Invariants for QCD algebra

- Some basics
- Calculation and squaring of amplitudes
- Various bases: Trace bases, DDM bases, Color flow bases, Multiplet bases
- Calculating using basic group invariants, Wigner 6js and 3js

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Motivation

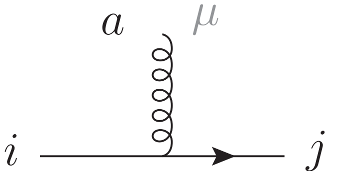
- With the LHC there is an increased interest in the treatment of color structure for processes with many colored partons
- This is applicable to fixed order calculations as well as parton showers and resummation
- I will talk about QCD (SU(N_c)), but group invariants for other groups can be treated similarly

The QCD Lagrangian

The QCD Lagrangian

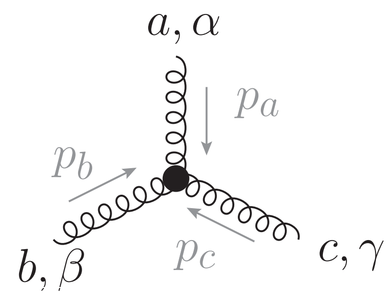
$$\mathcal{L} = \overline{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 + gA_\mu^a \overline{\psi}\gamma^\mu t^a \psi$$
$$-gf^{abc}(\partial_\mu A_\nu^a)A^{\mu b}A^{\nu c} - \frac{1}{4}g^2(f^{eab}A_\mu^a A_\nu^b)(f^{ecd}A^{\mu c}A^{\nu d})$$

contains:

- quark-gluon vertex, 

$= (t^a)^i{}_j$

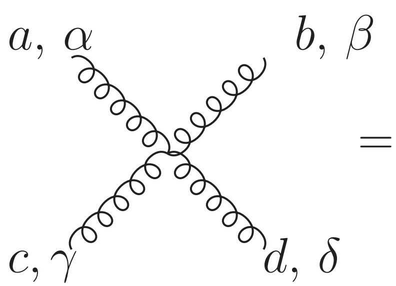
  Here $(t^a)^i{}_j$ are SU(3) generators and I take the graph to represent the color structure alone, no $ig\gamma^\mu$

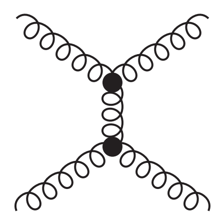
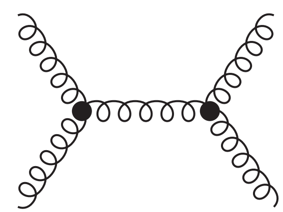
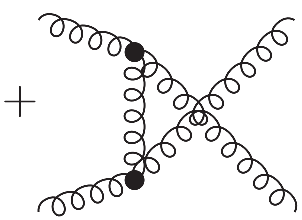
- triple-gluon vertex, 

$= i f^{abc}$

  Here we use the convention of reading the indices counter clockwise in the SU(3) structure constants f^{abc}, and again I only mean the color structure, no $-ig(g^{\alpha\beta}(p_a - p_b)^\gamma +$ cyclic)

- four-gluon vertex, here color and kinematic factors are correlated (so I cannot draw the color structure alone)



$=$  $+$  $+$ 

$\times i g_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\delta})$ $\quad \times i g_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\beta} g^{\gamma\delta})$ $\quad \times i g_s^2 (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta})$

$$= i f^{aeb}\, i f^{cde} + i f^{ace}\, i f^{bed} + i f^{aed}\, i f^{cbe}$$

$\times i g_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\delta})$ $\quad \times i g_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\beta} g^{\gamma\delta})$ $\quad \times i g_s^2 (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta})$

but the color structure is just a linear combination of triple-gluon vertices

Generators and structure constants

$$t^1 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} t^2 = \frac{1}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} t^3 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$t^4 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} t^5 = \frac{1}{2}\begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}$$

$$t^6 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} t^7 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} t^8 = \frac{1}{2\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

with $\mathrm{Tr}[t^a t^b] = \frac{1}{2}\delta^{ab} = T_R \delta^{ab}$, i.e. $T_R = \frac{1}{2}$

The structure constants f^{abc}, defined by

$$[t^a, t^b] = i f^{abc} t^c,$$

are totally antisymmetric. The non-zero structure constants are

$$f^{123} = 1,\ f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2},\ f^{458} = f^{678} = \frac{\sqrt{3}}{2}$$

and structure constants related by permutations.

But the last two slides are the most useless slides of this presentation...

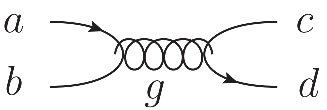
Dealing with color space

Due to confinement we never observe individual colors

- We average over incoming colors
- We sum over outgoing colors
- $\rightarrow$ we sum over the colors of all external partons
- As always in quantum mechanics we also sum over all degrees of freedom that can interfere with each other $\rightarrow$ we sum over the colors of all internal particles
- $\rightarrow$ We sum over all colors of all particles

So, if we for example consider

$q\bar{q} \to q\bar{q}$



,

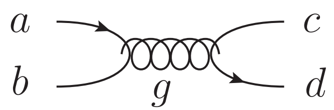
(let's pretend we have different flavors so we only have one Feynman diagram) we need the color sum

$$\frac{1}{3}\sum_{a=1}^{3}\frac{1}{3}\sum_{b=1}^{3}\sum_{c=1}^{3}\sum_{d=1}^{3}\left|\sum_{g=1}^{8}(t^g)^a{}_b(t^g)^c{}_d\right|^2$$

One way of dealing with this sum is to pick a particular representation of the generators, and sum over $3^4 * 8 = 648$ terms. Luckily there are better ways...

The color structures, for example

A sum over color for internal lines is always implicit

$$\sum_g (t^g)^a{}_b (t^g)^c{}_d =$$  ,

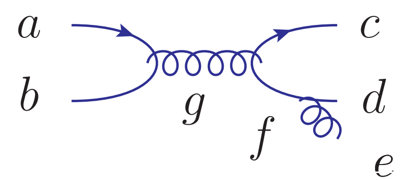
we can view as vectors living in some vector space — the overall color singlet vector space, where outgoing plus incoming colors form a total singlet. The physical observables are given by summing over all external colors, i.e., for the interference between two different color amplitudes $A_{a,b,c,\ldots}$ $B_{a,b,c,\ldots}$ we always want

$$\sum_{a,b,c,\ldots} (A_{a,b,c,\ldots})^* B_{a,b,c,\ldots}$$

It is easy to convince oneself about that the above sum is a scalar product on the vector space of total color singlet color structures with the external indices $a, b, c...$, i.e.,

$$\langle A, B\rangle = \sum_{a,b,c,\ldots} (A_{a,b,c,\ldots})^* B_{a,b,c,\ldots}$$

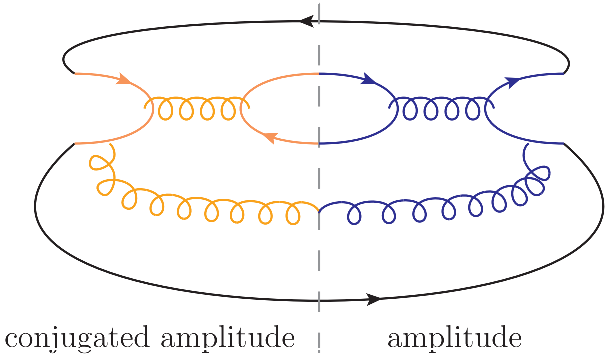
$\rightarrow$ We can use all our knowledge of vector spaces and scalar products

Example: If $A = (t^g)^a{}_b (t^g)^f{}_c (t^e)^d{}_f =$ 

, then

$$\langle A|A\rangle = \sum_{a,b,c,d,e,f,g,h,i} \left[(t^h)^a{}_b (t^h)^i{}_c (t^e)^d{}_i\right]^* (t^g)^a{}_b (t^g)^f{}_c (t^e)^d{}_f$$

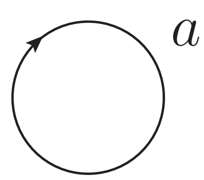
$$= \sum_{a,b,c,d,e,f,g,h,i} (t^h)^b{}_a (t^h)^c{}_i (t^e)^i{}_d (t^g)^a{}_b (t^g)^f{}_c (t^e)^d{}_f$$

$=$ 

The first equality holds since the generators are Hermitian, and the last holds since we always sum over the color of internal lines

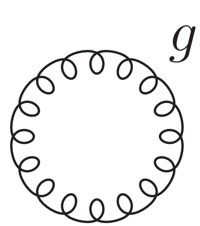
As seen above we can represent the squared amplitude with a picture. We can also calculate in pictures! To do so we need just a few rules

