecture notes (recommended!)]
  - Highly energetic particle scattering off potential
  - (regained use in scattering on nucleus)
- 2D example
  - scattering on square potential of limited range
QM eikonal scattering in 2D

Glauber, 1958 Boulder lectures

\[ \psi(x) = e^{ikz} + \frac{\sqrt{k}}{\sqrt{2\pi}} e^{i\frac{\pi}{4}} f(\theta) \frac{e^{ikr}}{\sqrt{r}} \]

Incoming wave

Scattered wave

Scattering amplitude

Factor high-momentum part

\[ \psi(\vec{r}) = e^{ikz} \phi(\vec{r}) \]

Substitute in Schrodinger equation

\[ e^{ikz} (-k^2 + 2ik\partial_z + \nabla^2 + k^2) \phi(\vec{r}) = e^{ikz} \frac{2m}{\hbar^2} V(\vec{r}) \phi(\vec{r}) \]

Solve

\[ \phi(\vec{r}) = e^{-\frac{i}{\hbar v} \int_{-\infty}^{z} dz' V(x,z')} \]

Exponential form!

\[ E \gg V \quad k \gg 1/a \]
QM eikonal scattering in 2D

- Approximation at amplitude level, some “wave” information is preserved. If potential is “black”:
  - Optical theorem
  - Scattering cross section

  \[ \sigma_{tot} = 2 \text{Im} f(\theta = 0) = 2a \]
  \[ \sigma_{scattered} = \frac{k}{2\pi} \int d\theta |f(\theta)|^2 = a \]

  ✔ Factor 2 is due to diffraction, fill in shadow of target
Eikonal QFT: QED

- Charged particle emits soft photon
  - Propagator: expand numerator & denominator in soft momentum, keep lowest order
  - Vertex: expand in soft momentum, keep lowest order

\[
\frac{(p + k)^\mu + p^\mu}{2p \cdot k + k^2} \rightarrow \frac{2p^\mu}{2p \cdot k}
\]
Basics of eikonal approximation in QED

\[ p \cdot (k_1 + k_2) p \cdot k_2 = \frac{1}{p \cdot k_1} \frac{1}{p \cdot k_2} \]

Exact:
\[ \frac{1}{(p + K_1)^2} (2p + K_2 + K_1)^{\mu_1} \ldots \frac{1}{(p + K_n)^2} (2p + K_n)^{\mu_n}, \quad K_i = \sum_{m=i}^{n} k_m. \]

Approx:
\[ \frac{1}{2p K_1} 2p^{\mu_1} \ldots \frac{1}{2p K_n} 2p^{\mu_n} \]

Eikonal identity:
\[ \prod_{i} \frac{p^{\mu_i}}{p \cdot k_i}. \]

Independent, uncorrelated emissions, Poisson process
Eikonal approximation: no dependence on emitter spin

- Emitter spin becomes irrelevant in eikonal approximation
  - Fermion

\[
\begin{align*}
    & p + k \\
    & \rightarrow p \\
    & \mathcal{M} \frac{i(p + k)}{(p + k)^2} (-i g_s \gamma^\mu) u(p)
\end{align*}
\]

- Approximate, and use Dirac equation
  \[ \not\!p u(p) = 0 \]

- Result:
  \[ g \left( \mathcal{M} u(p) \right) \times \frac{p^\mu}{p \cdot k} \]

- Notice
  - No sign of emitter spin anymore (= scalar emitter)
  - Coupling of photon proportional to emitter momentum \( p^\mu \)
Another eikonal effect: coherence in emission

- Eikonal approximation in amplitude, coherence possible
  - First in QED

Square the amplitude, take the eikonal approximation, and combine with phase. Result

\[
d\sigma_R = d\sigma \frac{\alpha_s}{2\pi} \frac{dE}{E} d\cos \theta d\phi \ E^2 \frac{p \cdot \bar{p}}{p \cdot k \bar{p} \cdot k}
\]

- Only non-zero when \( \theta' < \theta \): angular ordering after azimuthal integral
  ✓ photon that is too soft only see the sum of the charges, which is zero here.

- In QCD very similar result (after being a little bit more careful with color charges). Radiation function

\[
W_{ij} = \frac{\omega^2 p_i \cdot p_j}{p_i \cdot q \ p_j \cdot q} = \frac{1 - v_i v_j \cos \theta_{ij}}{(1 - v_i \cos \theta_{iq})(1 - v_j \cos \theta_{jq})}
\]

✓ clearly has eikonal form. Notice, it is an interference effect:
After eikonal approximation, we suddenly see interesting patterns.

One loop vertex correction, in eikonal approximation

\[ \mathcal{A}_0 \int d^n k \frac{1}{k^2} \frac{p \cdot \bar{p}}{(p \cdot k)(\bar{p} \cdot k)} \]

Two loop vertex correction, in eikonal approximation

\[ \mathcal{A}_0 \frac{1}{2} \left( \int d^n k \frac{1}{k^2} \frac{p \cdot \bar{p}}{(p \cdot k)(\bar{p} \cdot k)} \right)^2 \]

Exponential series! A really beautiful result

\[ \exp \left[ \right] \]

Yennie, Frautschi, Suura '61
QCD exponentiation: webs

- Not immediately generalizable to QCD, seemingly
  - Vertices terms have color charges, which don’t commute
  - Still, an exponentiation theorem holds

\[ \sum_D \mathcal{F}_D C_D = \exp \left[ \sum_i \bar{C}_i w_i \right] \]

\[ \text{Gatheral; Frenkel, Taylor; Sterman; EL, Stavenga, White} \]
Eikonal approximation from QM path integrals

