Next-to-soft QCD corrections to hadronic cross sections

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Outline

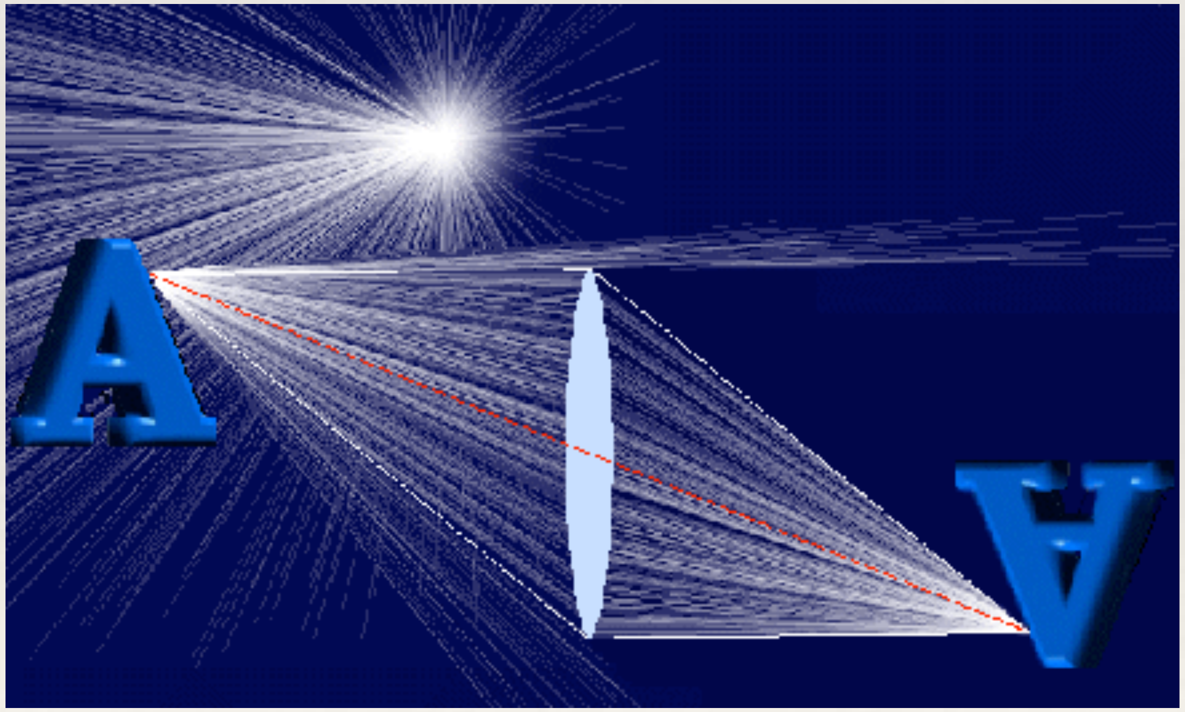
- Introduction to leading power soft approximation
  - Soft exponentiation
- Next-to-leading power threshold logarithms
  - Regions, factorization at NNLO
- Next-leading power at NLO
  - colour singlet, prompt photon
- LL resummation at NLP
  - Drell-Yan, prompt photon

A brief history of the eikonal approximation

- “Eikon” originally from Greek εικεναι [to resemble]
  - leading to εικον [icon, image]
- Predates quantum mechanics, and even Maxwell
  - also known in optics as “ray optics”
    - ✓ Rays are straight lines, perpendicular to wave fronts

Eikonal optics: rays

- Can describe formation of images/eikons
  - wavelength << size of scatterer
- Cannot describe diffraction, polarization etc
  - these are wave phenomena

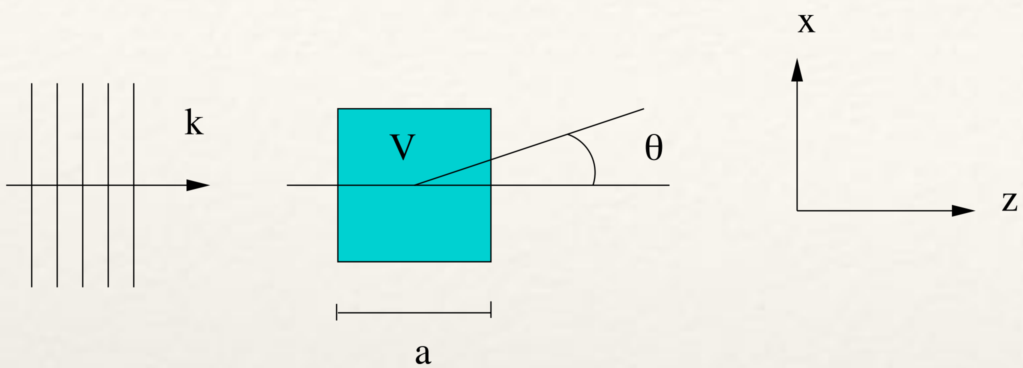


Eikonal quantum mechanics

- Eikonal approximation in QM scattering
  - R.J. Glauber [1959 lecture notes (recommended!)]
  - Highly energetic particle scattering off potential
  - (regained use in scattering on nucleus)
- 2D example
  - scattering on square potential of limited range

QM eikonal scattering in 2D

Glauber, 1958 Boulder lectures



Incoming wave

Scattered wave

$$\psi(x) = e^{ikz} + \frac{\sqrt{k}}{\sqrt{2\pi}} e^{i\frac{\pi}{4}} f(\theta) \frac{e^{ikr}}{\sqrt{r}}$$

Scattering amplitude

Factor high-momentum part

$$\psi(\vec{r}) = e^{ikz}\phi(\vec{r})$$

$$E \gg V \qquad k \gg 1/a$$

Substitute in Schrodinger equation

$$e^{ikz}(-k^2 + 2ik\partial_z + \cancel{\nabla^2} + k^2)\phi(\vec{r}) = e^{ikz}\frac{2m}{\hbar^2}V(\vec{r})\phi(\vec{r})$$

Solve

$$\phi(\vec{r}) = e^{-\frac{i}{\hbar v}\int_{-\infty}^{z} dz' V(x,z')}$$

Exponential form!

QM eikonal scattering in 2D

- Approximation at amplitude level, some "wave" information is preserved. If potential is "black":
  - Optical theorem

$$\sigma_{tot} = 2\,\mathrm{Im}\,f(\theta = 0) = 2a$$

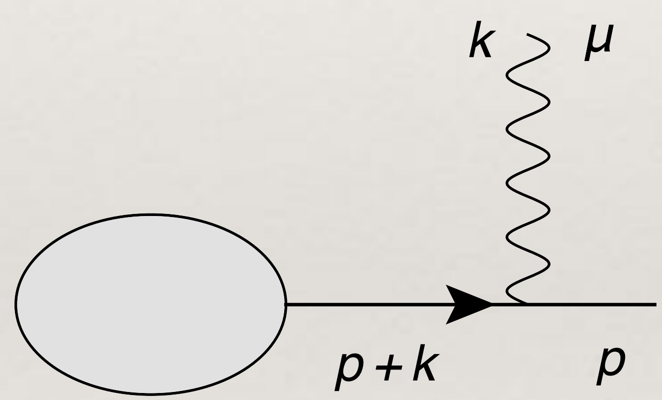
  - Scattering cross section

$$\sigma_{scattered} = \frac{k}{2\pi}\int d\theta\,|f(\theta)|^2 = a$$

    - ✓ Factor 2 is due to diffraction, fill in shadow of target

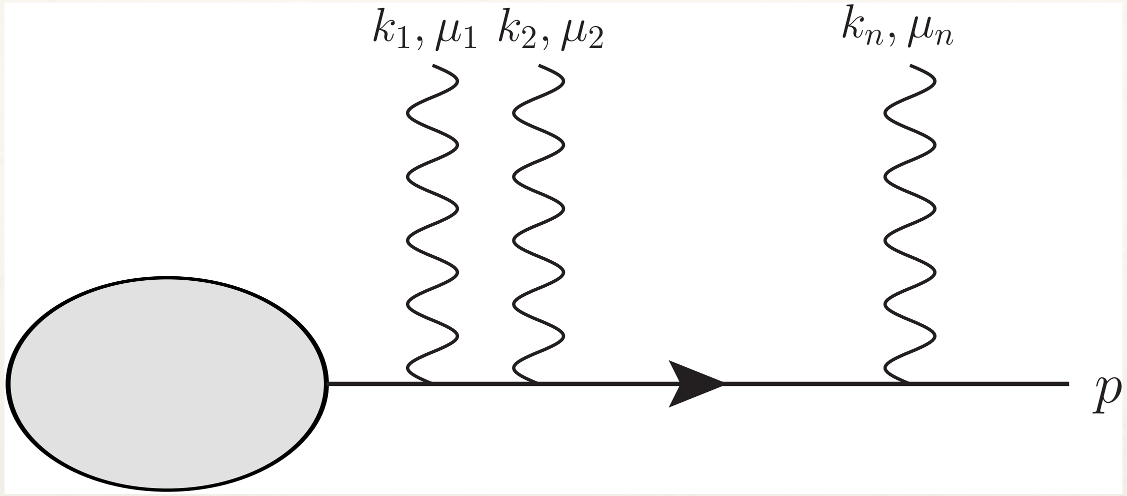
Eikonal QFT: QED

- Charged particle emits soft photon
  - Propagator: expand numerator & denominator in soft momentum, keep lowest order
  - Vertex: expand in soft momentum, keep lowest order



$$\frac{(p+k)^\mu + p^\mu}{2p\cdot k + k^2} \longrightarrow \frac{2p^\mu}{2p\cdot k}$$

Basics of eikonal approximation in QED



Exact: $$\frac{1}{(p+K_1)^2}(2p+K_2+K_1)^{\mu_1}\dots\frac{1}{(p+K_n)^2}(2p+K_n)^{\mu_n}, \quad K_i=\sum_{m=i}^{n}k_m.$$

Approx: $$\frac{1}{2pK_1}2p^{\mu_1}\dots\frac{1}{2pK_n}2p^{\mu_n}$$

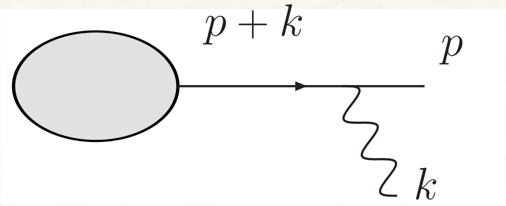
Eikonal identity: $$\frac{1}{p\cdot(k_1+k_2)\,p\cdot k_2}+\frac{1}{p\cdot(k_1+k_2)\,p\cdot k_1}=\frac{1}{p\cdot k_1\,p\cdot k_2}$$

Sum over all perm's: $$\prod_i \frac{p^{\mu_i}}{p\cdot k_i}.$$

Independent, uncorrelated emissions, Poisson process

Eikonal approximation: no dependence on emitter spin

- Emitter spin becomes irrelevant in eikonal approximation
  - Fermion



$$M \frac{i(\not{p}+\not{k})}{(p+k)^2}(-ig_s\gamma^\mu)u(p)$$

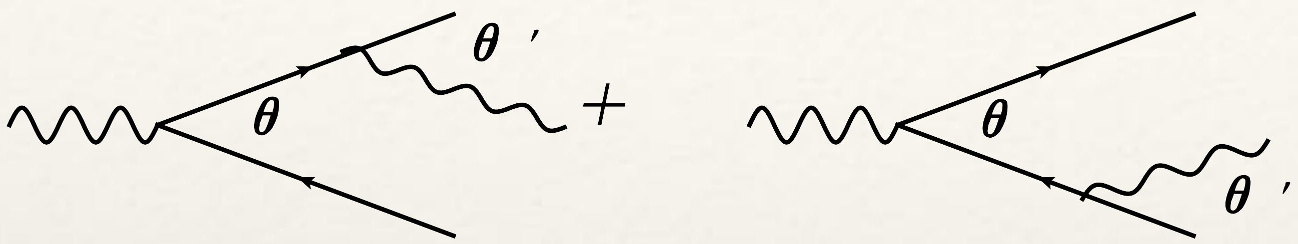
  - Approximate, and use Dirac equation $\not{p}u(p) = 0$
  - Result:

$$g\,(M\,u(p)) \times \frac{p^\mu}{p\cdot k}$$

  - Notice
    - ✓ No sign of emitter spin anymore (= scalar emitter)
    - ✓ Coupling of photon proportional to emitter momentum p^μ !