- There are N_c possible quark colors



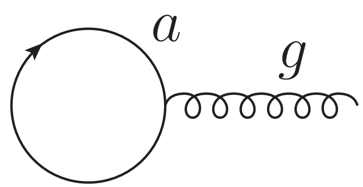
$$= N_c \qquad \sum_{a=1}^{N_c} \delta^a{}_a = N_c$$

- There are $N_g = N_c^2 - 1$ possible gluon colors



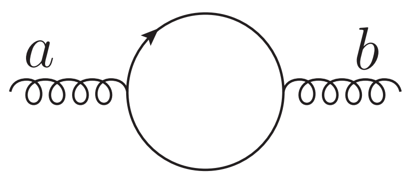
$$= N_c^2 - 1 \qquad \sum_{g=1}^{N_c^2-1} \delta^{gg} = N_c^2 - 1$$

- The generators are traceless

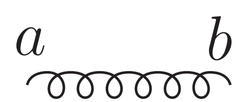


$= 0 \qquad \sum_{a=1}^{N_c} (t^g)^a{}_a = 0$

- Generator normalization

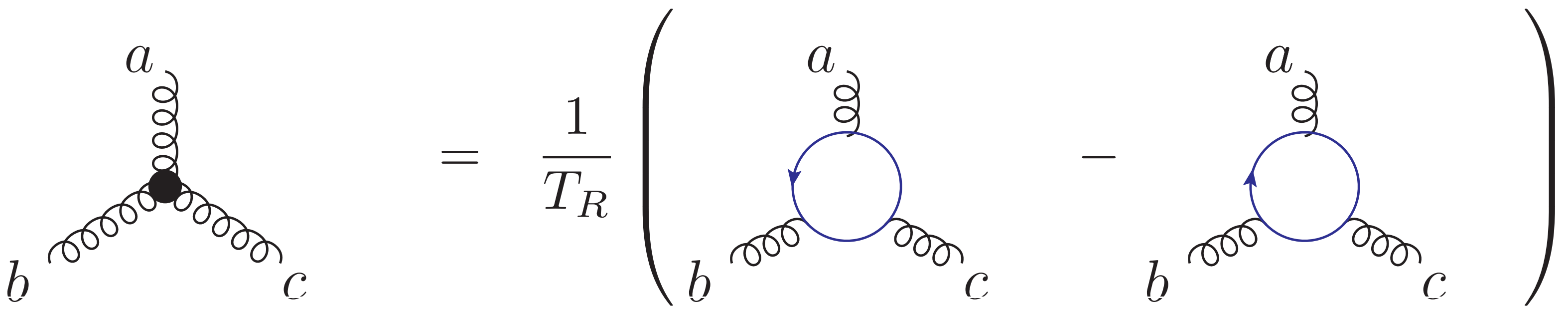


$= T_R$



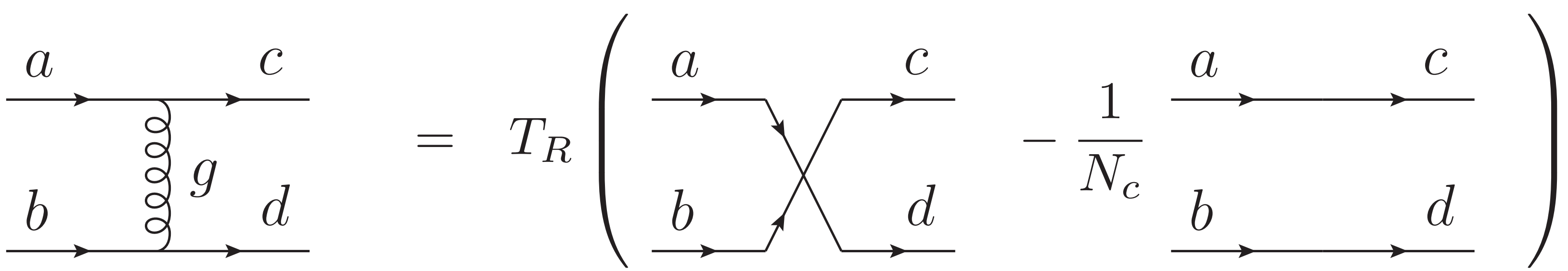
$\mathrm{Tr}[t^a t^b] = T_R \delta^{ab}$

- The algebra $[t^a, t^b] = i f^{abc} t^c \Rightarrow$



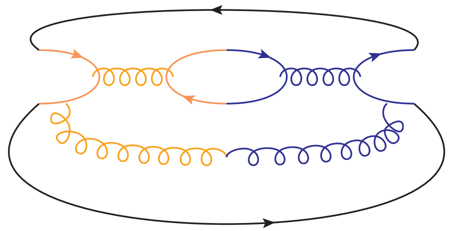
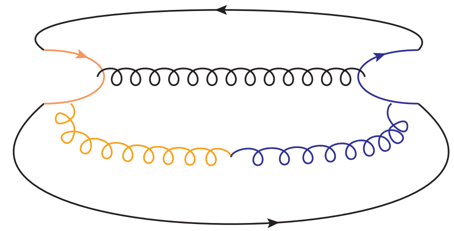
$$i f^{abc} = \frac{1}{T_R}\left[\text{Tr}[t^a t^b t^c] - \text{Tr}[t^b t^a t^c]\right]$$

- The Fierz identity (the completeness relation)

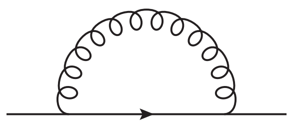
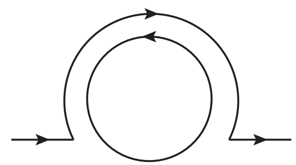


$$(t^g)^a{}_c (t^g)^b{}_d = T_R\left[\delta^a{}_d \delta^b{}_c - \frac{1}{N_c}\delta^a{}_c \delta^b{}_d\right]$$

Let's apply the rules to our example

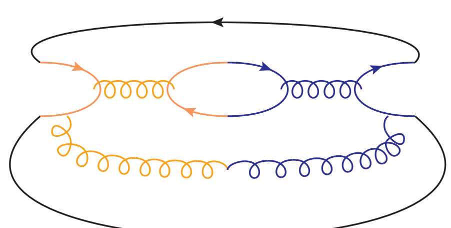
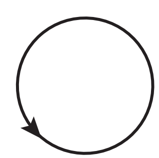
 $= T_R$ 

To further simplify the color structure we note using Fierz

 $= T_R \Bigg($  $- \frac{1}{N_c}$  $\Bigg) = T_R \left(N_c - \frac{1}{N_c} \right)$ 

$= T_R \frac{N_c^2 - 1}{N_c}$  $\equiv C_F$ 

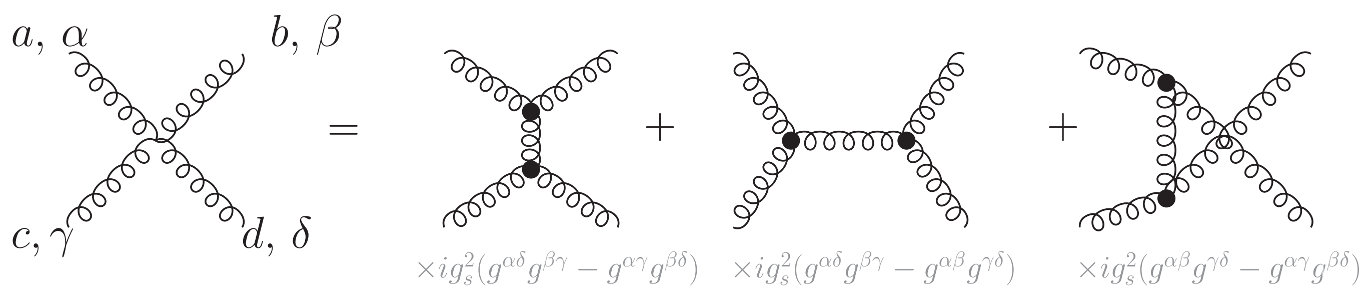
Giving, for the squared amplitude

 $= T_R C_F^2$  $= T_R C_F^2 N_c$

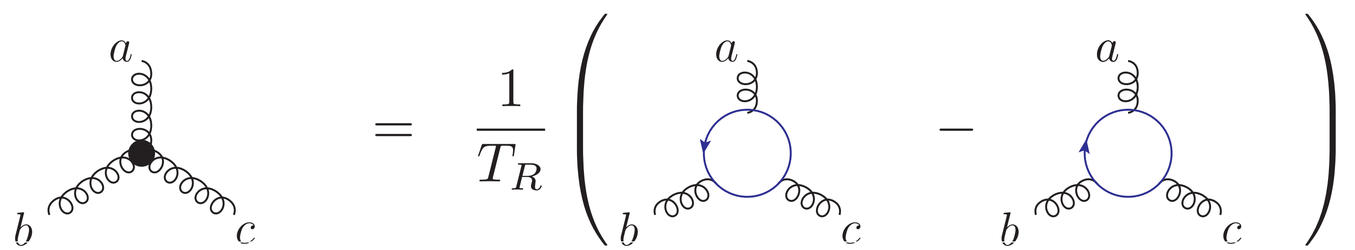
- In this way we can square any color amplitude and calculate any interference term. In general we have interference terms between different Feynman diagrams/color structures, but these are treated in precisely the same way.
- One way of dealing with color space is to just square the amplitudes one by one as one encounters them
- Alternatively, we may use any basis (spanning set)