Another way to exponentiate: use textbook QFT result

\[
\text{Sum of all diagrams} = \exp \left( \text{Connected diagrams} \right)
\]

Write scattering amplitude as path integral

\[
M(p_1, p_2, \{k\}) = \int \mathcal{D}A_s \mathcal{D}x(t) H[x] f_1[A_s, x(t)] f_2[A_s, x(t)] e^{iS[A_s]}
\]

Eikonal vertices are sources for gauge bosons along line v

\[
v \cdot A(x(t))
\]

Disconnected

Connected

x(t): path of charged particle

EL, Stavenga, White

\[
f = e^{i \int dt \left( \frac{1}{2} \dot{x}^2 + p \cdot A + \ldots \right)}
\]
Path integral method, non-abelian

- Not immediately obvious how this could work (the path integral must be an actual exponential), since
  - Source terms have non-abelian SU(3)-valued charges, so don’t commute
  - External line factors are path-ordered exponentials
  - Nevertheless

\[
\sum_D \mathcal{F}_D C_D = \exp \left[ \sum_i \bar{C}_i w_i \right]
\]

- To prove, use replica trick (from statistical physics)
Replica trick

- Relates exponentiation of soft gauge fields to that of connected diagrams in QFT.
- Consider a $N$ copies of a scalar theory

- If $Z$ is exponential, find out what contributes to $\log Z$

$$Z^N = 1 + N \log Z + \mathcal{O}(N^2)$$

- Amounts to diagrams that allow only one replica $\rightarrow$ connected!

$$Z[J]^N = \int \mathcal{D}\phi_1 \cdots \mathcal{D}\phi_N e^{iS[\phi_1] + \cdots + iS[\phi_N] + J\phi_1 + \cdots + J\phi_N}$$
Replica method and QCD

Amplitude for two colored lines

\[ S(p_1, p_2) = H(p_1, p_2) \int \mathcal{D} A_s f(\infty) e^{iS[A_s]} \]

Replicate, and introduce **replica ordering operator** \( R \)

\[ f(\infty) = \mathcal{P} \exp \left[ \int dx \cdot A(x) \right] \prod_{i=1}^{N} \mathcal{P} \exp \left[ \int dx \cdot A_i(x) \right] = R \mathcal{P} \exp \left[ \sum_{i=1}^{N} \int dx \cdot A_i(x) \right] \]

Look for diagrams of replica multiplicity \( N \). These will go into exponent

(a) is order \( N \)

(b) for equal replica number \((i=j): C_F^2 \). For \( i \neq j \) also \( C_F^2 \). Sum:

\[ NC_F^2 + N(N-1)C_F^2 = N^2 C_F^2 \]

(c) for equal replica number \((i=j): C_F^2 - C_F C_A / 2 \). For \( i \neq j \) \( C_F^2 \). Term linear in \( N \):

\[ N \left( C_F^2 - \frac{C_F C_A}{2} \right) + (-N)C_F^2 = N \left( -\frac{C_F C_A}{2} \right) \]
**Multiple colored lines**

- **Structure**

\[ \sum \mathcal{F}(D)C(D) = \exp\left[ \sum_{d,d'} \mathcal{F}(d)R_{dd'}C(d') \right] \]

- multi-parton webs are “closed sets” of diagrams, with modified color factors

![Diagrams](image)

- Closed form solution for modified color factor

\[ \frac{1}{6} \left[ C(3a) - C(3b) - C(3c) + C(3d) \right] \times \left[ M(3a) - 2M(3b) - 2M(3c) + M(3d) \right] \]

- Interesting properties of projector matrix (reduces degree of divergence)
Perturbative series for cross sections in QFT

Typical perturbative behavior of observable

\[ \hat{O}_2 = 1 + \alpha (L^2 + L + 1) + \alpha^2 (L^4 + L^3 + L^2 + L + 1) + \ldots \]

- \( \alpha \) is the coupling of the theory (QCD, QED, ..)
- \( L \) is some numerically large logarithm
- \( "1" = \pi^2, \ln(2), \text{anything not-logarithmic} \)
- Notice: **effective expansion parameter is} \( \alpha L^2 \) i.e. a problem when >1!!
- **Fix:** reorganize/resum terms such that

\[ \hat{O} = 1 + \alpha_s (L^2 + L + 1) + \alpha_s^2 (L^4 + L^3 + L^2 + L + 1) + \ldots \]

\[ = \exp \left( L g_1 (\alpha_s L) + g_2 (\alpha_s L) + \alpha_s g_3 (\alpha_s L) + \ldots \right) C(\alpha_s) \]

- \( + \text{constants} \)
- \( + \text{suppressed terms} \)

Notice the definition of LL, NLL, etc
Threshold logarithms

Log of “energy excess above production threshold”

\[ L^2 = \ln^2 \left( 1 - \frac{Q^2}{s} \right) \equiv \ln^2 (1 - z) \]

\[ S \geq s \geq Q^2 \]
Threshold resummed Drell-Yan (or Higgs) cross section

Threshold logarithms can be resummed to all orders

\[
\frac{d\sigma^{\text{resum}}}{dQ^2}(z) = \int_C \frac{dN}{2\pi i} z^{-N} \hat{\sigma}(N)
\]

\[
\sigma(N) = \exp \left[ - \int_0^1 dx \frac{x^{N-1} - 1}{1-x} \left\{ \int_{Q^2}^{Q^2(1-x)^2} \frac{d\mu}{\mu} A(\alpha_s(\mu)) \right. \right.
\]

\[
+ D(\alpha_s((1-x)Q)) \left\} \right] \times (1 + \alpha_s(Q^2) \frac{C_F}{\pi} + \ldots)
\]