Another eikonal effect: coherence in emission

- Eikonal approximation in amplitude, coherence possible
  - First in QED



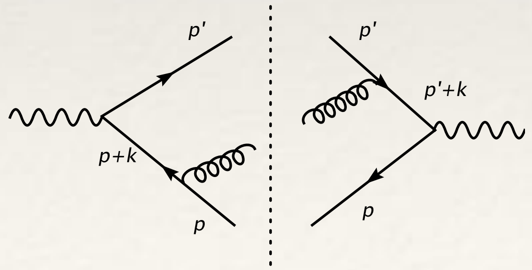
  - Square the amplitude, take the eikonal approximation, and combine with phase. Result

$$d\sigma_R = d\sigma \frac{\alpha_s}{2\pi} \frac{dE}{E} d\cos\theta d\phi \; E^2 \frac{p \cdot \bar{p}}{p \cdot k \, \bar{p} \cdot k}$$

  - Only non-zero when θ'<θ : angular ordering after azimuthal integral
    - ✓ photon that is too soft only see the sum of the charges, which is zero here.
  - In QCD very similar result (after being a little bit more careful with color charges). Radiation function

$$W_{ij} = \frac{\omega^2 p_i \cdot p_j}{p_i \cdot q \, p_j \cdot q} = \frac{1 - v_i v_j \cos\theta_{ij}}{(1 - v_i \cos\theta_{iq})(1 - v_j \cos\theta_{jq})}$$

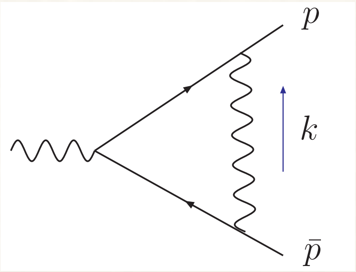
    - ✓ clearly has eikonal form. Notice, it is an interference effect:



Eikonal exponentiation

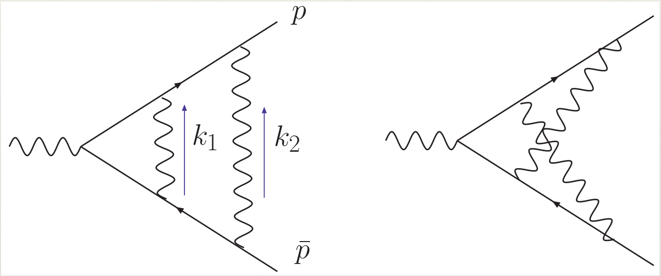
- After eikonal approximation, we suddenly see interesting patterns.

One loop vertex correction, in eikonal approximation



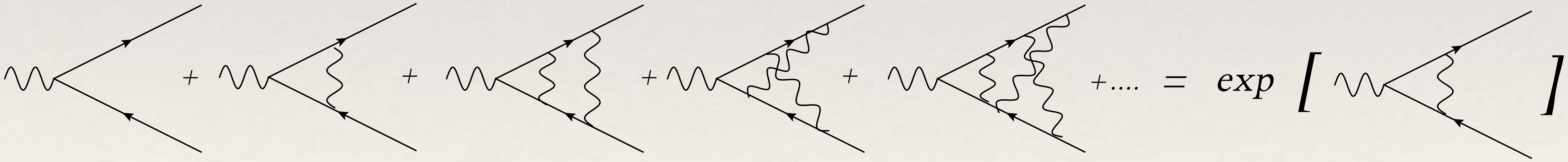
$$\mathcal{A}_0 \int d^n k \frac{1}{k^2} \frac{p \cdot \bar{p}}{(p \cdot k)(\bar{p} \cdot k)}$$

Two loop vertex correction, in eikonal approximation



$$\mathcal{A}_0 \, \frac{1}{2} \left(\int d^n k \frac{1}{k^2} \frac{p \cdot \bar{p}}{(p \cdot k)(\bar{p} \cdot k)} \right)^2$$

Exponential series! A really beautiful result



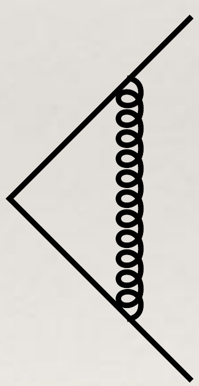
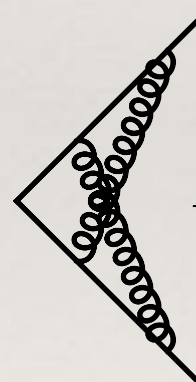
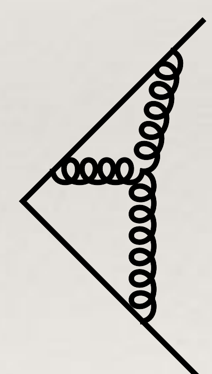
Yennie, Frautschi, Suura '61

QCD exponentiation: webs

- Not immediately generalizable to QCD, seemingly
  - Vertices terms have color charges, which don't commute
  - Still, an exponentiation theorem holds

$$\sum_D \mathcal{F}_D C_D = \exp\left[\sum_i \bar{C}_i w_i\right]$$

Webs

- Exp $\Big[$ C_F  $+\left(-\frac{1}{2}C_A C_F\right)$  $+\left(-\frac{1}{2}C_A C_F\right)$  $\Big]$

Gatheral; Frenkel, Taylor; Sterman

EL, Stavenga, White

Eikonal approximation from QM path integrals

EL, Stavenga, White

Another way to exponentiate: use textbook QFT result

$$\text{Sum of all diagrams} = \exp\left(\text{Connected diagrams}\right)$$

$$f = e^{i \int dt (\frac{1}{2}\dot{x}^2 + p \cdot A + ..)}$$

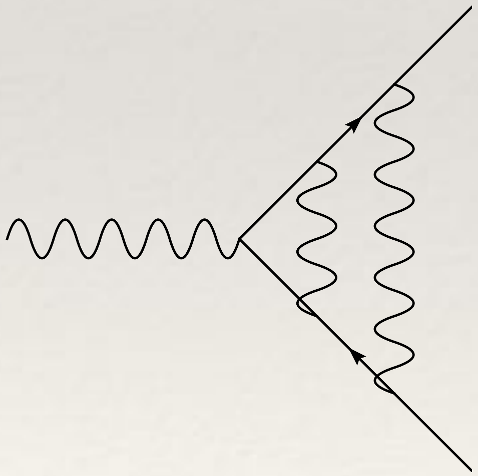
Write scattering amplitude as path integral

$$M(p_1, p_2, \{k\}) = \int \mathcal{D}A_s \, \mathcal{D}x(t) \, H[x] \, f_1[A_s, x(t)] \, f_2[A_s, x(t)] \, e^{iS[A_s]}$$

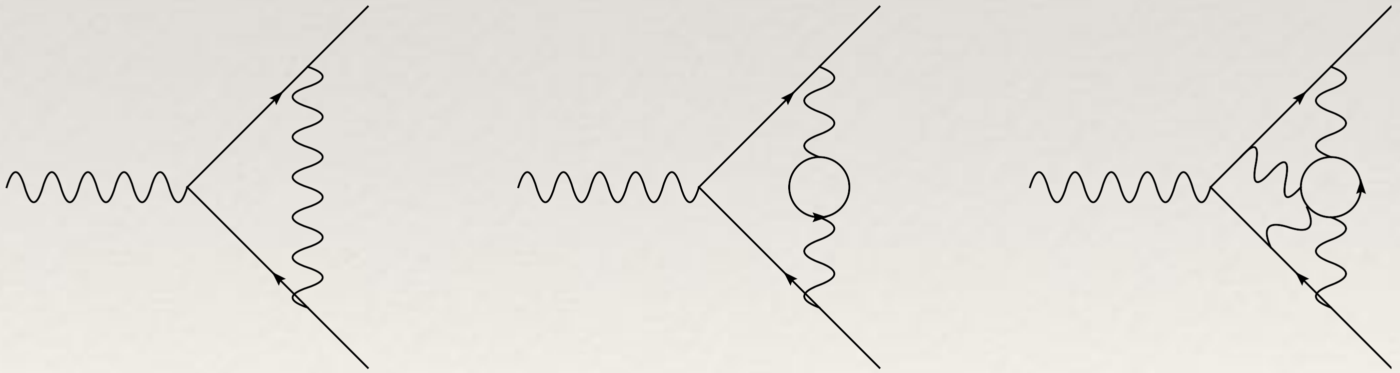
x(t): path of charged particle

Eikonal vertices are sources for gauge bosons along line v

$$v \cdot A(x(t))$$

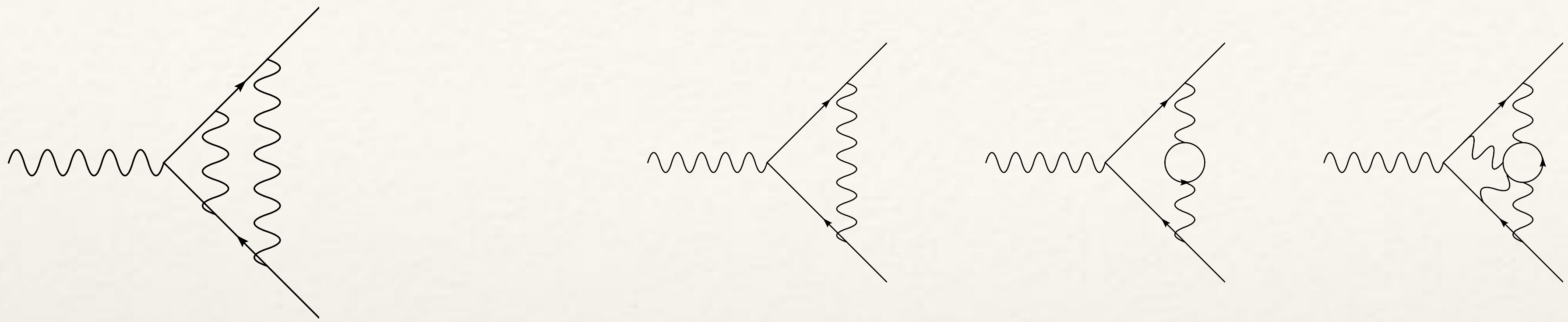


Disconnected



Connected

Path integral method, non-abelian



- Not immediately obvious how this could work (the path integral must be an actual exponential), since
  - Source terms have non-abelian SU(3)-valued charges, so don't commute
  - External line factors are path-ordered exponentials
  - Nevertheless

$$\sum_D \mathcal{F}_D C_D = \exp\left[\sum_i \bar{C}_i w_i\right]$$

Gatheral; Frenkel, Taylor; Sterman

- To prove, use replica trick (from statistical physics)

Replica trick

EL, Stavenga, White

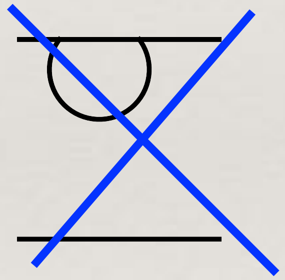
- Relates exponentiation of soft gauge fields to that of connected diagrams in QFT.
- Consider a N copies of a scalar theory

  - If Z is exponential, find out what contributes to log Z

$$Z[J]^N = \int \mathcal{D}\phi_1 \ldots \mathcal{D}\phi_N e^{iS[\phi_1]+..+iS[\phi_N]+J\phi_1+..J\phi_N}$$

