Probabilities & Signalling in Quantum Field Theory

Jeff Forshaw

Based on work with Robert Dickinson & Peter Millington:

Phys. Rev. D93 (2016) 065054 [arXiv: 1601.07784]
Phys. Lett. B774 (2017) 706-709 [arXiv:1702.04131]

Thanks to Peter for help with the slides.

$$S = T\exp\left[\frac{1}{i}\int_{-\infty}^{+\infty} dt\, H_{\text{int}}(t)\right]$$

S-matrix theory = technology for calculating and dealing with **amplitudes**.

Amplitudes are not physical observables, suffering artefacts like gauge dependence, ghosts, IR singularities and superficially acausal behaviour.

These artefacts are eliminated only when we combine individual amplitudes together to obtain physical probabilities.

Dream: develop the technology for calculating these **probabilities** directly in the hope that such artefacts never appear explicitly.

Causality is built into QFT through the vanishing of the equal-time commutator (bosons) or anti-commutator (fermions) of field operators:

$$\big[\phi(x), \phi(y)\big] \equiv \big[\phi_x, \phi_y\big] = 0 \qquad \text{if} \qquad (x-y)^2 < 0 \qquad \text{(space-like)}$$

Yet, it is the **Feynman propagator** that is ubiquitous in S-matrix theory:

$$\Delta^{(\mathrm{F})}(x,y) \equiv \Delta^{(\mathrm{F})}_{xy} = \frac{1}{2}\,\mathrm{sgn}(x_0 - y_0)\,\underbrace{\langle\big[\phi_x, \phi_y\big]\rangle}_{\text{causal}} + \frac{1}{2}\,\underbrace{\langle\big\{\phi_x, \phi_y\big\}\rangle}_{\text{a-causal}}$$

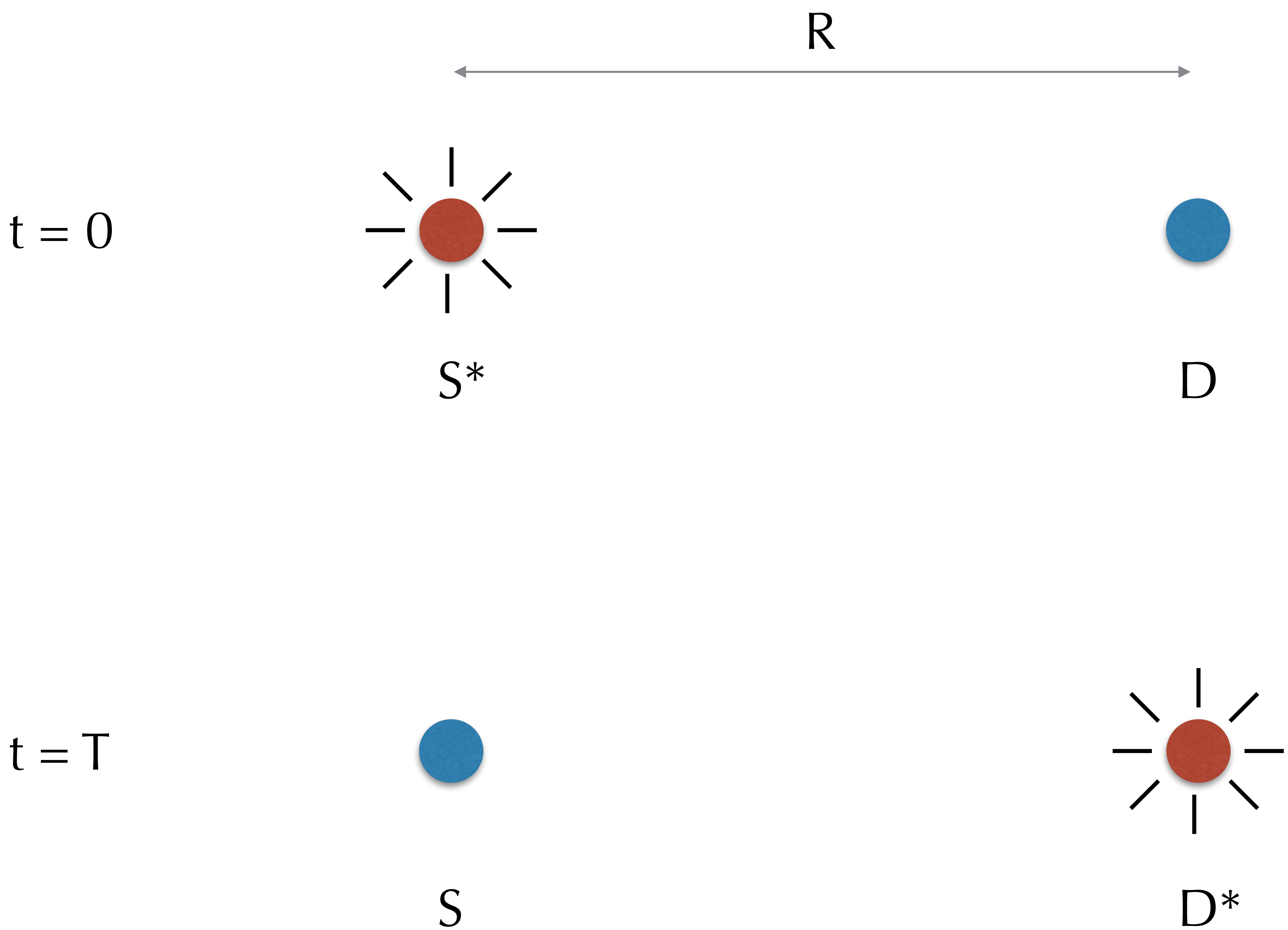
The S-matrix is not a good place to start: infinite plane waves in infinite past/future.