Note: functions in exponent only depend on $\alpha_s$

A similar case: top quark pair production, much smaller uncertainty

Concurrent uncertainties:

- Scales \( \sim 3\% \)
- pdf (at 68\%cl) \( \sim 2-3\% \)
- $\alpha_s$ (parametric) \( \sim 1.5\% \)
- $m_{\text{top}}$ (parametric) \( \sim 3\% \)

Soft gluon resummation makes a difference

5\% $\rightarrow$ 3\%
NLP threshold behavior

- For Drell-Yan, DIS, Higgs, singular behavior in perturbation theory when $z \rightarrow 1$

\[ \delta(1 - z) \left[ \frac{\ln^i(1 - z)}{1 - z} \right] + \ln^i(1 - z) \]

- plus distributions have been organized to all orders (="resummation"), also possible for $\ln(1-z)$?

- "Zurich" method of threshold expansion allows computation (for NNNLO Higgs production)

\[ (1 - z)^p \ln^q(1 - z) \]

- done to $p=37$..

- Much development in SCET

- Useful also for improving NNLO slicing (N-jettiness) methods

- Alternative terminology to “NLP”
  - Next-to-soft
  - Next-to-eikonal

Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Mistlberger

Larkoski, Neill, Stewart, Moult, Kolodrubetz, Rothen, Zhu, Tackmann, Vita, Feige
Beneke, Campanario, Mannel, Peckja
Numerical effects of NLP logarithms

✦ General power expansion

\[
\frac{d\sigma}{dz} = \sum_{n=0}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^n \sum_{m=0}^{2n-1} \left\{ c_{nm}^{(-1)} \log^m (1 - z) \right\} + c_{nm}^{(0)} \log^m (1 - z) + \ldots
\]

Kramer, EL, Spira, 1998

Anastasiou, Duhr, Dulat, Herzog, Mistlberger, 2015

✦ NLP logs can be quite important
Next-to-eikonal Feynman rules

- Keep 1 term more in k expansion beyond eikonal approximation

Scalar:
\[
\frac{2p^\mu + k^\mu}{2p \cdot k + k^2} \rightarrow \frac{2p^\mu}{2p \cdot k} - \frac{k^\mu}{2p \cdot k} - \frac{k^2}{(2p \cdot k)^2} \quad 2p^\mu
\]

Fermion:
\[
\frac{\slashed{p} + \slashed{k}}{2p \cdot k + k^2} \gamma^\mu u(p) \rightarrow \left[ \frac{2p^\mu}{2p \cdot k} - \frac{k \gamma^\mu}{2p \cdot k} - \frac{k^2}{(2p \cdot k)^2} \right] u(p)
\]

- Becomes emitter-spin dependent, recoil now included
- Is there predictive power for the next-to-eikonal terms?
Classic NLP result: Low’s theorem

- These rules are good for emissions from external lines. At NLP order, also 1 “internal” emission contributes

\[ \Gamma^\mu = \Gamma^\mu + \Gamma^\mu + \Gamma^\mu \]

- Low’s theorem (scalars, generalization to spinors by Burnett-Kroll, to massless particles by Del Duca): **LBKD theorem**

  ✓ Work to order \( k \), and use Ward identity

  \[
  \Gamma^\mu = \left[ \frac{(2p_1 - k)^\mu}{-2p_1 \cdot k} + \frac{(2p_2 + k)^\mu}{2p_2 \cdot k} \right] \Gamma + \left[ \frac{p_1^\mu (k \cdot p_2 - k \cdot p_1)}{p_1 \cdot k} + \frac{p_2^\mu (k \cdot p_1 - k \cdot p_2)}{p_2 \cdot k} \right] \frac{\partial \Gamma}{\partial p_1 \cdot p_2}
  \]

- Elastic amplitude still determines the emission to NLP accuracy,
  - note the derivative
  - detailed knowledge of “internal part” not needed
NLP logarithms for Drell-Yan

- Goal: combine (N)LP matrix elements with (N)LP phase space to \textbf{predict} \( \ln^i(1-z) \) for NNLO Drell-Yan

\[
\frac{1}{\sigma^{(0)}} \frac{d\hat{\sigma}}{dz} \sim \int d\Phi_{\text{LP}} |\mathcal{M}|^2_{\text{LP}} + \int d\Phi_{\text{LP}} |\mathcal{M}|^2_{\text{NLP}} + \int d\Phi_{\text{NLP}} |\mathcal{M}|^2_{\text{LP}} + \ldots
\]

- We pursue two methods:
  - 1. Method of regions
  - 2. Factorization
- NLO is “easy”, real test at NNLO
NLP logs in Drell-Yan at NNLO

- Check NLP Feynman rules for NNLO Drell-Yan double real emission

\[ K_{\text{NE}}^{(2)}(z) = \left( \frac{\alpha_s}{4\pi} C_F \right)^2 \left[ -\frac{32}{e^2} D_0(z) + \frac{128}{e^2} D_1(z) - \frac{128}{e^2} \log(1-z) \right. \\
\left. - \frac{256}{e} D_2(z) + \frac{256}{e} \log^2(1-z) - \frac{320}{e} \log(1-z) \right. \\
\left. + \frac{1024}{3} D_3(z) - \frac{1024}{3} \log^3(1-z) + 640 \log^2(1-z) \right], \]

\[ D_i = \left[ \frac{\log^i(1-z)}{1-z} \right] + \]

- Result at NLP level, agrees with equivalent exact result. $C_F^2$ terms e.g.

