Massive Quark Jet Function at Two Loops

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Doktoratskolleg
Particles and Interactions

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Outline

- Introduction & Motivation
- Calculation: General Setup
- Topology A
  - IBP Reduction and Solution of the Master Integrals
  - Imaginary Part and Renormalization
- Topology B
  - Information Extraction and Numerics
- Secondary Mass Effects
- Summary & Outlook

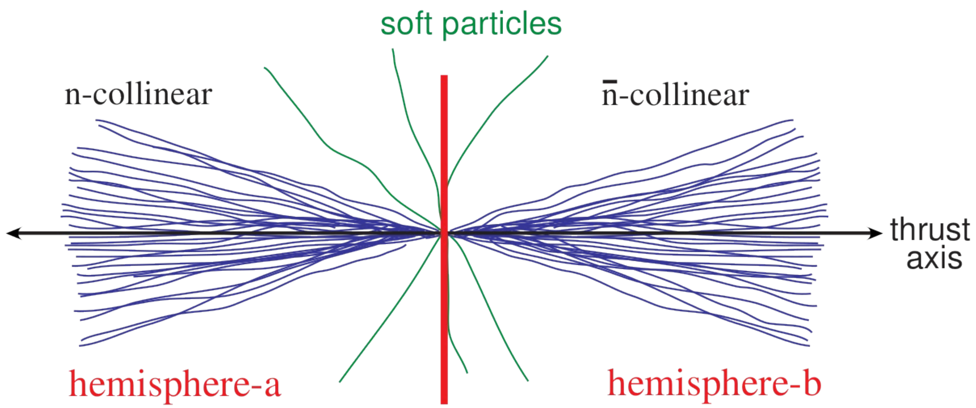
Introduction

- Collider physics:
  - Large scale hierarchy if final state consists of
    - Jets: Highly energetic, collimated, strongly interacting particles
    - Soft radiation

- Large scale hierarchies: Large logarithms of the scale ratios spoil perturbative expansion

$$1+\alpha_s(\log^2+\log+1)+\alpha_s^2(\log^4+\log^3+\log^2+\log+1)+\ldots$$

$$\alpha_s \log \sim 1$$



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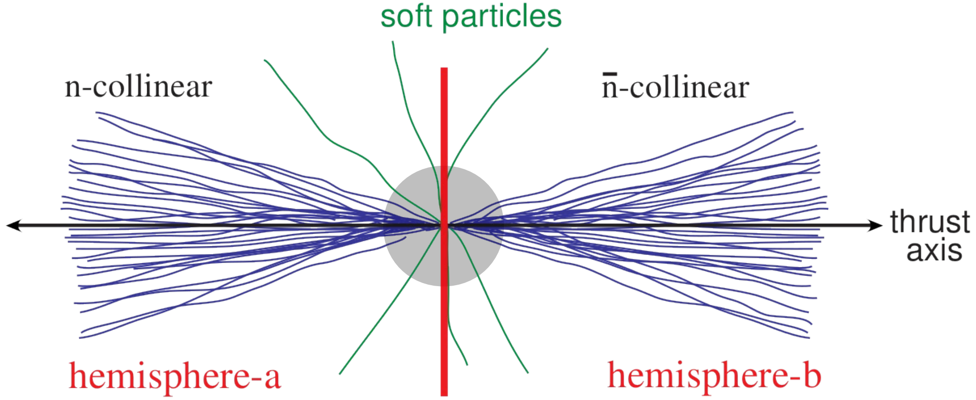
  - Hard scale $\leftrightarrow$ COM energy



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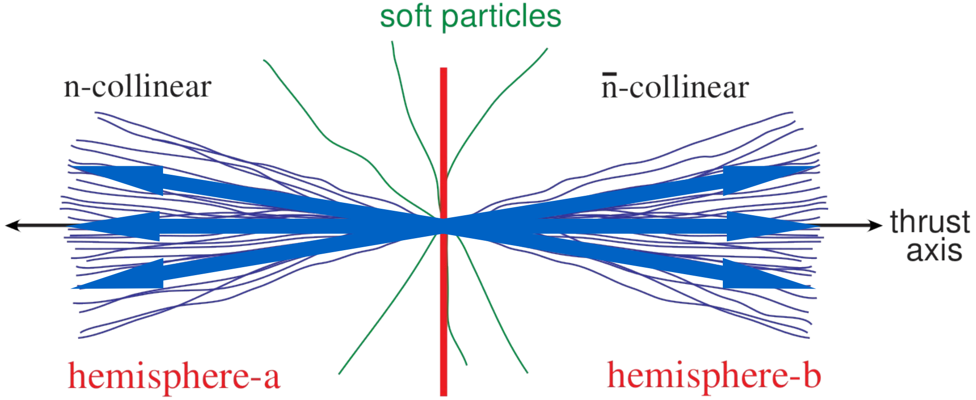

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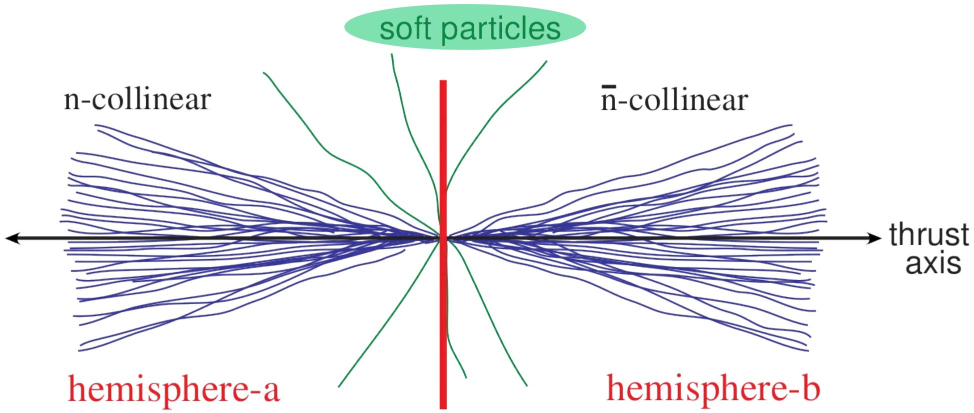
  - Soft scale $\leftrightarrow$ Scale of low energy radiation
  - (Non-perturbative scale)



Introduction and Motivation

$$\sigma \sim H \times f_p \otimes f_p \otimes J_1 \otimes J_2 \otimes \cdots \otimes J_n \otimes S$$

- Solution: Factorization formula
  - Separates short and long distance physics
  - Enables resummation of large logs
- Can be derived by using Soft-Collinear Effective Theory (SCET)

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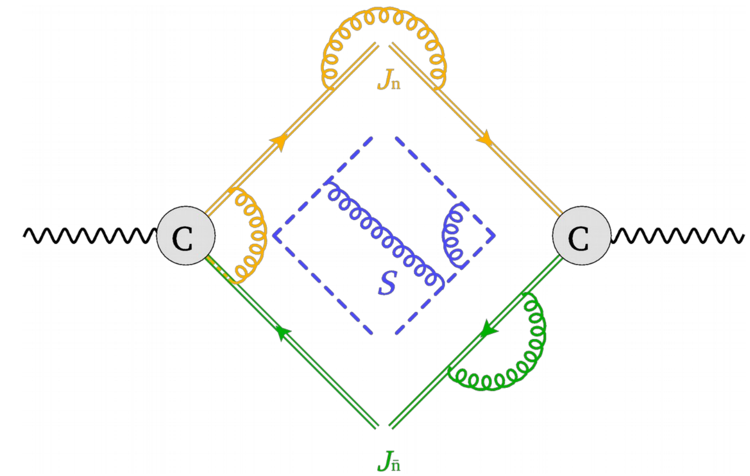

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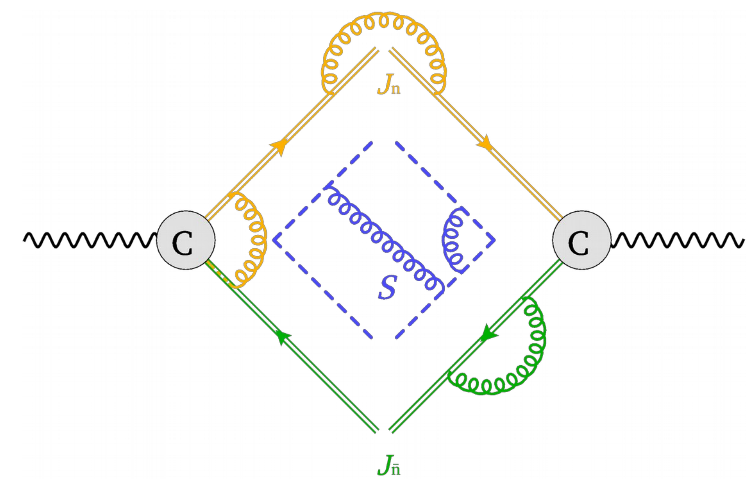

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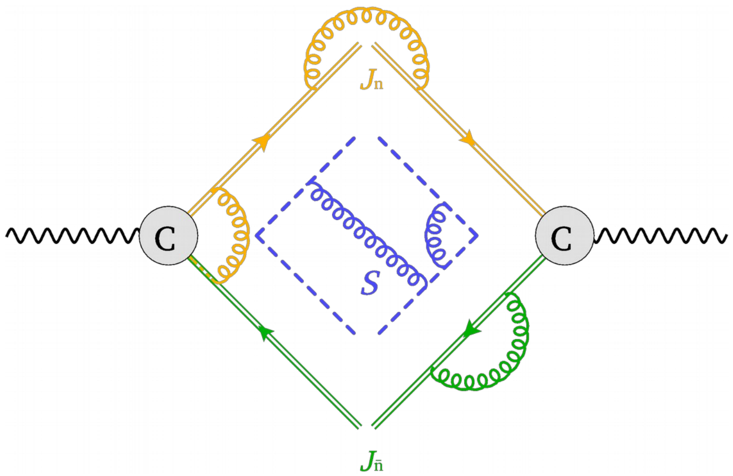

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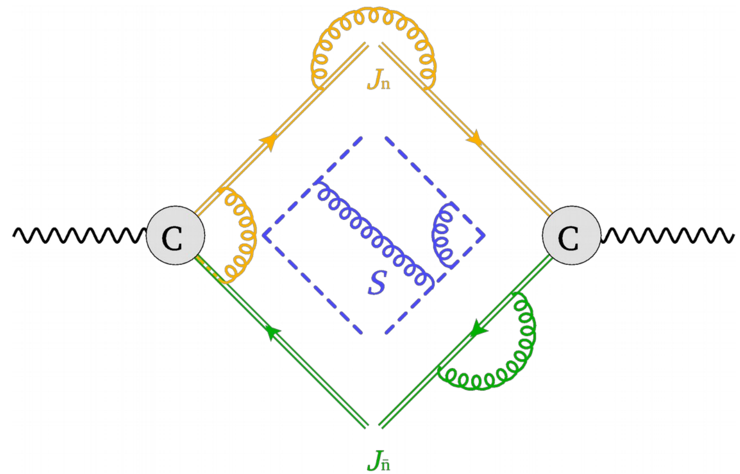

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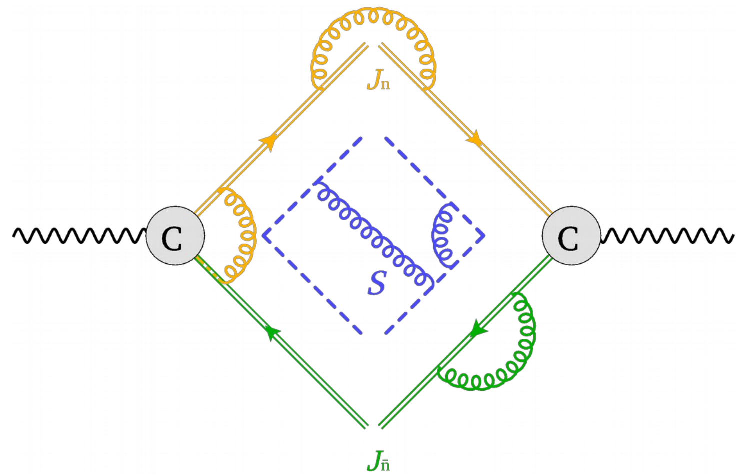

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- Every part corresponds to one of the characteristic scales
  - Hard scale ↔ Hard function
  - Jet scale ↔ Jet function
  - Soft scale ↔ Soft function
  - Nonperturbative scale ↔ Parton Distribution Functions (PDFs)

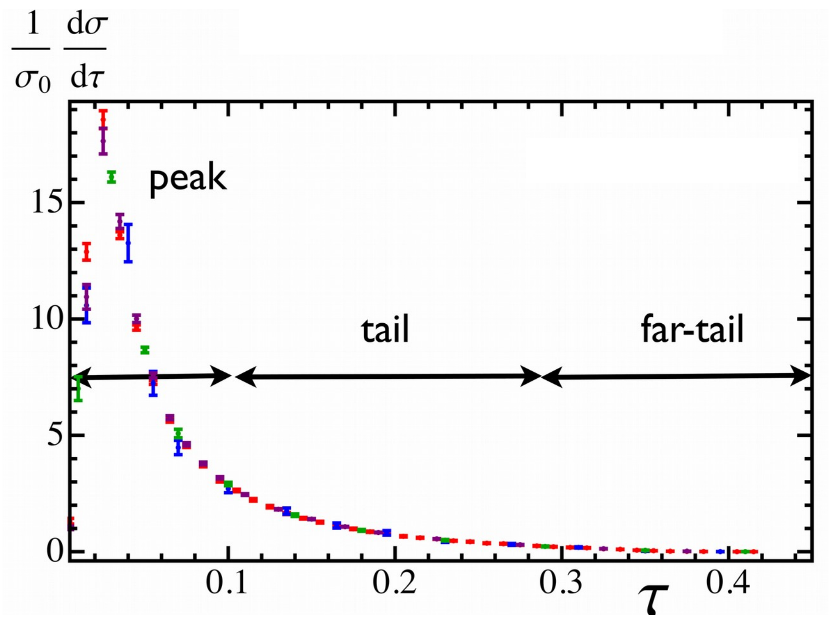
Introduction and Motivation

$$\sigma \sim H \times f_p \otimes f_p \otimes J_1 \otimes J_2 \otimes \cdots \otimes J_n \otimes S$$

- Broad range of applications
  - Many QCD processes
  - Flavor physics (B decays, ...)
  - Collider physics (event shapes, α_s determination, ...)
  - ...

Introduction and Motivation

- Mass effects of primary produced quarks:
  - In SCET region mainly contained in jet function
- Example: Tail region in thrust distribution of $e^+ e^- \rightarrow t\,\bar{t}$

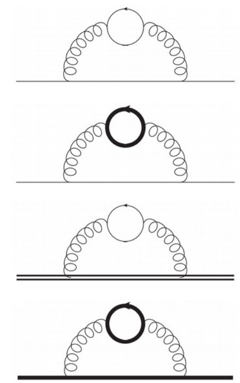


[Talk bei B. Dehnadi ('15)]

Introduction and Motivation

- Status of two-loop quark jet function (jet mass measurement):
  - Massless [Becher, Neubert ('06)]
  - Massive (secondary) [Gritschacher et al. ('13)]
  - bHQET [Jain et al. ('08)]
  - Massive (primary) → Missing

    [Hoang, CL, Stahlhofen (Soon)]



- Primary massive:
  - Last missing piece for N³LL resummation with full quark mass dependence for some event shapes: top mass calibration, α_s
  - Universal → also used in DIS, pp, ...
  - Interesting to study SCET ↔ bHQET transition

Jet function regimes:

$\frac{p^2 - m^2}{m^2} \ll 1$ bHQET

$\frac{p^2 - m^2}{Q^2} \ll 1$ Massive

$\frac{m^2}{p^2 - m^2} \ll 1$ Massless

Calculation

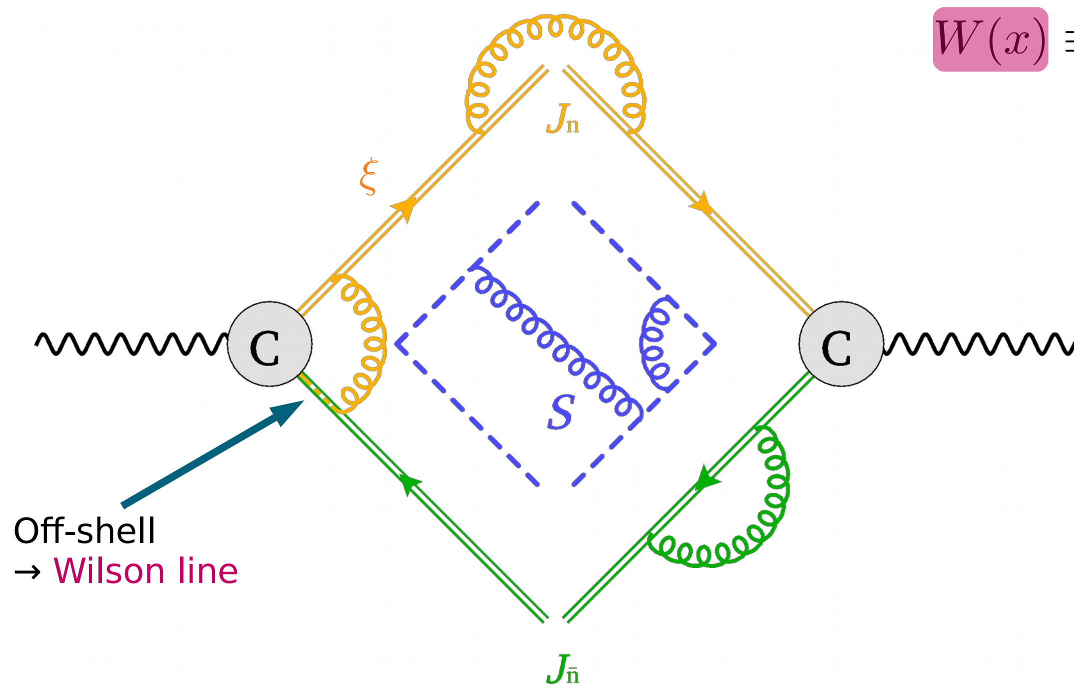
- Definition of SCET jet function in n-direction:

$$\boxed{J(s,m^2,\mu)} = \frac{1}{\pi}\mathrm{Im}\left[i\mathcal{J}(s,m^2,\mu)\right]$$

$$\frac{\not{n}}{2}\bar{n}\cdot p\,\mathcal{J}(s,m^2,\mu) = \int \mathrm{d}x\; \mathrm{e}^{-ip\cdot x}\,\langle 0|\mathrm{T}\left\{\boxed{W^\dagger(0)}\xi(0)\,\bar{\xi}(x)\boxed{W(x)}\right\}|0\rangle$$

$$s \equiv p^2 - m^2 + i0$$

$$\boxed{W(x)} \equiv \mathrm{P}\exp\left[ig\int_{-\infty}^{0}\mathrm{d}s\,\bar{n}\cdot A(x+s\bar{n})\right]$$