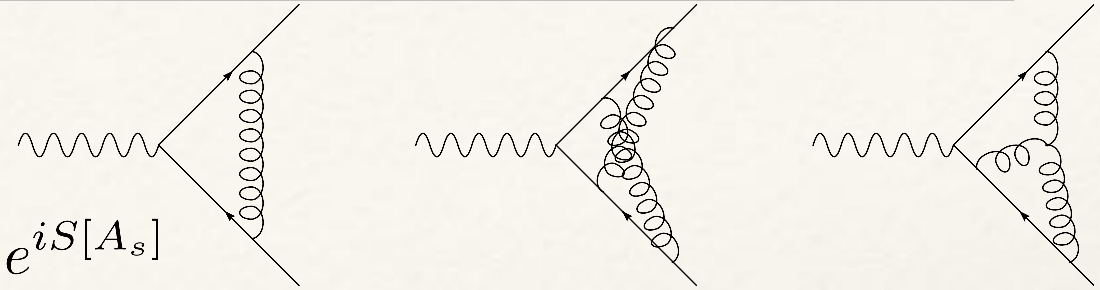
  - Amounts to diagrams that allow only one replica $\longrightarrow$ connected!

$$Z^N = 1 + N \log Z + \mathcal{O}(N^2)$$



Replica method and QCD

Amplitude for two colored lines

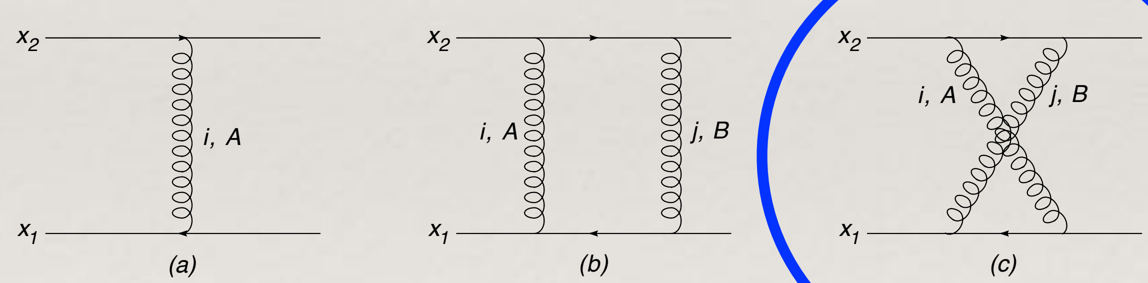


$$S(p_1,p_2) = H(p_1,p_2)\int \mathcal{D}A_s f(\infty) e^{iS[A_s]}$$

Replicate, and introduce **replica ordering operator R**

$$f(\infty) = \mathcal{P}\exp\left[\int dx\cdot A(x)\right] \qquad \prod_{i=1}^{N}\mathcal{P}\exp\left[\int dx\cdot A_i(x)\right] = \mathcal{R}\mathcal{P}\exp\left[\sum_{i=1}^{N}\int dx\cdot A_i(x)\right]$$

Look for diagrams of replica multiplicity N. These will go into exponent



Web
Modified color factor

(a) is order N

(b) for equal replica number (i=j): C_F^2. For i≠j also C_F^2. Sum: $NC_F^2 + N(N-1)C_F^2 = N^2C_F^2$

(c) for equal replica number (i=j): C_F^2-C_F C_A /2.
For i≠j C_F^2. Term linear in N:

$$N\left(C_F^2 - \frac{C_FC_A}{2}\right) + (-N)C_F^2 = N\left(-\frac{C_FC_A}{2}\right)$$

Multiple colored lines

- Structure

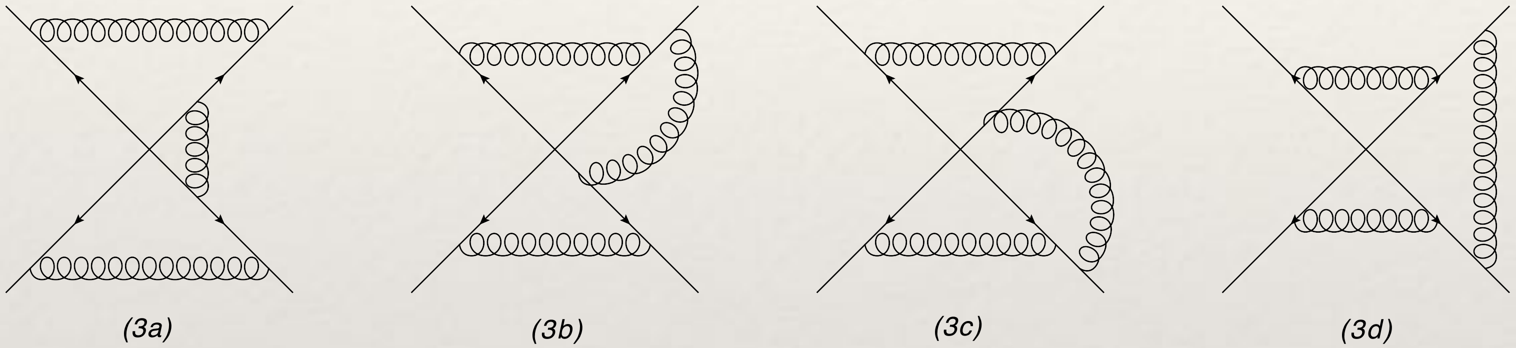
Projector matrix

$$\sum \mathcal{F}(D)C(D) = \exp[\sum_{d,d'} \mathcal{F}(d)R_{dd'}C(d')]$$

$$\sum_{d'} R_{dd'} = 0$$

Eigenvalues 0 or 1

  - multi-parton webs are “closed sets” of diagrams, with modified color factors



= Web

- Closed form solution for modified color factor

$$\frac{1}{6}\Big[C(3a) - C(3b) - C(3c) + C(3d)\Big] \times \Big[M(3a) - 2M(3b) - 2M(3c) + M(3d)\Big]$$

  - Interesting properties of projector matrix (reduces degree of divergence)

Perturbative series for cross sections in QFT

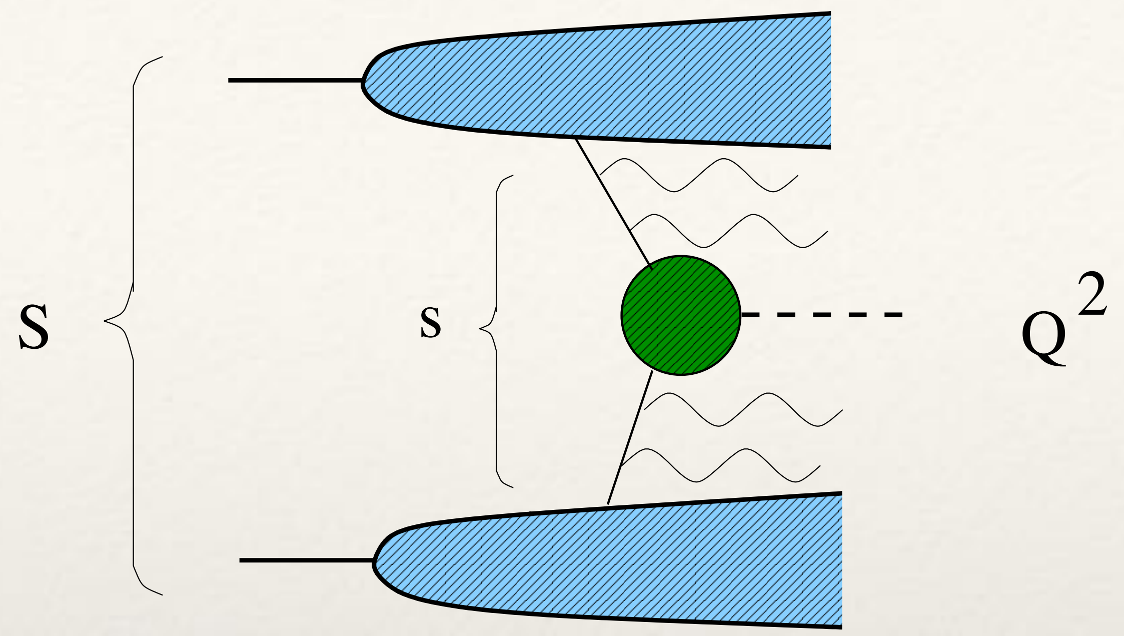
- Typical perturbative behavior of observable
  - α is the coupling of the theory (QCD, QED, ..)
  - L is some numerically large logarithm
  - “1” = π^2, ln(2), anything not-logarithmic
  - Notice: ***effective* expansion parameter is αL^2 i.e. a problem when >1!!**
  - **Fix**: reorganize/resum terms such that

$$\hat{O}_2 = 1 + \alpha(L^2 + L + 1) + \alpha^2(L^4 + L^3 + L^2 + L + 1) + \ldots$$

$$\begin{aligned}\hat{O} &= 1 + \alpha_s(L^2 + L + 1) + \alpha_s^2(L^4 + L^3 + L^2 + L + 1) + \ldots \\ &= \exp\left(\underbrace{\underbrace{Lg_1(\alpha_s L)}_{LL} + g_2(\alpha_s L)}_{NLL} + \alpha_s g_3(\alpha_s L) + \ldots\right)\underbrace{C(\alpha_s)}_{\text{constants}} \\ &\quad + \text{ suppressed terms}\end{aligned}$$

- Notice the definition of LL, NLL, etc

Threshold logarithms



Log of "energy excess above production threshold"

$S \geq s \geq Q^2$

$$L^2 = \ln^2\left(1 - \frac{Q^2}{s}\right) \equiv \ln^2(1-z)$$

Threshold resummed Drell-Yan (or Higgs) cross section

Sterman; Catani, Trentadue; Ahmed, Makahud, Rana, Ravindran

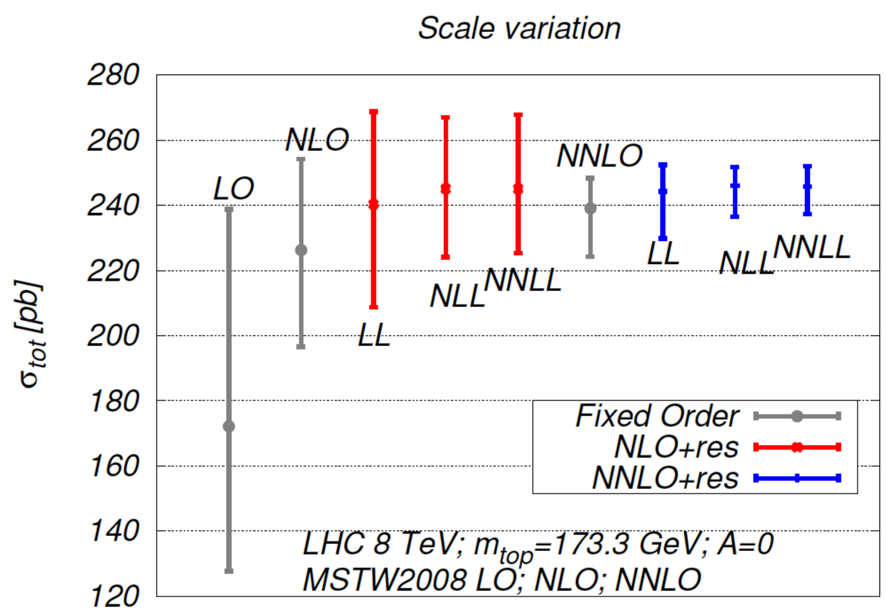
Threshold logarithms can be resummed to all orders

$$\frac{d\sigma^{\rm resum}}{dQ^2}(z) = \int_C \frac{dN}{2\pi\, i}\, z^{-N}\, \hat{\sigma}(N)$$

$$\sigma(N) = \exp\left[-\int_0^1 dx\, \frac{x^{N-1}-1}{1-x}\left\{\int_{Q^2}^{Q^2(1-x)^2} \frac{d\mu}{\mu} A(\alpha_s(\mu)) + D(\alpha_s((1-x)Q))\right\}\right] \times \left(1+\alpha_s(Q^2)\frac{C_F}{\pi}+\ldots\right)$$

Note: functions in exponent only depend on α_s

A similar case: top quark pair production, much smaller uncertainty



Concurrent uncertainties:

Scales	~ 3%
pdf (at 68%cl)	~ 2-3%
α_s (parametric)	~ 1.5%
m_{top} (parametric)	~ 3%

Soft gluon resummation makes a difference

5% -> 3%

NLP threshold behavior

- For Drell-Yan, DIS, Higgs, singular behavior in perturbation theory when z⟶ 1

$$\delta(1-z) \qquad \left[\frac{\ln^i(1-z)}{1-z}\right]_+ \qquad \ln^i(1-z)$$

  - plus distributions have been organized to all orders (="resummation"), also possible for ln(1-z)?