Surely, it is the **retarded propagator** that should be ubiquitous:

$$\Delta^{(\mathrm{R})}_{xy} = \Delta^{(\mathrm{A})}_{yx} = \frac{1}{i}\,\theta(x_0 - y_0)\,\langle\big[\phi_x, \phi_y\big]\rangle$$

An archetypal signalling process: **Fermi's two-atom problem**

[**E. Fermi**, Rev. Mod. Phys. 4 (**1932**) 87]



Fermi calculated that $P(D^*S|DS^*) = 0$ for T < R/c

but he made a mistake

Fermi should have obtained a non-zero result for all T:

- Vacuum can excite D at any time (R independent)
- Even the R dependent part of *P* is non-zero for T < R/c

There is no paradox though because Fermi's observable is **non-local**.

Resolution finally came via Shirokov (1967) and Ferretti (1968).

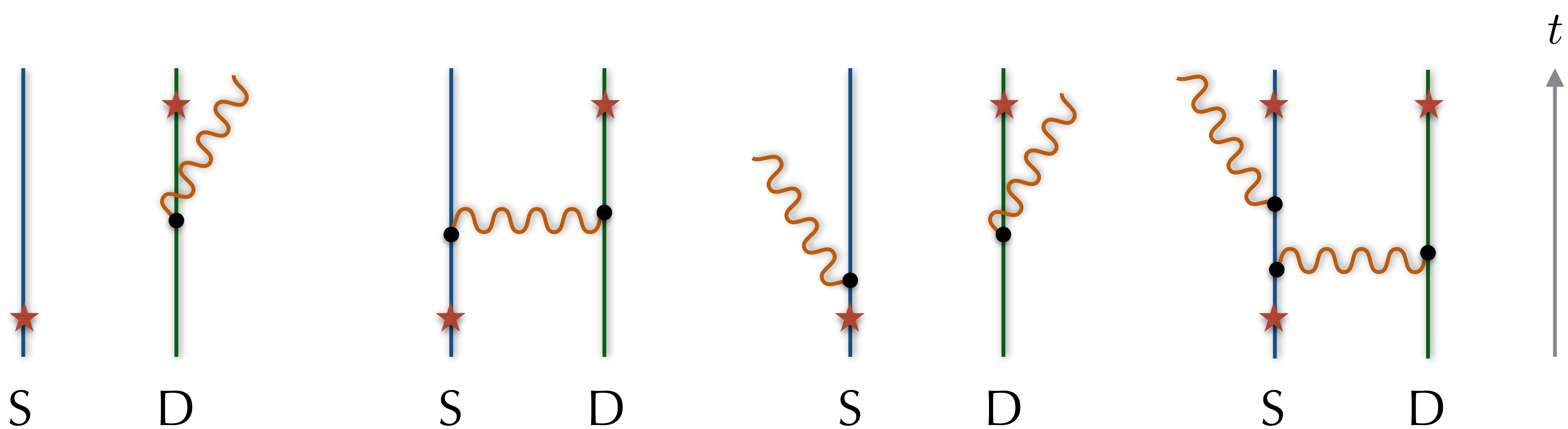
Think of **measuring only D** and not S (or the electromagnetic field) at time T.

In that case:

$$\frac{\mathrm{d}P(D^*|DS^*)}{\mathrm{d}R} = 0 \qquad \text{for } T < R/c$$

[**M. I. Shirokov**, Sov. J. Nucl. Phys. 4 (**1967**) 774; **B. Ferretti**, in *Old and new problems in elementary particles*, ed. Puppi, G., Academic Press, New York (**1968**); **E. A. Power** and **T. Thirunamachandran**, Phys. Rev. A56 (**1997**) 3395; for a summary of the history of the Fermi problem, see **R. Dickinson**, **J. Forshaw** and **P. Millington**, Phys. Rev. D93 (**2016**) 065054.]

Amplitude-level analysis: the relevant Feynman graphs



1 2 3 4

Acausal terms cancel in the sum of

$$1 \times 4^* \;+\; 2 \times 2^* \;+\; 3 \times 3^*$$

+ c.c. crossed

Causality emerges only at the level of probabilities

"In this paper I will not say anything new; but I hope that it will not be completely useless because, even if already known or immediately deducible from known facts, it does not seem to be clearly remembered." **Ferretti 1967**

- 1932 Fermi's original paper

PHYSICAL REVIEW
LETTERS

VOLUME 73 | 1 AUGUST 1994 | NUMBER 5

There Are No Causality Problems for Fermi's Two-Atom System

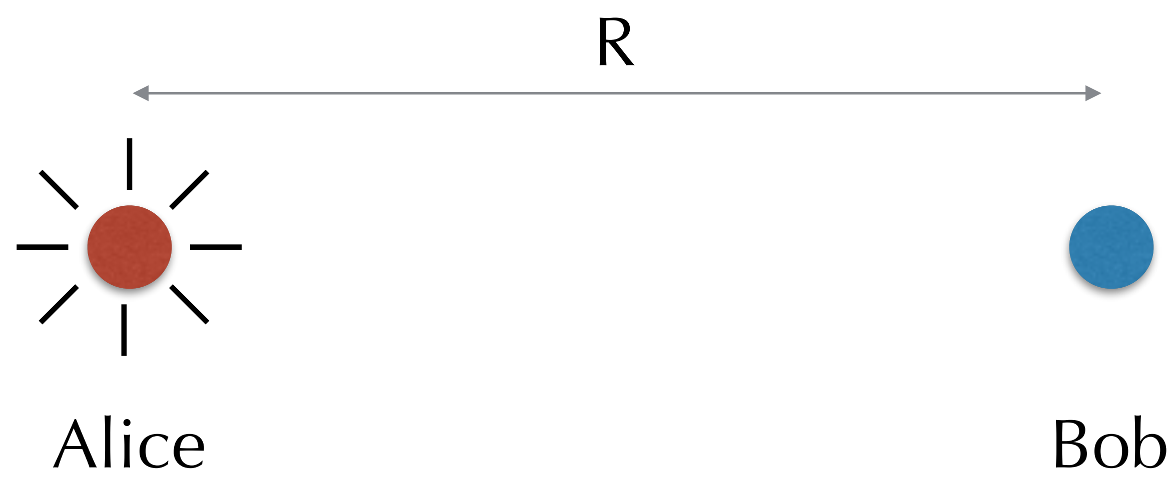
Detlev Buchholz[1] and Jakob Yngvason[2]

[1] *II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, D-22761 Hamburg, Germany*
[2] *Science Institute, University of Iceland, Dunhaga 3, IS-107 Reykjavik, Iceland*
(Received 2 March 1994)

A repeatedly discussed gedanken experiment, proposed by Fermi to check Einstein causality, is reconsidered. It is shown that, contrary to a recent statement made by Hegerfeldt, there appears no causality paradox in a proper theoretical description of the experiment.