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$$W(x) \equiv \mathrm{P}\exp\left[ig\int_{-\infty}^{0}\mathrm{d}s\,\bar{n}\cdot A(x+s\bar{n})\right]$$

- First simplification (for the SCET experts):
  - Collinear quark field in SCET:

$$\psi_c(x) = \frac{\not{n}\bar{\not{n}}}{4}\psi_c(x) + \frac{\bar{\not{n}}\not{n}}{4}\psi_c(x) \equiv \xi(x) + \eta(x)$$

$$\xi(x) = \frac{\not{n}\bar{\not{n}}}{4}\psi_c(x)$$

Power suppressed
→ Usually integrated out

$$\langle 0 \big| \mathrm{T}\left\{ W^\dagger(0) \xi(0)\, \bar{\xi}(x) W(x) \right\} \big| 0 \rangle = \left\langle 0 \middle| \mathrm{T}\left\{ \frac{\not{n}\not{\bar{n}}}{4} W^\dagger(0) \psi_c(0)\, \bar{\psi}_c(x) W(x) \frac{\not{\bar{n}}\not{n}}{4} \right\} \middle| 0 \right\rangle$$

- SCET Lagrangian:

$$\mathcal{L}_c = \bar{\xi}\left(in \cdot D_c + (i\not{D}_{c,\perp} - m) \frac{1}{i\bar{n} \cdot D_c} (i\not{D}_{c,\perp} + m) \right) \frac{\not{\bar{n}}}{2} \xi$$

→ SCET Feynman rules

- QCD Lagrangian:

$$\mathcal{L}_c = \bar{\psi}_c (i\not{D}_c - m) \psi_c$$

→ QCD Feynman rules

→ Less complicated

→ Less diagrams

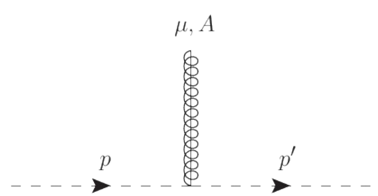
- Subleading terms removed by additional Projector

$$\langle 0 | \mathrm{T}\left\{ W^\dagger(0)\xi(0)\,\bar{\xi}(x) W(x)\right\} | 0 \rangle = \left\langle 0 \middle| \mathrm{T}\left\{ \frac{\not{n}\bar{\not{n}}}{4} W^\dagger(0)\psi_c(0)\,\bar{\psi}_c(x) W(x)\frac{\bar{\not{n}}\not{n}}{4}\right\} \middle| 0 \right\rangle$$

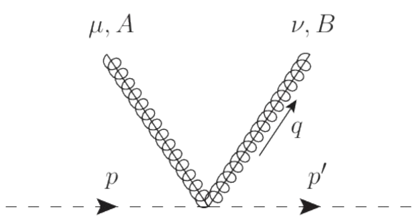
- SCET Lagrangian:

$$\mathcal{L}_c = \bar{\xi}\left(in\cdot D_c + (i\not{D}_{c,\perp} - m)\frac{1}{i\bar{n}\cdot D_c}(i\not{D}_{c,\perp} + m)\right)\frac{\bar{\not{n}}}{2}\xi$$

→ SCET Feynman rules



$$= \quad (ig_s T^A \tilde{\mu}^{\varepsilon})\left[n^\mu + \frac{\gamma_\perp^\mu\left(\not{p}_\perp + m\right)}{p^-} + \frac{\left(\not{p}'_\perp - m\right)\gamma_\perp^\mu}{p'^-} - \frac{\left(\not{p}'_\perp - m\right)\left(\not{p}_\perp + m\right)}{p^- p'^-}\bar{n}^\mu \right]\frac{\bar{\not{n}}}{2}$$



$$= \quad \frac{ig_s^2 T^A T^B \tilde{\mu}^{2\varepsilon}}{p^- - q^-}\left[\gamma_\perp^\mu \gamma_\perp^\nu - \frac{\gamma_\perp^\mu\left(\not{p}_\perp + m\right)}{p^-}\bar{n}^\nu - \frac{\left(\not{p}'_\perp - m\right)\gamma_\perp^\nu}{p'^-}\bar{n}^\mu + \frac{\left(\not{p}'_\perp - m\right)\left(\not{p}_\perp + m\right)}{p^- p'^-}\bar{n}^\mu\bar{n}^\nu \right]\frac{\bar{\not{n}}}{2}$$
$$+ \frac{ig_s^2 T^B T^A \tilde{\mu}^{2\varepsilon}}{q^- + p'^-}\left[\gamma_\perp^\nu \gamma_\perp^\mu - \frac{\gamma_\perp^\nu\left(\not{p}_\perp + m\right)}{p^-}\bar{n}^\mu - \frac{\left(\not{p}'_\perp - m\right)\gamma_\perp^\mu}{p'^-}\bar{n}^\nu + \frac{\left(\not{p}'_\perp - m\right)\left(\not{p}_\perp + m\right)}{p^- p'^-}\bar{n}^\mu\bar{n}^\nu \right]\frac{\bar{\not{n}}}{2}$$

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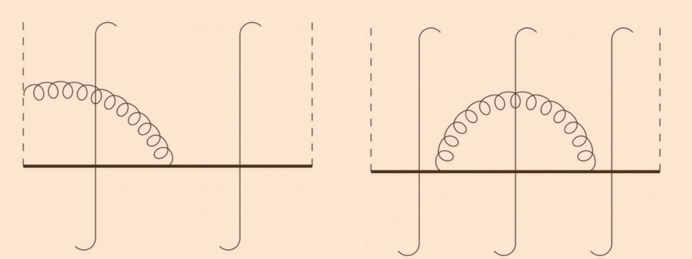
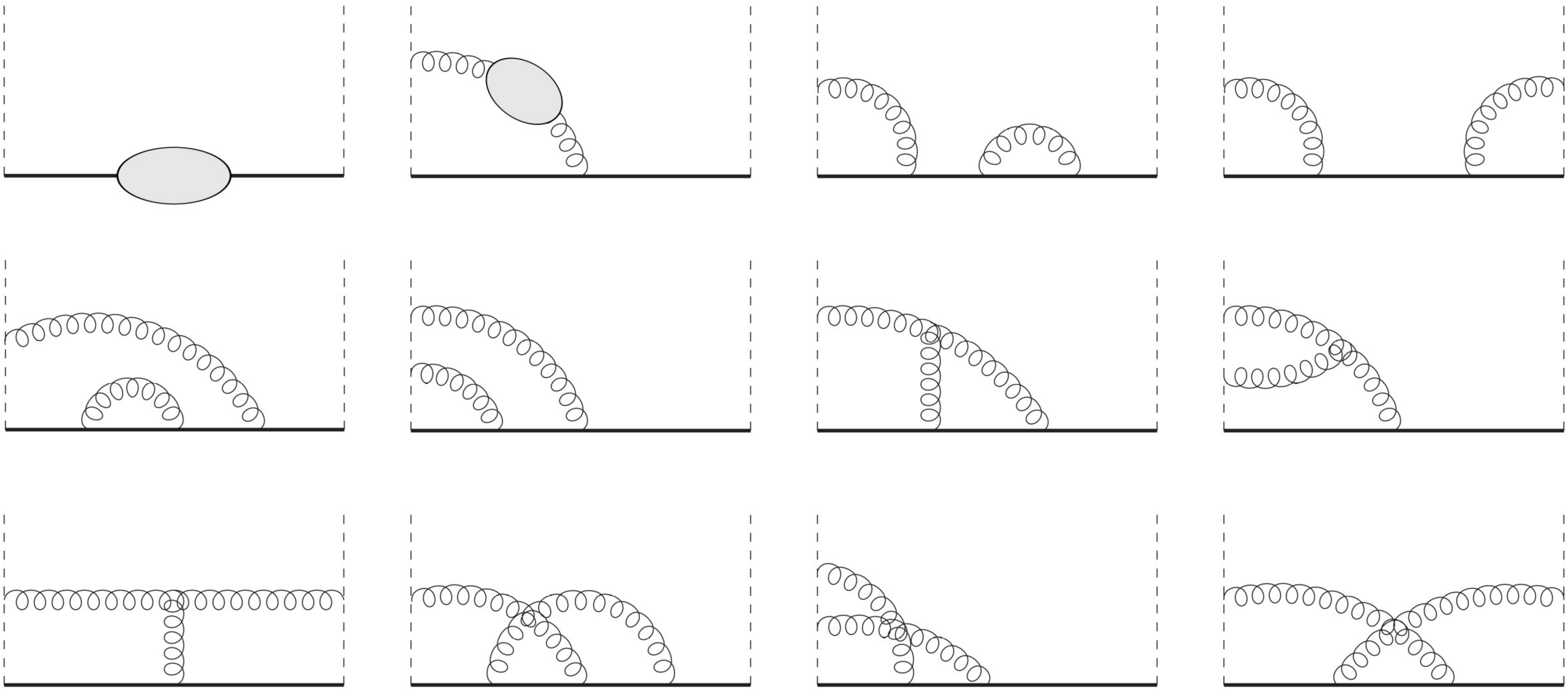
1-loop:



2-Loop: 23 Diagrams in Feynman Gauge



+ Mirror

Calculation – Two Integral Topologies

- Dirac structure and other simplifications: FORM [Ruijl et al. ('17)]
- Most general integral:

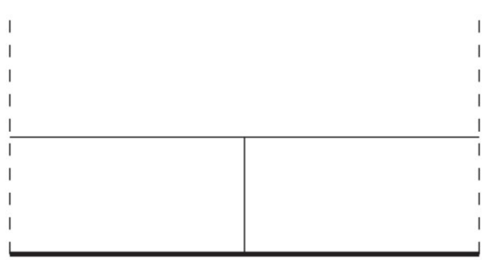
$$\int \mathrm{d}^d k\, \mathrm{d}^d \ell\, \frac{1}{[-k^2]^{a_1}[-\ell^2]^{a_2}[-(k-\ell)^2]^{a_3}[-(p+k)^2+m^2]^{b_1}[-(p+\ell)^2+m^2]^{b_2}[-(p+k+\ell)^2+m^2]^{b_3}[-\bar{n}\cdot k]^{c_1}[-\bar{n}\cdot \ell]^{c_2}}$$

$$= -\pi^d (-s)^{d-a_1-a_2-a_3-b_1-b_2-b_3} (\bar{n}\cdot p)^{-c_1-c_2} J(a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2)$$

$$k^2+l^2-(p^2-m^2)-(k-l)^2+((p+k)^2-m^2)+((p+l)^2-m^2)-((p+k+l)^2-m^2)=0$$

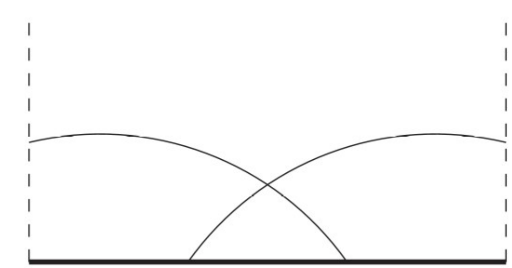
"Topology A"

$$\int \mathrm{d}^d k\, \mathrm{d}^d \ell\, \frac{1}{[-k^2]^{a_1}[-\ell^2]^{a_2}[-(k-\ell)^2]^{a_3}[-(p+k)^2+m^2]^{b_1}[-(p+\ell)^2+m^2]^{b_2}[-\bar{n}\cdot k]^{c_1}[-\bar{n}\cdot \ell]^{c_2}}$$



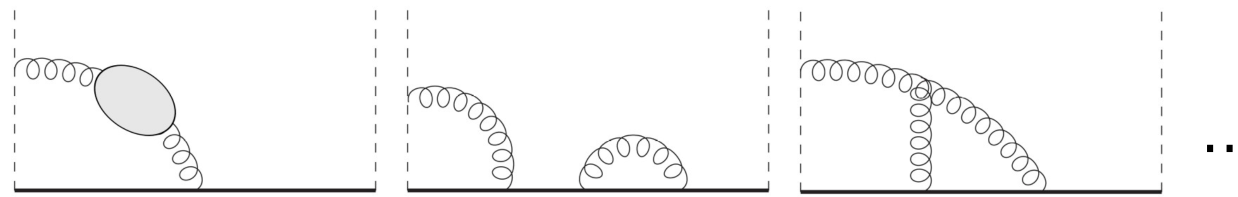
"Topology B"

$$\int \mathrm{d}^d k\, \mathrm{d}^d \ell\, \frac{1}{[-k^2]^{a_1}[-\ell^2]^{a_2}[-(p+k)^2+m^2]^{b_1}[-(p+\ell)^2+m^2]^{b_2}[-(p+k+\ell)^2+m^2]^{b_3}[-\bar{n}\cdot k]^{c_1}[-\bar{n}\cdot \ell]^{c_2}}$$

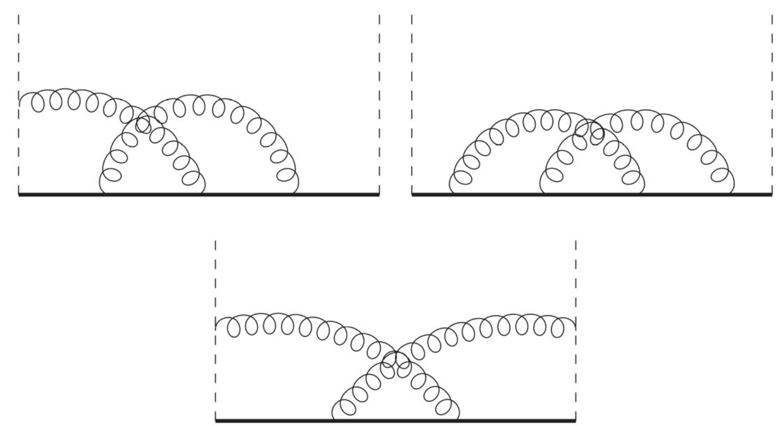


Calculation – Two Integral Topologies

- Topology A:
  - Present in all 23 diagrams
    19 pure Topology A diagrams
  - Color structures:
    $C_F^2,\ C_F C_A,\ C_F T_F n_\ell \sim \mathcal{O}(N_c^2)$
  - Solution in terms of PolyLogs



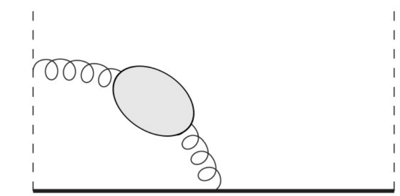
- Topology B:
  - Present in 4 diagrams
  - Color structure: $C_F^2 - C_F C_A/2 \sim \mathcal{O}(N_c^0)$
  - More general functions
  - Much harder to compute



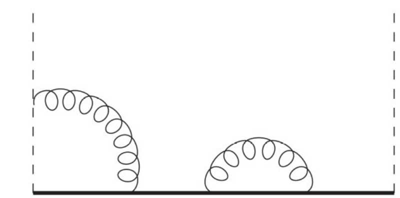
→ Consider Topology A first

Topology A – Roadmap

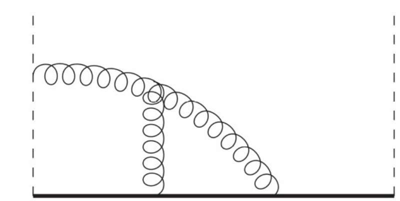
- Reduce number of integrals
  → IBP reduction to Master Integrals (MIs)



- Solving the MIs
  → Method of differential equations



- Take Imaginary part / Discontinuity



- Renormalization, Results and cross checks

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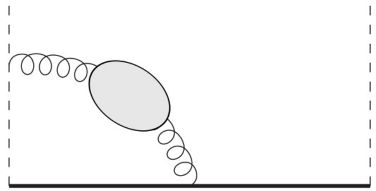

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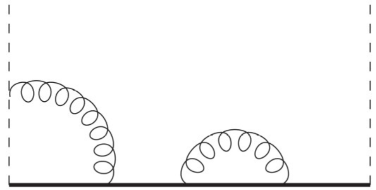

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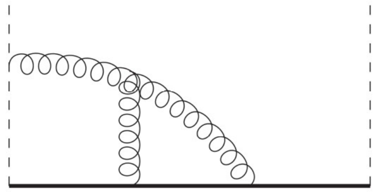

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Topology A – Calculation

- 19 diagrams lead to ~110 different scalar two-loop integrals
  - Step one: Reduce number of integrals by using "Integration By Parts (IBP) reduction" [Tkachov ('81)]

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- How does this work?
  - Based on a fundamental property of DimReg integrals:

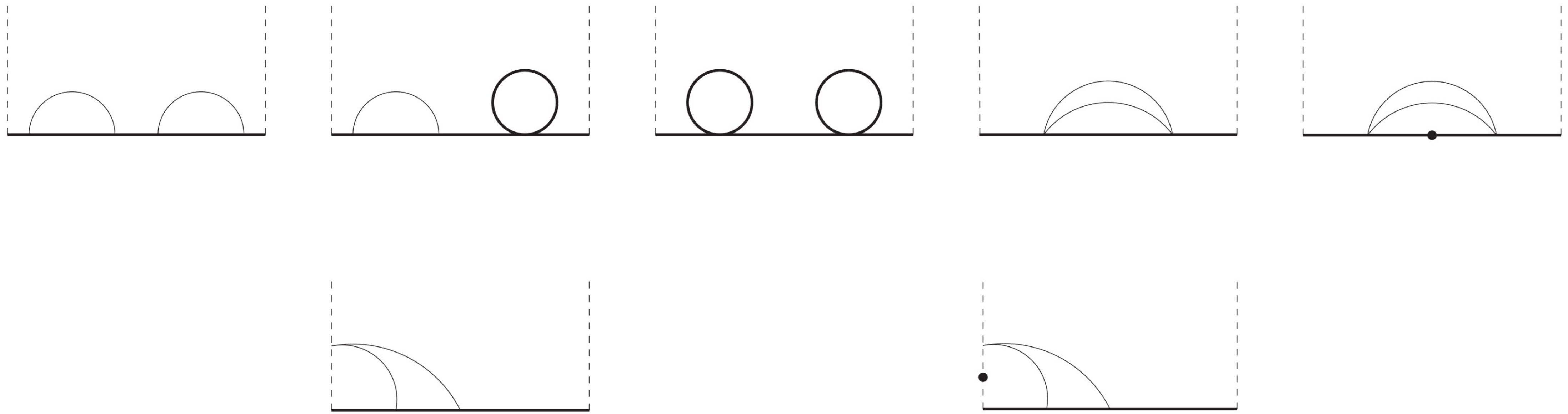
$$\int \mathrm{d}^d k \, \frac{\partial}{\partial k^\mu} f^\mu(k) = 0$$

  - Use arising set of relations to reduce the integrals of interest

$$0 = \int \mathrm{d}^d k \, \mathrm{d}^d \ell \, \frac{\mathrm{d}}{\mathrm{d}\{k^\mu, \ell^\mu\}} \frac{\{p^\mu, k^\mu . \ell^\mu\}}{[-k^2]^{a_1}[-\ell^2]^{a_2}[-(k-\ell)^2]^{a_3}[-(p+k)^2+m^2]^{b_1}[-(p+\ell)^2+m^2]^{b_2}[-\bar{n}\cdot k]^{c_1}[-\bar{n}\cdot\ell]^{c_2}}$$

Topology A – Calculation

- In practice: IBP reduction automatized (FIRE [Smirnov ('13)])
  - Outcome: ~110 Integrals → 7 Master Integrals (MIs)



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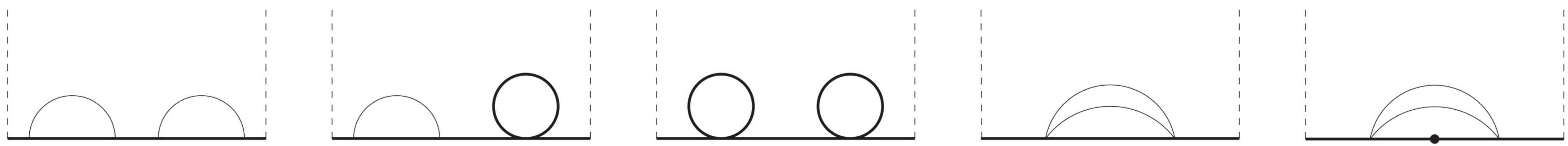

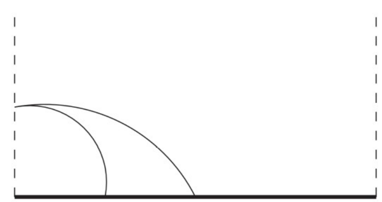

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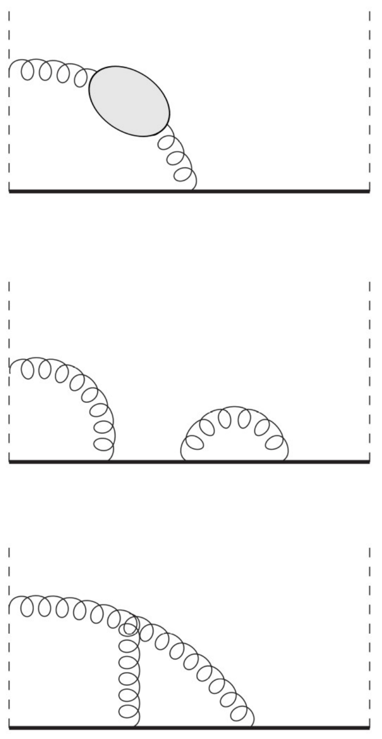

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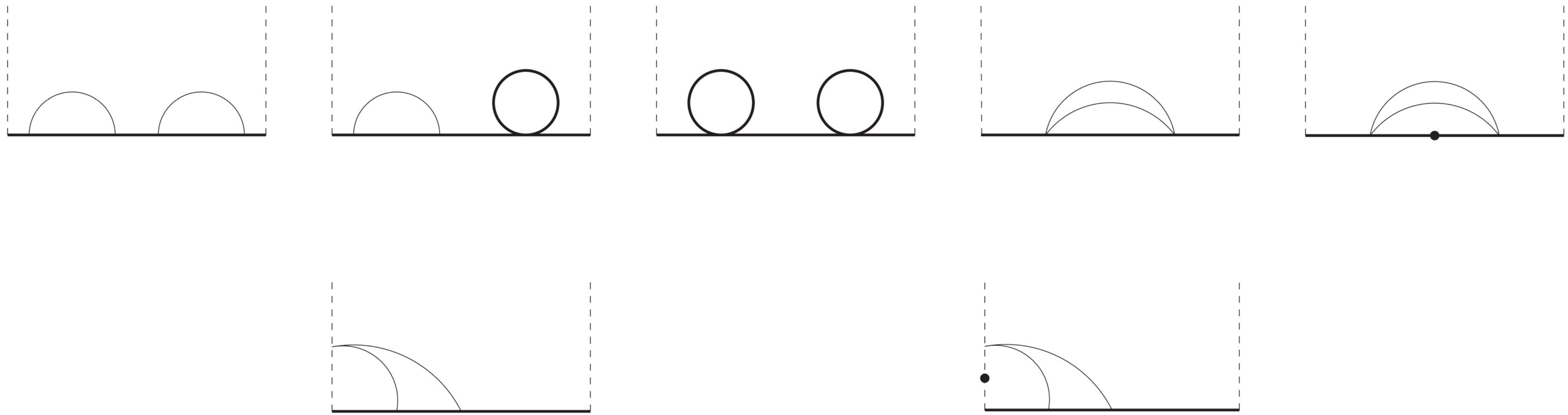

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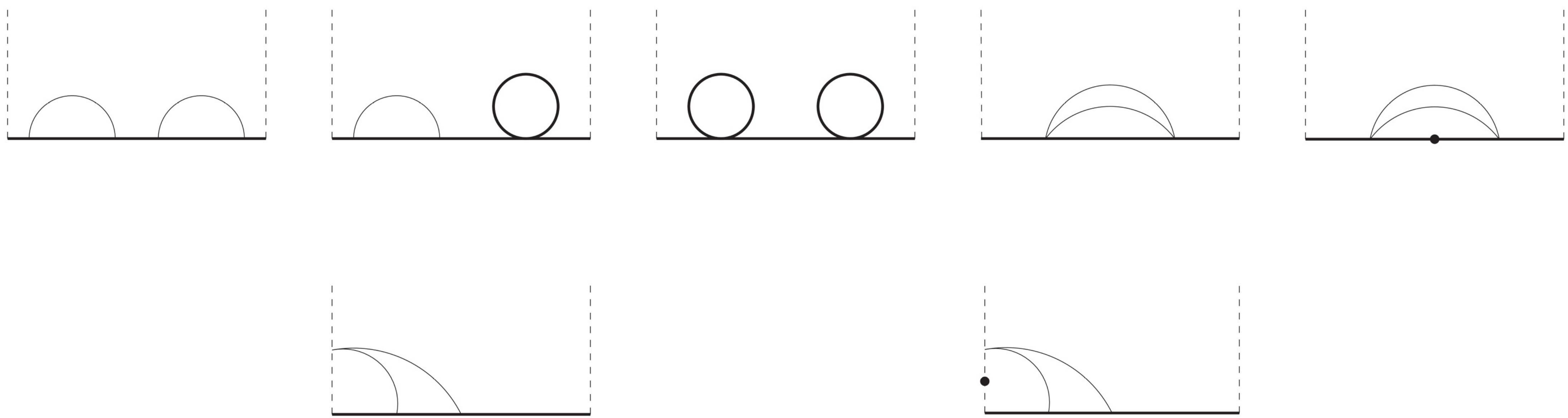

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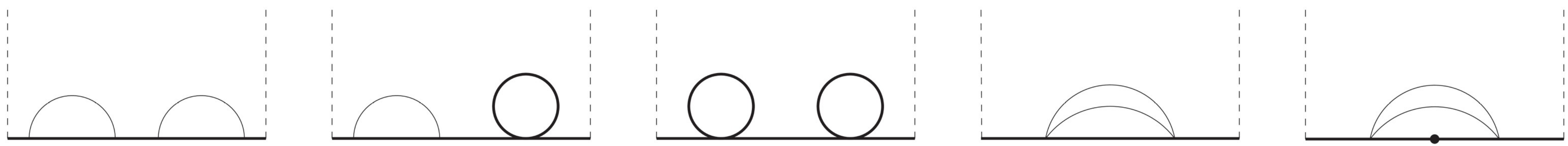

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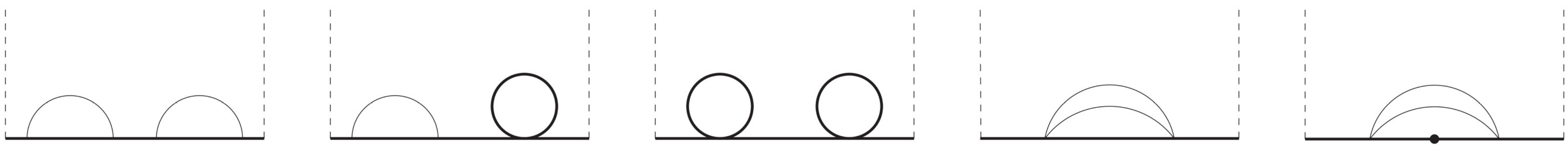

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Intermezzo: Harmonic PolyLogs (HPLs)

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  - Idea: Mix different kernels $$K_0(t) = \boxed{\frac{1}{t}} \qquad K_1(t) = \boxed{\frac{1}{1-t}} \qquad K_{-1}(t) = \boxed{\frac{1}{1+t}}$$

Intermezzo: Harmonic PolyLogs (HPLs)

- Idea: Mix different kernels
  → Definition of HPLs

$$K_0(t) = \boxed{\frac{1}{t}} \qquad K_1(t) = \boxed{\frac{1}{1-t}} \qquad K_{-1}(t) = \boxed{\frac{1}{1+t}}$$

$$H(a;x) = \int_0^x \mathrm{d}t\, K_a(t) \qquad H(a, a_1, \ldots, a_k; x) = \int_0^x \mathrm{d}t\, K_a(t) H(a_1, \ldots, a_k; x)$$

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- Some special values:

$$H(1,1;x) = \frac{1}{2}\log^2(1-x)$$

$$H(-1,-1,0;x) = -\mathrm{Li}_2(-x)$$

$$H(-1,1,1;x) = \frac{1}{2}\log\frac{1+x}{2}\log^2(1-x) + \mathrm{Li}_3\left(\frac{1}{2}\right) + \log(1-x)\mathrm{Li}_2\left(\frac{1-x}{2}\right) - \mathrm{Li}_3\left(\frac{1-x}{2}\right)$$

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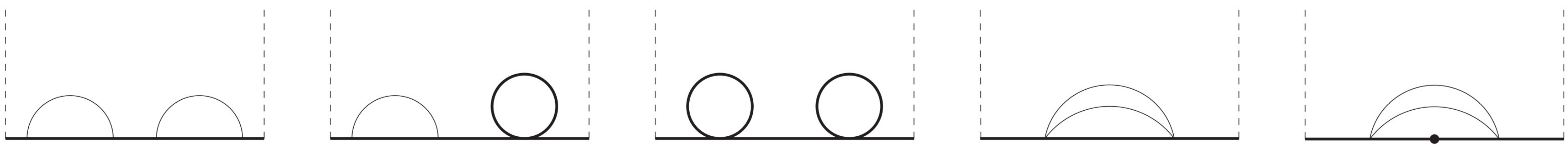

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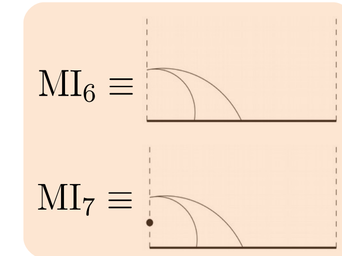


- Solving MIs by using differential equations:
  - Differentiate MIs of interest wrt. $x \equiv -m^2/p^2$ and reduce outcome to MIs

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{MI}_6(x) = \sum_j c_j \mathrm{I}_j(x) \overset{\text{IBP}}{=} \sum_j \tilde{c}_j \mathrm{MI}_j(x) \qquad \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{MI}_7(x) = \sum_j d_j \mathrm{I}_j(x) \overset{\text{IBP}}{=} \sum_j \tilde{d}_j \mathrm{MI}_j(x)$$

  - All MI solutions except for MI_6 and MI_7 known from Feynman parametrization → insert
  - Decouple ODEs to get separate equation for MI_6 (we were lucky here...)

$$\hat{c}_2 \mathrm{MI}_6''(x) + \hat{c}_1 \mathrm{MI}_6'(x) + \hat{c}_0 \mathrm{MI}_6(x) + f(x) = 0$$

  - Solve ODEs order by order in ε: $\mathrm{MI}_6 = \sum_{j=-2} \mathrm{MI}_6^{[j]} \varepsilon^j$

  - Simple example for $\mathrm{MI}_6^{[-1]}$:

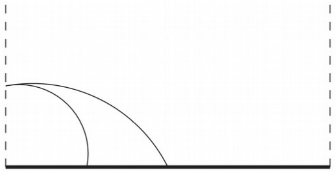
$$-\pi^4 \frac{H(-1;x)}{1+x} + \pi^4 \frac{H(0;x)}{1+x} + \frac{1+5x}{1+x} \underbrace{\mathrm{MI}_6^{[-2]\prime}(x)}_{=0} + \mathrm{MI}_6^{[-1]\prime}(x) + x\mathrm{MI}_6^{[-1]\prime\prime}(x) = 0$$

$$\Rightarrow \mathrm{MI}_6^{[-1]}(x) = \pi^4 \underbrace{\int_0^x \mathrm{d}x' \, \frac{1}{x'} \int_0^{x'} \mathrm{d}x'' \, \frac{H(-1;x'')}{1+x''}}_{H(0,-1,-1;x)} - \pi^4 \underbrace{\int_0^x \mathrm{d}x' \, \frac{1}{x'} \int_0^{x'} \mathrm{d}x'' \, \frac{H(0;x'')}{1+x''}}_{H(0,-1,0;x)} + k_1 \int_0^x \mathrm{d}x' \, \frac{1}{x'} + k_2$$

Integration constants fixed by massless limit $x \to 0$

Topology A – Calculation

- Solution:



$$= J(0,1,1,1,1,0,1,0) = \mathrm{MI}_6 = \mathrm{e}^{-2\varepsilon\gamma_E}\Bigg[\frac{1}{\varepsilon^2}\frac{\pi^2}{12} + \frac{1}{\varepsilon}\Bigg(\frac{1}{6}\big(12\mathrm{Li}_3(-x) + 6\mathrm{Li}_3(x+1)$$

$$- 6\mathrm{Li}_2(x+1)\log(x+1) - 6\mathrm{Li}_2(-x)\log(x) - 3\log(-x)\log^2(x+1) + \pi^2\log(x+1) + 15\zeta(3)\big)\Bigg)$$

$$+ \Bigg(\frac{1}{360}\big(900\mathrm{Li}_2(-x)^2 + 240\pi^2\mathrm{Li}_2(-x) + 2160\mathrm{Li}_4(-x) + 2160\mathrm{Li}_4(x+1)$$

$$+ 540\mathrm{Li}_2(-x)\log^2(x) + 360\mathrm{Li}_2(x+1)\log^2(x+1) - 720\mathrm{Li}_2(x+1)\log(x+1)\log(x)$$

$$- 1800\mathrm{Li}_3(-x)\log(x) + 1440\mathrm{Li}_3(x+1)\log(x) + 1440\mathrm{Li}_3(-x)\log(x+1)$$

$$- 1440\mathrm{Li}_3(x+1)\log(x+1) - 360H(0,0,1,1;-x) - 1440\zeta(3)\log(x) + 1800\zeta(3)\log(x+1)$$

$$+ 60\pi^2\log^2(x+1) - 120\pi^2\log(x+1)\log(x) + 31\pi^4\big)\Bigg) + \mathcal{O}(\varepsilon)\Bigg]$$

$$x \equiv -m^2/p^2$$

Topology A – Roadmap

✓ • Reduce number of integrals
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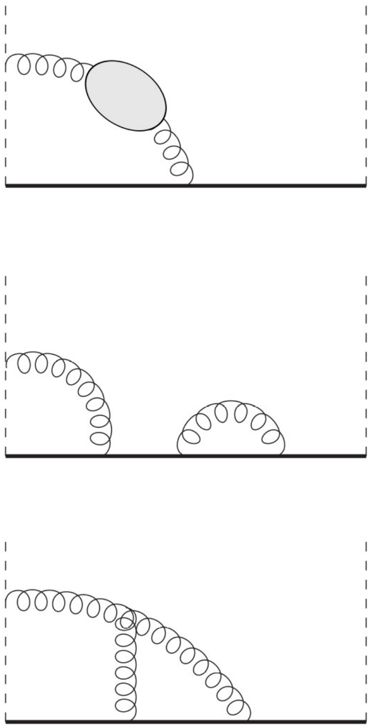
✓ • Solving the MIs
→ Method of differential equations