- Next, 1 Real-1 Virtual
Diagnosis: method of regions

- How does it work?
  - Divide up $k_1$ (=loop-momentum) integral into hard, 2 collinear and a soft region, by appropriate scaling
    
    \[
    \text{Hard: } k_1 \sim \sqrt{s} (1, 1, 1) ; \quad \text{Soft: } k_1 \sim \sqrt{s} (\lambda^2, \lambda^2, \lambda^2) ; \\
    \text{Collinear: } k_1 \sim \sqrt{s} (1, \lambda, \lambda^2) ; \quad \text{Anticollinear: } k_1 \sim \sqrt{s} (\lambda^2, \lambda, 1).
    \]
  - expand integrand in $\lambda$, to leading and next-to-leading order
  - but then integrate over all $k_1$ anyway!
  - Treat emitted momentum as soft and incoming momenta as hard

\[k_2^\mu = (\lambda^2, \lambda^2, \lambda^2)\]
Results

- Hard region (expansion in $\lambda^2$): $LP + some\ NLP$
- Soft region (expansion in $\lambda^2$): $ZERO$
- (anti-)collinear regions (expansion in $\lambda$): $NLP\ only$

Result:

- the full $K^{(1)}_{1r,1v}$ is reproduced, including constants

For **predictive power**, need factorization

Bonocore, EL, Magnea, Vernazza, White
Can we predict the $\ln(1-z)$ logarithms from lower orders?

- Factorize the cross section,
  - $H$: the hard and the soft function
  - $J$: incoming-jet functions
- Next, add one extra soft emission. Let every blob radiate!

- Compute each new "blob + radiation", and put it together. New: radiative jet function

$$J_{\mu}(p, n, k, \alpha_s(\mu^2), \epsilon) \, u(p) = \int d^d y \, e^{-i(p-k) \cdot y} \, \langle 0 \mid \Phi_n(y, \infty) \, \psi(y) \, j_{\mu}(0) \mid p \rangle$$
Factorization approach to NLP logarithms

- Upshot: a factorization formula for the emission amplitude

\[
A_{\mu,a}(p_j, k) = \sum_{i=1}^{2} \left( \frac{1}{2} \tilde{S}_{\mu,a}(p_j, k) + g T_{i,a} T_{i,\mu} \frac{\partial}{\partial p_i^\nu} + J_{\mu,a}(p_i, n_i, k) \right) A(p_j) - A_{\mu,a}(p_j, k)
\]

- $J_\mu$ is needed at one-loop level
**Predicted NLP threshold logs vs exact result**

- Compute blobs, one-loop radiative jet function, contract with cc amplitude and integrate over phase space. Exact calculation gives

\[
K_{rv}^{(2)}(z) = \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F^2 \left[ \frac{32D_0(z) - 32}{\epsilon^3} + \frac{-64D_1(z) + 48D_0(z) + 64L(z) - 96}{\epsilon^2} \right. \right.
\]
\[
+ \frac{64D_2(z) - 96D_1(z) + 128D_0(z) - 64L^2(z) + 208L(z) - 196}{\epsilon} - \frac{128}{3} D_3(z) \left. \right. \right.
\]
\[
+ 96D_2(z) - 256D_1(z) + 256D_0(z) + \frac{128}{3} L^3(z) - 232L^2(z) + 412L(z) - 408 \left. \right] \right.
\]
\[
+ C_A C_F \left[ \frac{8D_0(z) - 8}{\epsilon^3} + \frac{-32D_1(z) + 32L(z) - 16}{\epsilon^2} + \frac{64D_2(z) - 64L^2(z) + 64L(z) + 20}{\epsilon} \right.
\]
\[
- \frac{256}{3} D_3(z) + \frac{256}{3} L^3(z) - 128L^2(z) - 60L(z) + 8 \right\} ,
\]

\( (4.6) \)

- Result: **perfect agreement** for 4 powers of the next-to-eikonal/soft logarithms at NNLO

\[
L(z) = \ln(1 - z)
\]

\[
\ln^3(1 - z), \; \ln^2(1 - z), \; \ln(1 - z), \; \ln^0(1 - z),
\]
Generalize NLP factorization (the LBKD theorem) beyond Drell-Yan, to arbitrary colour-singlet final states

- look at NLO only, i.e. predict

\[ D_1 = \left[ \frac{\ln(1 - z)}{1 - z} \right]_+ \quad D_0 = \left[ \frac{1}{1 - z} \right]_+ \quad L_1 = \ln(1 - z) \quad L_0 = \ln^0(1 - z) \]

- where “1-z” can take different forms for 2 -> 2,3 etc scattering

- apply to Drell-Yan, (multi-)Higgs, (vector boson pairs)

- for inclusive and fully differential cross sections
Previous factorization at NLO

\[ A_{\mu,a}^{(1)} \{p_i\}, k) = \sum_{l=1}^{2} \left[ g_s T_{l,a} G_{i,\mu}^{\nu} \frac{\partial}{\partial p_{l}^{\nu}} + J_{\mu,a}^{(1)}(p_l, n_l, k) \right] A^{(0)} \{p_i\} \]