- "Zurich" method of threshold expansion allows computation (for NNNLO Higgs production)

$$(1-z)^p \ \ln^q(1-z)$$

Anasthasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Mistlberger

  - done to p=37..

- Much development in SCET

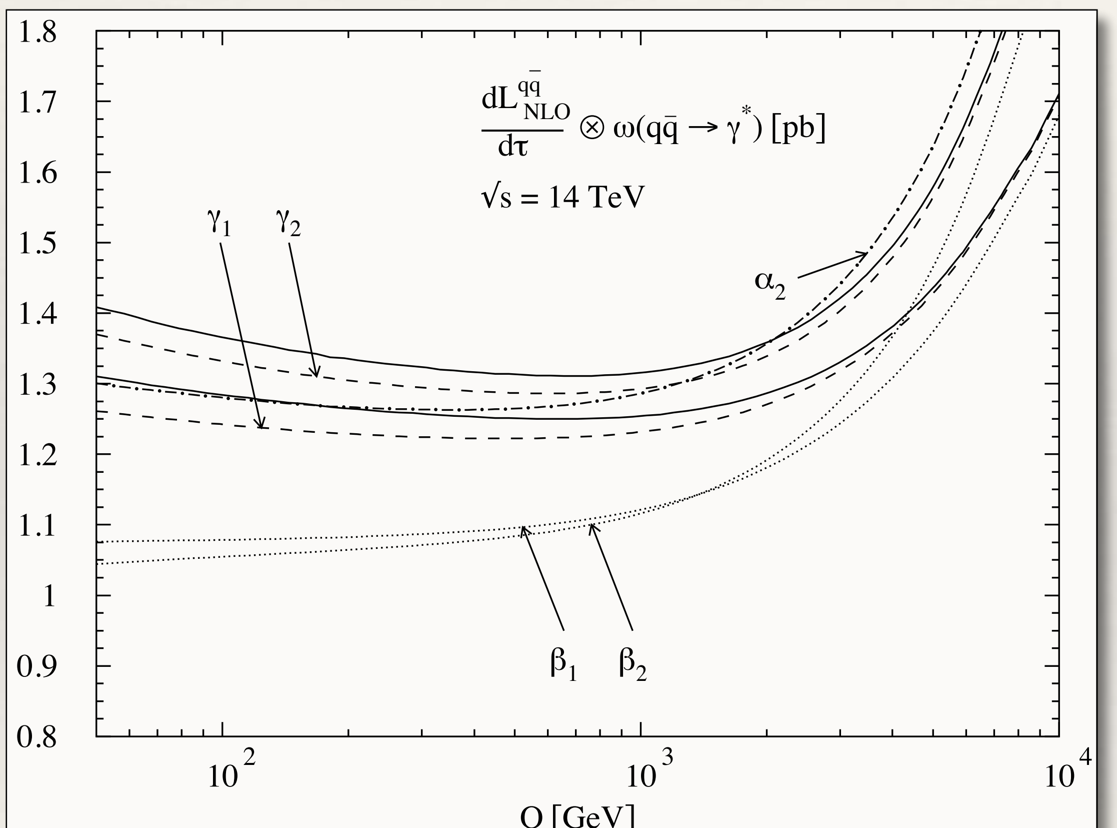
Larkoski, Neill, Stewart, Moult, Kolodrubetz, Rothen, Zhu, Tackmann, Vita, Feige ; Beneke, Campanario, Mannel, Peckja

- Useful also for improving NNLO slicing (N-jettiness) methods
- Alternative terminology to "NLP"
  - Next-to-soft
  - Next-to-eikonal

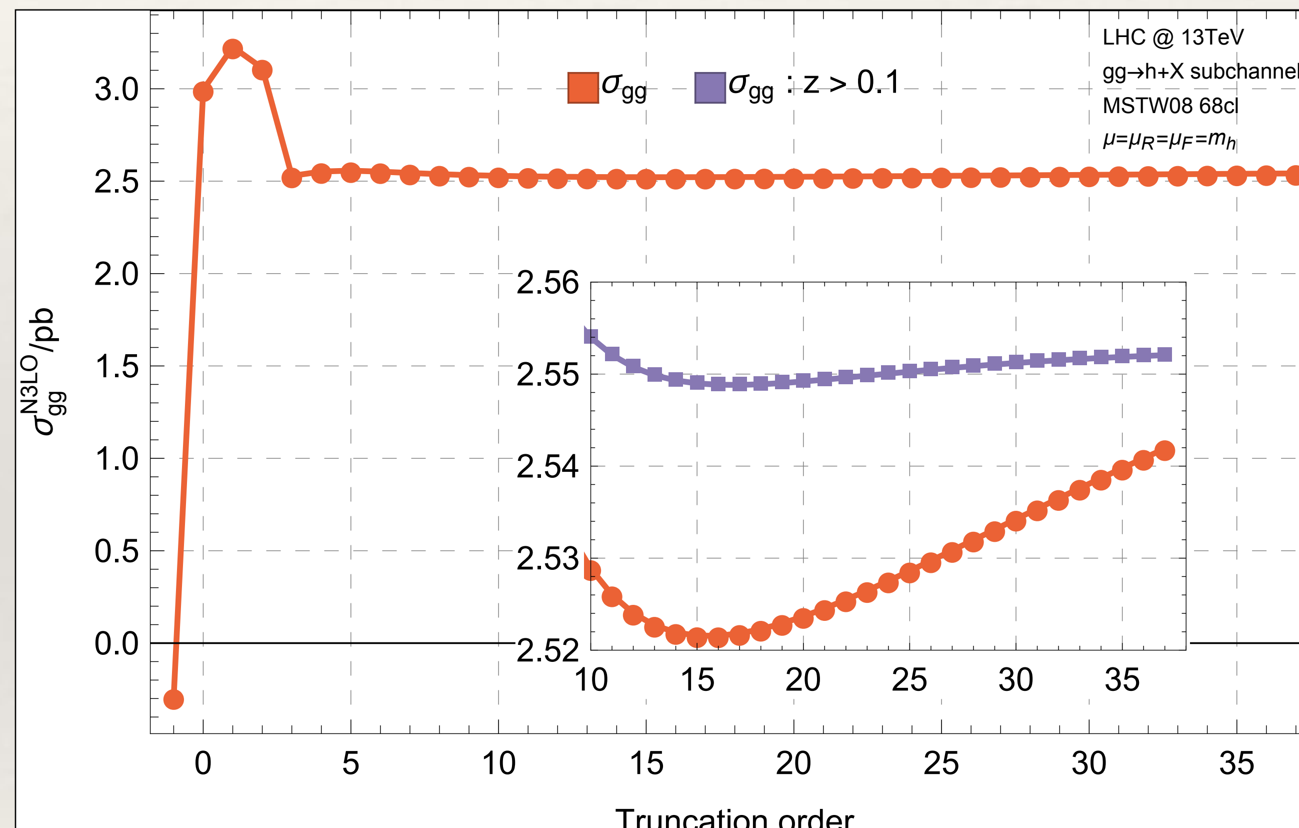
Numerical effects of NLP logarithms

- General power expansion

$$\frac{d\sigma}{dz} = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \sum_{m=0}^{2n-1} \left\{ c_{nm}^{(-1)} \left.\frac{\log^m(1-z)}{1-z}\right|_+ + c_{nm}^{(0)} \log^m(1-z) + \ldots \right\}$$



Kramer, EL, Spira, 1998



Anastasiou, Duhr, Dulat, Herzog, Mistlberger, 2015

- NLP logs can be quite important

Next-to-eikonal Feynman rules

- Keep 1 term more in k expansion beyond eikonal approximation

$$\text{scalar}: \quad \frac{2p^\mu + k^\mu}{2p\cdot k + k^2} \longrightarrow \frac{2p^\mu}{2p\cdot k} + \frac{k^\mu}{2p\cdot k} - k^2 \frac{2p^\mu}{(2p\cdot k)^2}$$

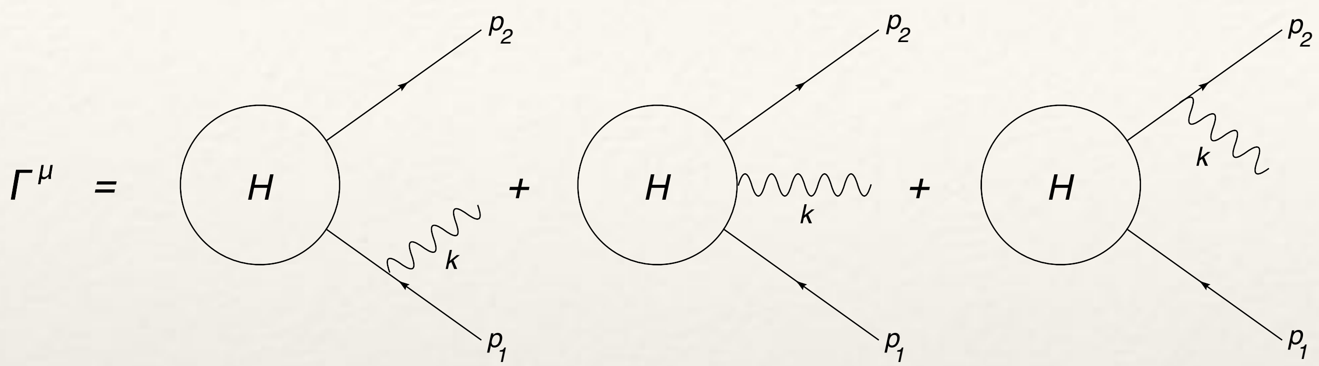
$$\text{fermion}: \quad \frac{\not{p} + \not{k}}{2p\cdot k + k^2}\gamma^\mu u(p) \longrightarrow \left[\frac{2p^\mu}{2p\cdot k} + \frac{\not{k}\gamma^\mu}{2p\cdot k} - k^2 \frac{2p^\mu}{(2p\cdot k)^2}\right] u(p)$$

  - Becomes emitter-spin dependent, recoil now included
  - Is there predictive power for the next-to-eikonal terms?

Eikonal term

Classic NLP result: Low's theorem

- These rules are good for emissions from external lines. At NLP order, also 1 "internal" emission contributes



- Low's theorem (scalars, generalization to spinors by Burnett-Kroll, to massless particles by Del Duca): **LBKD theorem**
  - ✓ Work to order k, and use Ward identity

$$\Gamma^\mu = \left[\frac{(2p_1 - k)^\mu}{-2p_1 \cdot k} + \frac{(2p_2 + k)^\mu}{2p_2 \cdot k}\right]\Gamma + \left[\frac{p_1^\mu (k \cdot p_2 - k \cdot p_1)}{p_1 \cdot k} + \frac{p_2^\mu (k \cdot p_1 - k \cdot p_2)}{p_2 \cdot k}\right]\boxed{\frac{\partial \Gamma}{\partial p_1 \cdot p_2}}$$

- Elastic amplitude still determines the emission to NLP accuracy,
  - note the derivative
  - detailed knowledge of "internal part" not needed