• Take Imaginary part / Discontinuity

• Renormalization, Results and cross checks



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Topology A – Calculation

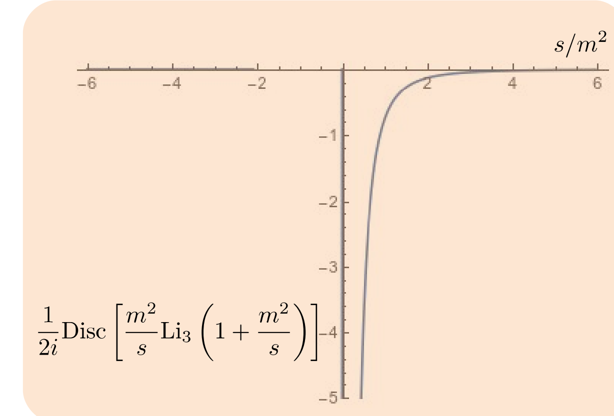
$s \equiv p^2 - m^2 + i0$

- Known that Jet function contains δ- and plus-distributions
  - Imaginary part in two steps
    1) s>0: Looking for branch cuts
       - only present in logs, can always be extracted from PolyLogs

       $\rightarrow$ $\mathrm{Im}[\log(-s - i0)] = -\pi\,\Theta(s)$
    2) s=0: Fix distributional structure
       - Integrate full expression over small region around 0 and take discontinuity

         $\rightarrow$ deduce distributional structure in terms of delta and plus-distributions

  - Check all intermediate steps numerically



Example

$$\frac{1}{s}\mathrm{Li}_3\left(1+\frac{m^2}{s}\right) = \frac{1}{s}\left[-\frac{1}{6}\pi^2\log\left(-1-\frac{m^2}{s}\right) - \frac{1}{6}\log^3\left(-1-\frac{m^2}{s}\right) + \mathrm{Li}_3\left(\frac{s}{s+m^2}\right)\right]$$

No branch cut for s>0
Goes linearly to 0 for s→0

s>0: Log branch cuts

$$\Rightarrow \frac{1}{2i}\mathrm{Disc}\left[\frac{1}{s}\mathrm{Li}_3\left(1+\frac{m^2}{s}\right)\right]_{s>0} = -\frac{\pi\log^2\left(\frac{m^2}{\mu^2}\right)}{2s} + \frac{\pi\log\left(\frac{m^2}{\mu^2}\right)\log\left(\frac{s}{\mu^2}\right)}{s} - \frac{\pi\log^2\left(\frac{s}{\mu^2}\right)}{2s} - \frac{\pi\log\left(\frac{m^2}{\mu^2}\right)\log\left(\frac{s}{m^2}+1\right)}{s} + \frac{\pi\log\left(\frac{s}{m^2}+1\right)\log\left(\frac{s}{\mu^2}\right)}{s} - \frac{\pi\log^2\left(\frac{s}{m^2}+1\right)}{2s}$$

s=0: Integrate

$$\frac{1}{2i}\mathrm{Disc}\left[\int_{-\Lambda_1}^{\Lambda_2}\mathrm{d}s\,\frac{1}{s}\left(-\frac{1}{6}\pi^2\log\left(-1-\frac{m^2}{s}\right)-\frac{1}{6}\log^3\left(-1-\frac{m^2}{s}\right)\right)\right] = \frac{1}{6}\pi\log^3\left(\frac{m^2}{\mu^2}\right) - \frac{1}{2}\pi\log\left(\frac{\Lambda_2}{\mu^2}\right)\log^2\left(\frac{m^2}{\mu^2}\right) + \frac{1}{2}\pi\log^2\left(\frac{\Lambda_2}{\mu^2}\right)\log\left(\frac{m^2}{\mu^2}\right) - \frac{1}{6}\pi\log^3\left(\frac{\Lambda_2}{\mu^2}\right) + \mathcal{O}(\Lambda_2)$$

$\Lambda_1, \Lambda_2 > 0$

Deduce full distributional structure:

$$\frac{1}{2i}\mathrm{Disc}\left[\frac{1}{s}\mathrm{Li}_3\left(1+\frac{m^2}{s}\right)\right] = \frac{1}{6}\pi\log^3\left(\frac{m^2}{\mu^2}\right)\mathcal{L}_{-1} - \frac{1}{2}\pi\mathcal{L}_0(s)\log^2\left(\frac{m^2}{\mu^2}\right) + \pi\mathcal{L}_1(s)\log\left(\frac{m^2}{\mu^2}\right) - \frac{1}{2}\pi\mathcal{L}_2(s)$$
$$+\Theta(s)\left[-\frac{\pi\log\left(\frac{m^2}{\mu^2}\right)\log\left(\frac{s}{m^2}+1\right)}{s} + \frac{\pi\log\left(\frac{s}{m^2}+1\right)\log\left(\frac{s}{\mu^2}\right)}{s} - \frac{\pi\log^2\left(\frac{s}{m^2}+1\right)}{2s}\right]$$

$\mathcal{L}_{-1}(s) \equiv \delta(s), \quad \mathcal{L}_{n\geq 0} \equiv \frac{1}{\mu^2}\left[\frac{\Theta(s)\log^n s/\mu^2}{s/\mu^2}\right]_+$ $\quad \int_{-\Lambda_1}^{\Lambda_2}\mathrm{d}s\,\mathcal{L}_{-1}(s) = 1$ $\quad \int_{-\Lambda_1}^{\Lambda_2}\mathrm{d}s\,\mathcal{L}_{n\geq 0}(s) = \frac{\log^{n+1}(\Lambda_2/\mu^2)}{n+1}$ $\quad s \equiv p^2 - m^2 + i0$

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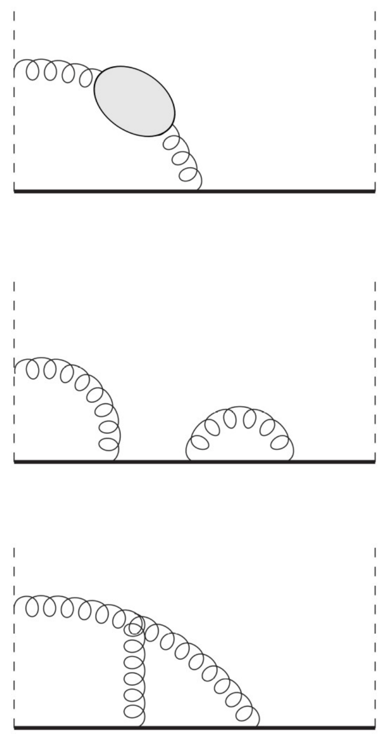

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Topology A – Calculation

- Reconsider color structure:
  - Topo A: C_F^2, $C_F C_A$, $C_F T_F \sim \mathcal{O}(N_c^2)$
  - Topo B: $C_F^2 - C_F C_A/2 \sim \mathcal{O}(N_c^0)$

- Topo A contains all (primary massive) contributions in first two orders in N_c
  - Renormalization and cross checks in $1/N_c \ll 1$ expansion

Topology A/Large N_c – Calculation

- Renormalization:
  - Mass (pole) and coupling ($\overline{\text{MS}}$) renormalization straightforward (before taking imaginary part)
  - Jet function renormalization

$$J(s, m^2, \mu) = \int \mathrm{d}s'\, Z(s, s', \mu) J_{\text{bare}}(s', m^2, \mu)$$

Known from massless result (same UV behavior)
→ good cross check!

→ Two-loop contribution:

$$J_{[2]} = J_{[2],\text{bare}} + Z_{[2]} + J_{[1],\text{bare}} \otimes Z_{[1]}$$

Topology A/Large N_c – Results

- Large N_c result:

$$J(s,m^2,\mu) = \mathcal{L}_{-1}(s) + \frac{\alpha_s(\mu)}{4\pi} N_c \bigg[\mathcal{L}_{-1}(s)\Big(4 - \frac{\pi^2}{6} + \frac{1}{2}L_m + L_m^2\Big) + \mathcal{L}_0(s)\Big(-2 - 2L_m\Big) + 8\mathcal{L}_1(s) + \Theta(s)\Big(\frac{s}{2(m^2+s)^2} - \frac{2L_r}{s}\Big)\bigg]$$
$$+ \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 N_c^2 \bigg\{\mathcal{L}_{-1}(s)\bigg[\frac{1}{2}L_m^4 + \frac{1}{2}L_m^3 + \left(\frac{1031}{72} - \frac{5\pi^2}{6}\right)L_m^2 - \frac{1}{72}\left(288\zeta(3) - 799 + 242\pi^2\right)L_m - \frac{359\zeta(3)}{18} - \frac{23\pi^4}{120} - \frac{89\pi^2}{54} + \frac{155515}{2592}\bigg]$$
$$+ \mathcal{L}_0(s)\bigg[\frac{1}{27}\left(-54L_m^3 - 180L_m^2 + \left(63\pi^2 - 843\right)L_m + 702\zeta(3) + 129\pi^2 - 1310\right)\bigg]$$
$$+ \mathcal{L}_1(s)\bigg[\frac{2}{9}\left(36L_m^2 + 111L_m - 21\pi^2 + 290\right)\bigg] + \mathcal{L}_2(s)\bigg[-\frac{4}{3}\left(9L_m + 20\right)\bigg] + 8\,\mathcal{L}_3(s) + \mathcal{R}_{N_c^2}\bigg\}$$
$$+ \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 N_c T_F\, n_\ell \bigg\{\mathcal{L}_{-1}(s)\bigg[\frac{1}{324}\left(-1044L_m^2 + 36\left(8\pi^2 - 37\right)L_m + 288\zeta(3) + 132\pi^2 - 6037\right)\bigg]$$
$$+ \mathcal{L}_0(s)\bigg[-\frac{4}{27}\left(-9L_m^2 - 48L_m + 6\pi^2 - 94\right)\bigg] + \mathcal{L}_1(s)\bigg[-\frac{16}{9}\left(3L_m + 8\right)\bigg] + \frac{16}{3}\mathcal{L}_2(s) + \mathcal{R}_{N_c}\bigg\} + \mathcal{O}(N_c^0, \alpha_s^3)$$

Regular parts

Topology B contributions

$$s \equiv p^2 - m^2 + i0 \qquad \mathcal{L}_{-1}(s) \equiv \delta(s), \quad \mathcal{L}_{n\geq 0} \equiv \frac{1}{\mu^2}\left[\frac{\Theta(s)\log^n s/\mu^2}{s/\mu^2}\right]_+, \quad L_m \equiv \log\frac{m^2}{\mu^2}$$

Topology A/Large N$_c$ – Results

$$L_m \equiv \log\frac{m^2}{\mu^2}, \quad L_s \equiv \log\frac{s}{\mu^2}, \quad L_r \equiv \log\frac{m^2+s}{m^2}$$

- Regular parts:

$$\mathcal{R}_{N_c^2} = \Theta(s)\left\{\frac{\left(25m^4+50sm^2+28s^2+12\left(m^2+s\right)^2 L_r\right)\mathrm{Li}_2\left(-\frac{s}{m^2}\right)}{3s\left(m^2+s\right)^2} - \frac{8\mathrm{Li}_3\left(\frac{s}{m^2+s}\right)}{s}\right.$$

$$\left. - \frac{4\left(\left(L_m-L_s\right)\mathrm{Li}_2\left(-\frac{s}{m^2}\right)+4\mathrm{Li}_3\left(-\frac{s}{m^2}\right)\right)}{s} + \frac{1}{\left(m^2+s\right)^3}\left[-\frac{7m^6(L_r m^2-s)}{2s^2}+\frac{4L_r^3 m^6}{3s}+\frac{3L_r^2 m^6}{2s}+\ldots\right]\right\}$$

$$\mathcal{R}_{N_c} = \Theta(s)\frac{1}{9}\left[-\frac{24\,\mathrm{Li}_2\left(-\frac{s}{m^2}\right)}{s}+\frac{36(m^2L_r-s)}{s^2}+L_m\left(\frac{24L_r}{s}-\frac{6s}{\left(m^2+s\right)^2}\right)-\frac{48L_rL_s}{s}+\frac{12sL_s}{\left(m^2+s\right)^2}+\frac{12}{m^2+s}-\frac{19s}{\left(m^2+s\right)^2}+\frac{58L_r}{s}\right]$$

Topology A/Large Nc – Crosschecks

- RGE: Known $Z(s', s, \mu)$ [Becher et al. ('06)] reproduces correct $1/\varepsilon$ divergences (+logs) ✓

- Massless limit:
  - No spurious singularities due to off-shellness of quark propagator

$$\lim_{m\to 0} J(s, m^2, \mu) + \mathcal{O}(N_c^0, \alpha_s^3) = J(s, \mu) + \mathcal{O}(N_c^0, \alpha_s^3) \quad ✓$$

Massless Jet Function [Becher et al. ('06)]

- Heavy quark (bHQET) limit:

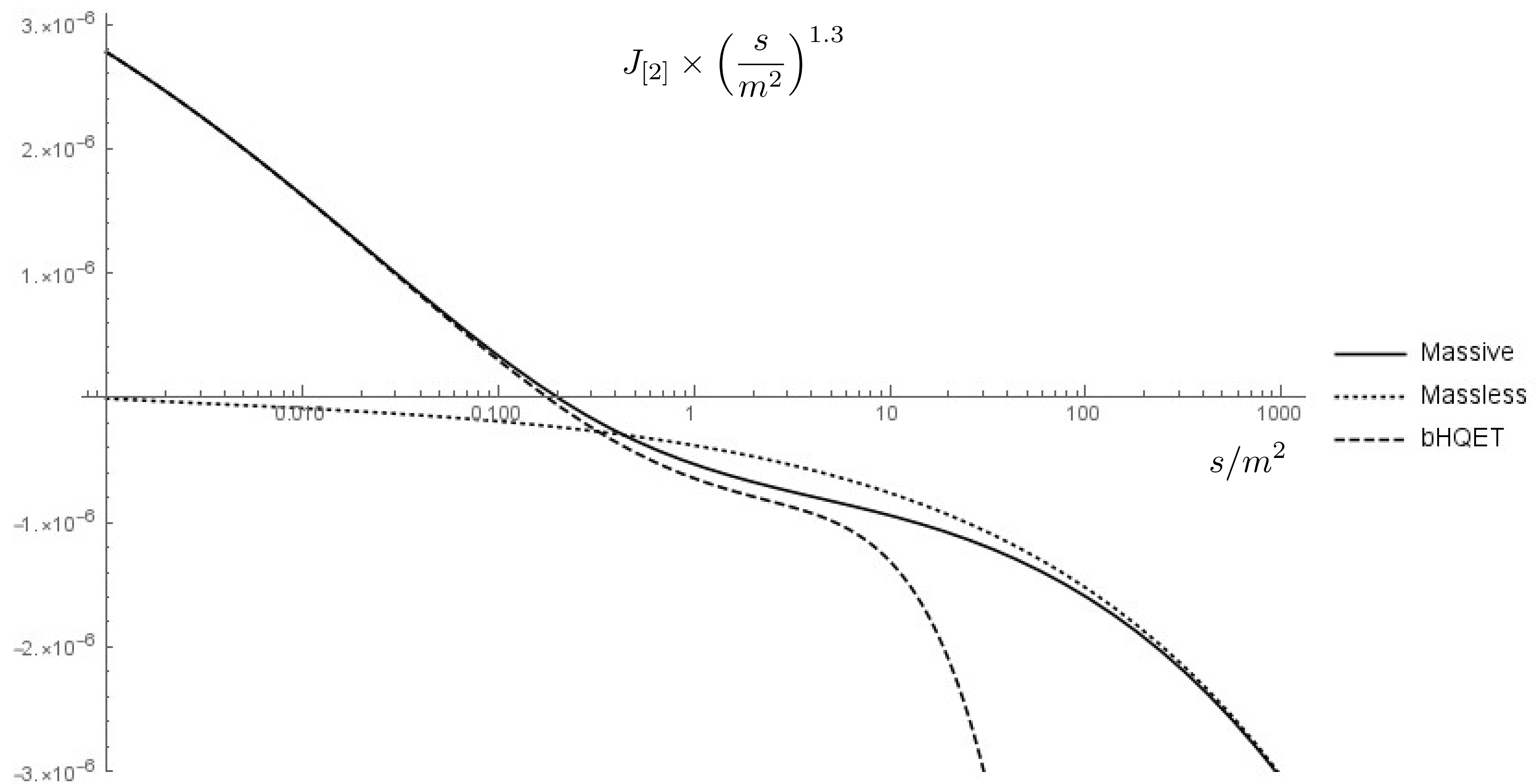
bHQET Jet Function [Jain et al. ('08)]

$$J(s, m^2, \mu) + \mathcal{O}(N_c^0, \alpha_s^3) = C_m J_B(\hat{s}, \mu) + \mathcal{O}(N_c^0, \alpha_s^3, 1/m^2) \quad ✓✓$$

$\hat{s} \equiv s/m$

Matching coefficient [Hoang et al. ('16)]

Topology A/Large Nc – Crosschecks



$m = 173\,\mathrm{GeV}$

$\mu = 5\,\mathrm{GeV} + \sqrt{s}$

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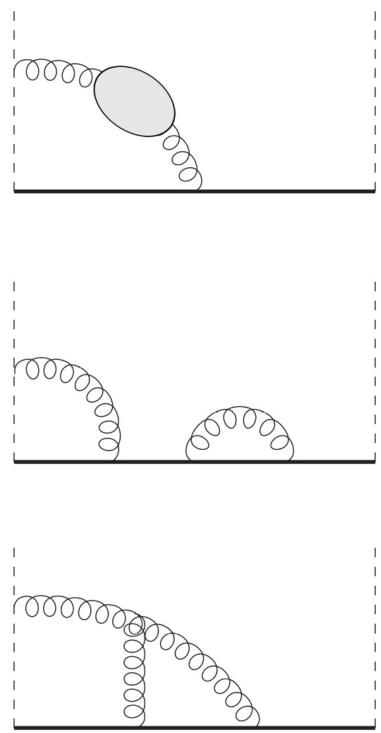