- **G** is a projector, **T** a color matrix
- **initial quarks:**
  \[ J_{\mu}^{a}(p, n, k) = g_s T^{a} \left[ \frac{(2p - k)_{\mu}}{2p \cdot k} + i k^{\beta} p^{\cdot k} S_{\beta \mu} \right] \]
  \[ S_{\beta \mu} = \frac{i}{4} [\gamma_{\beta}, \gamma_{\mu}] \]

- **initial gluons:**
  \[ J_{\mu, \rho\sigma}^{a}(p, n, k) = g_s T^{a} \left[ \frac{(2p - k)_{\mu}}{2p \cdot k} \eta_{\rho\sigma} - \frac{i k^{\beta} p^{\cdot k}}{p \cdot k} M_{\beta \mu, \rho\sigma} \right] \]
  \[ M_{\beta \mu, \rho\sigma} = i (\eta_{\beta \rho} \eta_{\mu\sigma} - \eta_{\beta \sigma} \eta_{\mu\rho}) \]

- notice the spin-dependent Lorentz generator ("next-to-soft theorem")
- notice derivative term (Low’s theorem)
Lorentz generator

- The derivative term can be written as the **orbital part** of Lorentz generator

\[
G_{l,\mu}^{\nu} \frac{\partial}{\partial p_l^\nu} = \frac{k^\nu}{p_l \cdot k} \left[ p_{l,\nu} \frac{\partial}{\partial p_l^\mu} - p_{l,\mu} \frac{\partial}{\partial p_l^\nu} \right] = -\frac{i k^\nu L_{\nu \mu}^{(l)}}{p_i \cdot k}
\]

- so that

\[
A_{\mu, a}^{(1)} \left( \{p_i\}, k \right) = \sum_{l=1}^{2} g_s T_{l,a} \left[ \frac{(2p_l - k)_\mu}{2p_l \cdot k} - \frac{i k^\nu}{p_l \cdot k} \left( L_{\nu \mu}^{(l)} + \Sigma_{\nu \mu}^{(l)} \right) \right] A^{(0)} \left( \{p_i\} \right)
\]

\[
= \sum_{l=1}^{2} g_s T_{l,a} \left[ \frac{p_{l,\mu}}{p_l \cdot k} - \frac{i k^\nu}{p_l \cdot k} J_{\nu \mu}^{(l)} \right] A^{(0)} \left( \{p_i\} \right)
\]

- leads to **Scalar + Orbital + Spin** part of the NLP amplitude
Colour singlet production in gg channel

- **Square amplitude**

\[
|A_{\text{NLP}}|^2 = \sum_{\text{colours}} \left( A_{\text{scal.}}^{\sigma_1,\mu_1 \nu_1} + A_{\text{spin}}^{\sigma_1,\mu_1 \nu_1} + A_{\text{orb.}}^{\sigma_1,\mu_1 \nu_1} \right)^* \mathcal{P}_{\mu_1 \mu_2}(p_1, l_1) \mathcal{P}_{\nu_1 \nu_2}(p_2, l_2) \mathcal{P}_{\sigma_1 \sigma_2}(k, l_3) \\
\times \left( A_{\text{scal.}}^{\sigma_2,\mu_2 \nu_2} + A_{\text{spin}}^{\sigma_2,\mu_2 \nu_2} + A_{\text{orb.}}^{\sigma_2,\mu_2 \nu_2} \right),
\]

where

\[
\mathcal{P}_{\alpha \beta}(p, l) \equiv \sum_{\lambda} \epsilon^{(\lambda)}_\alpha(p) \epsilon^{(\lambda)*}_\beta(p) = -\eta_{\alpha \beta} + \frac{p_\alpha l_\beta + p_\beta l_\alpha}{p \cdot l}
\]

- Can be done using \(-\eta_{\alpha \beta}\) only (external ghosts are beyond NLP)
- Truncate to NLP, leads to

\[
|A_{\text{NLP}}|^2 = \sum_{\text{colours}} \left\{ |A_{\text{scal.}}^{\sigma,\mu \nu}|^2 + 2 \text{Re} \left[ \left( A_{\text{spin}}^{\sigma,\mu \nu} + A_{\text{orb.}}^{\sigma,\mu \nu} \right)^* A_{\text{scal.},\sigma,\mu \nu} \right] \right\}
\]

- Easy part: scalar (eikonal) part

\[
\sum_{\text{colours}} |A_{\text{scal.}}^{\sigma,\mu \nu}|^2 = 2 g_s^2 N_c (N_c^2 - 1) \frac{p_1 \cdot p_2}{p_1 \cdot k p_2 \cdot k} |A_{\mu \nu}|^2
\]
CS production in gg channel

- Spin * scalar vanishes (anti-symmetry in \(\mu\nu\))
- Orbital part leads to shifts in momentum dependence

\[
\sum_{\text{colours}} 2 \text{Re} \left[ A_{\text{orb.}}^{\sigma,\mu\nu} A_{\text{scal.},\sigma,\mu\nu} \right] = \frac{2g_s^2 N_c (N_c^2 - 1) p_1 \cdot p_2}{p_1 \cdot k p_2 \cdot k} \left[ \frac{\partial}{\partial \vec{p}_1^\alpha} + \frac{\partial}{\partial \vec{p}_2^\alpha} \right] |A_{\mu\nu}|^2
\]

where

\[
\delta p_1^\alpha = -\frac{1}{2} \left( \frac{p_2 \cdot k}{p_1 \cdot p_2} p_1^\alpha - \frac{p_1 \cdot k}{p_1 \cdot p_2} p_2^\alpha + k^\alpha \right), \quad \delta p_2^\alpha = -\frac{1}{2} \left( \frac{p_1 \cdot k}{p_1 \cdot p_2} p_2^\alpha - \frac{p_2 \cdot k}{p_1 \cdot p_2} p_1^\alpha + k^\alpha \right)
\]