NLP logarithms for Drell-Yan

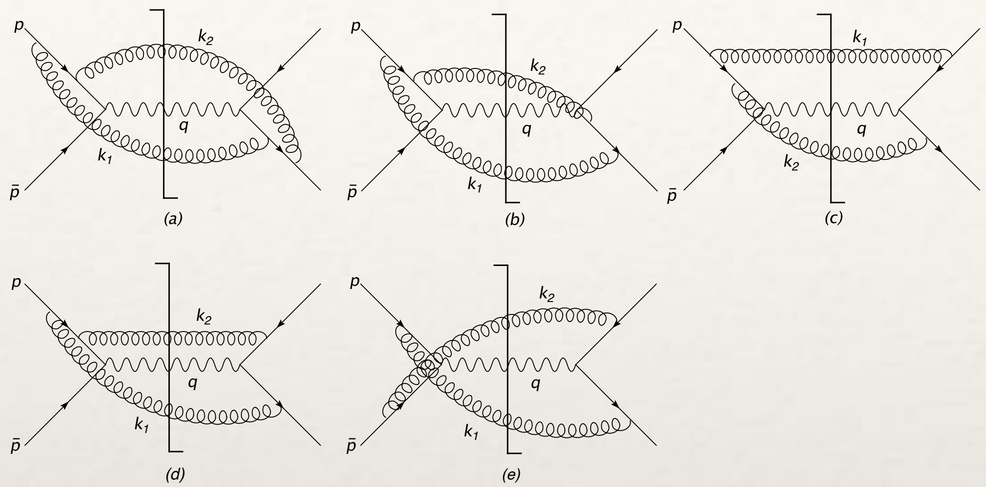
- Goal: combine (N)LP matrix elements with (N)LP phase space to **predict** $\ln^i(1-z)$ for NNLO Drell-Yan

$$\frac{1}{\sigma^{(0)}}\frac{d\hat{\sigma}}{dz} \sim \int d\Phi_{\rm LP}|\mathcal{M}|^2_{\rm LP} + \int d\Phi_{\rm LP}|\mathcal{M}|^2_{\rm NLP} + \int d\Phi_{\rm NLP}|\mathcal{M}|^2_{\rm LP} + \ldots$$

- We pursue two methods:
  - ✓ 1. Method of regions
  - ✓ 2. Factorization
- NLO is “easy”, real test at NNLO

NLP logs in Drell-Yan at NNLO

- Check NLP Feynman rules for NNLO Drell-Yan *double real* emission

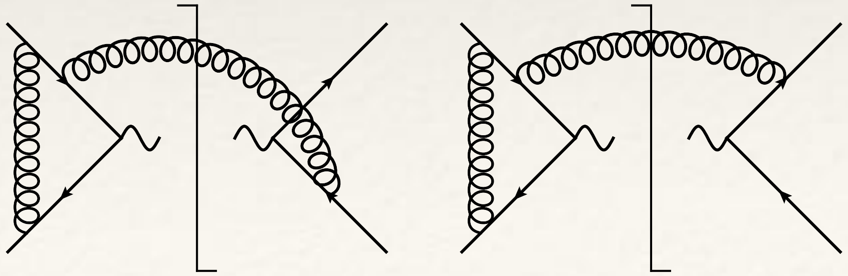


  - Result at NLP level, agrees with equivalent exact result. C_F^2 terms e.g.

$$K^{(2)}_{\rm NE}(z) = \left(\frac{\alpha_s}{4\pi}C_F\right)^2\left[-\frac{32}{\epsilon^3}\mathcal{D}_0(z) + \frac{128}{\epsilon^2}\mathcal{D}_1(z) - \frac{128}{\epsilon^2}\log(1-z)\right.$$
$$-\frac{256}{\epsilon}\mathcal{D}_2(z) + \frac{256}{\epsilon}\log^2(1-z) - \frac{320}{\epsilon}\log(1-z)$$
$$\left.+\frac{1024}{3}\mathcal{D}_3(z) - \frac{1024}{3}\log^3(1-z) + 640\log^2(1-z)\right],$$

$$\mathcal{D}_i = \left[\frac{\log^i(1-z)}{1-z}\right]_+$$

- Next, 1 Real- 1 Virtual



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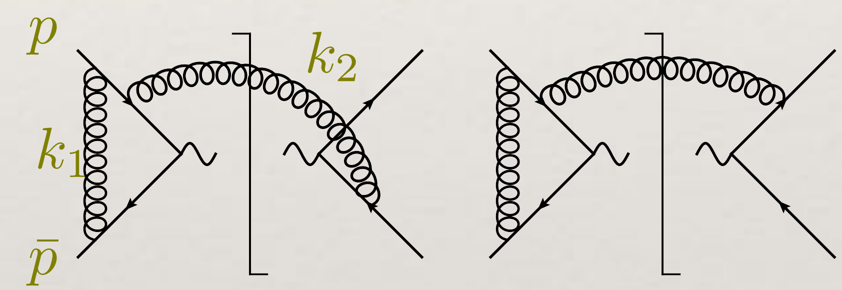
Diagnosis: method of regions

Beneke, Smirnov

- How does it work?
  - Divide up k_1 (=loop-momentum) integral into hard, 2 collinear and a soft region, by appropiate scaling

$$\text{Hard}: \quad k_1 \sim \sqrt{\hat{s}}\,(1,1,1)\;; \qquad \text{Soft}: \quad k_1 \sim \sqrt{\hat{s}}\,(\lambda^2,\lambda^2,\lambda^2)\;;$$
$$\text{Collinear}: \quad k_1 \sim \sqrt{\hat{s}}\,(1,\lambda,\lambda^2)\;; \quad \text{Anticollinear}: \quad k_1 \sim \sqrt{\hat{s}}\,(\lambda^2,\lambda,1)\;.$$



  - expand integrand in λ, to leading and next-to-leading order
  - but then integrate over *all* k_1 anyway!
  - Treat emitted momentum as soft and incoming momenta as hard

$$k_2^\mu = (\lambda^2, \lambda^2, \lambda^2)$$

Method of region: result

- Results

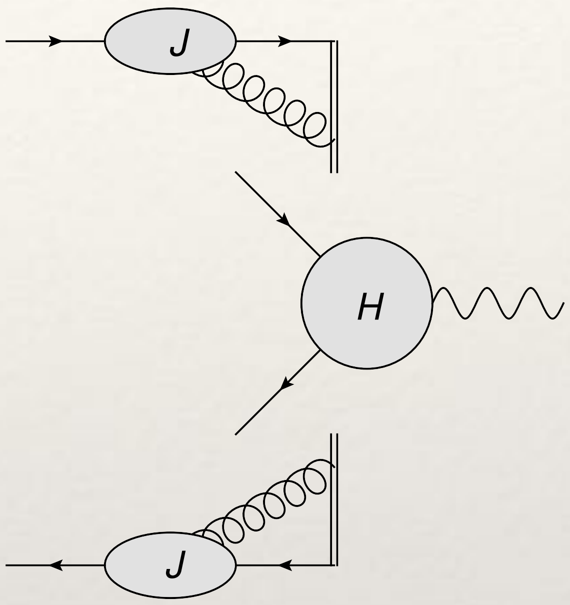
Bonocore, EL, Magnea, Vernazza, White

  - Hard region (expansion in λ^2): *LP + some NLP*
  - Soft region (expansion in λ^2): *ZERO*
  - (anti-)collinear regions (expansion in λ): *NLP only*
- Result:
  - the full $K^{(1)}_{1r,1v}$ is reproduced, including constants
- For **predictive power**, need factorization

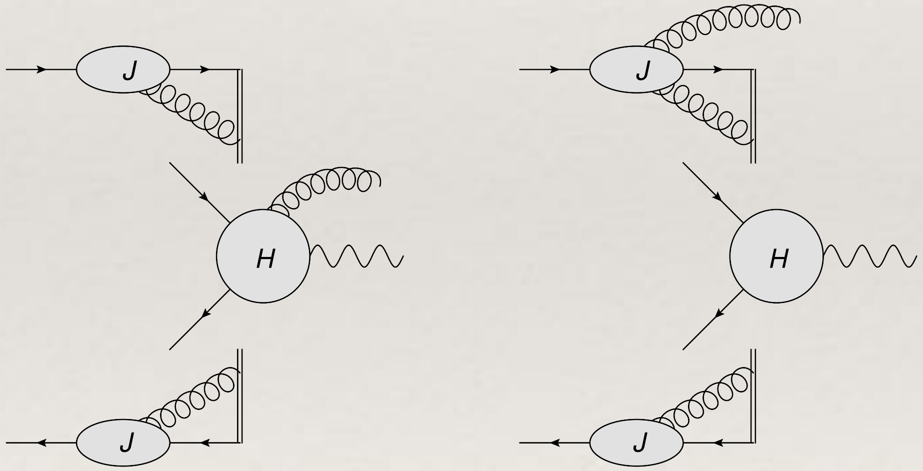
A factorization approach from Low's theorem

Bonocore, EL, Magnea, Melville,Vernazza, White

- Can we *predict* the ln(1-z) logarithms from lower orders?
  - Factorize the cross section,
    - ✓ H: the hard and the soft function
    - ✓ J: incoming-jet functions



- Next, add one extra soft emission. Let every blob radiate!



Del Duca

  - Compute each new "blob + radiation", and put it together. New: **radiative jet function**

$$J_\mu\left(p, n, k, \alpha_s(\mu^2), \epsilon\right) u(p) = \int d^d y \; \mathrm{e}^{-\mathrm{i}(p-k)\cdot y} \; \langle 0 \,|\, \Phi_n(y,\infty)\, \psi(y)\, j_\mu(0) \,|\, p \rangle$$

Factorization approach to NLP logarithms

- Upshot: a factorization formula for the emission amplitude

$$\mathcal{A}_{\mu,a}(p_j,k) = \sum_{i=1}^{2}\left(\frac{1}{2}\,\widetilde{\mathcal{S}}_{\mu,a}(p_j,k) + g\,\mathbf{T}_{i,a}\,G^{\nu}_{i,\mu}\,\frac{\partial}{\partial p_i^{\nu}} + J_{\mu,a}\,(p_i,n_i,k)\right)\mathcal{A}(p_j) - \mathcal{A}^{\widetilde{\mathcal{J}}}_{\mu,a}(p_j,k)$$

Soft function   Orbital term   Jet function   Overlap

  - J_μ is needed at one-loop level

Predicted NLP threshold logs vs exact result

- Compute blobs, one-loop radiative jet function, contract with cc amplitude and integrate over phase space. Exact calculation gives

$$
\begin{aligned}
K^{(2)}_{\rm rv}(z) \; = \; & \left(\frac{\alpha_s}{4\pi}\right)^2 \Bigg\{ C_F^2 \Bigg[\frac{32\mathcal{D}_0(z) - 32}{\epsilon^3} + \frac{-64\mathcal{D}_1(z) + 48\mathcal{D}_0(z) + 64L(z) - 96}{\epsilon^2} \\
& + \frac{64\mathcal{D}_2(z) - 96\mathcal{D}_1(z) + 128\mathcal{D}_0(z) - 64L^2(z) + 208L(z) - 196}{\epsilon} - \frac{128}{3}\mathcal{D}_3(z) \\
& + 96\mathcal{D}_2(z) - 256\mathcal{D}_1(z) + 256\mathcal{D}_0(z) + \frac{128}{3}L^3(z) - 232L^2(z) + 412L(z) - 408 \Bigg] \\
& + C_A C_F \Bigg[\frac{8\mathcal{D}_0(z) - 8}{\epsilon^3} + \frac{-32\mathcal{D}_1(z) + 32L(z) - 16}{\epsilon^2} + \frac{64\mathcal{D}_2(z) - 64L^2(z) + 64L(z) + 20}{\epsilon} \\
& - \frac{256}{3}\mathcal{D}_3(z) + \frac{256}{3}L^3(z) - 128L^2(z) - 60L(z) + 8 \Bigg] \Bigg\}, \qquad (4.6)
\end{aligned}
$$

$$L(z) = \ln(1-z)$$

- **Result**: *perfect agreement* for 4 powers of the next-to-eikonal/soft logarithms at NNLO

$$\ln^3(1-z), \;\; \ln^2(1-z), \;\; \ln^1(1-z), \;\; \ln^0(1-z),$$

Colour-singlet final states

- Generalize NLP factorization (the LBKD theorem) beyond Drell-Yan, to arbitrary colour-singlet final states
  - look at NLO only, i.e. predict

$$D_1 = \left[\frac{\ln(1-z)}{1-z}\right]_+ \qquad D_0 = \left[\frac{1}{1-z}\right]_+ \qquad L_1 = \ln(1-z) \qquad L_0 = \ln^0(1-z)$$

    - ✓ where "1-z" can take different forms for 2 -> 2,3 etc scattering
  - apply to Drell-Yan, (multi-)Higgs, (vector boson pairs)
  - for inclusive and fully differential cross sections