- 1994: Buchholz and Yngvason restore order

"Weak causality"



1. Alice prepares her atom at $t = 0$ (excited = 1, ground = 0)
   Bob prepares his atom at $t = 0$.

2. Bob measures his atom at $t = T$.

3. Go to step 1 and repeat.

4. Bob can determine Alice's choice only after accumulating sufficient statistics.

Schlieder (1971)
Buchholz & Yngvason (1994)
Hegerfeld (1998)

A manifestly causal way to compute probabilities

e.g. $$P = \langle i|U^\dagger|f\rangle\langle f|U|i\rangle = \mathrm{Tr}(\underbrace{|f\rangle\langle f|}_{E}U\underbrace{|i\rangle\langle i|}_{\rho_0}U^\dagger)$$

$$U = \mathrm{T}\exp\left[\frac{1}{i}\int_{t_i}^{t_f} \mathrm{d}t\ H_{\mathrm{int}}(t)\right]$$

To see causality: commute E through U and use BCH

The BCH formula leads to an expansion of nested commutators:
[see also **M. Cliche** and **A. Kempf**, Phys. Rev. A81 (**2010**) 012330; **J. D. Franson** and **M. M. Donegan**, Phys. Rev. A65 (**2002**) 052107; **R. Dickinson**, **J. Forshaw**, **P. Millington** and **B. Cox**, JHEP 1406 (**2014**) 049.]

$$\boxed{P = \sum_{j=0}^{\infty}\int_{t_i}^{t_f} \mathrm{d}t_1\,\mathrm{d}t_2\cdots\mathrm{d}t_j\ \Theta_{12\cdots j}\ \langle i|\mathcal{F}_j|i\rangle}$$

where $$\mathcal{F}_0 = E$$

$\Theta_{12\cdots j}$ enforces $t_1 > t_2 > \cdots t_j$

$$\mathcal{F}_j = \frac{1}{i}\left[\mathcal{F}_{j-1}, H_{\mathrm{int}}(t_j)\right]$$

e.g. Fermi problem in scalar field theory

$$H_0 = \sum_n \omega_n^S \, |n^S\rangle\langle n^S| + \sum_n \omega_n^D |n^D\rangle \, \langle n^D| + \int \mathrm{d}^3\mathbf{x} \, \left(\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \tfrac{1}{2}m^2\phi^2 \right)$$

$$H_{\mathrm{int}}(t) = M^S(t)\phi(\mathbf{x}^S, t) + M^D(t)\phi(\mathbf{x}^D, t) \qquad |\mathbf{x}^S - \mathbf{x}^D| = R$$

$$M^X(t) = \sum_{m,n} \mu_{mn}^X \, e^{i\omega_{mn}^X t} \, |m^X\rangle\langle n^X| \qquad X = S, D \qquad \omega_{mn} \equiv \omega_m - \omega_n$$

$$P = \mathrm{Tr}(E\rho_T) \qquad E = E^S \otimes E^D \otimes \mathcal{E} \qquad \text{e.g.} \;\; E = |f\rangle\langle f|$$

$$E = \sum_{n,\alpha} |n^S, q^D, \alpha^\phi\rangle\langle n^S, q^D, \alpha^\phi|$$

$$\rho_T = U_{T,0}\rho_0 U_{T,0}^\dagger \qquad \text{e.g.} \;\; \rho_0 = |i\rangle\langle i|$$

$$U_{T,0} = \mathrm{Texp}\left[\frac{1}{i} \int_0^T \mathrm{d}t \, H_{\mathrm{int}} \right]$$

Notation: e.g. $\{[E^D, M_1^D], M_2^D\} = E^D_{1\underline{2}}$

$$E^X_{\dots k} \equiv \tfrac{1}{i}\left[E^X_{\dots}, M^X_k\right] \quad E^X_{\dots \underline{k}} \equiv \left\{E^X_{\dots}, M^X_k\right\} \quad \mathcal{E}^{\cdots X}_{\dots k} \equiv \tfrac{1}{i}\left[\mathcal{E}^{\cdots}_{\dots}, \phi^X_k\right] \quad \mathcal{E}^{\cdots X}_{\dots \underline{k}} \equiv \left\{\mathcal{E}^{\cdots}_{\dots}, \phi^X_k\right\}$$

and

$$E_{\underset{\circ}{k}\underset{\circ}{l}}\mathcal{E}_{\underset{\bullet}{k}\underset{\bullet}{l}} \equiv E_{kl}\mathcal{E}_{\underline{k}\underline{l}} + E_{k\underline{l}}\mathcal{E}_{\underline{k}l} + E_{\underline{k}l}\mathcal{E}_{k\underline{l}} + E_{\underline{k}\underline{l}}\mathcal{E}_{kl}$$

Can then write down any F operator:

$$\mathcal{F}_1 = \tfrac{1}{2}\left(E^S_1 E^D \mathcal{E}^S_{\underline{1}} + E^S_{\underline{1}} E^D \mathcal{E}^S_1 + E^S E^D_1 \mathcal{E}^D_{\underline{1}} + E^S E^D_{\underline{1}} \mathcal{E}^D_1\right) = \tfrac{1}{2}\left(E^S_{\underset{\circ}{1}} E^D \mathcal{E}^S_{\underset{\bullet}{1}} + E^S E^D_{\underset{\circ}{1}} \mathcal{E}^D_{\underset{\bullet}{1}}\right)$$