The most popular bases: Trace bases

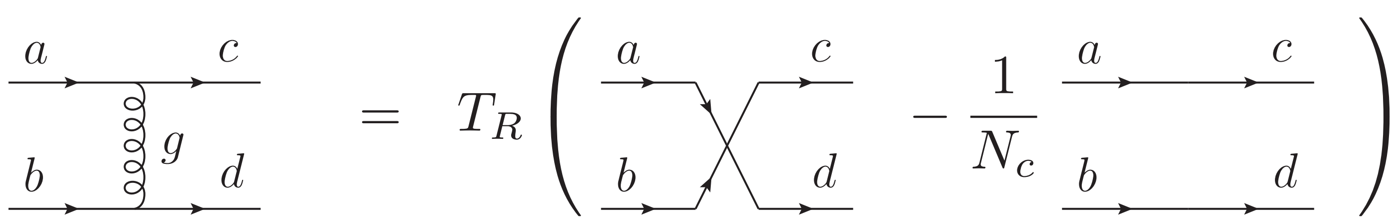
- Every 4g vertex can be replaced by 3g vertices:



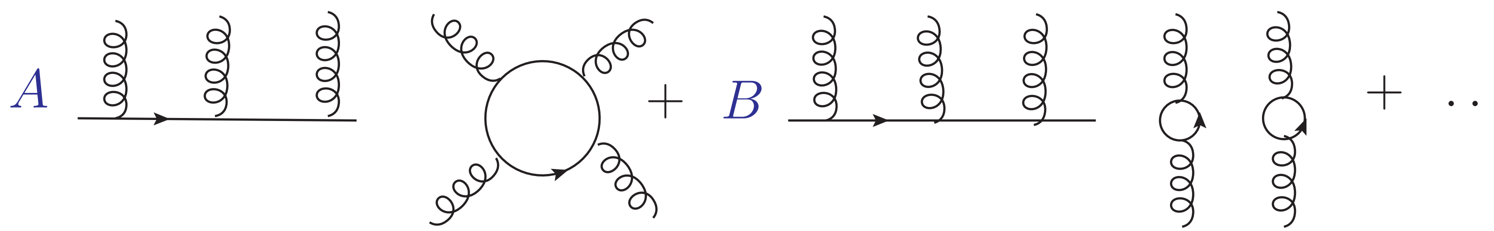
- Every 3g vertex can be replaced using:



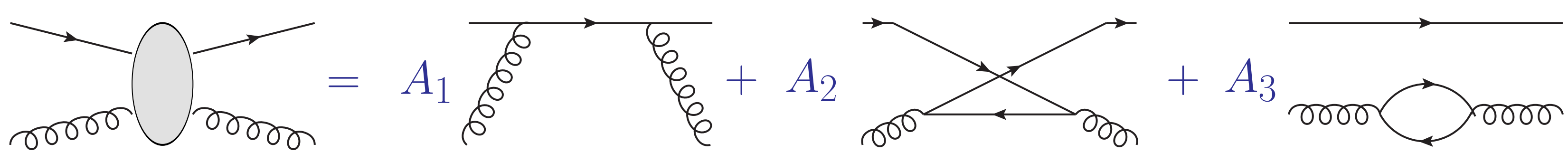
- After this every internal gluon can be removed using Fierz:



- This can be applied to any QCD amplitude, tree level or beyond
- In general an amplitude can be written as linear combination of different color structures, like



- For example for 2 (incoming + outgoing) gluons and one $q\bar{q}$ pair



(an incoming quark is the same as an outgoing anti-quark)

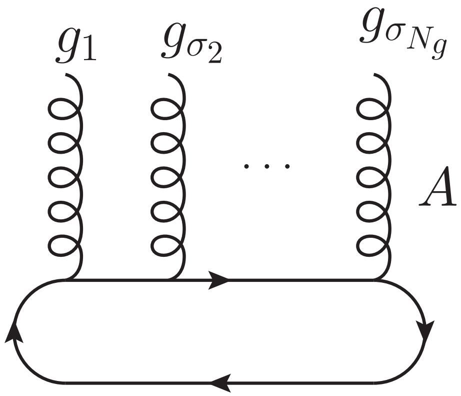
- The above type of color structure can be used as a spanning set, a trace basis

These bases have some nice properties

- Conceptual simplicity
- Can be reduced for a given *order* in perturbation theory, for example, for tree-level N_g-gluon amplitudes we have $(N_g - 1)!$ color structures of form

$$\mathcal{M}(g_1, g_2, \ldots, N_g) = \sum_{\sigma \in S_{N_g-1}} \mathrm{Tr}(t^{g_1} t^{g_{\sigma_2}} \ldots t^{g_{\sigma_{N_g}}}) A(\sigma)$$

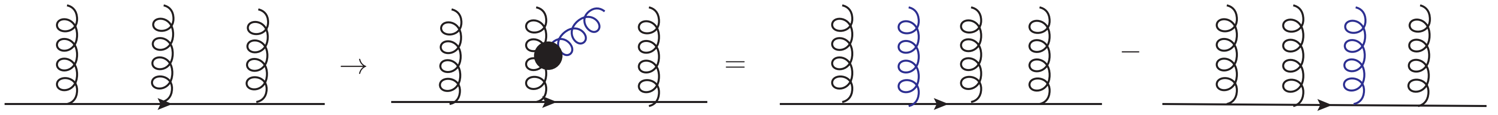
$$= \sum_{\sigma \in S_{N_g-1}}$$



$$A(\sigma),$$

whereas for higher orders we also have products of traces.

- Taking the leading N_c limit is trivial and results in a flow of colors
- The basis vectors are orthogonal when $N_c \to \infty$
- The effect of gluon emission is easily described:



We get just one new basis vector if the emitter is an (anti-)quark and two if the emitter is a gluon

- So is the effect of gluon exchange

For these reasons trace bases are commonly used:

- MadGraph (fixed order calculations)
  (J. Alwall, M. Herquet, F. Maltoni, O. Mattelaer, T. Stelzer, JHEP 1106 (2011) 128, 1106.0522)
- ColorFull (C++ code for color space, more later)
  (M.S., Eur.Phys.J. C75 (2015) 5, 236, 1412.3967, hepforge since 2013, http://colorfull.hepforge.org/)
- $N_c = 3$ parton showers by M.S. and S. Plätzer, and by D. Soper and Z. Nagy
  (D. Soper and Z. Nagy JHEP 0709 (2007) 114, 0706.0017,
  S. Plätzer and MS, JHEP 07(2012)042, 1201.0260,
  S. Plätzer, MS, J. Thorén, JHEP 1811 (2018) 009, 1809.05002)
- Resummation
  (M.S., JHEP 0909 (2009) 087, 0906.1121,
  E. Gerwick, S Höche, S. Marzani, S. Schumann, JHEP 1502 (2015) 106, 1411.7325)

ColorMath

- I have written a Mathematica package, ColorMath,
  (Eur. Phys. J. C 73:2310 (2013), 1211.2099)
- ColorMath is an easy to use Mathematica package for color summed calculations in QCD, SU(N_c)
- Repeated indices are implicitly summed

```
In[2]:= Amplitude = I f[g1, g2, g] t[{g}, q1, q2]
Out[2]= i t^{g}q1_q2 f^{g1,g2,g}

In[3]:= CSimplify[Amplitude Conjugate[Amplitude /. g → h]]
Out[3]= 2 Nc (-1 + Nc^2) TR^2
```

- ColorMath does not automatically construct bases, but given a basis (constructed by the user) it can calculate the soft anomalous dimension matrix automatically

- The ColorMath package and tutorial can be downloaded from http://library.wolfram.com/infocenter/MathSource/8442/ or www.thep.lu.se/~malin/ColorMath.html

ColorFull

For the purpose of treating a general QCD color structure (any number of partons, any order) I have written a C++ color algebra code, ColorFull, which:

- Automatically creates trace bases for any number and kind of partons, and to arbitrary order in α_s
- Squares color amplitudes in various ways
- Describes the effect of gluon emission, calculates "radiation matrices", $\mathbf{T_i}$, which gives the vectors obtained when emitting a gluon from parton i decomposed in the larger basis

- Describes the effect of gluon exchange, automatically calculates soft anomalous dimension matrices
- Is shipped with Herwig++ ($\geq$ 7)

ColorFull can be downloaded from `colorfull.hepforge.org`,
(M.S., Eur.Phys.J. C75 (2015) 5, 236, 1412.3967)

There are also drawbacks with trace bases

- Not orthogonal
  $\rightarrow$ When squaring amplitudes almost all cross terms have to be taken into account $\rightarrow N^2_{\text{basis}}$ terms
- Overcomplete
  For $N_g + N_{q\bar{q}} > N_c$ the bases are also overcomplete
- The size of the vector space asymptotically grows as an exponential in the number of gluons/$q\bar{q}$-pairs for finite N_c

- For general N_c the basis size grows as a factorial

$$N_{\text{vec}}[N_q, N_g] \quad = \quad N_{\text{vec}}[N_q, N_g - 1](N_g - 1 + N_q) + N_{\text{vec}}[N_q, N_g - 2](N_g - 1)$$

where

$$\begin{aligned} N_{\text{vec}}[N_q, 0] &= N_q! \\ N_{\text{vec}}[N_q, 1] &= N_q N_q! \end{aligned}$$

(S. Keppeler & M.S. JHEP09(2012)124, 1207.0609)

- For general N_c and gluon only amplitudes (to all order) the size is given by Subfactorial(N_g)$\approx N_g!/e$
- For tree-level gluon amplitudes traces may be used as spanning vectors giving $(N_g - 1)!$ spanning vectors

Example: Number of spanning vectors for N_g gluons (without imposing charge conjugation invariance). These numbers are representative also for N_g gluons plus $q\bar{q}$-pairs.

N_g	Vectors $N_c = 3$	Vectors $N_c \to \infty$	LO Vectors $N_c \to \infty$
4	8	9	3!=6
5	32	44	4!=24
6	145	265	120
7	702	1 854	720
8	3 598	14 833	5 040
9	19 280	133 496	40 320
10	107 160	1 334 961	362 880
11	614 000	14 684 570	3 628 800
12	3 609 760	176 214 841	39 916 800

(Y. Du, M.S. & J. Thorén, JHEP 1505 (2015) 119, 1503.00530)

The dimension of the full vector space (all orders) for $N_c = 3$

N_g	$N_{q\bar{q}} = 0$	N_g	$N_{q\bar{q}} = 1$	N_g	$N_{q\bar{q}} = 2$
4	8	3	10	2	13
5	32	4	40	3	50
6	145	5	177	4	217
7	702	6	847	5	1 024
8	3 598	7	4 300	6	5 147
9	19 280	8	22 878	7	27 178
10	107 160	9	126 440	8	149 318
11	614 000	10	721 160	9	847 600
12	3 609 760	11	4 223 760	10	4 944 920

(M.S. & J. Thorén HEP 1509 (2015) 055, 1507.03814)

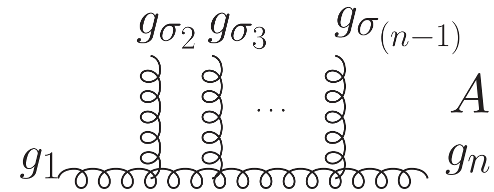
- For tree-level gluon processes, we can get away with the tree-level color structures giving $(N_g - 1)!^2$ terms when squaring amplitudes.
- For NLO gluon processes we need more color structures.
- For all order resummation all color structures will appear $\rightarrow N^2_{\text{basis}} \approx (N_g!/e)^2$ when squaring. On the other hand if we really want to exponentiate the soft anomalous dimension matrix this scales as $N^3_{\text{basis}} \approx (N_g!/e)^3$
- Numbers for processes with quarks are comparable. (For every gluon you can alternatively treat one $q\overline{q}$-pair)
- Hard to go beyond $\sim$ 8 gluons plus $q\overline{q}$-pairs

DDM bases

- The DDM bases (adjoint bases) are based on the observation that tree-level gluon-only color structures can be expressed as

$$\mathcal{M}(g_1, g_2, \ldots, g_n) = \sum_{\sigma \in S_{N_g-2}} i f^{g_1 g_{\sigma_2} i_1} i f^{i_1 g_{\sigma_3} i_2} \ldots i f^{i_{n-3} g_{\sigma_{n-1}} g_n} A(\sigma)$$

$$= (-1)^{N_g} \sum_{\sigma \in S_{N_g-2}}$$



$$A(\sigma).$$

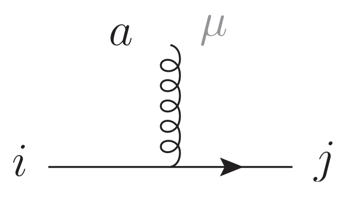
V. Del Duca, L. J. Dixon, and F. Maltoni, Nucl. Phys. B 571(2000) 51-70, hep-ph/9910563

- In this way we only need $(N_g - 2)!$ spanning vectors
- Charge conjugation symmetry is manifest
- For higher order color structures additional basis vectors are needed
- These bases have been generalized to processes with quarks by Melia
  T. Melia, Phys.Rev.D88(2013), no. 1014020, 1304.7809
  T. Melia, Phys.Rev.D89(2014), no. 7 074012, 1312.0599

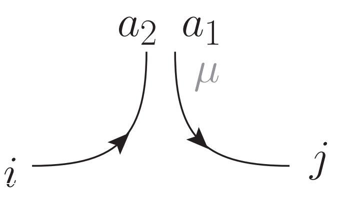
Color flow bases

- One way out is to give up exact treatment of color structure and run a Monte Carlo over colors
- This is particularly efficient in the color flow basis
- Here the adjoint representation indices are rewritten in terms of fundamental representation indices and new color flow Feynman rules are derived (Maltoni, Stelzer, Paul, Willenbrock, Phys.Rev. D67 (2003), hep-ph/0209271)
- Explicit colors (r, g, or b) are then assigned to the lines, and one may run a Monte Carlo sum over colors to sample color space
- This is not exact but the color structure treatment is much quicker (Comix, T. Gleisberg, S. Hoeche, JHEP 0812 (2008) 039, 0808.3674; S. Plätzer, Eur.Phys.J. C74 (2014) 6, 2907, 1312.2448; S. Prestel and J. Isaacson 1806.10102)

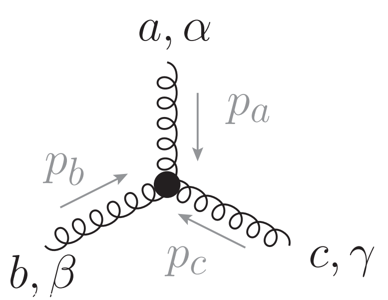
- quark-gluon vertex,



$$= i g_s \gamma^\mu (t^a)^i{}_j \rightarrow \quad i g_s \gamma^\mu \delta^i{}_{a_2} \delta^{a_1}{}_j =$$

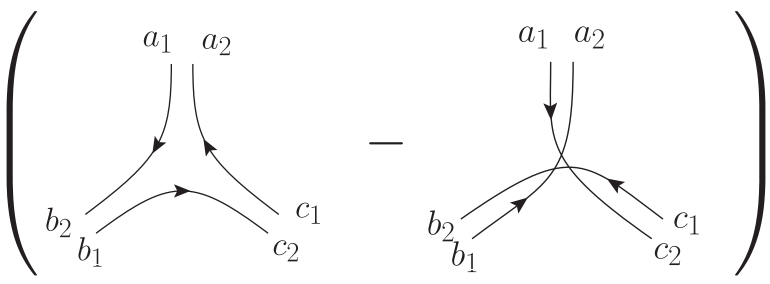


- triple-gluon vertex,



$$= i f^{abc}(-i g_s (g^{\alpha\beta}(p_a - p_b)^\gamma + \text{cyclic}))$$

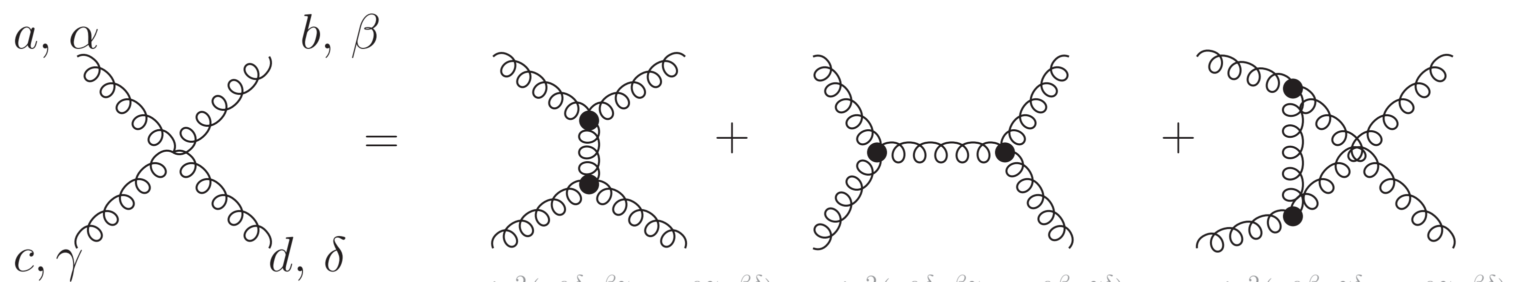
$$\rightarrow \frac{1}{T_R}$$



$$(-i g_s (g^{\alpha\beta}(p_a - p_b)^\gamma + \text{cyclic}))$$

can easily be written in completely symmetric form...