- Result is simple: dipole times shifted squared LO amplitude

\[
|A_{\text{NLP}}|^2 = \frac{2g_s^2 N_c (N_c^2 - 1) p_1 \cdot p_2}{p_1 \cdot k p_2 \cdot k} |A_{\mu\nu}(p_1 + \delta p_1, p_2 + \delta p_2)|^2
\]
CS production in q\bar{q} channel

- Scalar plus orbital part very similar to gg case
  
  $$|A^\sigma_{\text{NLP}}|_{\text{scal.}+\text{orb.}}^2 = \frac{g_s^2 C_F}{z} \frac{s}{p_1 \cdot k} \frac{s}{p_2 \cdot k} |A(p_1 + \delta p_1, p_2 + \delta p_2)|^2$$

- except for the $1/z$, which is due to the kinematic shift
  
  $$s \rightarrow (p_1 + p_2 + \delta p_1 + \delta p_2)^2 = s + 2(\delta p_1 + \delta p_2) \cdot (p_1 + p_2)$$

- which is the same as
  
  $$s \rightarrow z s$$

- But the spin part now does not cancel:
  
  $$\sum_{\text{colours}} 2 \text{Re} \left[ A^\dagger_{\text{scal.}} A_{\text{spin}} \right]_{\text{NLP}} = -g_s^2 N_c C_F \frac{2p_1 \cdot p_2}{p_1 \cdot k} \frac{2p_2 \cdot p_1}{k} \frac{k \cdot (p_1 + p_2)}{p_1 \cdot p_2} |A(p_1, p_2)|^2$$

- precisely compensates $1/z \approx 1 + (1-z)!!$
Squared amplitudes and cross sections

- In summary
  - **gluons**
    \[ |\mathcal{A}_{\text{NLP}}|^2 = \frac{2g_s^2 N_c (N_c^2 - 1) p_1 \cdot p_2}{p_1 \cdot k p_2 \cdot k} |\mathcal{A}_{\mu\nu} (p_1 + \delta p_1, p_2 + \delta p_2)|^2 \]
  - **quarks**
    \[ |\mathcal{A}_{\text{NLP}}|^2 = g_s^2 C_F \frac{s}{p_1 \cdot k p_2 \cdot k} |A(p_1 + \delta p_1, p_2 + \delta p_2)|^2 \]
  - **Up to colour factors the same:**
    - eikonal (dipole) factor times shifted Born cross section
    - Born can be loop-induced, have complex parts etc.
  - **Combine carefully with phase space for general inclusive formula**

\[
\frac{d\hat{\sigma}_{\text{NLP}}^{(gg)}}{dz} = C_A K_{\text{NLP}} (z, \epsilon) \hat{\sigma}_{\text{Born}}^{(gg)} (zs) \quad K_{\text{NLP}} (z, \epsilon) = \frac{\alpha_s}{\pi} \left( \frac{4\pi \mu^2}{s} \right)^\epsilon z (1 - z)^{-1 - 2\epsilon} \frac{\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)\Gamma(1 - \epsilon)}
\]
Single Higgs production

\[
\frac{d\sigma_{\text{NLP}}^h}{dz} = \frac{\alpha_s^3 C_A}{288\pi^2 v^2} F(z, \epsilon) \left( \frac{2 - D_0(z)}{\epsilon} + 2D_1(z) - D_0(z) - 4\log(1 - z) + 2 \right)
\]

with \( F \) the well-known Born function. D’s and L’s agree with exact calculation, but also with full top mass dependence!

Dawson; Spira, Djouadi, Graudenz, Zerwas
Di-Higgs production

- Double Higgs production at NLO-NLP

\[ z \frac{d\sigma_{NLO}}{dz} = \frac{\alpha_s}{3\pi} C_A \left( \frac{\overline{\mu}^2}{s} \right) \epsilon \left[ \frac{12 - 6D_0(z)}{\epsilon} + 12D_1(z) - 24\log(1 - z) \right] \sigma_{Born}^{hh} (zs) \]

- where

\[ \frac{d\hat{\sigma}_{Born}}{dt} = \frac{\alpha_s^2}{8\pi^3} \frac{1}{512v^4} \left[ |C_{\triangle}F_{\triangle} + C_{\square}F_{\square}|^2 + |C_{\square}G_{\square}|^2 \right] \]

- with triangle and box graphs, again for full top mass dependence

- Should be useful for numerical evaluations, and seeing new patterns

- Similar result for triple-Higgs production

Borowka, Greiner, Heinrich, Jones, Kerner, Schlenk, Zirke

De Florian, Mazzitelli
Final state partons: Prompt photon production

Beenakker, van Beekveld, EL, White to appear
With final state partons: prompt photon

- Two LO channels: $q\bar{q}$ and $qg$
  
  ![Diagram of LO channels](image)