$$\mathcal{F}_2 = \tfrac{1}{4}\left(E^S_{\underset{\circ}{1}\underset{\circ}{2}} E^D \mathcal{E}^{SS}_{\underset{\bullet}{1}\underset{\bullet}{2}} + E^S_{\underset{\circ}{1}} E^D_{\underset{\circ}{2}} \mathcal{E}^{SD}_{\underset{\bullet}{1}\underset{\bullet}{2}} + E^S_{\underset{\circ}{2}} E^D_{\underset{\circ}{1}} \mathcal{E}^{DS}_{\underset{\bullet}{1}\underset{\bullet}{2}} + E^S E^D_{\underset{\circ}{1}\underset{\circ}{2}} \mathcal{E}^{DD}_{\underset{\bullet}{1}\underset{\bullet}{2}}\right)$$

$$\boxed{\mathcal{F}_n = 2^{-n} \sum_{a=0}^{n} E^S_{(\underset{\circ}{1}\dots\underset{\circ}{a}} \, E^D_{\underset{\circ}{a+1}\dots\underset{\circ}{n})} \, \mathcal{E}^{(S\dots S \;\; D\dots D)}_{(\underset{\bullet}{1}\,\dots\,\underset{\bullet}{a}\;\underset{\bullet}{a+1}\dots\underset{\bullet}{n})}}$$

(…) = permutations subject to time ordering within each operator e.g. **the Fermi case** (only D is observed to be in state with energy ω_q)

$$E = \sum_{n,\alpha} |n^S, q^D, \alpha^\phi\rangle\langle n^S, q^D, \alpha^\phi|$$

$$= \sum_{n,\alpha} \mathbb{1}^S \mathbb{1}^\phi |q^D\rangle\langle q^D|$$

$$|i\rangle = |p^S, g^D, 0^\phi\rangle$$

Unit E operator in field space implies 1 index must be underlined on $\mathcal{E}_{\underline{1}\ldots}$

Implies 1 index is never underlined on $E^X_{1\ldots}$

Since the E operator in S space is also the unit operator, the latest time must always reside on $E^D_{1\ldots}$

Lowest order:

$$\langle i|\,\mathcal{F}_2\,|i\rangle = \langle p^S g^D 0^\phi|\,\tfrac{1}{4}\Big(E^D_{12}\mathcal{E}^{DD}_{\underline{1}\underline{2}} + E^D_{1\underline{2}}\mathcal{E}^{DD}_{\underline{1}2} + E^D_1 E^S_{\underline{2}}\mathcal{E}^{DS}_{\underline{1}2}\Big)\,|p^S g^D 0^\phi\rangle$$

$$= |\mu^D_{qg}|^2\Big(\Delta^{DD(\mathrm{H})}_{12}\cos\omega^D_{qg}t_{12} + \Delta^{DD(\mathrm{R})}_{12}\sin\omega^D_{qg}t_{12}\Big)$$

$$\Delta^{XY(H)}_{ij} = \langle 0|\{\phi^X_i, \phi^Y_j\}|0\rangle$$

$$\Delta^{XY(R)}_{ij} = -i\langle 0|[\phi^X_i, \phi^Y_j]|0\rangle\;\Theta_{ij}$$

No dependence on source atom, S.

$$
\begin{aligned}
\langle i|\,\mathcal{F}_4\,|i\rangle \;\supset\; & \langle p^S g^D 0^\phi|\;\tfrac{1}{16}\Big(E^D_{12}E^S_{\underline{3}\underset{\circ}{4}}\mathcal{E}^{DDSS}_{\underline{1}\underline{2}3\underset{\bullet}{\underline{4}}} + E^D_{13}E^S_{\underline{2}\underset{\circ}{4}}\mathcal{E}^{DSDS}_{\underline{1}2\underline{3}\underset{\bullet}{4}} + E^D_{1\underset{\circ}{4}}E^S_{\underline{2}3}\mathcal{E}^{DSSD}_{\underline{1}2\underline{3}\underset{\bullet}{4}}\Big)\;|p^S g^D 0^\phi\rangle \\
=\; & \tfrac{1}{16}\,\langle E^D_{12}\rangle\,\Big(\,\langle E^S_{\underline{3}4}\rangle\,\langle \mathcal{E}^{DDSS}_{\underline{1}\underline{2}3\underline{4}}\rangle + \langle E^S_{\underline{3}\underline{4}}\rangle\,\langle \mathcal{E}^{DDSS}_{\underline{1}\underline{2}34}\rangle\,\Big) \\
& + \tfrac{1}{16}\,\langle E^D_{13}\rangle\,\Big(\,\langle E^S_{\underline{2}4}\rangle\,\langle \mathcal{E}^{DSDS}_{\underline{1}2\underline{3}\underline{4}}\rangle + \langle E^S_{\underline{2}\underline{4}}\rangle\,\langle \mathcal{E}^{DSDS}_{\underline{1}2\underline{3}4}\rangle\,\Big) \\
& + \tfrac{1}{16}\,\langle E^D_{14}\rangle\,\langle E^S_{\underline{2}3}\rangle\,\langle \mathcal{E}^{DSSD}_{\underline{1}2\underline{3}\underline{4}}\rangle + \tfrac{1}{16}\,\langle E^D_{1\underline{4}}\rangle\,\langle E^S_{\underline{2}3}\rangle\,\langle \mathcal{E}^{DSSD}_{\underline{1}2\underline{3}4}\rangle
\end{aligned}
$$

$$
\begin{aligned}
=\; 2\sum_n |\mu^S_{pn}|^2\,|\mu^D_{qg}|^2\Big\{ & \cos\omega^D_{qg}t_{12}\Big(\sin\omega^S_{pn}t_{34}\,\Delta^{DS(\mathrm{H})}_{24} + \cos\omega^S_{pn}t_{34}\,\Delta^{DS(\mathrm{R})}_{24}\Big)\Delta^{DS(\mathrm{R})}_{13} \\
& + \cos\omega^D_{qg}t_{12}\Big(\sin\omega^S_{pn}t_{34}\,\Delta^{DS(\mathrm{H})}_{14} + \cos\omega^S_{pn}t_{34}\,\Delta^{DS(\mathrm{R})}_{14}\Big)\Delta^{DS(\mathrm{R})}_{23} \\
& + \cos\omega^D_{qg}t_{13}\Big(\sin\omega^S_{pn}t_{24}\,\Delta^{DS(\mathrm{H})}_{34} + \cos\omega^S_{pn}t_{24}\,\Delta^{DS(\mathrm{R})}_{34}\Big)\Delta^{DS(\mathrm{R})}_{12} \\
& + \sin\omega^S_{pn}t_{23}\Big(\cos\omega^D_{qg}t_{14}\,\Delta^{SD(\mathrm{H})}_{34} + \sin\omega^D_{qg}t_{14}\,\Delta^{SD(\mathrm{R})}_{34}\Big)\Delta^{DS(\mathrm{R})}_{12}\Big\}
\end{aligned}
$$