- four-gluon vertex



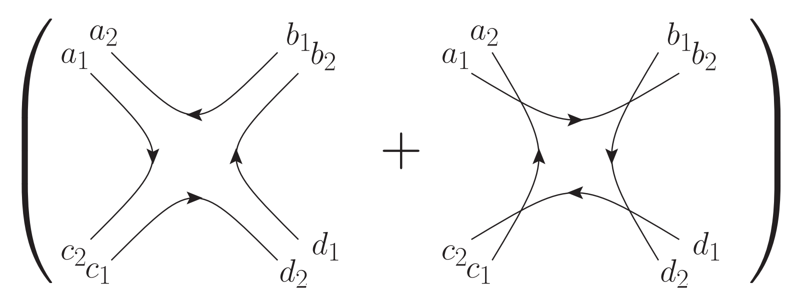
$\times i g_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\delta})$ $\quad \times i g_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\beta} g^{\gamma\delta})$ $\quad \times i g_s^2 (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta})$

$$= \; i f^{aeb}\, i f^{cde} \; + \; i f^{ace}\, i f^{bed} \; + \; i f^{aed}\, i f^{cbe}$$

$\times i g_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\delta})$ $\quad \times i g_s^2 (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\beta} g^{\gamma\delta})$ $\quad \times i g_s^2 (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta})$

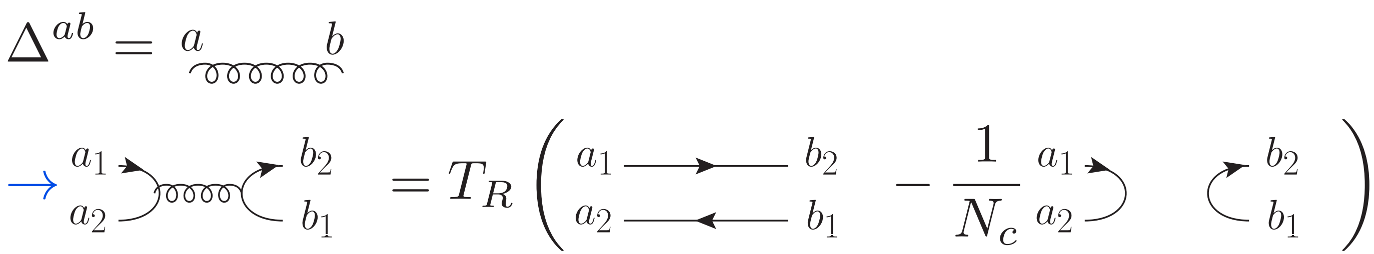
$\rightarrow$

$$i g_s^2 \left(2 g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\beta} g^{\gamma\delta}\right) \frac{1}{T_R}$$



$$+[c \leftrightarrow d] \quad + [b \leftrightarrow d]$$

- Color structure of propagator

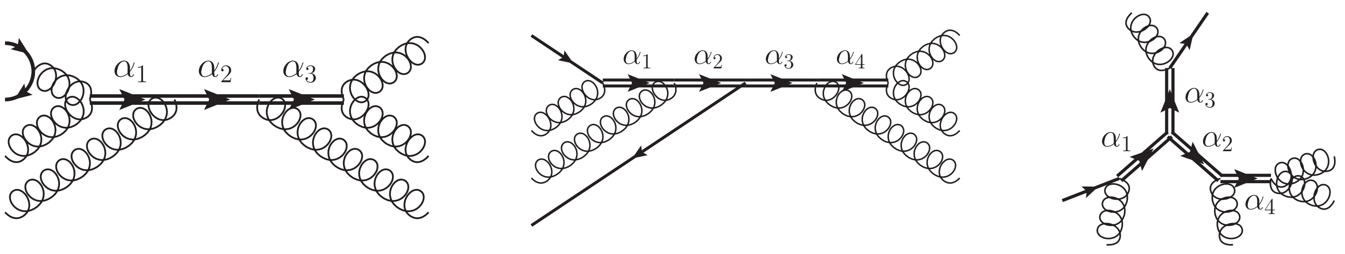


- Similarly the $q\bar{q}$-pairs corresponding to external gluons have to be forced to be in octets when squaring amplitudes

Warning: Conventions differ from those in hep-ph/0209271

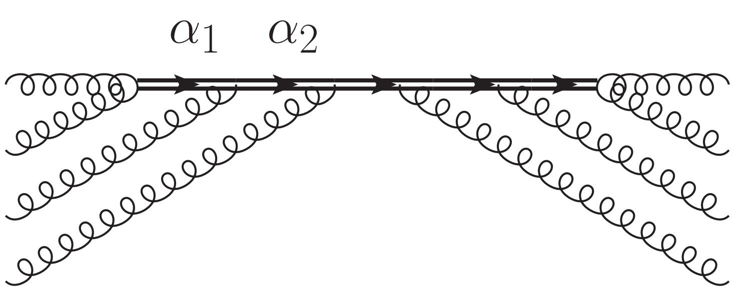
Multiplet bases

- QCD is based on SU(3) $\rightarrow$ the color space may be decomposed into irreducible representations
- Orthogonal basis vectors corresponding to irreducible representations may be constructed, in may different ways...



- The construction of the corresponding basis vectors is non-trivial, and a general strategy was presented relatively recently, (S. Keppeler & M.S. JHEP09(2012)124, 1207.0609, generalized by MS and J.Thorén in 1809.05002)
- With general, I mean general: general number of quarks and gluons, general order in α_s and general N_c

- In this presentation I will – for comparison – often talk about processes with gluons only, however, processes with quarks can be treated similarly
- The gluon basis vectors are of form



and can thus be characterized by a chain of representations $\alpha_1, \alpha_2, \ldots$ (In principle we have to differentiate between different vertices as well)

- These vectors are orthogonal ($\rightarrow$ minimal) by construction

For many partons the size of the vector space is much smaller for $N_c = 3$ (exponential), than for $N_c \to \infty$ (factorial)

N_g	Vectors $N_c = 3$	Vectors $N_c \to \infty$ trace bases	LO Vectors $N_c \to \infty$ LO trace bases
4	8	9	3!=6
5	32	44	4!=24
6	145	265	120
7	702	1 854	720
8	3 598	14 833	5 040
9	19 280	133 496	40 320
10	107 160	1 334 961	362 880

Number of basis vectors for N_g gluons *without* imposing vectors to appear in charge conjugation invariant combinations

... but the real advantage comes when squaring as the multiplet bases are orthogonal and the trace bases are not

N_g	Vectors $N_c = 3$	Vectors $N_c \to \infty$ trace bases	LO Vectors $N_c \to \infty$ LO trace bases
4	8	$(9)^2$	$(6)^2$
5	32	$(44)^2$	$(24)^2$
6	145	$(265)^2$	$(120)^2$
7	702	$(1\,854)^2$	$(720)^2$
8	3 598	$(14\,833)^2$	$(5\,040)^2$
9	19 280	$(133\,496)^2 \sim 10^{10}$	$(40\,320)^2 \sim 10^{9}$
10	107 160	$(1\,334\,961)^2 \sim 10^{12}$	$(362\,880)^2 \sim 10^{11}$

Number of terms from color when squaring for N_g gluons *without imposing charge conjugation invariant combinations*

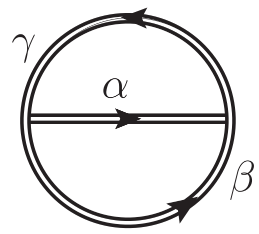
- Multiplet bases can potentially speed up exact calculations in color space very significantly, as squaring amplitudes becomes much quicker
- But before squaring, amplitudes must be decomposed in multiplet bases
- How quickly can amplitudes be expressed in multiplet bases?