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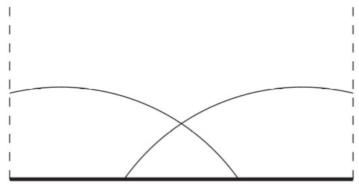
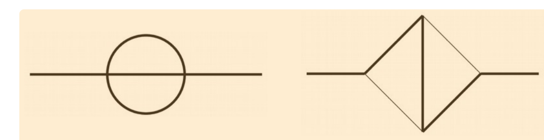
Topology B – Calculation



$$\int \mathrm{d}^d k\, \mathrm{d}^d \ell\, \frac{1}{[-k^2]^{a_1}[-\ell^2]^{a_2}[-(p+k)^2+m^2]^{b_1}[-(p+\ell)^2+m^2]^{b_2}[-(p+k+\ell)^2+m^2]^{b_3}[-\bar{n}\cdot k]^{c_1}[-\bar{n}\cdot \ell]^{c_2}}$$

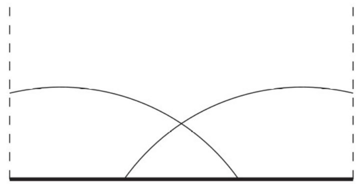


- Two issues related to topology B integrals:
  - Generalized sunrise and kite-type integrals

    → complicated, sunrise/kite contain elliptic functions, active area of research

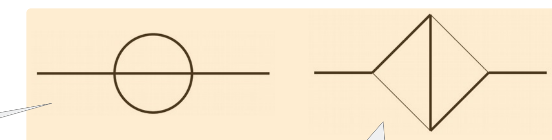
    [Broadhurst et al. (‘93), Adams et al. (‘16), ..., Sabry (‘62), Adams et al. (‘16)]

Topology B – Calculation



$$\int \mathrm{d}^d k \, \mathrm{d}^d \ell \, \frac{1}{[-k^2]^{a_1}[-\ell^2]^{a_2}[-(p+k)^2+m^2]^{b_1}[-(p+\ell)^2+m^2]^{b_2}[-(p+k+\ell)^2+m^2]^{b_3}[-\bar{n}\cdot k]^{c_1}[-\bar{n}\cdot\ell]^{c_2}}$$

- Two issues related to topology B integrals:
  - Generalized sunrise and kite-type integrals



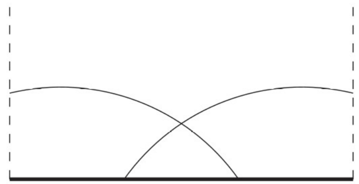
→ complicated, sunrise/ki... ...ain elliptic functions, active ... of research

...: al. ('93), Adams et al. ('16), ..., ...), Adams et al. ('16)]

$$\int \mathrm{d}^d k \, \mathrm{d}^d \ell \, \frac{1}{[-(p+k)^2+m^2]^2[-(p+\ell)^2+m^2][-(p+k+\ell)^2+m^2]}$$

$$\int \mathrm{d}^d k \, \mathrm{d}^d \ell \, \frac{1}{[-k^2][-\ell^2][-(p+k)^2+m^2][-(p+\ell)^2+m^2][-(p+k+\ell)^2+m^2]}$$

Topology B – Calculation



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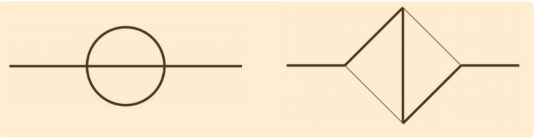


- Two issues related to topology B integrals:
  - Generalized sunrise and kite-type integrals

    → complicated, sunrise/kite contain elliptic functions, active area of research

    [Broadhurst et al. (‘93), Adams et al. (‘16), ..., Sabry (‘62), Adams et al. (‘16)]
  - Rapidity divergences in individual integrals (sum finite)
    - Has to be taken into account in IBP reduction (additional regulator)

      → Blows up number of MIs
    - Harder to compute

Topology B – IBPs and Rapidity Regulators

- Potential rapidity divergences and IBP reduction
  - Example: Consider analytic regulator

$$\frac{\mathrm{d}^d k}{[-\bar{n}\cdot k]^c} \to \frac{\nu^\eta\, \mathrm{d}^d k}{[-\bar{n}\cdot k]^{\eta+c}}$$

- IBP relation:

Rapidity finite

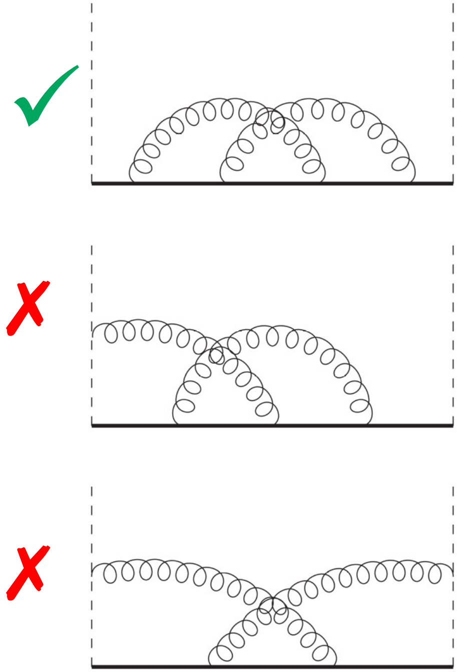
$$\int \mathrm{d}^d k\, \mathrm{d}^d \ell \, \frac{1}{[-\ell^2][-(p+k)^2+m^2][-(p+\ell)^2+m^2][-(p+k+\ell)^2+m^2][-\bar{n}\cdot k]^{1+\eta}}$$
$$= -(1+\eta)(\bar{n}\cdot p)\int \mathrm{d}^d k\, \mathrm{d}^d \ell \, \frac{1}{[-\ell^2][-(p+k)^2+m^2][-(p+\ell)^2+m^2][-(p+k+\ell)^2+m^2][-\bar{n}\cdot k]^{2+\eta}} + \ldots$$

$\mathcal{O}(1/\eta)$ rapidity divergent

- Crucial to include regulator in the whole reduction → high number of MIs
- In general $\mathcal{O}(\eta^{n>0})$ terms are needed

Topology B – Calculation

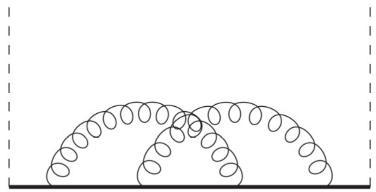
- Reasons not to go through this
  - High amount of work…
    - Additional regulator → high number of very complicated integrals
  - … but very small contribution
    - 2 out of 23 diagrams
    - $1/N_c$ suppressed by two orders relative to Topo A
    - A lot is already known:
      - Distributions, massless limit, logs, …



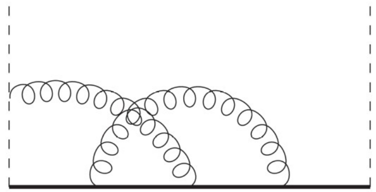
- Reasonable to choose partially numerical approach

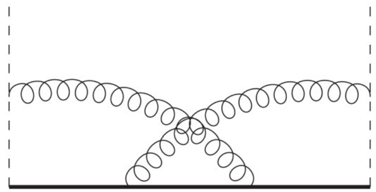
Topology B – Roadmap

- Solve diagram without Wilson line analytically
  - Usual setup: IBP, solve/find MIs, ...



- Get analytic information about Wilson line diagrams where possible (imaginary part)
  - Evaluate rest numerically by Sector Decomposition
  - Fit outcome





- Put results everything together and do crosschecks

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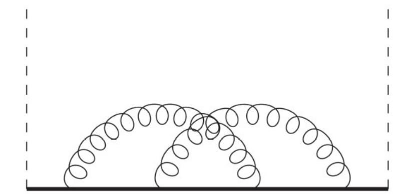

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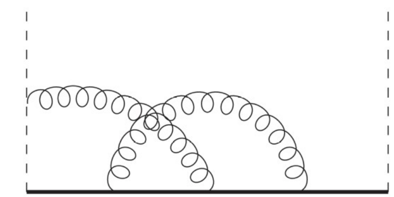

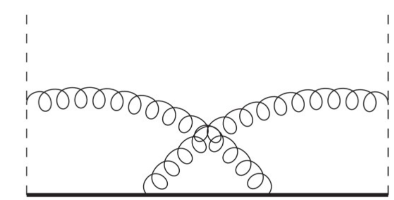

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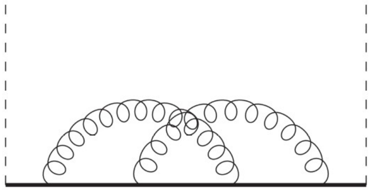
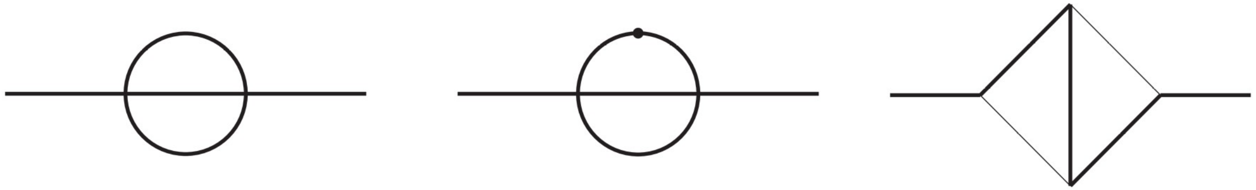
Topology B – Calculation

- Solve non-Wilson line diagram:
  - MIs from IBP reduction: Known from Topo A, except for

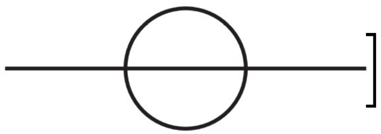




→ Sunrise and Kite integrals
→ Luckily other people did the work!

Topology B – Calculation

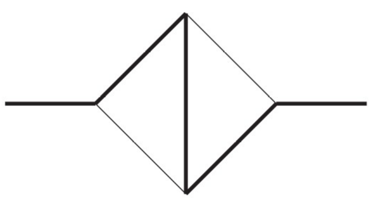
- Some expressions:

$\mathrm{Im}[$  $]$

$$\overset{\mathcal{O}(\varepsilon^0)}{\sim} -\frac{\pi}{p^2}\sqrt{(p-m)(p+3m)}\times\Big\{-4m^2p\,\mathrm{K}(\kappa)+\frac{(p-m)(p^2+3m^2)}{2}\,\mathrm{E}(\kappa)\Big\}\,\Theta(p^2-9m^2),$$

$$\text{with}\quad \kappa^2 := \frac{(p+m)^3(p-3m)}{(p-m)^3(p+3m)}\,,$$

[Bauberger et al. ('95)]



$$\overset{\mathcal{O}(\varepsilon^0)}{\sim} -\frac{\pi^2}{6}H(1;p^2/m^2)-2H(0,1,1;p^2/m^2)+H(1,0,1;p^2/m^2)$$
$$+\frac{2}{3}\int_9^\infty \mathrm{d}t\left(1-\frac{8}{t-1}\right)\log\left(1-\frac{p^2}{t\,m^2}\right)\frac{2}{\sqrt{(\sqrt{t}+3)(\sqrt{t}-1)^3}}\mathrm{K}\left(\frac{(\sqrt{t}-3)(\sqrt{t}+1)^3}{(\sqrt{t}+3)(\sqrt{t}-1)^3}\right)$$

[Remiddi, Tancredi ('16)]

$$\mathrm{E}(m)\equiv\int_0^{\frac{\pi}{2}}\mathrm{d}t\,\sqrt{1-m\sin^2 t},\qquad \mathrm{K}(m)\equiv\int_0^{\frac{\pi}{2}}\mathrm{d}t\,\frac{1}{\sqrt{1-m\sin^2 t}},$$

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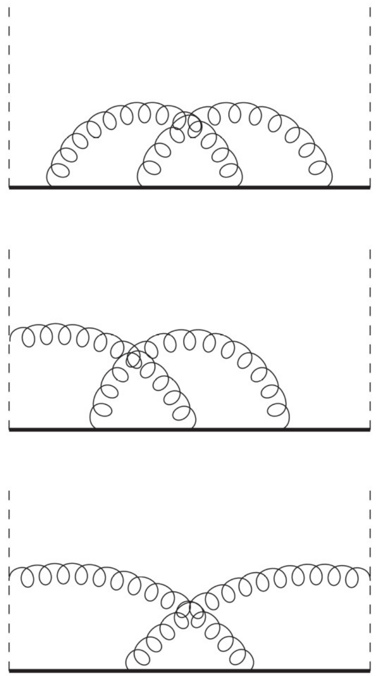
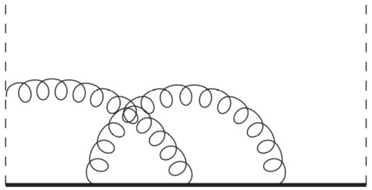


Topology B – Calculoation

- Wilson line diagrams: Only missing piece with color structures C_F^2, $C_F C_A$
  $\rightarrow$ What is known?
  - Heavy Quark limit / distributional structure

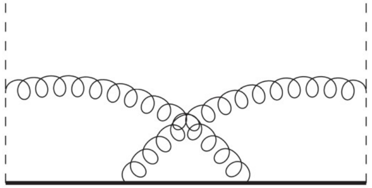
$$J(s, m^2, \mu) = C_m J_B(\hat{s}, \mu) + \mathcal{O}(1/m^2)$$

$\mathcal{O}(m^0/s)$



  - Massless limit

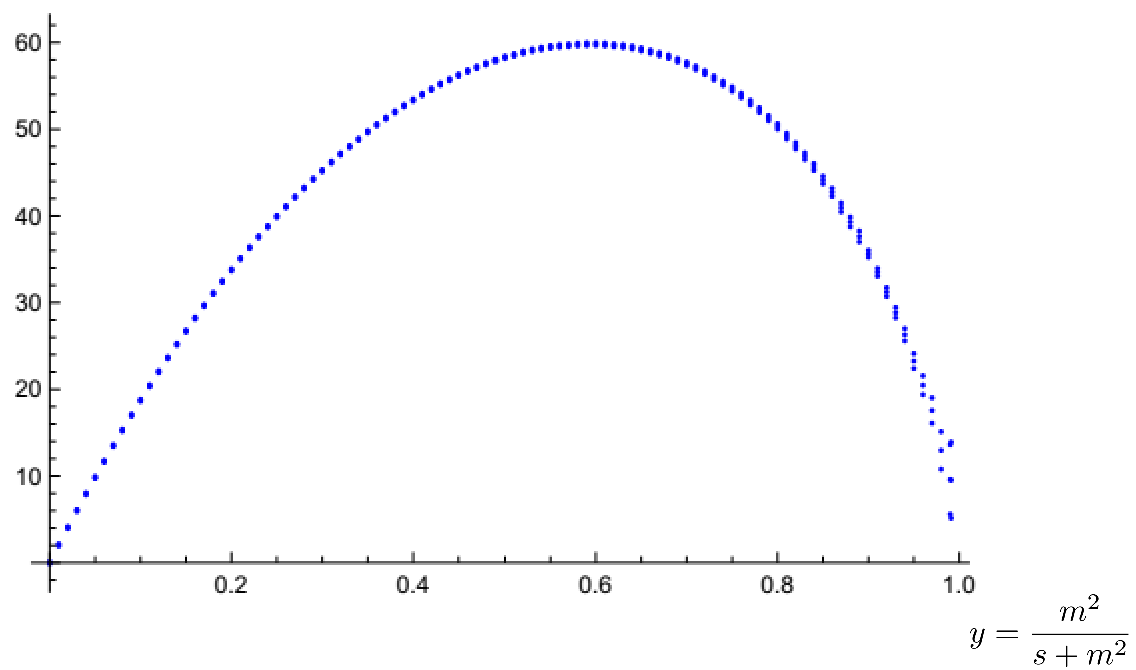
$$J(s, m^2, \mu) = J(s, \mu) + \mathcal{O}(m^2/s^2)$$



  - Renormalization constant / anomalous dimension (massless results)
    $\rightarrow$ Dependence on $\log^n(s/\mu^2)$

Topology B – Calculation

- Numerical evaluation and fit:
  - Put the missing part into Sector decomposition program (pySecDec)
  - Parametrization in $y = m^2/(s+m^2) \rightarrow$ physical region $0 < y < 1$
  - Evaluate at ~120 points with reasonable precision (~0.1% / max. runtime)



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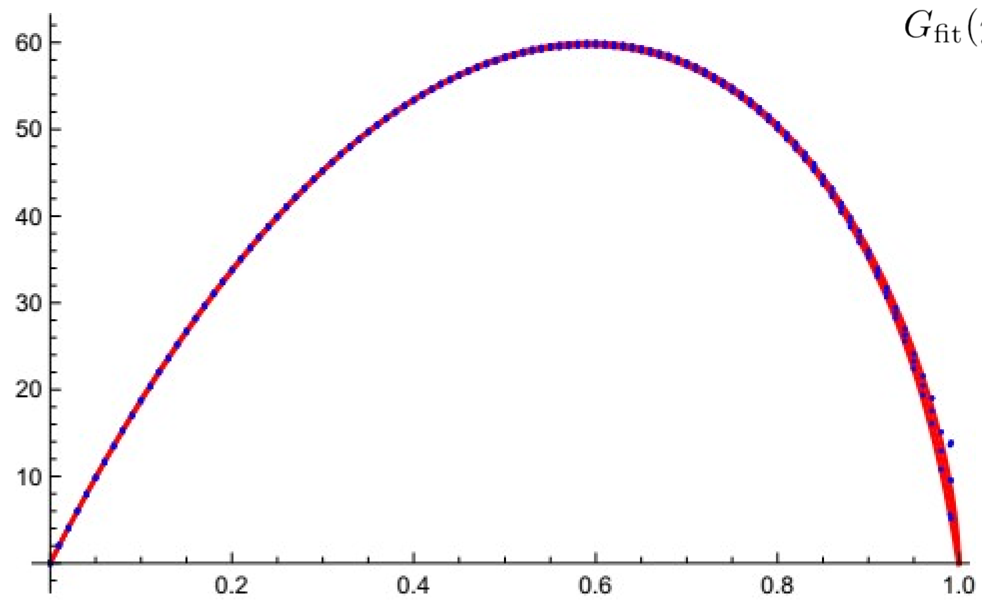

$$y = \frac{m^2}{s+m^2}$$

$$\begin{aligned} G_{\text{fit}}(y) = (1-y)\Big[&110.4155y + 44.3644\log(1-y) + 23.0489\log^2(1-y) \\ &+ 1.7781\log^3(1-y) - 13.0195(1-y)\log^3(1-y) \\ &+ \delta\Big(-17.1918y - 17.8114\log(1-y) - 11.2818\log^2(1-y) \\ &\quad - 2.3770\log^3(1-y) - 5.8021(1-y)\log^3(1-y)\Big)\Big] \end{aligned}$$