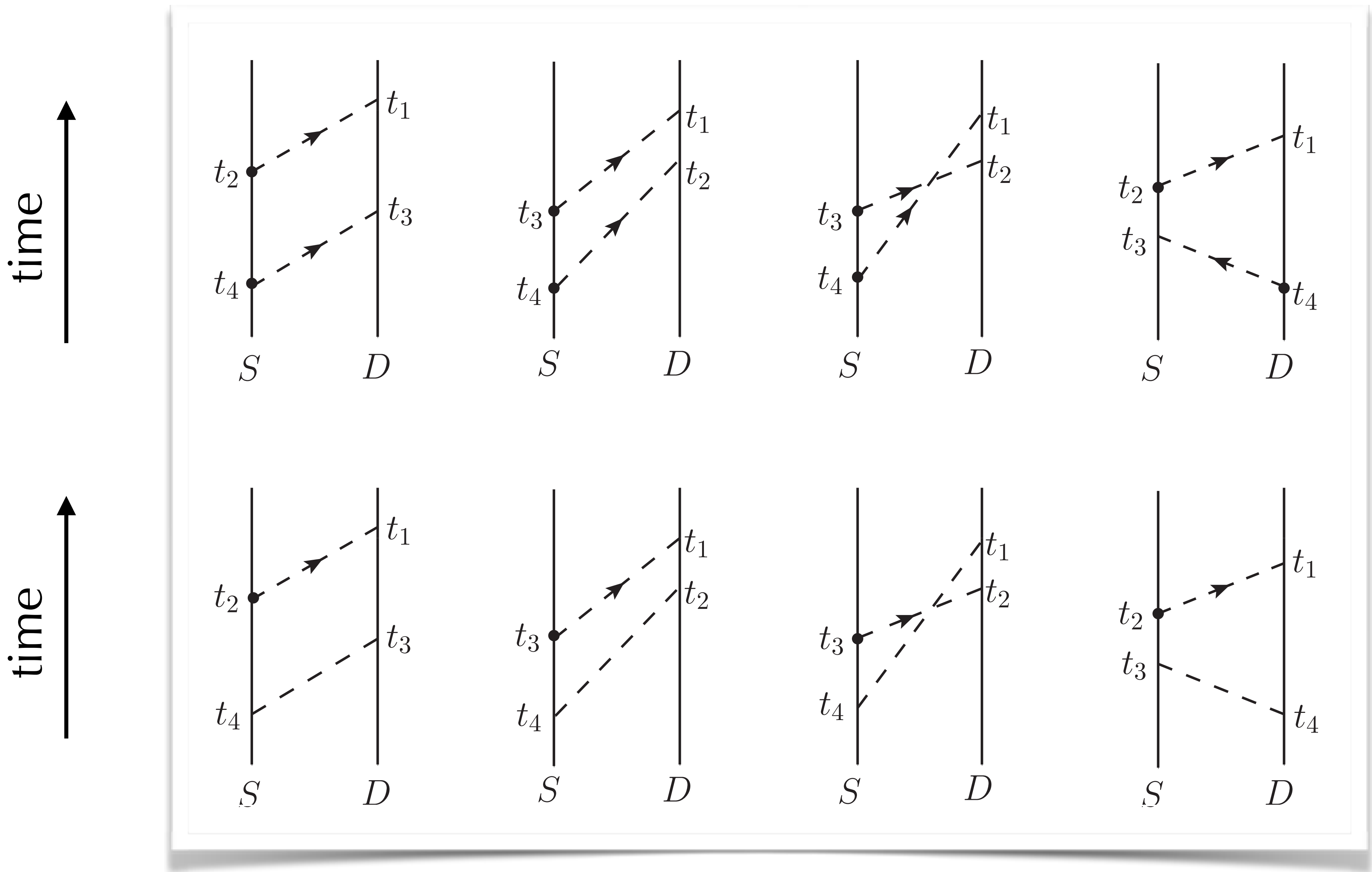
$$t_{ij} \;\equiv\; t_i - t_j$$

- Every term is purely real.

- Every term contains a retarded propagator linking S and D = manifestly causal.

- Just need expectation values of nested commutators & anti-commutators.

- Simple diagrammatic rules…..



The graphs relevant to the part of the probability that D is excited at time T that depends on the location of atom S.

These are <u>NOT</u> Feynman graphs

Latest vertex on S always connected to a future vertex on D by a retarded propagator.

Computing expectation values

1. **The field**

The vacuum expectation value of a general nesting of commutators and anti-commutators, i.e. $\mathcal{E}_{1\ldots(2p)}$ with any combination of underlinings, can be written as 2^p times the sum of all distinct products of p propagators subject to the following rule: every non-underlined (commutation) index must become the second index on a retarded propagator and all remaining indices are paired and associated with Hadamard propagators.

e.g.