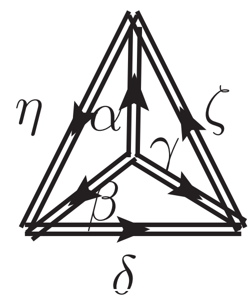
Decomposing color structure in multiplet bases

- One way of decomposing color structure into multiplet bases would be to simply evaluate the scalar product between each possible Feynman diagram and each possible vector as we have seen in the first half of this talk.
- The problem is that this scales very badly, a factorial from the number of diagrams, an exponential from the number of basis vectors and another (growing) factor from each single scalar product evaluation
- $\rightarrow$ no way
- We need a better strategy

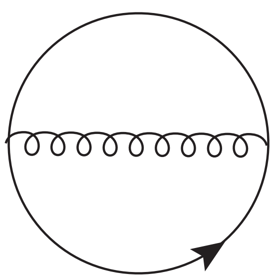
Group invariants!

- Fortunately there is one: Any group invariant quantity can be evaluated using Wigner 3j and 6j coefficients, respectively:

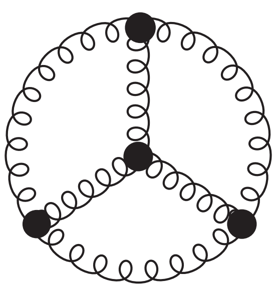




- For example



$= T_R (N_c^2 - 1)$



$= 2\, T_R^2 N_c^2 (N_c^2 - 1)$

Using standard normalization of vertices

- Using the multiplet basis we can evaluate the needed 3j and 6j coefficients for higher representations

- Furthermore, only a small number of such coefficients are needed, up to NLO

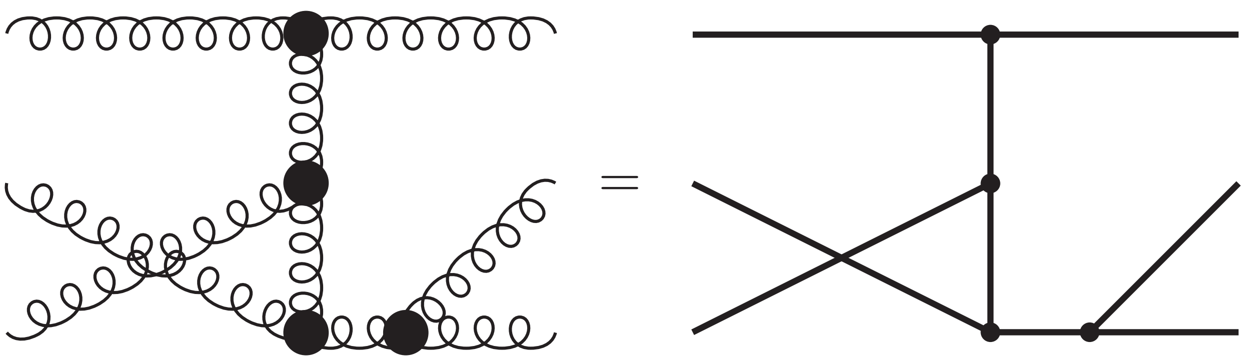
N_g	4	6	8	10	12
$N_c = 3$	29	120	272	476	733
$N_c \geq N_g$	44	389	2 023	8 077	27 631

and they can be evaluated once and for all
(Numbers could be slightly reduced by additional symmetries, and smart choices of vertices)

- As a test case, all coefficients needed for evaluation of processes with up to 6 gluons or 8 (quarks + antiquarks) have been explicitly calculated (M.S. & J. Thorén, JHEP 1509 (2015) 055, 1507.03814; 1809.05002)

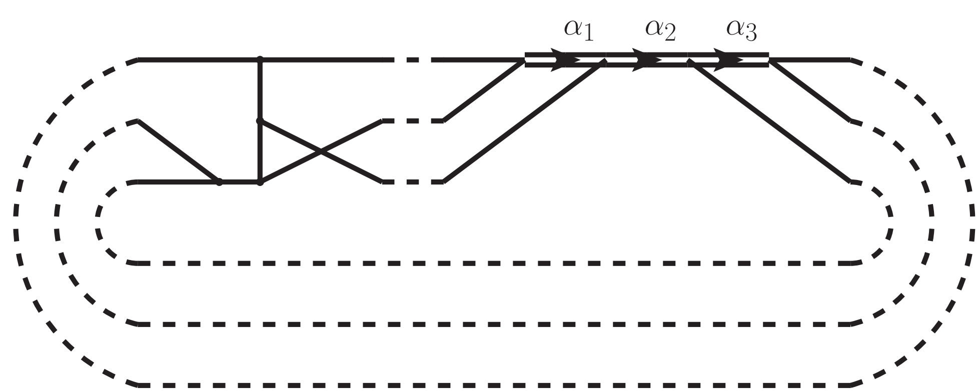
Decomposing color with 6j and 3j coefficients

As an example consider the color structure of the Feynman diagram:

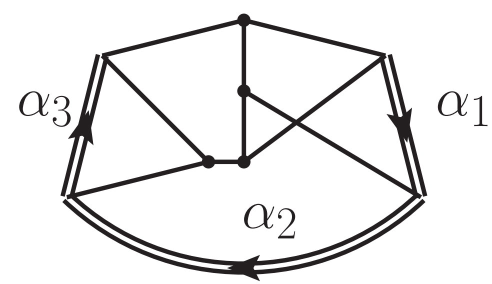


The scalar product between the color structure and a basis vector is given by:

$$A(\alpha_1, \alpha_2, \alpha_3) =$$

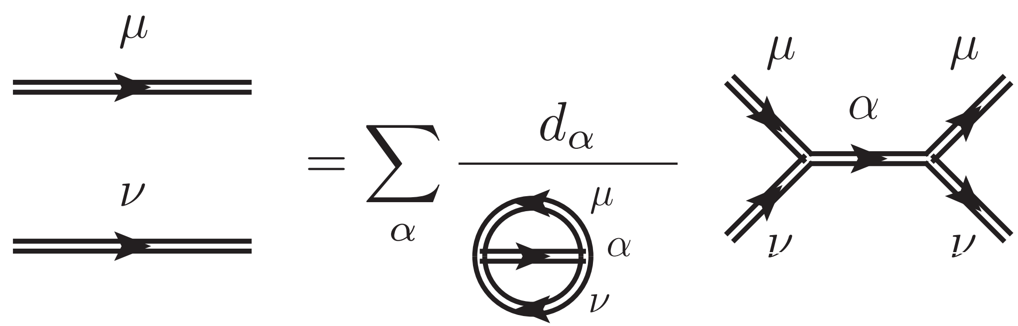


$$= \quad =$$

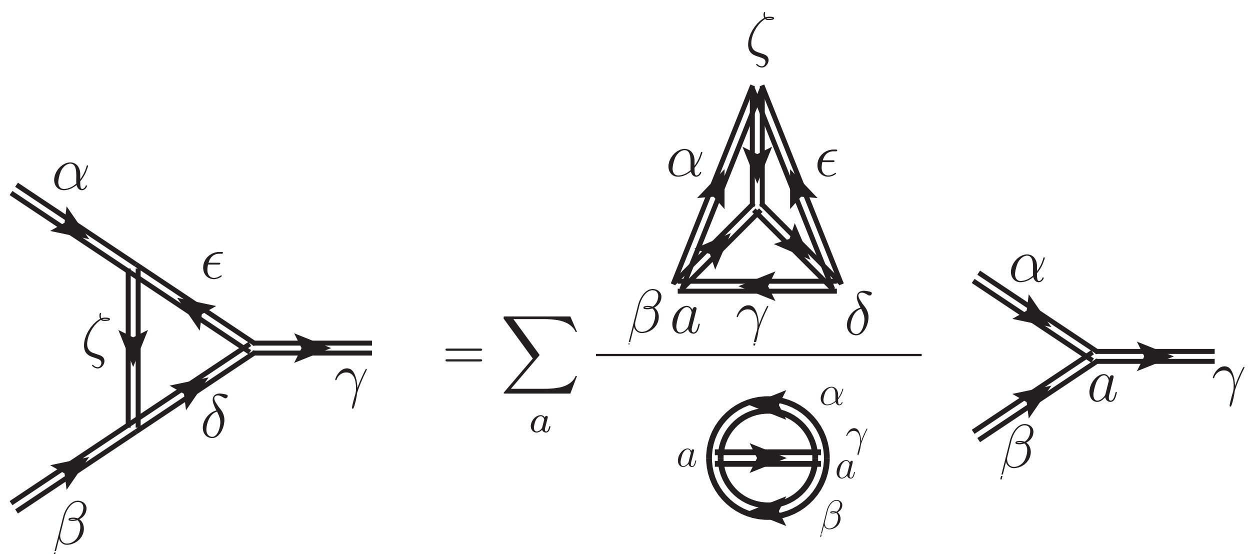


To simplify the color structure we need a few rules:

- The completeness relation

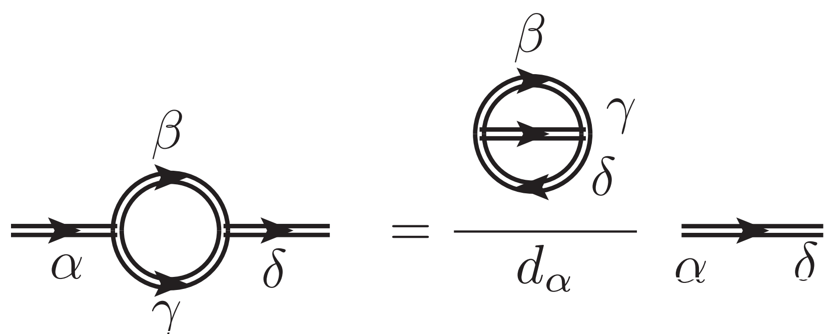


- and the vertex correction relation



Some other useful relations are:

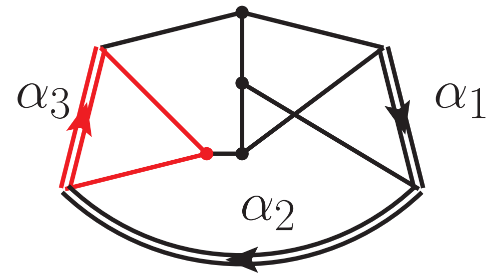
- two vertex loops give just a constant



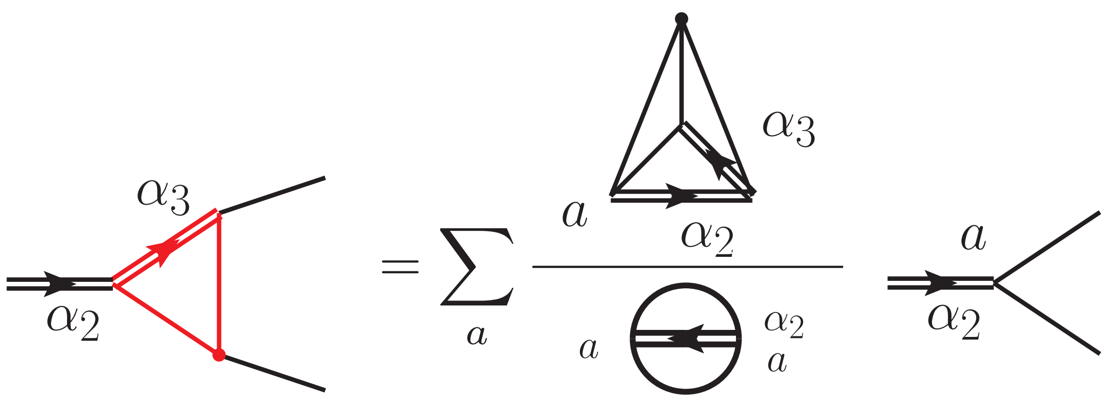
- dimension relation

In our color structure we note that we have a vertex correction:

$$A(\alpha_1, \alpha_2, \alpha_3) \quad =$$



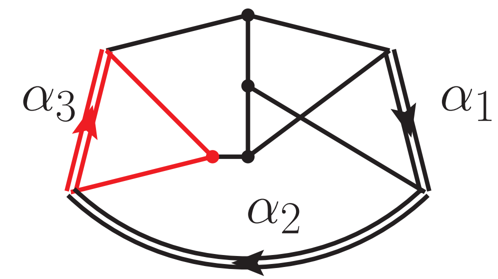
In our case the vertex correction is:



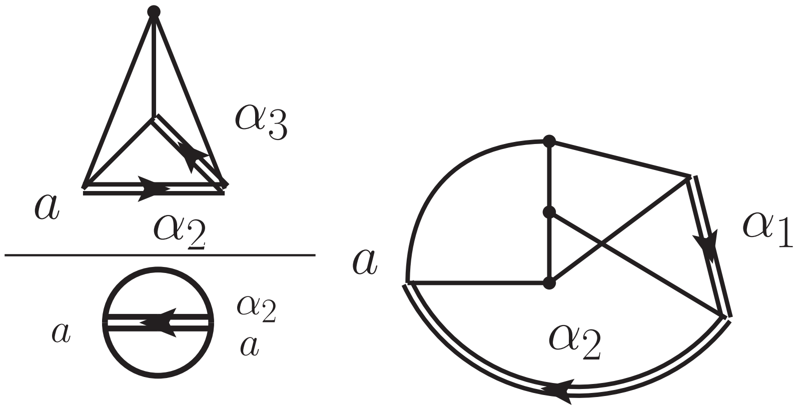
Where the sum runs over vertices a connecting the three representations α_1, α_3 and 8, and contains at most 2 terms.

Using the vertex correction results in:

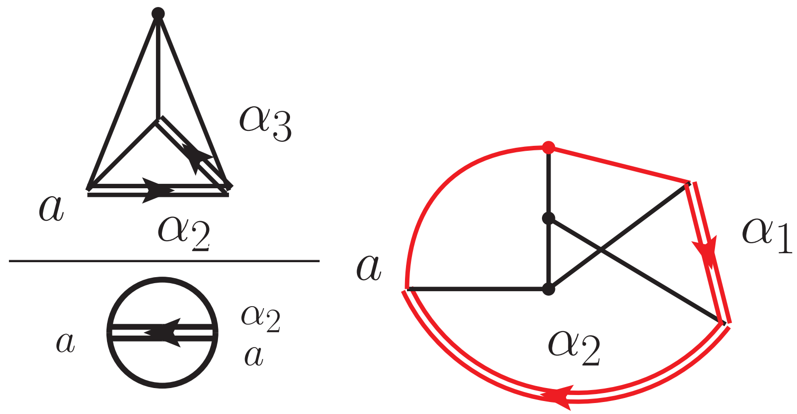
$$A(\alpha_1, \alpha_2, \alpha_3) =$$



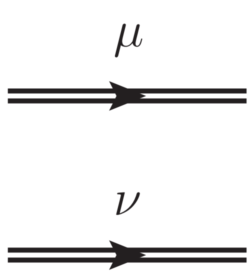
$$= \sum_a$$

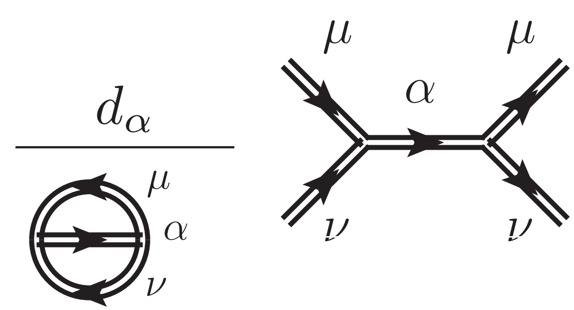


Now there is no trivial color structure, but we can pick any loop...