NLP terms in colorless final states @NLO

Bonocore, Del Duca, EL, Magnea,Vernazza, White
1706.04018

- Previous factorization at NLO

$$\mathcal{A}^{(1)}_{\mu,a}\left(\{p_i\},k\right) = \sum_{l=1}^{2}\left[g_s\,\mathbf{T}_{l,\,a}\,G^{\nu}_{l,\mu}\,\frac{\partial}{\partial p_l^{\nu}} + J^{(1)}_{\mu,\,a}\left(p_l,n_l,k\right)\right]\mathcal{A}^{(0)}\left(\{p_i\}\right)$$

  - ✓ G is a projector, **T** a color matrix

- initial quarks: $J^{a}_{\mu}(p,n,k) = g_s\mathbf{T}^{a}\left[\frac{(2p-k)_\mu}{2p\cdot k} + \frac{\mathrm{i}k^{\beta}}{p\cdot k}S_{\beta\mu}\right]$ $\quad S_{\beta\mu} = \frac{\mathrm{i}}{4}\left[\gamma_\beta,\gamma_\mu\right]$
- initial gluons: $J^{a}_{\mu,\,\rho\sigma}(p,n,k) = g_s\mathbf{T}^{a}\left[\frac{(2p-k)_\mu}{2p\cdot k}\eta_{\rho\sigma} - \frac{\mathrm{i}k^{\beta}}{p\cdot k}M_{\beta\mu,\,\rho\sigma}\right]$ $\quad M_{\beta\mu,\,\rho\sigma} = \mathrm{i}\Big(\eta_{\beta\rho}\eta_{\mu\sigma} - \eta_{\beta\sigma}\eta_{\mu\rho}\Big)$

- notice the spin-dependent Lorentz generator ("next-to-soft theorem")
- notice derivative term (Low's theorem)

Lorentz generator

- The derivative term can be written as the **orbital part** of Lorentz generator

$$G^{\nu}_{l,\mu}\frac{\partial}{\partial p_l^{\nu}} = \frac{k^{\nu}}{p_l\cdot k}\left[p_{l,\nu}\frac{\partial}{\partial p_l^{\mu}} - p_{l,\mu}\frac{\partial}{\partial p_l^{\nu}}\right] = -\frac{\mathrm{i}k^{\nu}\,\mathrm{L}^{(l)}_{\nu\mu}}{p_i\cdot k}$$

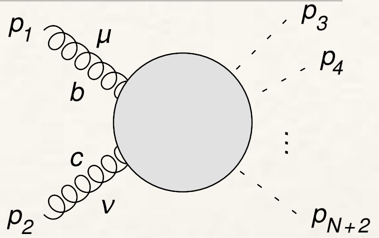
- so that

$$\mathcal{A}^{(1)}_{\mu,a}\left(\{p_i\},k\right) = \sum_{l=1}^{2} g_s\,\mathbf{T}_{l,a}\left[\frac{(2p_l-k)_{\mu}}{2p_l\cdot k} - \frac{\mathrm{i}k^{\nu}}{p_l\cdot k}\underbrace{\left(\mathrm{L}^{(l)}_{\nu\mu}+\Sigma^{(l)}_{\nu\mu}\right)}_{\mathrm{J}^{(l)}_{\nu\mu}}\right]\mathcal{A}^{(0)}\left(\{p_i\}\right)$$

$$= \sum_{l=1}^{2} g_s\,\mathbf{T}_{l,a}\left[\frac{p_{l,\mu}}{p_l\cdot k} - \frac{\mathrm{i}k^{\nu}\,\mathrm{J}^{(l)}_{\nu\mu}}{p_l\cdot k}\right]\mathcal{A}^{(0)}\left(\{p_i\}\right)$$

- leads to Scalar + Orbital + Spin part of the NLP amplitude

Colour singlet production in gg channel



- Square amplitude

$$|\mathcal{A}_{\rm NLP}|^2 = \sum_{\rm colours} \left(\mathcal{A}^{\sigma_1,\,\mu_1\nu_1}_{\rm scal.} + \mathcal{A}^{\sigma_1,\,\mu_1\nu_1}_{\rm spin} + \mathcal{A}^{\sigma_1,\,\mu_1\nu_1}_{\rm orb.}\right)^* \mathcal{P}_{\mu_1\mu_2}(p_1,l_1)\mathcal{P}_{\nu_1\nu_2}(p_2,l_2)\mathcal{P}_{\sigma_1\sigma_2}(k,l_3)$$
$$\times \left(\mathcal{A}^{\sigma_2,\,\mu_2\nu_2}_{\rm scal.} + \mathcal{A}^{\sigma_2,\,\mu_2\nu_2}_{\rm spin} + \mathcal{A}^{\sigma_2,\,\mu_2\nu_2}_{\rm orb.}\right), \qquad (3.7)$$

  - ✓ where $\mathcal{P}_{\alpha\beta}(p,l) \equiv \sum_\lambda \epsilon^{(\lambda)}_\alpha(p)\,\epsilon^{(\lambda)*}_\beta(p) = -\eta_{\alpha\beta} + \frac{p_\alpha l_\beta + p_\beta l_\alpha}{p\cdot l}$

- Can be done using $-\eta_{\alpha\beta}$ only (external ghosts are beyond NLP)
- Truncate to NLP, leads to

$$|\mathcal{A}_{\rm NLP}|^2 = \sum_{\rm colours}\left\{\left|\mathcal{A}^{\sigma,\,\mu\nu}_{\rm scal.}\right|^2 + 2{\rm Re}\left[\left(\mathcal{A}^{\sigma,\,\mu\nu}_{\rm spin} + \mathcal{A}^{\sigma,\,\mu\nu}_{\rm orb.}\right)^* \mathcal{A}_{{\rm scal.}\,\sigma,\,\mu\nu}\right]\right\}$$

- Easy part: scalar (eikonal) part

$$\sum_{\rm colours}\left|\mathcal{A}^{\sigma,\,\mu\nu}_{\rm scal.}\right|^2 = 2g_s^2 N_c\left(N_c^2-1\right)\frac{p_1\cdot p_2}{p_1\cdot k\, p_2\cdot k}\left|\mathcal{A}_{\mu\nu}\right|^2$$

CS production in gg channel

- Spin * scalar vanishes (anti-symmetriy in μν)
- Orbital part leads to shifts in momentum dependence

$$\sum_{\text{colours}} 2\,\mathrm{Re}\left[\mathcal{A}^{\sigma,\,\mu\nu}_{\text{orb.}}\mathcal{A}_{\text{scal.}\,\sigma,\,\mu\nu}\right] = \frac{2g_s^2 N_c\left(N_c^2-1\right)p_1\cdot p_2}{p_1\cdot k\,p_2\cdot k}\left[\delta p_1^\alpha\frac{\partial}{\partial p_1^\alpha}+\delta p_2^\alpha\frac{\partial}{\partial p_2^\alpha}\right]|\mathcal{A}_{\mu\nu}|^2$$

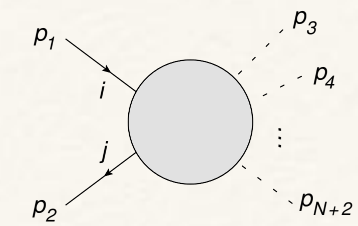
  - ✓ where

$$\delta p_1^\alpha = -\frac{1}{2}\left(\frac{p_2\cdot k}{p_1\cdot p_2}p_1^\alpha - \frac{p_1\cdot k}{p_1\cdot p_2}p_2^\alpha + k^\alpha\right),\quad \delta p_2^\alpha = -\frac{1}{2}\left(\frac{p_1\cdot k}{p_1\cdot p_2}p_2^\alpha - \frac{p_2\cdot k}{p_1\cdot p_2}p_1^\alpha + k^\alpha\right)$$

- Result is simple: dipole times shifted squared LO amplitude

$$\boxed{|\mathcal{A}_{\text{NLP}}|^2 = \frac{2g_s^2 N_c\left(N_c^2-1\right)p_1\cdot p_2}{p_1\cdot k\,p_2\cdot k}\left|\mathcal{A}_{\mu\nu}\left(p_1+\delta p_1, p_2+\delta p_2\right)\right|^2}$$

CS production in $q\bar{q}$ channel



- Scalar plus orbital part very similar to gg case

$$|A^{\sigma}_{\mathrm{NLP}}|^2_{\mathrm{scal.+orb.}} = \frac{g_s^2 C_F}{z} \frac{s}{p_1 \cdot k \, p_2 \cdot k} \, |A(p_1 + \delta p_1, p_2 + \delta p_2)|^2$$

  - except for the 1/z , which is due to the kinematic shift

$$s \;\to\; (p_1 + p_2 + \delta p_1 + \delta p_2)^2 \;=\; s + 2\,(\delta p_1 + \delta p_2) \cdot (p_1 + p_2)$$

  - which is the same as

$$s \;\to\; z s$$

- But the spin part now does *not* cancel:

$$\sum_{\mathrm{colours}} 2\,\mathrm{Re}\left[A^{\dagger}_{\mathrm{scal.}}\, A_{\mathrm{spin}}\right]_{\mathrm{NLP}} = -\, g_s^2 N_c C_F \frac{2 p_1 \cdot p_2}{p_1 \cdot k \, p_2 \cdot k} \underbrace{\frac{k \cdot (p_1 + p_2)}{p_1 \cdot p_2}}_{-(1-z)} \, |\mathcal{A}(p_1, p_2)|^2$$

  - precisely compensates 1/z ≅ 1 + (1-z)!!

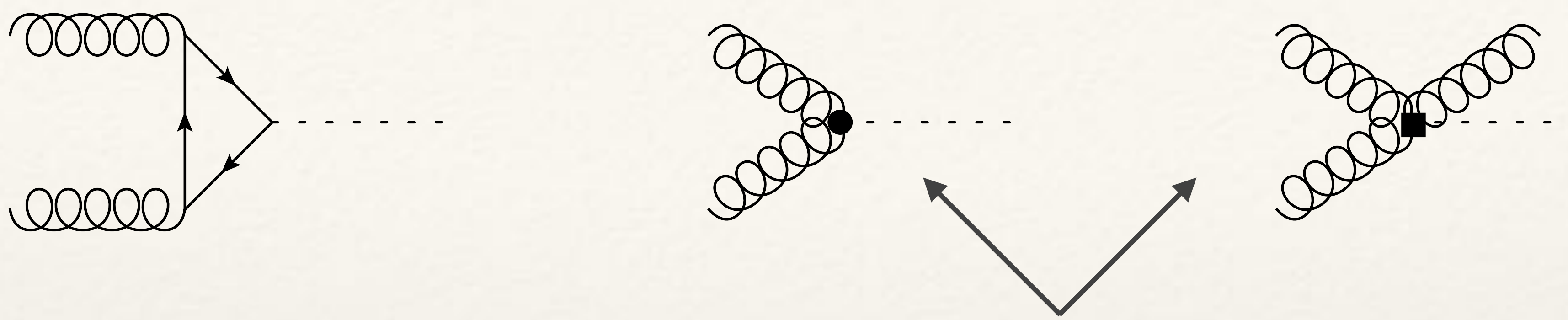
Squared amplitudes and cross sections

$$\delta p_1^\alpha = -\frac{1}{2}\left(\frac{p_2\cdot k}{p_1\cdot p_2}\,p_1^\alpha - \frac{p_1\cdot k}{p_1\cdot p_2}\,p_2^\alpha + k^\alpha\right),\quad \delta p_2^\alpha = -\frac{1}{2}\left(\frac{p_1\cdot k}{p_1\cdot p_2}\,p_2^\alpha - \frac{p_2\cdot k}{p_1\cdot p_2}\,p_1^\alpha + k^\alpha\right)$$