$$
\begin{aligned}
\mathcal{E} &= \mathbb{I} \\
\mathcal{E}_1 &= 0 \qquad \mathcal{E}_{\bar{1}} = 2\phi_1 \\
\langle 0|\mathcal{E}_{\bar{1}2}|0\rangle &= \tfrac{1}{i}\langle 0|[2\phi_1,\phi_2]|0\rangle = \langle 0|2\Delta_{12}|0\rangle = 2\Delta^{(\mathrm{R})}_{12} \\
\langle 0|\mathcal{E}_{\bar{1}\bar{2}}|0\rangle| &= \langle 0|\{2\phi_1,\phi_2\}|0\rangle = \langle 0|2\phi_{(1}\phi_{2)}|0\rangle = 2\Delta^{(\mathrm{H})}_{12} \\
\langle 0|\mathcal{E}_{\bar{1}2\bar{3}4}|0\rangle &= \langle 0|4\Delta_{12}\Delta_{34}|0\rangle = 4\Delta^{(\mathrm{R})}_{12}\Delta^{(\mathrm{R})}_{34} \\
\langle 0|\mathcal{E}_{\bar{1}2\bar{3}\bar{4}}|0\rangle &= \langle 0|4\Delta_{12}\phi_{(3}\phi_{4)}|0\rangle = 4\Delta^{(\mathrm{R})}_{12}\Delta^{(\mathrm{H})}_{34} \\
\langle 0|\,\mathcal{E}_{\underline{1}\underline{2}34}\,|0\rangle &= \langle 0|\,4\big(\Delta_{13}\Delta_{24}+\Delta_{23}\Delta_{14}\big)\,|0\rangle = 4\big(\Delta^{(\mathrm{R})}_{13}\Delta^{(\mathrm{R})}_{24}+\Delta^{(\mathrm{R})}_{23}\Delta^{(\mathrm{R})}_{14}\big)\ , \\
\langle 0|\,\mathcal{E}_{\underline{1}\underline{2}3\underline{4}}\,|0\rangle &= \langle 0|\,4\big(\Delta_{13}\phi_{(2}\phi_{4)}+\Delta_{23}\phi_{(1}\phi_{4)}\big)\,|0\rangle = 4\big(\Delta^{(\mathrm{R})}_{13}\Delta^{(\mathrm{H})}_{24}+\Delta^{(\mathrm{R})}_{23}\Delta^{(\mathrm{H})}_{14}\big)\ , \\
\langle 0|\,\mathcal{E}_{\underline{1}\underline{2}\underline{3}4}\,|0\rangle &= \langle 0|\,4\big(\phi_{(1}\phi_2\Delta_{3)4}\big)\,|0\rangle = 4\big(\Delta^{(\mathrm{H})}_{12}\Delta^{(\mathrm{R})}_{34}+\Delta^{(\mathrm{H})}_{13}\Delta^{(\mathrm{R})}_{24}+\Delta^{(\mathrm{H})}_{23}\Delta^{(\mathrm{R})}_{14}\big)\ , \\
\langle 0|\,\mathcal{E}_{\underline{1}\underline{2}\underline{3}\underline{4}}\,|0\rangle &= \langle 0|\,\tfrac{2}{3}\phi_{(1}\phi_2\phi_3\phi_{4)}\,|0\rangle = 4\big(\Delta^{(\mathrm{H})}_{12}\Delta^{(\mathrm{H})}_{34}+\Delta^{(\mathrm{H})}_{13}\Delta^{(\mathrm{H})}_{24}+\Delta^{(\mathrm{H})}_{23}\Delta^{(\mathrm{H})}_{14}\big)\ .
\end{aligned}
$$

2. **The atoms**

$$E = \epsilon_{mn}\, |m\rangle\langle n|$$

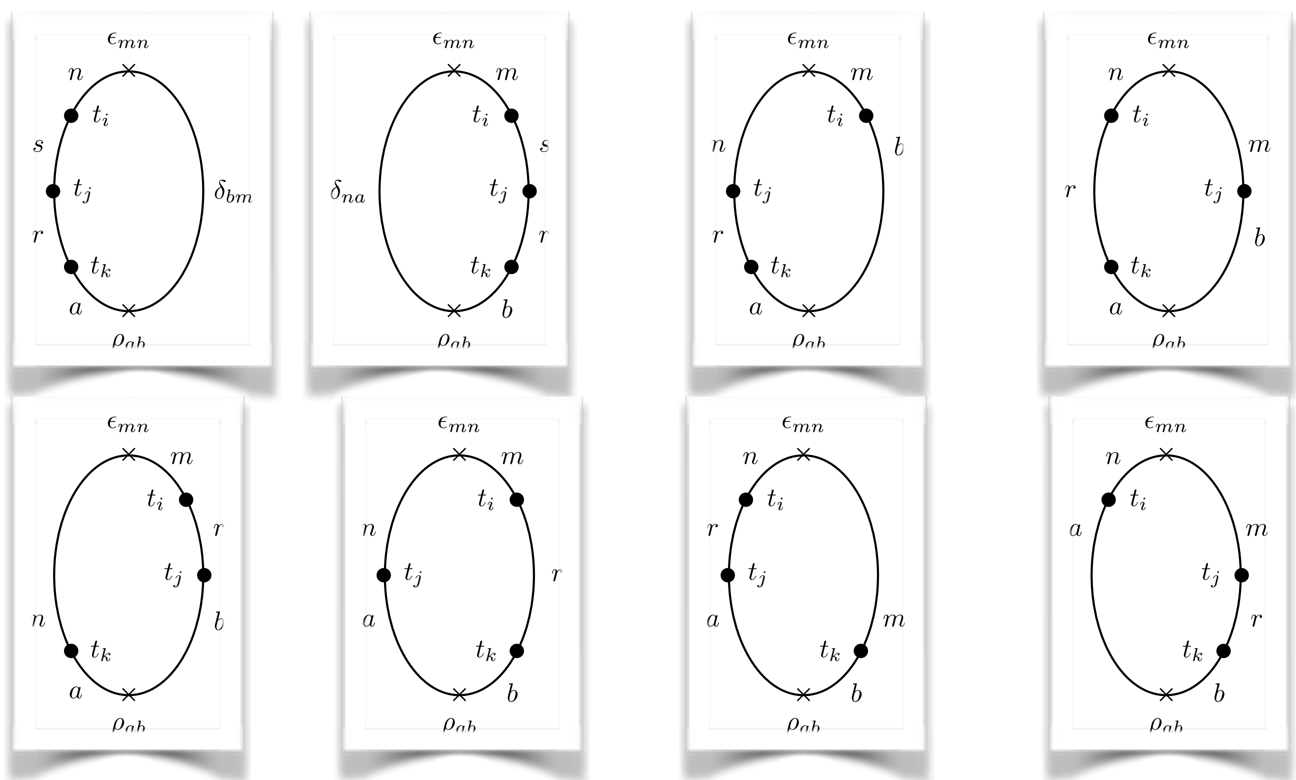
$$\mathrm{Tr}\Big(\rho_{ab}\, |a\rangle\langle b|\, \big[\big[\dots\big[\big[E, M_i\big]_{\eta_i}, M_j\big]_{\eta_j}, \dots\big], M_N\big]_{\eta_N}\Big)$$