- With extra radiation, different ways to define threshold. We shall use “w”→1

$$
\begin{align*}
  u_1 &= (p_1 - p_{\gamma})^2 \equiv -swv \\
  t_1 &= (p_2 - p_{\gamma})^2 \equiv s(v - 1) \\
  s_4 &= s + t_1 + u_1 = sv(1 - w)
\end{align*}
$$

- Two issues to deal with
  - shifting kinematics in 2 → 2 kinematics
  - soft fermion emission
Gluon emission

- For q̅q channel

- Can in fact write down general formula

\[
\mathcal{A}_{\text{NLP}} = \mathcal{A}_{\text{scal}} + \mathcal{A}_{\text{spin}} + \mathcal{A}_{\text{orb}} \\
= \sum_{j=1}^{n+2} \frac{g_s T_j}{2 p_j \cdot k} \left( \mathcal{O}_{\text{scal},j}^{\sigma} + \mathcal{O}_{\text{spin},j}^{\sigma} + \mathcal{O}_{\text{orb},j}^{\sigma} \right) \otimes i\mathcal{M}_{H}(p_1, \ldots, p_i, \ldots, p_{n+2}) \epsilon^*_\sigma(k),
\]

- color charge and spin generator depends on emitting IS or FS particle
- orbital part on IS or FS particle
Squared amplitude at NLP

- Result: again dipoles plus momentum shift
- Important to implement $2 \rightarrow 3$ momentum conservation in $2 \rightarrow 2$ matrix element
  - used Catani-Seymour dipoles (FKS is also possible)  

\[
|A_{\text{NLP},q\bar{q} \rightarrow \gamma gg}|^2 = \frac{C_F}{C_A} \left[ C_F \frac{2p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)} |M_{q\bar{q} \rightarrow \gamma g}(p_1 + \delta p_{1;2}, p_2 + \delta p_{2;1})|^2 \\
+ \frac{1}{2} C_A \frac{2p_1 \cdot p_R}{(p_1 \cdot k)(p_R \cdot k)} |M_{q\bar{q} \rightarrow \gamma g}(p_1 + \delta p_{1;R}, p_R - \delta p_{R;1})|^2 \\
+ \frac{1}{2} C_A \frac{2p_2 \cdot p_R}{(p_2 \cdot k)(p_R \cdot k)} |M_{q\bar{q} \rightarrow \gamma g}(p_2 + \delta p_{2;R}, p_R - \delta p_{R;2})|^2 \\
- \frac{1}{2} C_A \frac{2p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)} |M_{q\bar{q} \rightarrow \gamma g}(p_1 + \delta p_{1;2}, p_2 + \delta p_{2;1})|^2 \right].
\]

- Integrate over NLO phase, agrees with NLO calculation including $\ln(1-w)$ terms

Gervais

Gordon, Vogelsang
Soft fermions

- At NLP (not LP) one can have soft fermion emission

![Feynman diagrams](image)

- Effective feynman rule for left diagram (note that “u(k)” is of order \(\sqrt{k}\))

\[
i \mathcal{M}_{NLP,1,g} = \frac{g_s T^a c m c_j}{(p_1 - k)^2 + i\varepsilon} \epsilon^\mu (p_1) \bar{u}(k) \gamma_\mu \mathcal{M}_{c_j} (p_1, p_2, \ldots, p_{n+2})
\]

- Right diagram

\[
i \mathcal{M}_{NLP,1,g} = \frac{g_s T^b c m c_i}{(p_1 - k)^2 + i\varepsilon} \bar{u}(k) \gamma_\rho u(p_1) \mathcal{M}_{c_j} (p_1, p_2, \ldots, p_{n+2}).
\]

- Squaring amplitude and integration over phase space gives agreement with exact NLO

- Must keep careful track of singular regions
LL resummation of NLP logarithms

Bahjat-Abbas, Bonocore, EL, Magnea, Sinninghe Damsté, Vernazza, White

to appear
LL resummation of NLP logarithms

- We have organized NLP threshold logs at NLO and NNLO for Drell-Yan. Can one resum them?
- First resummation conjecture: just change kernel in regular resummation formula
  \[
  \frac{1 + z^2}{1 - z} \rightarrow \frac{2}{1 - z} - 2
  \]
  
  - reproduced NNLO NLP logs of van Neerven et al
- Physical kernel approach for inclusive quantities
  - using single log behaviour of kernel
- Recent LL resummation using SCET

\[
\Delta_{\text{NLP}}^{\text{LL}}(z, \mu) = \exp \left[ 4 S_{\text{LL}}(\mu_h, \mu) - 4 S_{\text{LL}}(\mu_s, \mu) \right] \times \frac{-8 C_F}{\beta_0} \ln \frac{\alpha_s(\mu)}{\alpha_s(\mu_s)} \theta(1 - z).
\]
NLP amplitude exponentiation via path integral

- Fluctuations around classical path are NE corrections
  - All NLP corrections from external lines exponentiate
  - Keep track via scaling variable $\lambda$
    \[ p^\mu = \lambda n^\mu \]

\[
f(\infty) = \int_{x(0)=0} D x \exp \left[ i \int_0^\infty dt \left( \frac{\lambda}{2} \dot{x}^2 + (n + \dot{x}) \cdot A(x_i + nt + x) + \frac{i}{2\lambda} \partial \cdot A(x_i + p_f t + x) \right) \right]
\]

- Exponentiation then in terms of NLP webs

\[
\sum C(D) F(D) = \exp \left[ \bar{C}(D) W_E(D) + \bar{C}'(D) W_{NE}(D) \right]
\]