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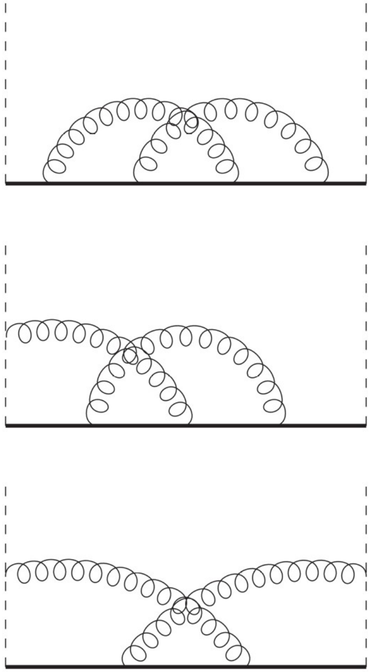

→ • Put results everything together and do crosschecks

Topology A+B – Results

$$\mathcal{L}_{-1}(s) = \delta(s)$$

$$\mathcal{L}_{n\geq 0} \equiv \frac{1}{\mu^2}\left[\frac{\Theta(s)\log^n s/\mu^2}{s/\mu^2}\right]_+$$

$$s = p^2 - m^2$$

$$L_m = \log\frac{m^2}{\mu^2}$$

- Some results: Distributional part

$$\begin{aligned}
J_2^m = C_F^2\Bigg[&\left(-8L_m^3 - 12L_m^2 + \left(\frac{20\pi^2}{3} - 36\right)L_m + 64\zeta_3 + \frac{20\pi^2}{3} - 32\right)\mathcal{L}_0(s)\\
&+\left(2L_m^4 + 2L_m^3 + \left(\frac{33}{2} - 2\pi^2\right)L_m^2 + L_m\left(-56\zeta_3 + \frac{13}{2} - \pi^2\right) - 38\zeta_3 - \frac{\pi^4}{2} + \frac{433}{8} + \pi^2\left(-\frac{1}{3} - 8\log 2\right)\right)\mathcal{L}_{-1}(s)\\
&+\left(32L_m^2 + 40L_m - \frac{40}{3}\left(\pi^2 - 6\right)\right)\mathcal{L}_1(s) + \left(-48L_m - 48\right)\mathcal{L}_2(s) + 32\mathcal{L}_3(s) + G_{C_F}\Bigg]\\
+ C_A C_F\Bigg[&\left(-\frac{22L_m^2}{3} + \frac{4}{9}\left(3\pi^2 - 100\right)L_m + \frac{4}{27}\left(135\zeta_3 - 547 + 42\pi^2\right)\right)\mathcal{L}_0(s)\\
&+\left(\left(\frac{367}{18} - \frac{2\pi^2}{3}\right)L_m^2 + L_m\left(20\zeta_3 + \frac{341}{18} - \frac{56\pi^2}{9}\right) - \frac{188\zeta_3}{9} - \frac{2\pi^4}{15} + \frac{60221}{648} + \pi^2\left(4\log 2 - \frac{169}{54}\right)\right)\mathcal{L}_{-1}(s)\\
&+\left(\frac{88L_m}{3} - \frac{8\pi^2}{3} + \frac{800}{9}\right)\mathcal{L}_1(s) - \frac{88}{3}\mathcal{L}_2(s) + G_{C_A}\Bigg]\\
+ C_F n_\ell T_F\Bigg[&\left(-\frac{58L_m^2}{9} + \frac{2}{9}\left(8\pi^2 - 37\right)L_m + \frac{1}{162}\left(288\zeta_3 - 6037 + 132\pi^2\right)\right)\mathcal{L}_{-1}(s)\\
&+\left(\frac{8L_m^2}{3} + \frac{128L_m}{9} - \frac{16}{27}\left(3\pi^2 - 47\right)\right)\mathcal{L}_0(s) + \left(-\frac{32L_m}{3} - \frac{256}{9}\right)\mathcal{L}_1(s) + \frac{32}{3}\mathcal{L}_2(s) + G_{T_F}\Bigg]
\end{aligned}$$

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\end{aligned}$$

Non-distributional parts

Topology A+B – Results

- Example of a non-distributional part

$$L_m \equiv \log \frac{m^2}{\mu^2}$$

$$L_s \equiv \log \frac{s}{\mu^2}$$

$$L_r \equiv \log \frac{m^2+s}{m^2}$$

$$s = p^2 - m^2$$

$$\mathrm{E}(m) \equiv \int_0^{\frac{\pi}{2}} \mathrm{d}t\, \sqrt{1 - m\sin^2 t},$$

$$\mathrm{K}(m) \equiv \int_0^{\frac{\pi}{2}} \mathrm{d}t\, \frac{1}{\sqrt{1 - m\sin^2 t},}$$

$$\begin{aligned}
G_{C_F} = \Theta(s)\Bigg\{ & -\frac{32L_r^3}{3s} + \frac{4\left(m^4+2sm^2+2s^2\right)L_r^2}{s\left(m^2+s\right)^2} + \left(\frac{6m^4+\left(-99+16\pi^2\right)sm^2+\left(-93+16\pi^2\right)s^2}{3s^2\left(m^2+s\right)} - \frac{8\mathrm{Li}_2\left(-\frac{s}{m^2}\right)}{s}\right)L_r \\
& -\frac{300m^4+56\left(6+\pi^2\right)sm^2+\left(45+86\pi^2\right)s^2}{6s\left(m^2+s\right)^2} + L_s^2\left(\frac{8\left(m^2+2s\right)}{\left(m^2+s\right)^2} - \frac{24L_r}{s}\right) + L_m^2\left(\frac{2\left(4m^2+7s\right)}{\left(m^2+s\right)^2} - \frac{16L_r}{s}\right) \\
& +\frac{4\left(-12m^4+16sm^2+7s^2\right)\mathrm{Li}_2\left(-\frac{s}{m^2}\right)}{s^2\left(m^2+s\right)} + L_m\left(-\frac{20L_r^2}{s} - \frac{4\left(-8m^6+3sm^4+14s^2m^2+s^3\right)L_r}{s^2\left(m^2+s\right)^2}\right. \\
& +\frac{-32m^6-72sm^4-39s^2m^2+7s^3}{s\left(m^2+s\right)^3} + L_s\left(\frac{32L_r}{s} - \frac{4\left(4m^2+7s\right)}{\left(m^2+s\right)^2}\right) - \left.\frac{16\mathrm{Li}_2\left(-\frac{s}{m^2}\right)}{s}\right) \\
& +L_s\left(\frac{12L_r^2}{s} + \frac{4\left(-8m^6+6sm^4+20s^2m^2+5s^3\right)L_r}{s^2\left(m^2+s\right)^2} + \frac{32m^6+72sm^4+32s^2m^2-14s^3}{s\left(m^2+s\right)^3} + \frac{16\mathrm{Li}_2\left(-\frac{s}{m^2}\right)}{s}\right) \\
& -\frac{16\mathrm{Li}_3\left(-\frac{s}{m^2}\right)}{s} + \frac{32\mathrm{Li}_3\left(\frac{s}{m^2+s}\right)}{s} + \frac{2(36+29\pi^2-156\zeta_3)}{3(m^2+s)} + \frac{G_{\text{fit}}\left(\frac{m^2}{m^2+s}\right)}{s}\Bigg\} \\
& +\Theta(s-8m^2)\Bigg\{ I_3(s,m^2)\frac{8m^2\left(2m^2+s\right)^2}{3s^2\left(m^2+s\right)^2} - \frac{4y^{3/4}\left(3y^2+44y-7\right)}{s(y-1)}\mathrm{E}\left(\frac{3y^2+8y^{3/2}+6y-1}{16y^{3/2}}\right) \\
& +\frac{2\left(9y^{7/2}+132y^{5/2}-21y^{3/2}+12y^3-11y^2+8y-1\right)}{3s(y-1)y^{3/4}}\mathrm{K}\left(\frac{3y^2+8y^{3/2}+6y-1}{16y^{3/2}}\right)\Bigg\}_{y=\frac{m^2}{s+m^2}}
\end{aligned}$$

Topology A+B – Results

- Example of a non-distributional part

$$
\begin{aligned}
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& -\frac{300m^4+56\left(6+\pi^2\right)sm^2+\left(45+86\pi^2\right)s^2}{6s\left(m^2+s\right)^2} + L_s^2\left(\frac{8\left(m^2+2s\right)}{\left(m^2+s\right)^2} - \frac{24L_r}{s}\right) + L_m^2\left(\frac{2\left(4m^2+7s\right)}{\left(m^2+s\right)^2} - \frac{16L_r}{s}\right) \\
& +\frac{4\left(-12m^4+16sm^2+7s^2\right)\mathrm{Li}_2\left(-\frac{s}{m^2}\right)}{s^2\left(m^2+s\right)} + L_m\left(-\frac{20L_r^2}{s} - \frac{4\left(-8m^6+3sm^4+14s^2m^2+s^3\right)L_r}{s^2\left(m^2+s\right)^2}\right. \\
& +\frac{-32m^6-72sm^4-39s^2m^2+7s^3}{s\left(m^2+s\right)^3} + L_s\left(\frac{32L_r}{s} - \frac{4\left(4m^2+7s\right)}{\left(m^2+s\right)^2}\right) - \left.\frac{16\mathrm{Li}_2\left(-\frac{s}{m^2}\right)}{s}\right) \\
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& +\Theta(s-8m^2)\Bigg\{ I_3(s,m^2)\frac{8m^2\left(2m^2+s\right)^2}{3s^2\left(m^2+s\right)^2} - \frac{4y^{3/4}\left(3y^2+44y-7\right)}{s(y-1)}\mathrm{E}\left(\frac{3y^2+8y^{3/2}+6y-1}{16y^{3/2}}\right) \\
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Topology A+B – Results

- Example of a non-distributional part

$$
\begin{aligned}
G_{C_F} = \Theta(s)\Bigg\{ & -\frac{32L_r^3}{3s} + \frac{4\left(m^4+2sm^2+2s^2\right)L_r^2}{s\left(m^2+s\right)^2} + \left(\frac{6m^4+\left(-99+16\pi^2\right)sm^2+\left(-93+16\pi^2\right)s^2}{3s^2\left(m^2+s\right)} - \frac{8\mathrm{Li}_2\left(-\frac{s}{m^2}\right)}{s}\right)L_r \\
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\end{aligned}
$$

$$
\int_9^{\frac{m^2+s}{m^2}} \mathrm{d}t\, \frac{2(t-9)}{(t-1)\sqrt{t^2-6t+8\sqrt{t}-3}}\,\mathrm{K}\left(\frac{\left(\sqrt{t}-3\right)\left(\sqrt{t}+1\right)^3}{t^2-6t+8\sqrt{t}-3}\right)
$$

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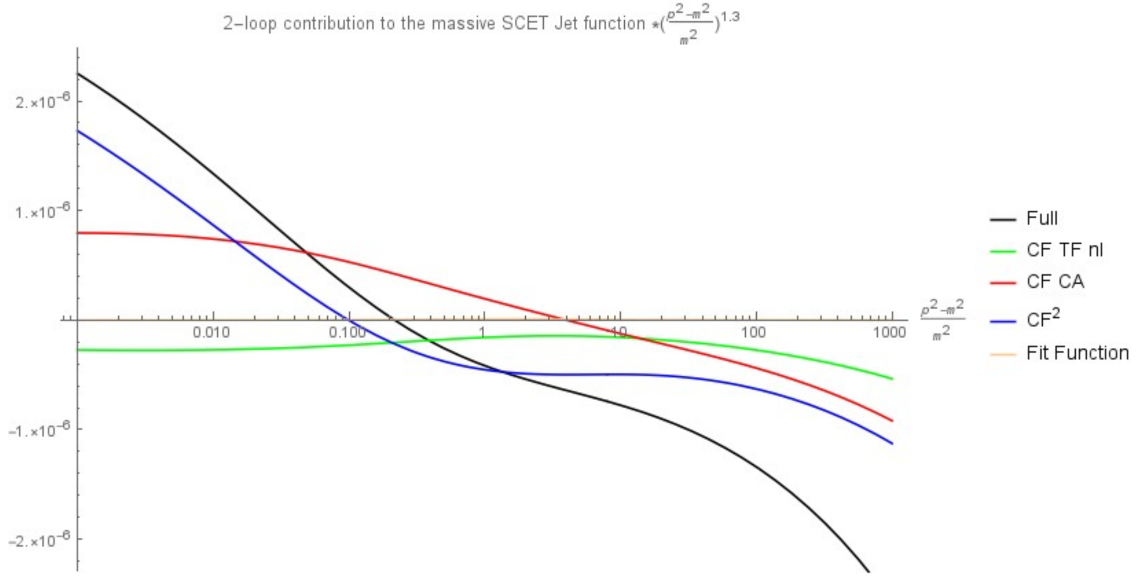
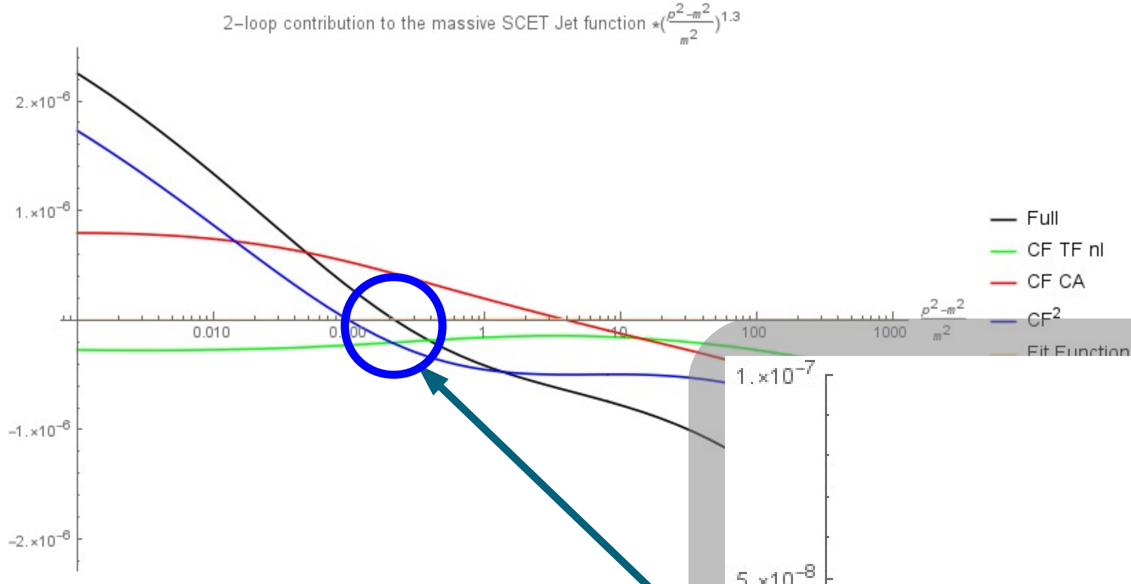
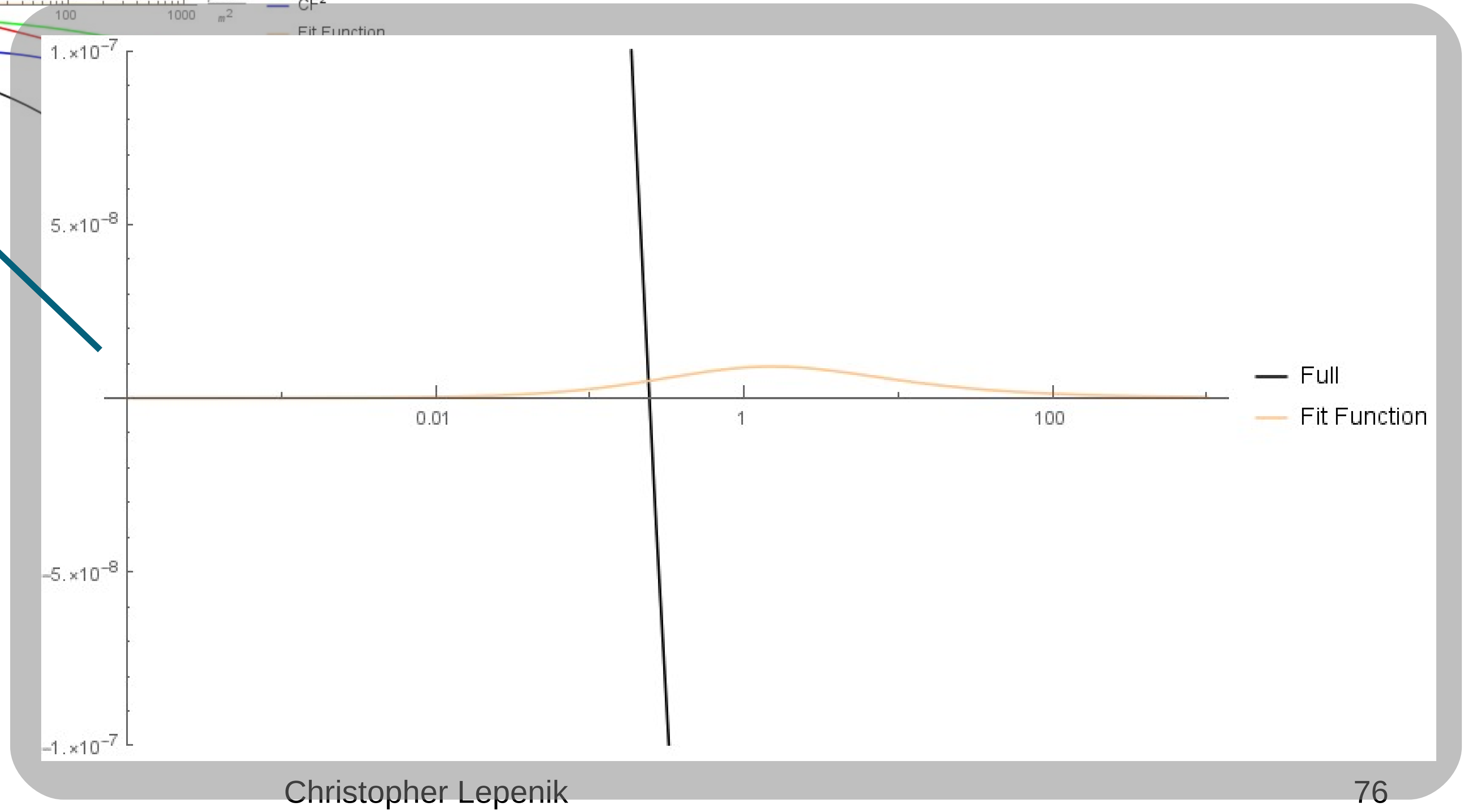
Topology A+B – Numerical Size of Fit