e.g. N = 3

$$\rho_{ab} = \delta_{ag}\delta_{gb}$$

$$\epsilon_{mn} = \delta_{mq}\delta_{qn}$$

for Fermi problem

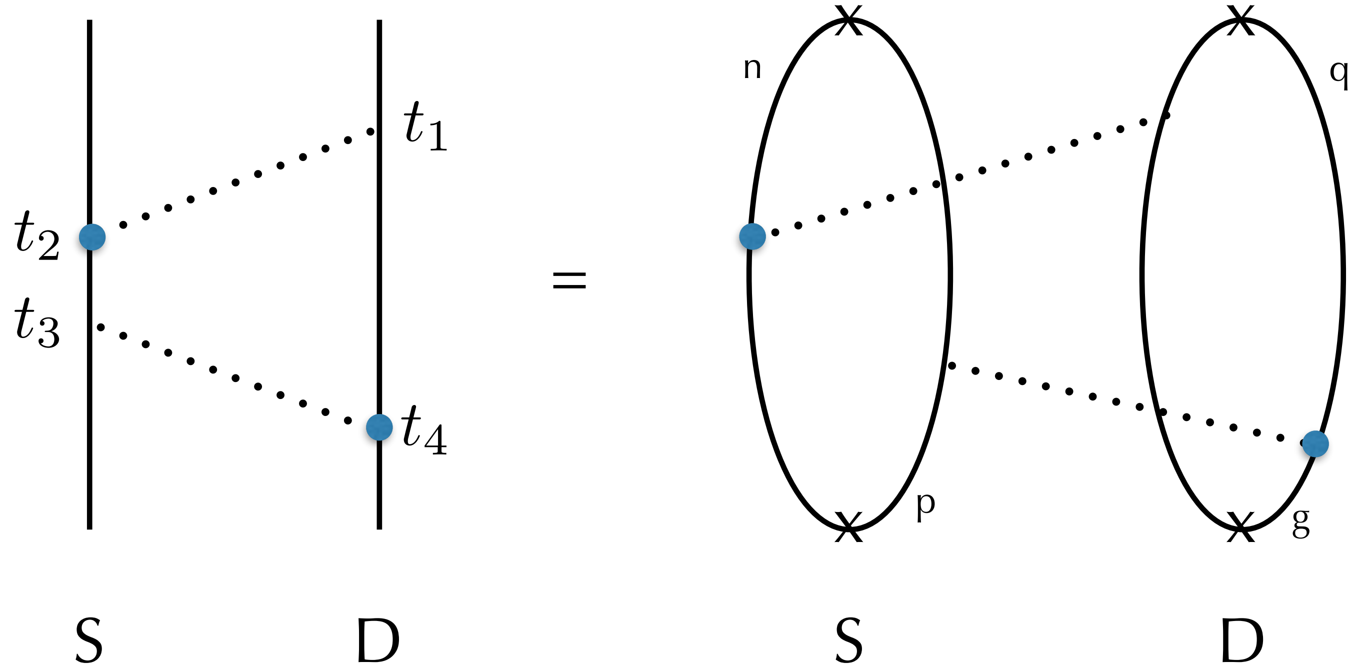


$$\epsilon_{mn}\, \rho_{ab}\, \mu_{bm}\mu_{ra}\mu_{nr}\, \Delta_{ij}^{r(>)} e^{-i\omega_a t_j} e^{i\omega_n t_i} e^{-i\omega_m t_k} e^{i\omega_b t_k}$$

$$\Delta_{ij}^{r(>)} = e^{-i\omega_r t_{ij}}$$

$$\Delta_>(x,y) = \langle\phi(x)\phi(y)\rangle = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3 2E} e^{-ip\cdot(x-y)}$$

1. Work clockwise around the ellipse and
   (a) assign a factor of μ_{rs} for each time,
   (b) connect consecutive times with atom Wightman propagators $\Delta_{ij}^{r(>)}$,
   (c) assign a factor of $e^{+(-)i\omega_r t_i}$ for the times t_i followed (preceded) by a cross.
2. Assign a factor of η_i for any time t_i appearing on the falling side of the ellipse.



$+$ 3 more

$$= \; 2 \sum_n |\mu^S_{pn}|^2 \, |\mu^D_{qg}|^2 \;\; \sin(\omega^S_{pn} t_{23}) \;\; \sin(\omega^D_{qg} t_{14}) \; \Delta^{SD(\mathrm{R})}_{34} \, \Delta^{DS(\mathrm{R})}_{12}$$

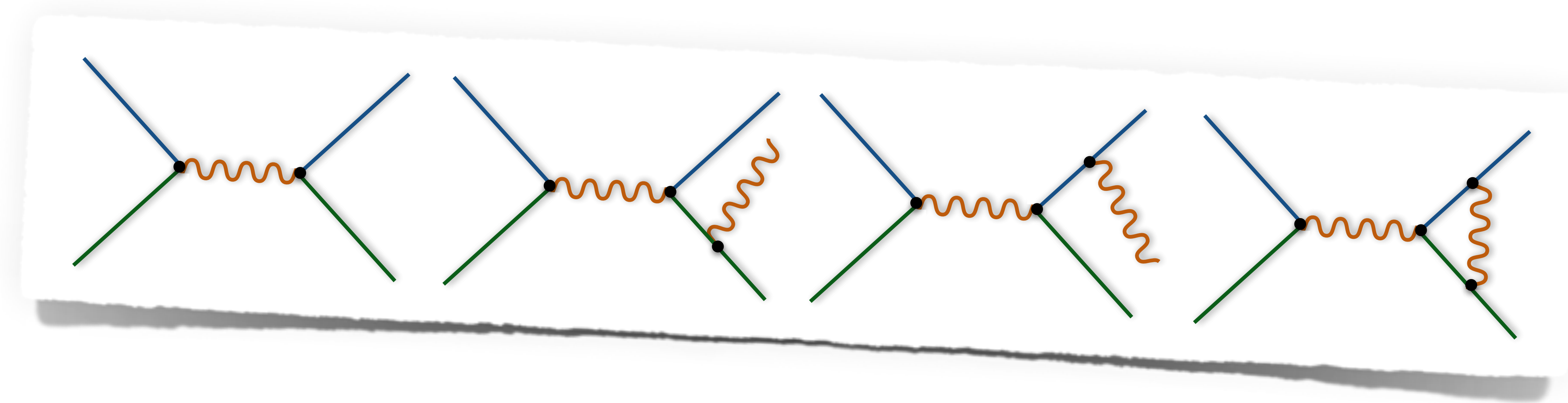
Since probabilities contain both **time-ordered** and **anti-time-ordered** contributions, the diagrammatic structure resembles that of the **closed-time-path formalism**.