- In summary
  - gluons $$|\mathcal{A}_{\rm NLP}|^2 = \frac{2g_s^2 N_c\left(N_c^2-1\right)p_1\cdot p_2}{p_1\cdot k\, p_2\cdot k}\left|\mathcal{A}_{\mu\nu}\left(p_1+\delta p_1, p_2+\delta p_2\right)\right|^2$$
  - quarks $$|A_{\rm NLP}|^2 = g_s^2\, C_F\,\frac{s}{p_1\cdot k\, p_2\cdot k}\left|A(p_1+\delta p_1, p_2+\delta p_2)\right|^2$$
- Up to colour factors the same:
  - eikonal (dipole) factor times shifted Born cross section
  - Born can be loop-induced, have complex parts etc.
- Combine carefully with phase space for general inclusive formula

$$\frac{d\hat{\sigma}^{(gg)}_{\rm NLP}}{dz} = C_A\, K_{\rm NLP}\left(z,\epsilon\right)\hat{\sigma}^{(gg)}_{\rm Born}\left(zs\right) \qquad K_{\rm NLP}\left(z,\epsilon\right) = \frac{\alpha_s}{\pi}\left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} z\left(1-z\right)^{-1-2\epsilon}\frac{\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)\Gamma(1-\epsilon)}$$

Single Higgs production



Infinite top mass limit not needed
extra operators = shift in kinematics

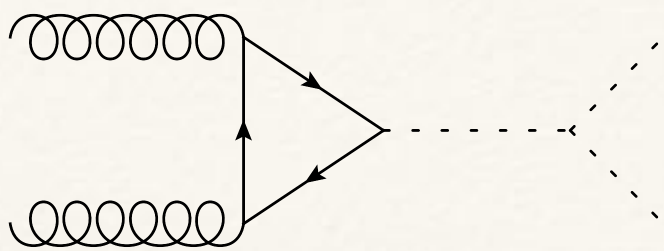
- Single Higgs production

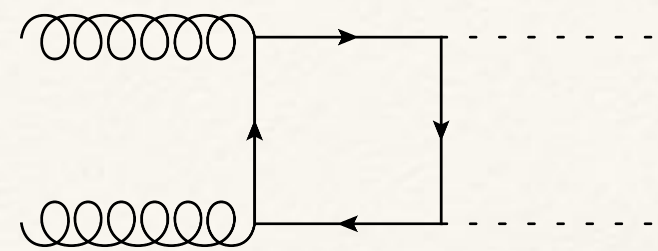
$$\frac{d\sigma^h_{\rm NLP}}{dz} = \frac{\alpha_s^3 C_A}{288\pi^2 v^2} F(z\tau,\epsilon)\left(\frac{2-\mathcal{D}_0(z)}{\epsilon} + 2\mathcal{D}_1(z) - \mathcal{D}_0(z) - 4\log(1-z) + 2\right)$$

- with F the well-known Born function. D’s and L’s agree with exact calculation, but also with full top mass dependence!

Dawson; Spira, Djouadi, Graudenz, Zerwas

Di-Higgs production





Borowka, Greiner, Heinrich, Jones, Kerner, Schlenk, Zirke

- Double Higgs production at NLO-NLP

$$z\,\frac{d\sigma^{hh}_{\rm NLP}}{dz} = \frac{\alpha_s}{3\pi}\,C_A\left(\frac{\overline{\mu}^2}{s}\right)^{\epsilon}\left[\frac{12-6\mathcal{D}_0(z)}{\epsilon}+12\mathcal{D}_1(z)-24\log(1-z)\right]\sigma^{hh}_{\rm Born}\,(zs)$$

  - where

$$\frac{d\hat{\sigma}^{hh}_{\rm Born}}{dt} = \frac{\alpha_s^2}{8\pi^3}\,\frac{1}{512\,v^4}\left[\left|C_\triangle F_\triangle + C_\Box F_\Box\right|^2 + \left|C_\Box G_\Box\right|^2\right]$$

    - ✓ with triangle and box graphs, again for full top mass dependence

  - Should be useful for numerical evaluations, and seeing new patterns

- Similar result for triple-Higgs production

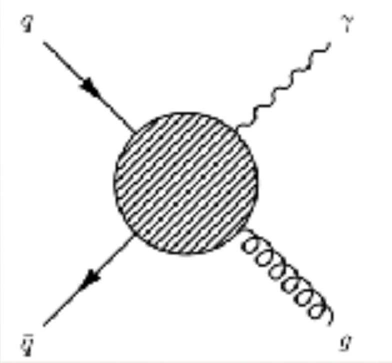
De Florian, Mazzitelli

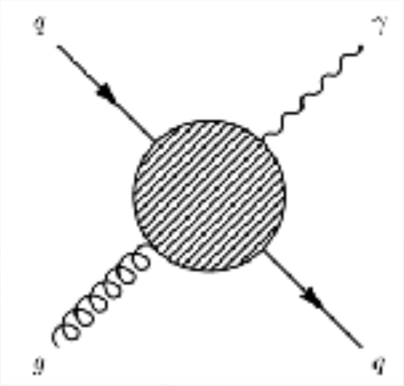
Final state partons:
Prompt photon production

Beenakker, van Beekveld, EL, White
to appear

With final state partons: prompt photon

- Two LO channels: $q\bar{q}$ and qg





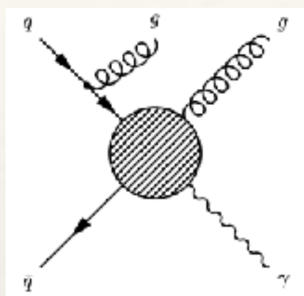
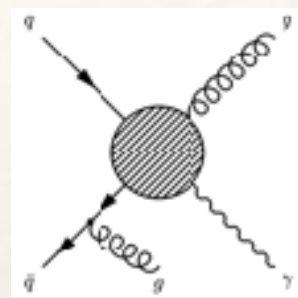
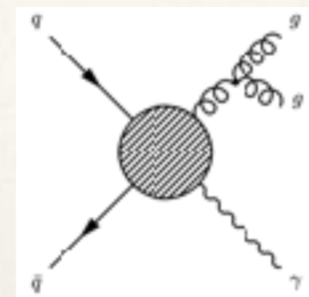
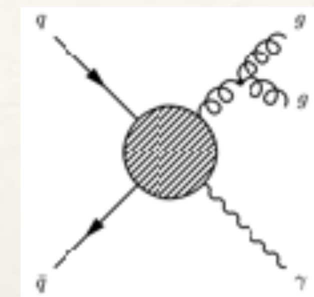
- With extra radiation, different ways to define threshold. We shall use "w"→1

$$\begin{aligned} u_1 &= (p_1 - p_\gamma)^2 \equiv -svw \\ t_1 &= (p_2 - p_\gamma)^2 \equiv s(v-1) \\ s_4 &= s + t_1 + u_1 = sv(1-w) \end{aligned}$$

- Two issues to deal with
  - shifting kinematics in $2 \to 2$ kinematics
  - soft fermion emission

Gluon emission

- For qq̄ channel

- Can in fact write down general formula

$$\begin{aligned} \mathcal{A}_{\rm NLP} &= \mathcal{A}_{\rm scal} + \mathcal{A}_{\rm spin} + \mathcal{A}_{\rm orb} \\ &= \sum_{j=1}^{n+2} \frac{g_s \mathbf{T}_j}{2p_j \cdot k} \left(\mathcal{O}^{\sigma}_{{\rm scal},j} + \mathcal{O}^{\sigma}_{{\rm spin},j} + \mathcal{O}^{\sigma}_{{\rm orb},j} \right) \otimes i\mathcal{M}_{\rm H}(p_1, \ldots, p_i, \ldots, p_{n+2}) \epsilon^*_\sigma(k), \end{aligned}$$

  - color charge and spin generator depends on emitting IS or FS particle
  - orbitral part on IS or FS particle

Squared amplitude at NLP

- Result: again dipoles plus momentum shift
- Important to implement $2 \to 3$ momentum conservation in $2 \to 2$ matrix element
  - used Catani-Seymour dipoles (FKS is also possible)

Gervais

$$
\begin{aligned}
|\mathcal{A}_{\mathrm{NLP},q\bar{q}\to\gamma gg}|^2 \;=\; \frac{C_F}{C_A}\Bigg[& C_F \frac{2p_1\cdot p_2}{(p_1\cdot k)(p_2\cdot k)}\,|\mathcal{M}_{q\bar{q}\to\gamma g}(p_1+\delta p_{1;2}, p_2+\delta p_{2;1})|^2 \\
& +\frac{1}{2}C_A \frac{2p_1\cdot p_R}{(p_1\cdot k)(p_R\cdot k)}\,|\mathcal{M}_{q\bar{q}\to\gamma g}(p_1+\delta p_{1;R}, p_R-\delta p_{R;1})|^2 \\
& +\frac{1}{2}C_A \frac{2p_2\cdot p_R}{(p_2\cdot k)(p_R\cdot k)}\,|\mathcal{M}_{q\bar{q}\to\gamma g}(p_2+\delta p_{2;R}, p_R-\delta p_{R;2})|^2 \\
& -\frac{1}{2}C_A \frac{2p_1\cdot p_2}{(p_1\cdot k)(p_2\cdot k)}\,|\mathcal{M}_{q\bar{q}\to\gamma g}(p_1+\delta p_{1;2}, p_2+\delta p_{2;1})|^2 \Bigg].
\end{aligned}
$$

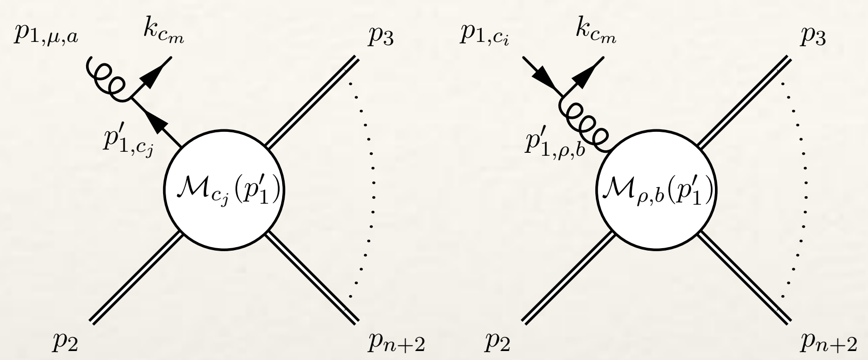
Note sign change for final state emitter

- Integrate over NLO phase, agrees with NLO calculation including ln(1-w) terms

Gordon, Vogelsang

Soft fermions

- At NLP (not LP) one can have soft fermion emission



  - Effective feynman rule for left diagram (note that “u(k)” is of order √k)

$$i\mathcal{M}_{\mathrm{NLP},1,g} = \frac{g_s T^a_{c_m c_j}}{(p_1-k)^2+i\varepsilon}\epsilon^\mu(p_1)\bar{u}(k)\gamma_\mu \not{p}_1 \mathcal{M}_{c_j}(p_1,p_2,\ldots,p_{n+2})$$

  - Right diagram

$$i\mathcal{M}_{\mathrm{NLP},1,g} = \frac{g_s T^b_{c_m c_i}}{(p_1-k)^2+i\varepsilon}\bar{u}(k)\gamma_\rho u(p_1)\mathcal{M}_{\rho,b}(p_1,p_2,\ldots,p_{n+2}).$$

- Squaring amplitude and integration over phase space gives agreement with exact NLO
  - Must keep careful track of singular regions

LL resummation of NLP logarithms

Bahjat-Abbas, Bonocore, EL, Magnea, Sinninghe Damsté, Vernazza, White
to appear

LL resummation of NLP logarithms

- We have organized NLP threshold logs at NLO and NNLO for Drell-Yan. Can one resum them?
- First resummation conjecture: just change kernel in regular resummation formula

Kraemer, EL, Spira; 1998
EL, Magnea, Stavenga

$$\frac{1+z^2}{1-z} \longrightarrow \frac{2}{1-z} - 2$$

  - reproduced NNLO NLP logs of van Neerven et al
- Physical kernel approach for inclusive quantities
  - using single log behaviour of kernel