Bonocore, EL, Magnea, Melville, Vernazza, White
LL resummation for cross section at NLP

- Can show that phase space NLP effects behave as
  \[ \varepsilon (1 - z) \]
  - i.e. softness suppression comes with singularity suppression
  - => phase space does not give leading logs
- Can show that there are no LL enhancements from purely collinear regions (single log)
  - => LL effects come then only from NLP soft function = NLP webs
Exponentiating NLP soft function

- Moments of cross section

\[
\int_0^1 d\tau \tau^{N-1} \frac{d\sigma_{DY}}{d\tau} \bigg|_{LL, NLP} = \sigma_0(Q^2) q_N(Q^2) \bar{q}_N(Q^2) \tilde{S}_{NLP}(N, Q^2, \epsilon),
\]

- with NLP soft function (f's are NLP Wilson lines)

\[
\tilde{S} = \frac{1}{N_c} \sum_n \text{Tr} \left[ \langle 0 | f_2^\dagger f_1 | n \rangle \langle n | f_1^\dagger f_2 | 0 \rangle \right] \delta \left( z - \frac{Q^2}{\tilde{s}} \right).
\]

- Exponentiation then gives

\[
\int_0^1 d\tau \tau^{N-1} \frac{d\sigma_{DY}}{d\tau} \bigg|_{LL, NLP} = \sigma_0(Q^2) q_{LL, NLP}(N, Q^2) \bar{q}_{LL, NLP}(N, Q^2) \times \exp \left[ \frac{\alpha_s C_F}{\pi} \left( 2 \log^2(N) + \frac{4 \log(N)}{N} \right) \right].
\]

- agrees with 1998 conjecture
LL resummation of NLP logarithms in prompt photon production

Basu, Beenakker, van Beekveld, EL, Misra, Motylinski

to appear
NLP resummation in prompt photon production at fixed $p_T$

- **Threshold resummation of powers of**
  
  $$\ln(1 - x_T^2) \quad x_T^2 = \frac{4p_T^2}{s}$$

- **Threshold resummation long known, to NNLL and even beyond**

- **Joint threshold+recoil resummation**
  
  EL, Sterman, Vogelsang

  - gives about 20% correction w.r.t. threshold

- **Two ways of including NLP logarithms**
  
  1) Extend kernel to NLP in Sudakov exponent => modified resummation exponent
  2) Extend PDF evolution to soft scale, automatically includes NLP terms

EL, Oderda, Sterman; Catani, Mangano, Nason, Oleari
Ridolfi; De Florian, Vogelsang+Sterman, Schaefer;
Becher, Schwartz + Lorentzen; Hinderer, Ringer, Sterman,
Vogelsang

\[\ln(1 - x_T^2) \quad x_T^2 = \frac{4p_T^2}{s}\]
Joint-resummation

**Joint-resummed formula**

\[
\frac{p_T^3}{8\pi S^2} \frac{d\sigma_{AB\rightarrow\gamma+X}^{(direct,\ joint)}}{dp_T} = \frac{p_T^4}{8\pi S^2} \sum_{ab} \int_C \frac{dN}{2\pi i} \int \frac{d^2Q_T}{(2\pi)^2} \left( \frac{S}{4|p_T - Q_T|/2} \right)^{N+1} f_{a/A}(N, \mu_F) f_{b/B}(N, \mu_F) \\
\times \int_0^1 d\tilde{x}_T^2(\tilde{x}_T^2)N \frac{\mathcal{M}_{ab\rightarrow\gamma d}(\tilde{x}_T^2)}{\sqrt{1 - \tilde{x}_T^2}} C_s^{(ab\rightarrow\gamma d)}(\alpha_s, Q^2/\mu_F^2, Q^2/\mu_R^2) \\
\times \int d^2b \ e^{ib\cdot Q_T} \theta(\mu - |Q_T|) P_{abd}(N, b, \mu_F, \mu_R, Q).
\]

**with resummation**

\[
P_{abd}(N, b, \mu_F, \mu_R, Q) = \exp [E_a^{PT}(N, b, \mu_F, \mu_R, Q) + E_b^{PT}(N, b, \mu_F, \mu_R, Q) + F_d(N, \mu_R, Q) + g_{abd}(N)]
\]

Initial state       Initial state       Final state       Soft

\[
E_a^{PT}(N, b, \mu_F, \mu_R, Q) = \int_0^{Q^2} \frac{dk_T^2}{k_T^2} A_a(\alpha_s(k_T^2)) \left[ J_0(bk_T) K_0 \left( \frac{2Nk_T}{Q} \right) + \ln \left( \frac{\tilde{N}k_T}{Q} \right) \right] \\
- \ln N \int_0^{Q^2} \frac{dk_T^2}{k_T^2} A_a(\alpha_s(k_T^2))
\]
Numerical results

- Effect of NLP logs, LL accuracy = about 10-20% positive
- Scale uncertainty reduces as well
Summary

- Soft approximation reveals patterns enabling all-order resummation
- Next-to-soft/NLP is also promising
- Factorization + LBDK theorem leads to strong predictive power for NLP threshold logs
  - Drell-Yan at NNLO
- Simply NLP formulae at NLO for colour singlet final states
  - and now also prompt photon
- LL resummation at NLP for Drell-Yan done
  - NLL seems much harder
- NLP corrections are becoming an interesting object of study
  - Dedicated recent workshops in Edinburgh and Amsterdam