[**J. S. Schwinger**, J. Math. Phys. 2 (**1961**) 407-432; **L. V. Keldysh**, Zh. Eksp. Teor. Fiz. 47 (**1964**) 1515-1527, Sov. Phys. JETP 20 (**1965**) 1018; **R. L. Kobes** and **G. W. Semenoff**, Nucl. Phys. B260 (**1985**) 714-746; B272 (1986) 329-364; **R. L. Kobes**, Phys. Rev. D43 (**1991**) 1269-1282; see also **R. Dickinson**, **J. Forshaw**, **P. Millington** and **B. Cox**, JHEP 1406 (2014) 049.]

In order to find a (weakly) causal result for the Fermi two-atom problem, we had to sum **inclusively** over the (unobserved) final state of the photon field.

By working directly with probabilities, summing inclusively over the states spanning a given Hilbert space corresponds to a unit operator, i.e. **we do not have to calculate the individual amplitudes for all possible emissions in the final state.**

What does this mean for the **Bloch-Nordsieck** or **Kinoshita-Lee-Nauenberg** theorems? Are they applied implicitly if we work directly with probabilities?



General observables

$$P = \langle i|U^\dagger E U|i\rangle = \mathrm{Tr}(EU\rho_i U^\dagger)$$

$$N_{\mathcal{R}_0} \equiv \sum_\lambda \int_{\mathcal{R}_0} \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\sqrt{\mathbf{k}^2+m^2}}\, a_\lambda^\dagger(\mathbf{k}) a_\lambda(\mathbf{k})$$

$$\Delta_{\mathcal{R}_0} \equiv \mathbb{I} + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} : \left(N_{\mathcal{R}_0}\right)^j : \quad = \quad \underline{\text{operator form of the Sudakov factor}}$$

$$= : e^{-N_{\mathcal{R}_0}} :$$

$$\Delta_{\mathcal{R}_0} |\mathbf{k}_1 \dots \mathbf{k}_N\rangle = \begin{cases} |\mathbf{k}_1 \dots \mathbf{k}_N\rangle & \text{if } n_0 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$n_0 =$ number of quanta in $\mathcal{R}_0$

e.g. $\Delta_{\mathbb{R}^3} = |0\rangle\langle 0|$

$\Delta_\emptyset = \mathbb{I}$

$$\Delta^{(j)}_{\mathcal{R}_0} \equiv : \frac{1}{j!}\left(N_{\mathcal{R}_0}\right)^j e^{-N_{\mathcal{R}_0}} : \quad = \quad \underline{\text{semi-inclusive projection operator}}$$

Projects onto the subspace of states in which exactly j particles have momenta in $\mathcal{R}_0$.

This generalises to

$$\Delta^{\{j_a\}}_{\{\mathcal{R}_a \subseteq \mathcal{R}_0\}} \equiv : \prod_a \left(\frac{1}{j_a!} \left(N_{\mathcal{R}_a}\right)^{j_a} \right) e^{-N_{\mathcal{R}_0}} :$$

Projects onto the subspace of states in which exactly $\sum_a j_a$ particles have momenta in $\mathcal{R}_0$, distributed so that exactly j_a particles have momenta in each disjoint subset $\mathcal{R}_a \subseteq \mathcal{R}_0$.

e.g. Pick $\mathcal{R}_0 = \mathbb{R}^3$ and one particle with momentum $\mathbf{k} \to \mathbf{k} + \mathrm{d}^3\mathbf{k}$. In this case we compute using

$$E = \; : N_{\mathbf{k}} \, e^{-N_{\mathbb{R}^3}} : \; = \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2E} : a^\dagger(\mathbf{k}) a(\mathbf{k}) \, |0\rangle\langle 0| : \; = \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2E} |\mathbf{k}\rangle\langle\mathbf{k}|$$

Can compute differential in any function of the final state momenta for observables that are fully inclusive over some region, i.e. the most general type of observable.

$$\frac{\mathrm{d}E}{\mathrm{d}V} = \sum_n \prod_{i=1}^n \int_{\mathcal{R}_0} \frac{\mathrm{d}^3\mathbf{k}_i}{(2\pi)^3} \frac{1}{2E_i} \, \delta\big(v_n(\{\mathbf{k}_i\}) - V\big) \frac{1}{n!} : \prod_{i=1}^n \left(a^\dagger(\mathbf{k}_i) a(\mathbf{k}_i)\right) e^{-N_{\mathcal{R}_0}} :$$

$\mathcal{R}_0$ is the region over which the observable is sensitive

Conclusions

- The S-matrix is (quite literally) only half the story.
- Einstein causality in the Fermi two-atom problem emerges only after we sum inclusively over the unobserved final states of the source atom and the electromagnetic field.
- There exists a way to compute directly at the level of probabilities where causality is explicit: How useful is it? What are the general graphical rules?
- What are the implications for dealing with soft and collinear IR divergences in gauge theories?
- There are parallels with the closed-time path formalism and diagrammatics of non-equilibrium QFT, including the Kobes-Semenoff unitarity cutting rules.