Soar, Moch, Vemaseren, Vogt; Moch, Vogt; Mattizelli, de Florian

- Recent LL resummation using SCET

Beneke, Broggio, Garny, Jaskiewicz, Szafron, LV, Wang, 2018

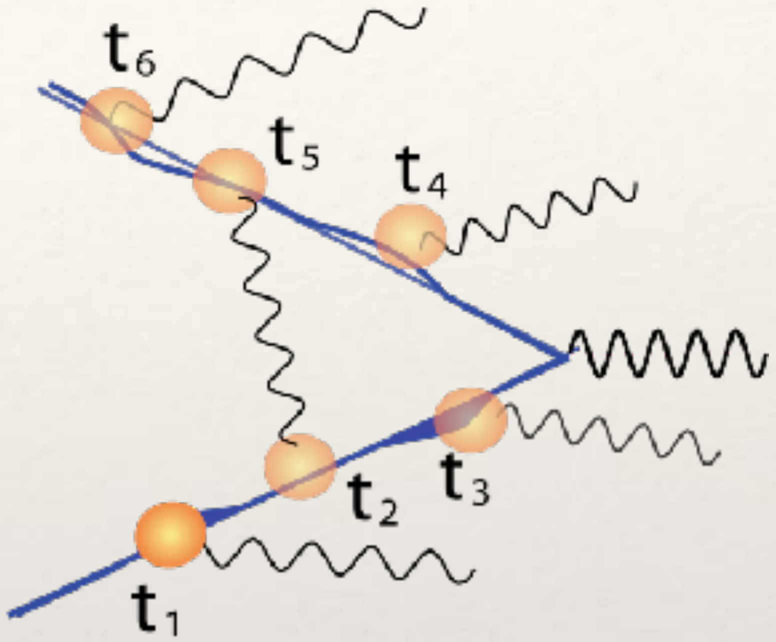
$$\Delta^{\rm LL}_{\rm NLP}(z,\mu) = \exp\left[4S^{\rm LL}(\mu_h,\mu) - 4S^{\rm LL}(\mu_s,\mu)\right] \times \frac{-8C_F}{\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_s)}\,\theta(1-z)\,.$$

NLP amplitude exponentiation via path integral

EL, Magnea, Stavenga, White

- Fluctuations around classical path are NE corrections
  - All NLP corrections from external lines exponentiate
  - Keep track via scaling variable λ

$$p^\mu = \lambda n^\mu$$



$$f(\infty) = \int_{x(0)=0} \mathcal{D}x \, \exp\left[i \int_0^\infty dt \left(\frac{\lambda}{2}\dot{x}^2 + (n+\dot{x})\cdot A(x_i + nt + x) + \frac{i}{2\lambda}\partial \cdot A(x_i + p_f t + x) \right) \right]$$

- Exponentiation then in terms of NLP webs

$$\sum C(D)\mathcal{F}(D) = \exp\left[\bar{C}(D)W_{\mathrm{E}}(D) + \bar{C}'(D)W_{\mathrm{NE}}(D)\right]$$

Bonocore, EL, Magnea, Melville, Vernazza, White

LL resummation for cross section at NLP

- Can show that phase space NLP effects behave as

$$\varepsilon\,(1-z)$$

  - i.e. softness suppression comes with singularity suppression
  - => phase space does not give leading logs
- Can show that there are no LL enhancements from purely collinear regions (single log)
  - => LL effects come then only from NLP soft function = NLP webs

Exponentiating NLP soft function

- Moments of cross section

$$\int_0^1 d\tau\, \tau^{N-1}\, \frac{d\sigma_{\rm DY}}{d\tau}\bigg|_{\rm LL,\, NLP} = \sigma_0(Q^2)\, q_N(Q^2)\bar{q}_N(Q^2)\, \bar{\mathcal{S}}_{\rm NLP}(N,Q^2,\epsilon),$$

- with NLP soft function (f's are NLP Wilson lines)

$$\tilde{\mathcal{S}} = \frac{1}{N_c}\sum_n \mathrm{Tr}\left[\langle 0|f_2^\dagger f_1|n\rangle\langle n|f_1^\dagger f_2|0\rangle\right]\delta\left(z-\frac{Q^2}{\hat{s}}\right).$$

- Exponentiation then gives

$$\int_0^1 d\tau\, \tau^{N-1}\, \frac{d\sigma_{\rm DY}}{d\tau}\bigg|_{\rm LL,\, NLP} = \sigma_0(Q^2)\, q_{\rm LL,\, NLP}(N,Q^2)\, \bar{q}_{\rm LL,\, NLP}(N,Q^2)$$
$$\times \exp\left[\frac{\alpha_s C_F}{\pi}\left(2\log^2(N) + \frac{4\log(N)}{N}\right)\right].$$

  - agrees with 1998 conjecture

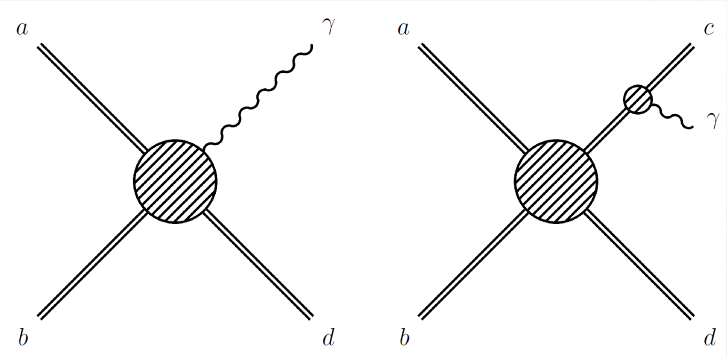
LL resummation of NLP logarithms in prompt photon production

Basu, Beenakker, van Beekveld, EL, Misra, Motylinski
to appear

NLP resummation in prompt photon production at fixed p_T

- Threshold resummation of powers of

$$\ln(1 - x_T^2) \qquad x_T^2 = \frac{4p_T^2}{s}$$



- Threshold resummation long known, to NNLL and even beyond

EL, Oderda, Sterman; Catani, Mangano, Nason, Oleari Ridolfi; De Florian,Vogelsang+Sterman, Schaefer; Becher, Schwartz + Lorentzen; Hinderer, Ringer, Sterman, Vogelsang

- Joint threshold+recoil resummation

EL, Sterman, Vogelsang

  - gives about 20% correction w.r.tthreshold

- Two ways of including NLP logarithms
  - 1) Extend kernel to NLP in Sudakov exponent => modified resummation exponent
  - 2) Extend PDF evolution to soft scale, automatically includes NLP terms

Joint-resummation

- Joint-resummed formula

$$
\frac{p_T^3 d\sigma^{(\text{direct,joint})}_{AB\to\gamma+X}}{dp_T} = \frac{p_T^4}{8\pi S^2}\sum_{ab}\int_{\mathcal{C}}\frac{dN}{2\pi i}\int\frac{d^2\mathbf{Q}_T}{(2\pi)^2}\left(\frac{S}{4|\mathbf{p}_T-\mathbf{Q}_T/2|^2}\right)^{N+1} f_{a/A}(N,\mu_F)f_{b/B}(N,\mu_F)
$$
$$
\times\int_0^1 \mathrm{d}\tilde{x}_T^2(\tilde{x}_T^2)^N\frac{|\mathcal{M}_{ab\to\gamma d}(\tilde{x}_T^2)|^2}{\sqrt{1-\tilde{x}_T^2}}C_\delta^{(ab\to\gamma d)}(\alpha_s,Q^2/\mu_F^2,Q^2/\mu_R^2)
$$
$$
\times\int d^2\mathbf{b}\ \mathrm{e}^{i\mathbf{b}\cdot\mathbf{Q}_T}\ \theta\left(\bar{\mu}-|\mathbf{Q}_T|\right)P_{abd}(N,b,\mu_F,\mu_R,Q). \qquad (13)
$$

  - with resummation

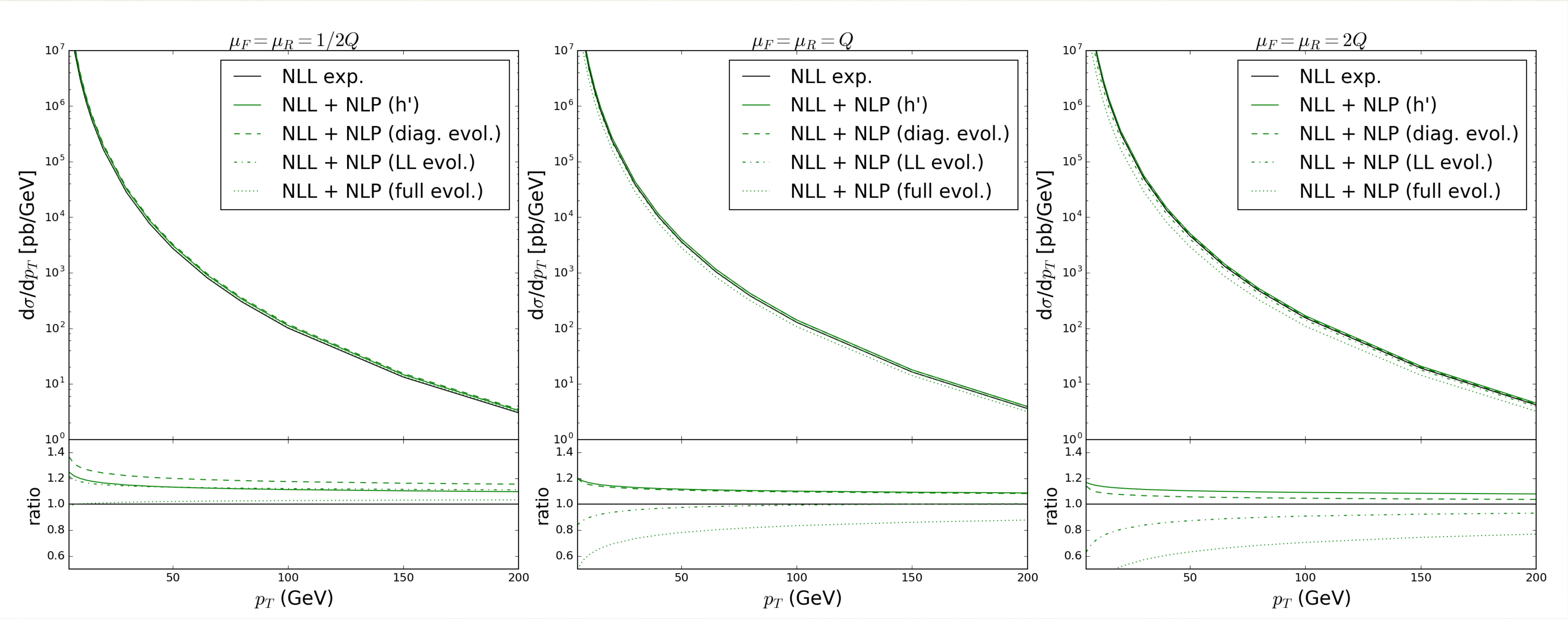
$$
P_{abd}(N,b,\mu_F,\mu_R,Q)=\exp\left[E_a^{\mathrm{PT}}(N,b,\mu_F,\mu_R,Q)+E_b^{\mathrm{PT}}(N,b,\mu_F,\mu_R,Q)+F_d(N,\mu_R,Q)+g_{abd}(N)\right]
$$

Initial state Initial state Final state Soft

$$
E_a^{\mathrm{PT}}(N,b,\mu_F,\mu_R,Q) = \int_0^{Q^2}\frac{\mathrm{d}k_T^2}{k_T^2}A_a(\alpha_s(k_T^2))\left[J_0(bk_T)K_0\left(\frac{2Nk_T}{Q}\right)+\ln\left(\frac{\bar{N}k_T}{Q}\right)\right]
$$
$$
-\ln\bar{N}\int_{\mu_F^2}^{Q^2}\frac{\mathrm{d}k_T^2}{k_T^2}A_a(\alpha_s(k_T^2))
$$

Numerical results

- Effect of NLP logs, LL accuracy = about 10-20% positive
- Scale uncertainty reduces as well



Summary

- Soft approximation reveals patterns enabling all-order resummation
- Next-to-soft/NLP is also promising
- Factorization + LBDK theorem leads to strong predictive power for NLP threshold logs
  - Drell-Yan at NNLO
- Simply NLP formulae at NLO for colour singlet final states
  - and now also prompt photon
- LL resummation at NLP for Drell-Yan done
  - NLL seems much harder
- NLP corrections are becoming an interesting object of study
  - Dedicated recent workshops in Edinburgh and Amsterdam