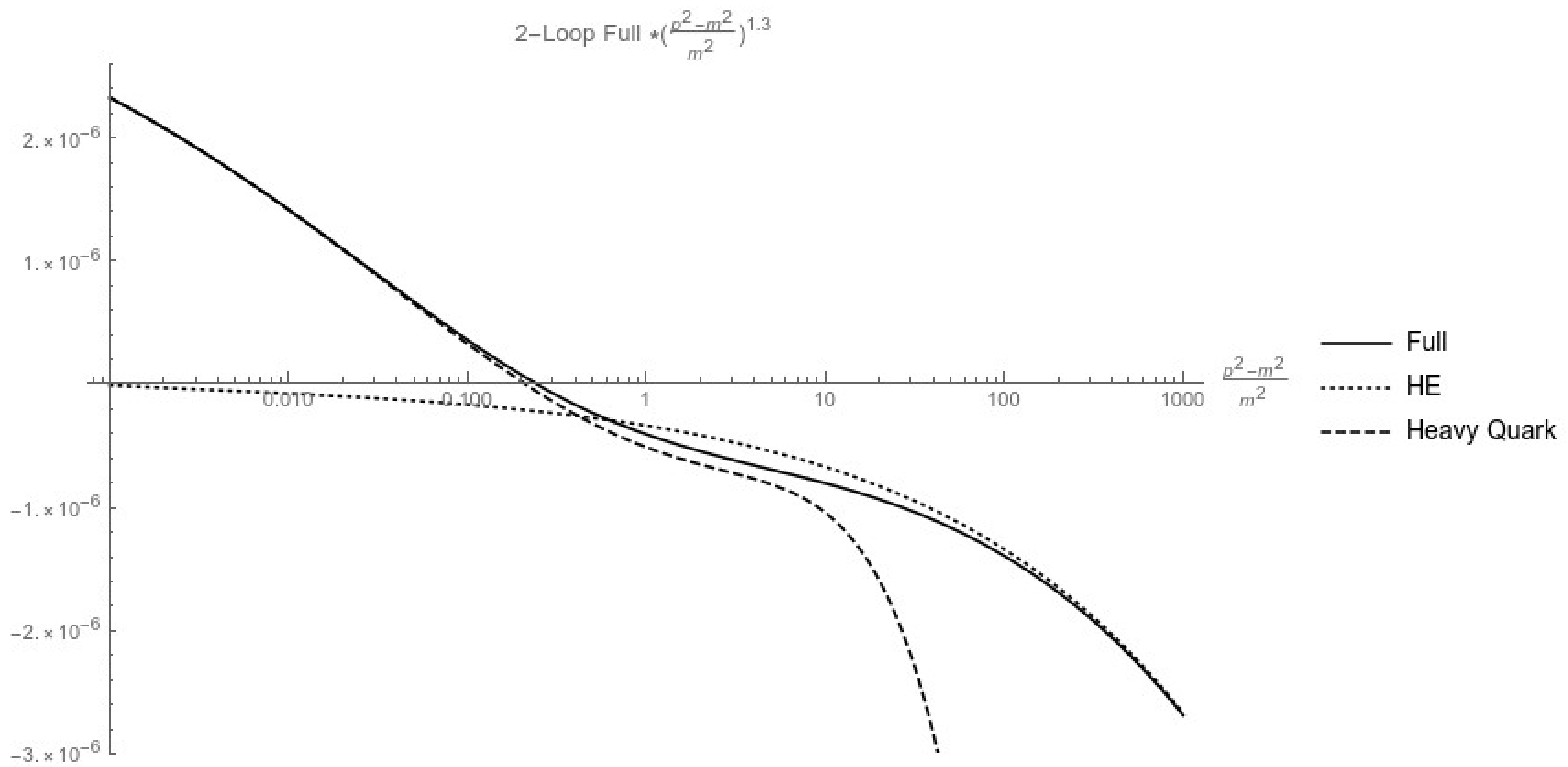
Topology A+B – Numerical Size of Fit





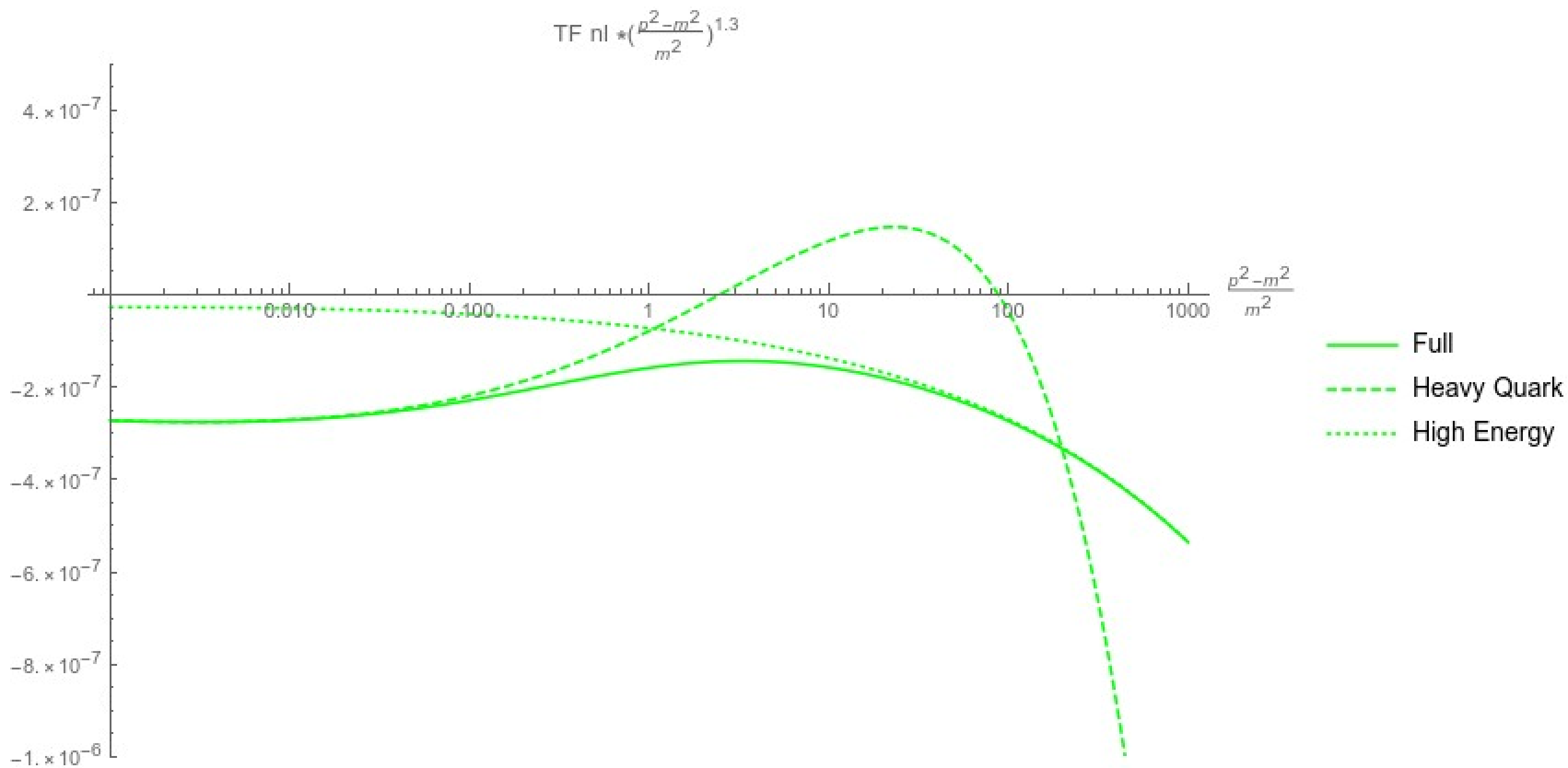
Topology A+B – Cross Checks

- Crosscheck: Logs, high energy- and heavy quark limit



Topology A+B – Cross Checks

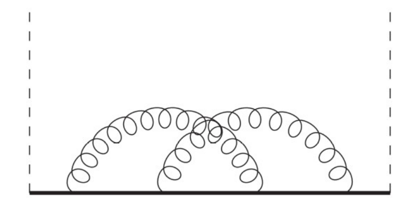
- Crosscheck: Logs, high energy- and heavy quark limit



Topology B – Roadmap

✓ • Solve diagram without Wilson line analytically

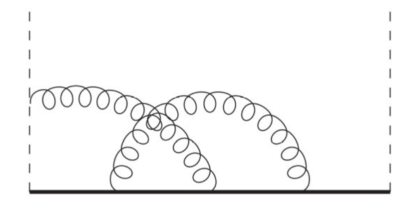
  – Usual setup: IBP, solve/find MIs, ...

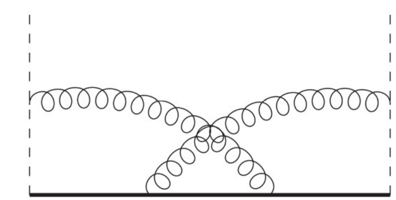


✓ • Get analytic information about Wilson line diagrams where possible (imaginary part)

  – Evaluate rest numerically by Sector Decomposition

  – Fit outcome

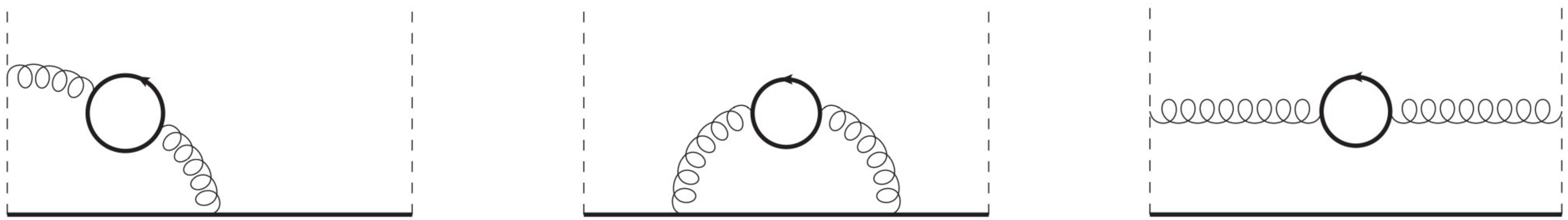




✓ • Put results everything together and do crosschecks

Secondary Mass Effects

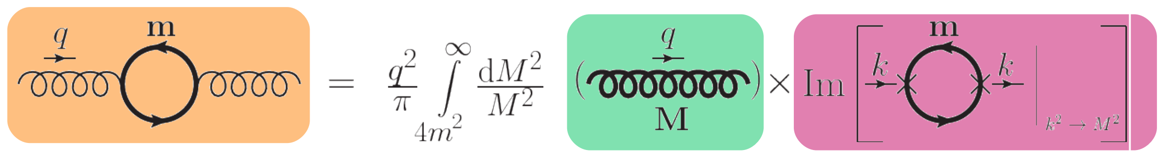
- One last contribution missing: Secondary produced heavy quarks ($C_F T_F$)



- Technique: Dispersion relation
  - Two loop $\rightarrow$ One loop with massive gluon + dispersion integral
  - Technically simpler
  - Many properties can be seen already at massive gluon level

Secondary Mass Effects

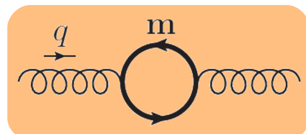
- “Subtracted” dispersion relation: On-shell, finite

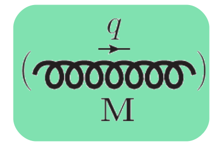


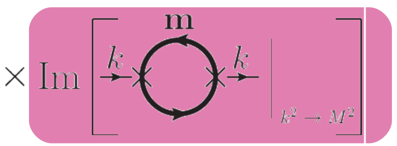
$$\frac{-i\,g^{\mu\rho}}{p^2+i\epsilon}\,\Pi^{\mathrm{OS}}_{\rho\sigma}(m^2,p^2)\,\frac{-i\,g^{\sigma\nu}}{p^2+i\epsilon} = \frac{1}{\pi}\int\frac{\mathrm{d}M^2}{M^2}\,\frac{-i\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)}{p^2-M^2+i\epsilon}\,\mathrm{Im}\left[\Pi(m^2,M^2)\right]$$

Secondary Mass Effects

- "Subtracted" dispersion relation: On-shell, finite



$= \frac{q^2}{\pi}\int_{4m^2}^{\infty}\frac{\mathrm{d}M^2}{M^2}$


$\times\,\mathrm{Im}$


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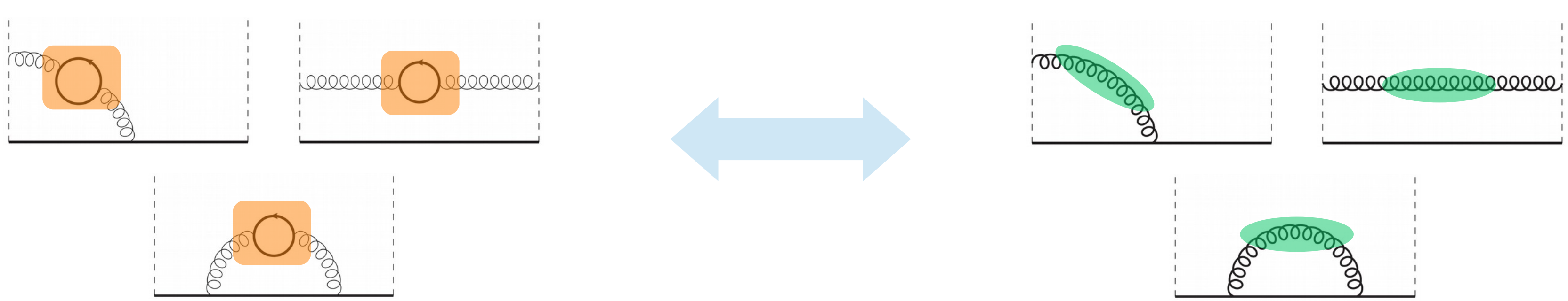
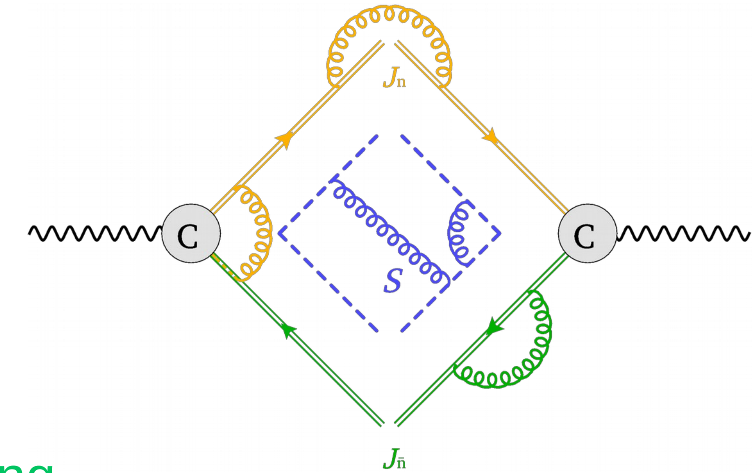
- Applied to the jet function:

$$J_2^{(n_\ell),\mathrm{sec},\mathrm{bare}}(s,m^2,\mu) = \frac{1}{\pi}\int\frac{\mathrm{d}M^2}{M^2}\,J_1^{M,\mathrm{bare}}(s,m^2,M^2,\mu)\,\mathrm{Im}[\Pi(m^2,M^2)]$$



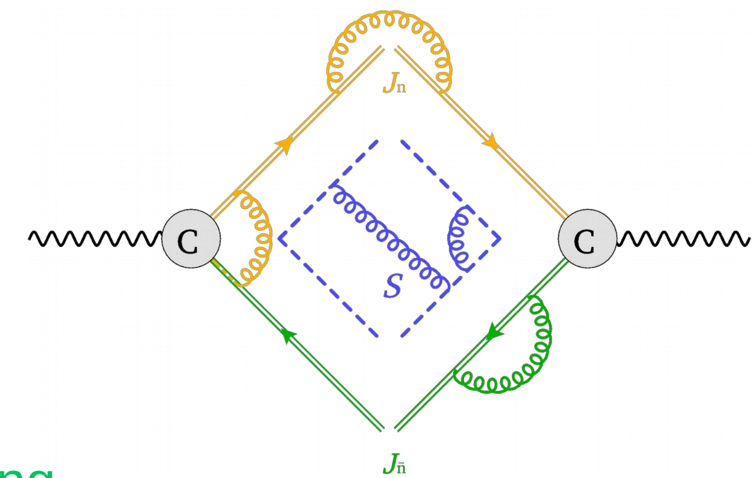
Secondary Mass Effects

- Problematic outcome after dispersion integral:
  - Rapidity divergences do not cancel
  - Massless limit wrong



- Reason: Soft modes in loop integral lead to double counting

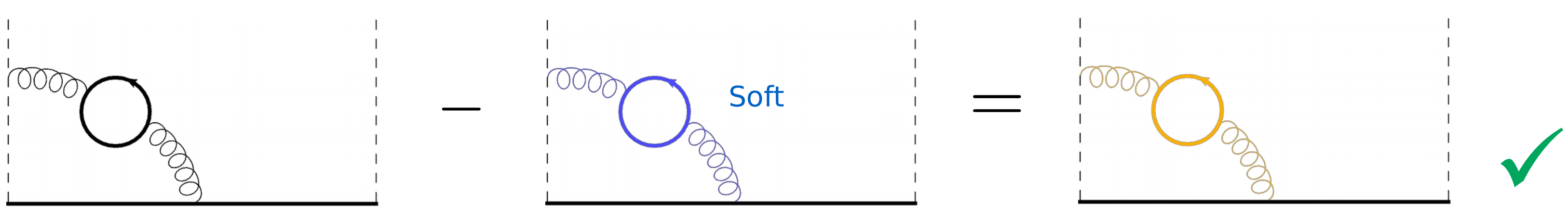
Secondary Mass Effects



- Problematic outcome after dispersion integral:
  - Rapidity divergences do not cancel
  - Massless limit wrong

- Reason: Soft modes in loop integral lead to double counting

- Solution: Soft mass bin subtraction
  - Subtract region where loop momentum is soft
  - Rapidity divergences cancel, correct massless limit is achieved



Secondary Mass Effects – Results

- One last time: Results

$$J_2^{m,\mathrm{sec}} = C_F T_F \Bigg[\left(-8L_m^2 - \frac{128L_m}{9} - \frac{224}{27}\right)\mathcal{L}_0(s) + \frac{32}{3} L_m \mathcal{L}_1(s)$$

$$+ \left(\frac{32L_m^3}{9} + \frac{70L_m^2}{9} + \frac{754L_m}{27} - \frac{32\zeta_3}{9} - \frac{46\pi^2}{27} + \frac{7075}{162}\right)\mathcal{L}_{-1}(s) + G_{m,\mathrm{sec}} \Bigg]$$

$$\mathcal{L}_{-1}(s) = \delta(s)$$

$$\mathcal{L}_{n\geq 0}(s) \equiv \frac{1}{\mu^2}\left[\frac{\Theta(s)\log^n s/\mu^2}{s/\mu^2}\right]_+$$

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$$\Pi(n;m) \equiv \int_0^{\frac{\pi}{2}} \mathrm{d}t\, \frac{1}{(1-n\sin^2 t)\sqrt{1-m\sin^2 t}}$$

$$G_{m,\mathrm{sec}} = \Theta(s)\frac{4}{3}L_m\left(\frac{s}{(m^2+s)^2} - \frac{4\log\left(\frac{m^2+s}{m^2}\right)}{s}\right)$$

$$+\Theta(s-8m^2)\frac{1}{s}\Bigg[-\frac{8y^{3/4}\left(81y^4+54y^3-24y^2-54y+71\right)}{27(y-1)^3}\mathrm{E}\left(\frac{3y^2+8y^{3/2}+6y-1}{16y^{3/2}}\right)$$

$$+\frac{2\left(252y^{9/2}+570y^{7/2}+822y^{5/2}+118y^{3/2}+45y^5+357y^4+720y^3+524y^2+3y+30\sqrt{y}+15\right)}{27\left(\sqrt{y}-1\right)\left(\sqrt{y}+1\right)^3 y^{3/4}}\mathrm{K}\left(\frac{3y^2+8y^{3/2}+6y-1}{16y^{3/2}}\right)$$

$$-\frac{2\left(\sqrt{y}-1\right)\left(15y^3+27y^2+25y+5\right)}{9\left(\sqrt{y}+1\right)y^{3/4}}\Pi\left(\frac{3y+2\sqrt{y}-1}{4y};\frac{3y^2+8y^{3/2}+6y-1}{16y^{3/2}}\right)+\frac{16}{3}\hat{I}(y)\Bigg]_{y=\frac{m^2}{s+m^2}}$$

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$$\Pi(n;m) \equiv \int_0^{\frac{\pi}{2}} \mathrm{d}t\,\frac{1}{(1-n\sin^2 t)\sqrt{1-m\sin^2 t}}$$

Secondary Mass Effects – Results

- One last time: Results

$$J_2^{m,\text{sec}} = C_F T_F \Bigg[\left(-8L_m^2 - \frac{128 L_m}{9} - \frac{224}{27} \right) \mathcal{L}_0(s) + \frac{32}{3} L_m \mathcal{L}_1(s)$$

$$+ \left(\frac{32 L_m^3}{9} + \frac{70 L_m^2}{9} + \frac{754 L_m}{27} - \frac{32\zeta_3}{9} - \frac{46\pi^2}{27} + \frac{7075}{162} \right) \mathcal{L}_{-1}(s) + G_{m,\text{sec}} \Bigg]$$

$$\mathcal{L}_{-1}(s) = \delta(s)$$

$$\mathcal{L}_{n\geq 0}(s) \equiv \frac{1}{\mu^2} \left[\frac{\Theta(s) \log^n s/\mu^2}{s/\mu^2} \right]_+$$

$$L_m \equiv \log \frac{m^2}{\mu^2}$$

$$L_s \equiv \log \frac{s}{\mu^2}$$

$$L_r \equiv \log \frac{m^2 + s}{m^2}$$

$$s = p^2 - m^2$$

$$\mathrm{E}(m) \equiv \int_0^{\frac{\pi}{2}} \mathrm{d}t \, \sqrt{1 - m \sin^2 t},$$

$$\mathrm{K}(m) \equiv \int_0^{\frac{\pi}{2}} \mathrm{d}t \, \frac{1}{\sqrt{1 - m \sin^2 t}},$$

$$\Pi(n; m) \equiv \int_0^{\frac{\pi}{2}} \mathrm{d}t \, \frac{1}{(1 - n \sin^2 t)\sqrt{1 - m \sin^2 t}}$$

$$G_{m,\text{sec}} = \Theta(s) \frac{4}{3} L_m \left(\frac{s}{(m^2+s)^2} - \cdots \right.$$

$$\int_y^{\frac{1}{4}(1-\sqrt{y})^2} \mathrm{d}z \, \frac{\sqrt{z-y}\, \log\left(\frac{y-4z-1-\sqrt{(y+4z-1)^2-16yz}}{y-4z-1+\sqrt{(y+4z-1)^2-16yz}} \right)}{z^{3/2}}$$

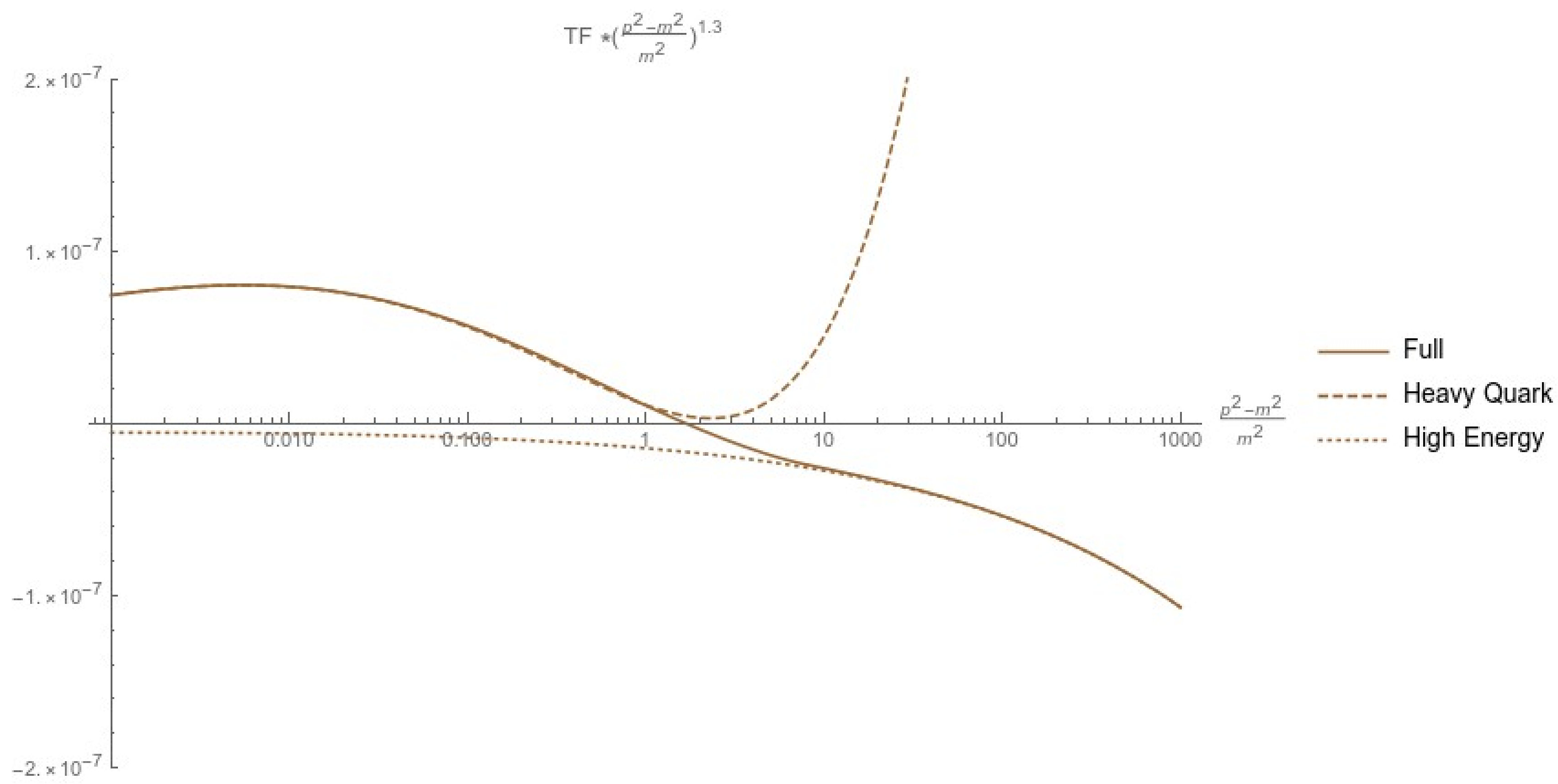
$$+ \Theta(s - 8m^2) \frac{1}{s} \Bigg[- \frac{8y^{3/4}\left(81y^4 + 54y^3 \cdots\right)}{27(y-1)^3} \mathrm{E}\left(\frac{3y^2 + \cdots}{16y^{\cdots}} \right)$$

$$+ \frac{2\left(252y^{9/2} + 570y^{7/2} + 822y^{5/2} + 118y^{3/2} + 45y^5 + 357y^4 + 720y^3 + 524y^2 + \cdots 30\sqrt{y} + 15\right)}{27\left(\sqrt{y}-1\right)\left(\sqrt{y}+1\right)^3 y^{3/4}} \mathrm{K}\left(\frac{3y^2 + 8y^{3/2} + 6y - 1}{16y^{3/2}} \right)$$

$$- \frac{2\left(\sqrt{y}-1\right)\left(15y^3 + 27y^2 + 25y + 5\right)}{9\left(\sqrt{y}+1\right) y^{3/4}} \Pi\left(\frac{3y + 2\sqrt{y} - 1}{4y}; \frac{3y^2 + 8y^{3/2} + 6y - 1}{16y^{3/2}} \right) + \frac{16}{3} \hat{I}(y) \Bigg]_{y = \frac{m^2}{s+m^2}}$$

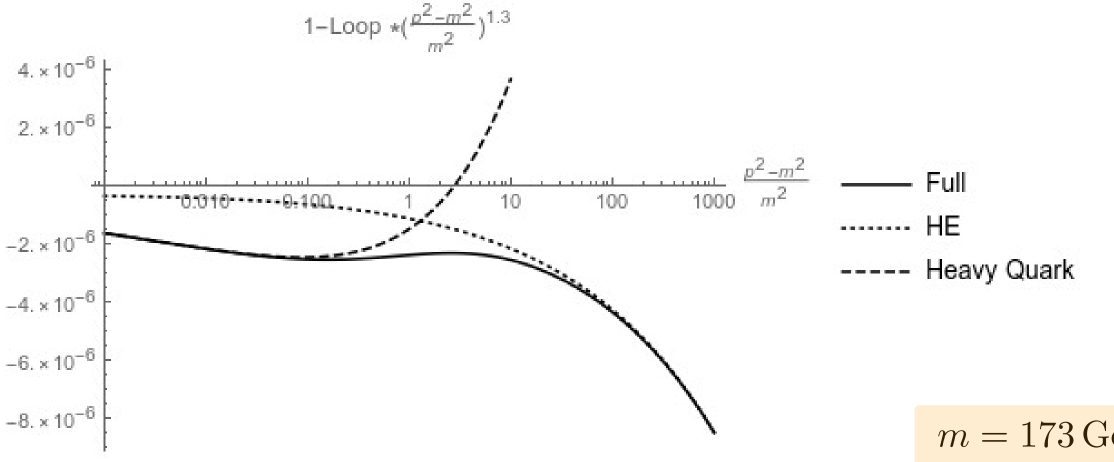
Secondary Mass Effects – Results

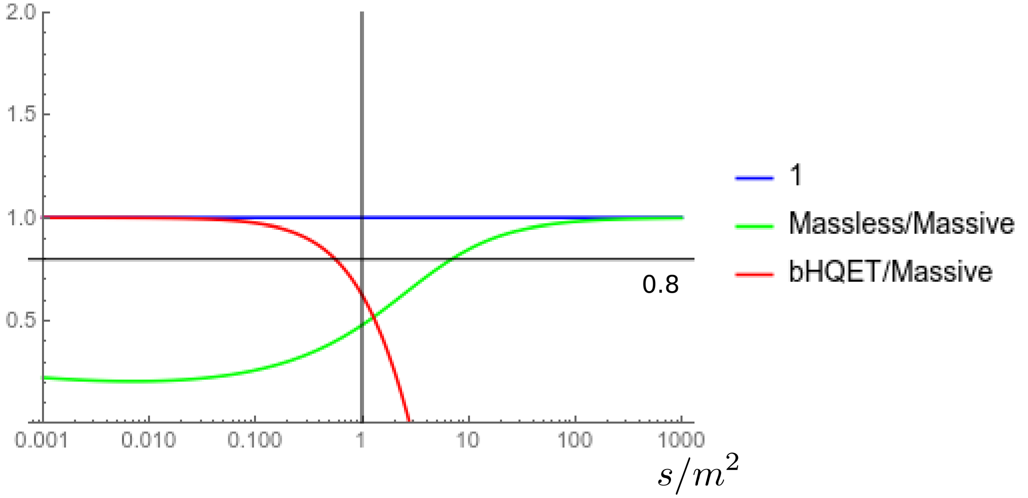
- Crosschecks



Full Result – Transition Region

- 1-Loop contribution:

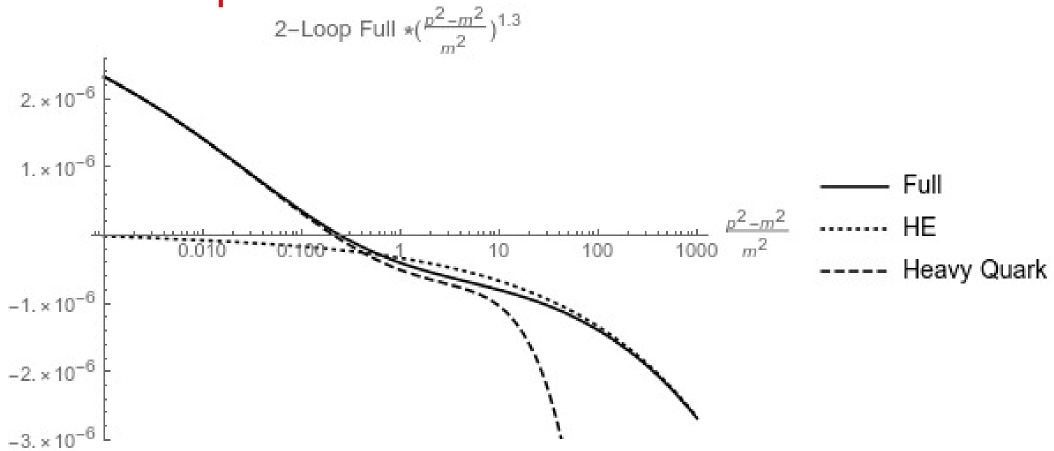


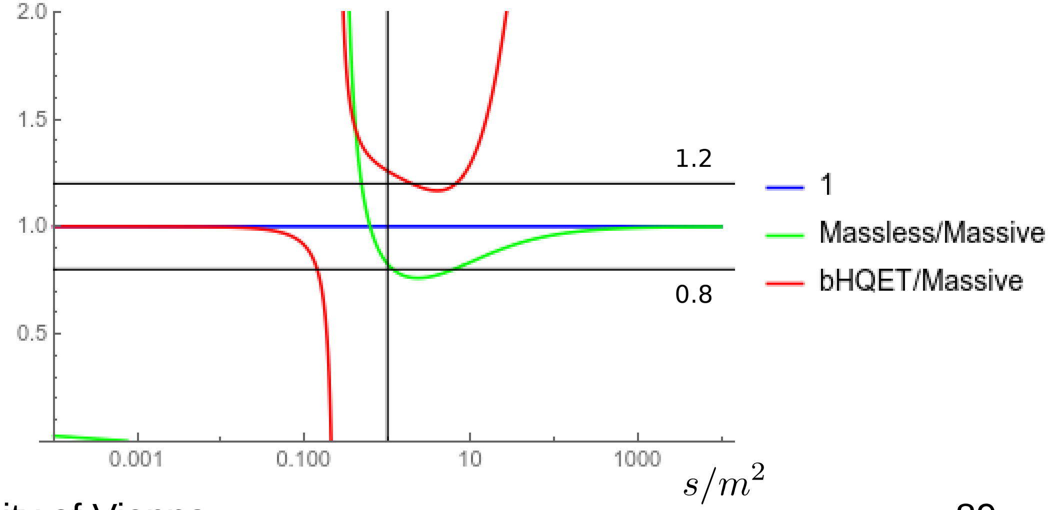


$m = 173\,\mathrm{GeV}$

$\mu = 5\,\mathrm{GeV} + \sqrt{s}$

- 2-Loop contribution:





Summary & Outlook

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Thank you for your attention!