$A(\alpha_1, \alpha_2, \alpha_3) = \sum_a$ 

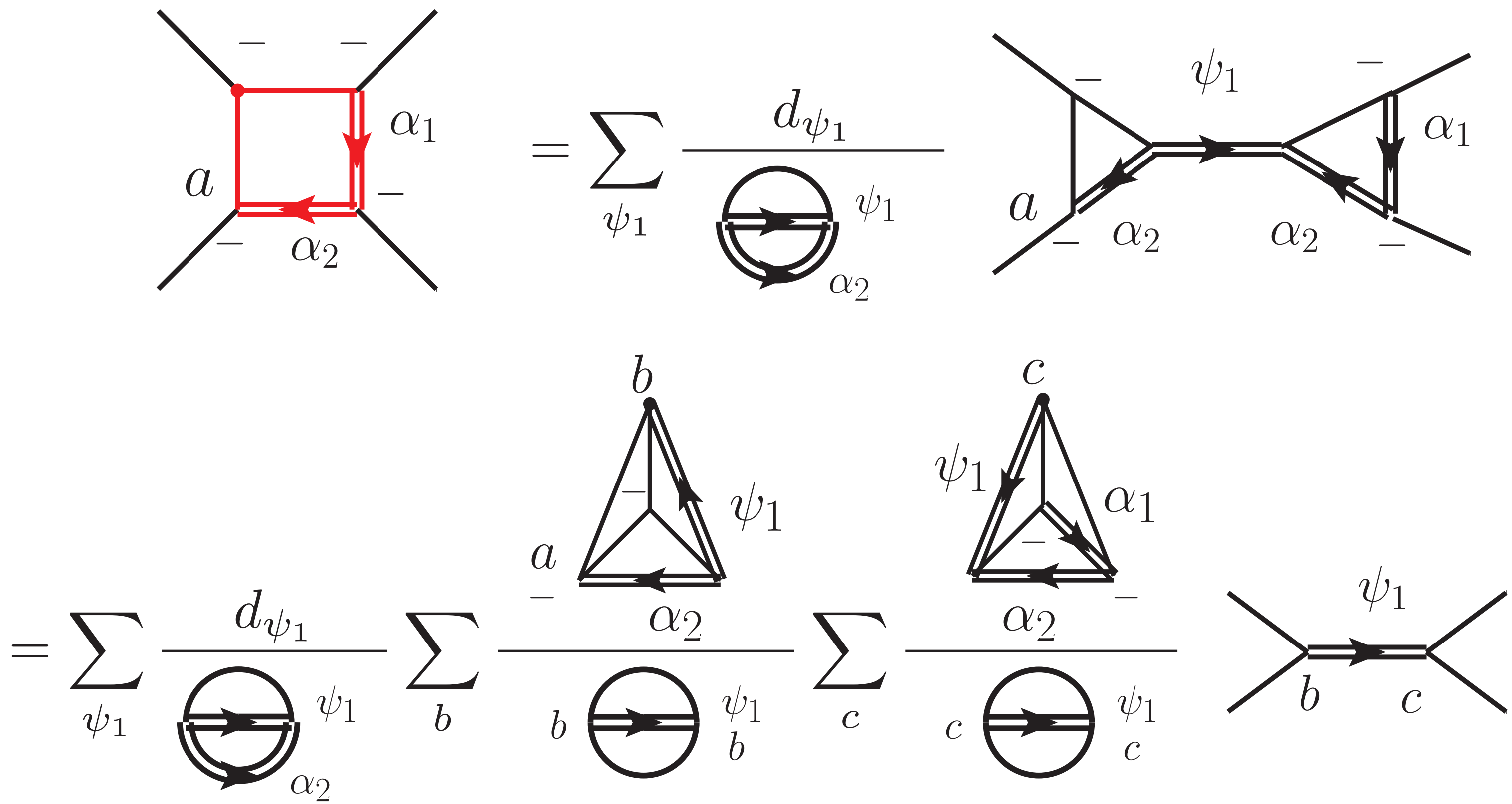
and use the completeness relation



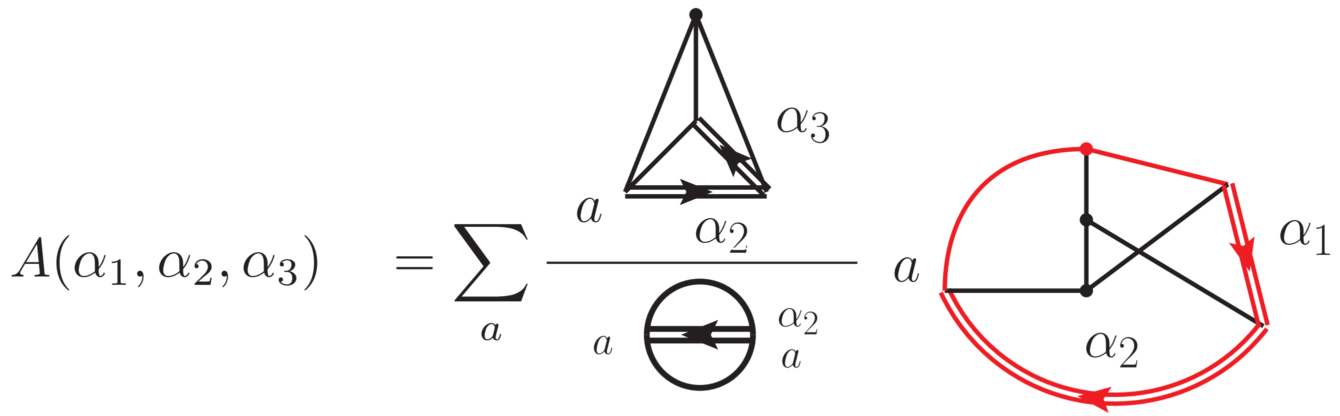
$= \sum_\alpha$ 

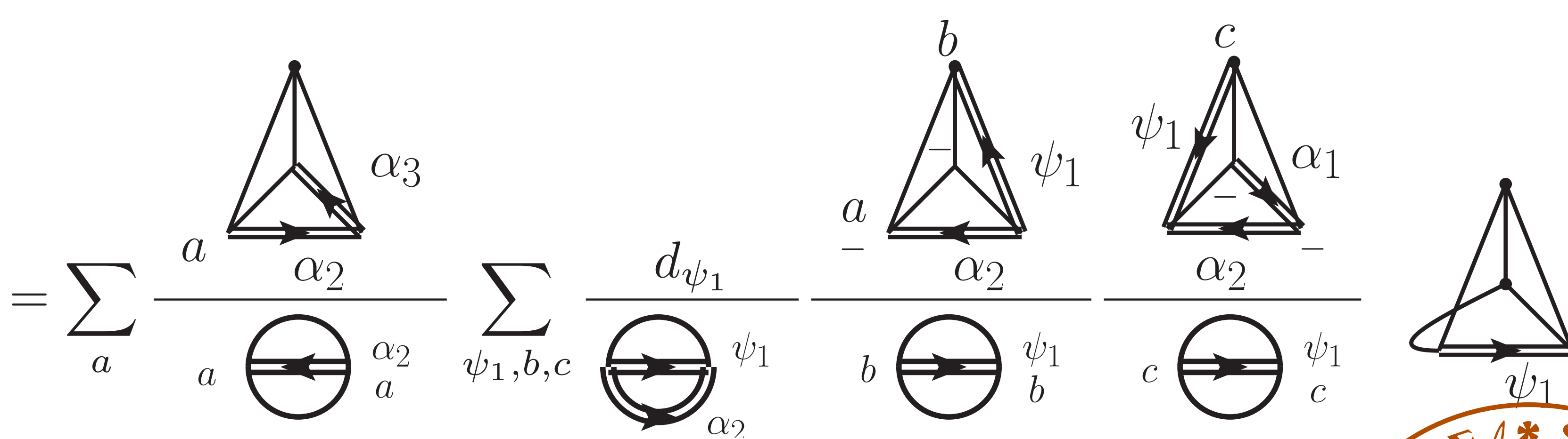
to remove it

Applying the completeness relation and removing vertex corrections:



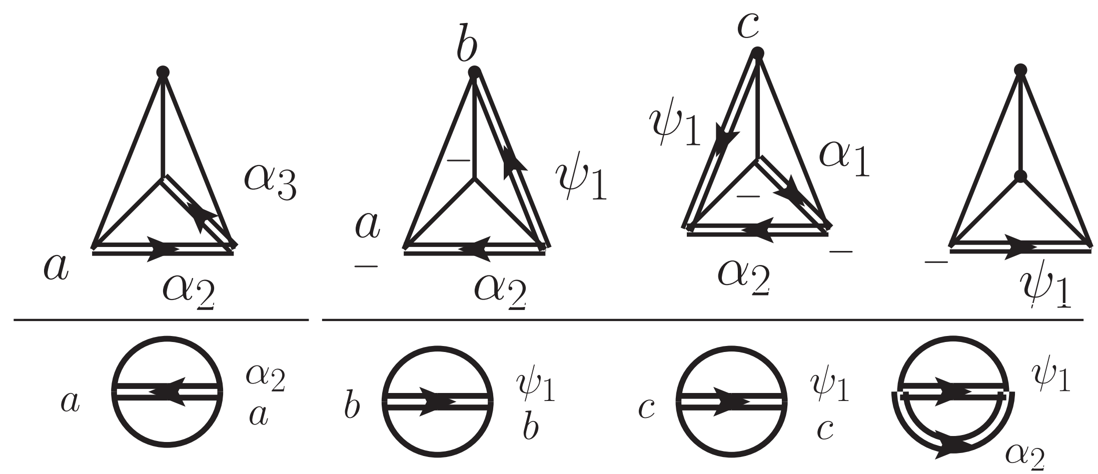
Removing the 4-vertex loop we get:





The final expression is:

$$A(\alpha_1, \alpha_2, \alpha_3) = \sum_{a,\psi_1,b,c} d_{\psi_1}$$

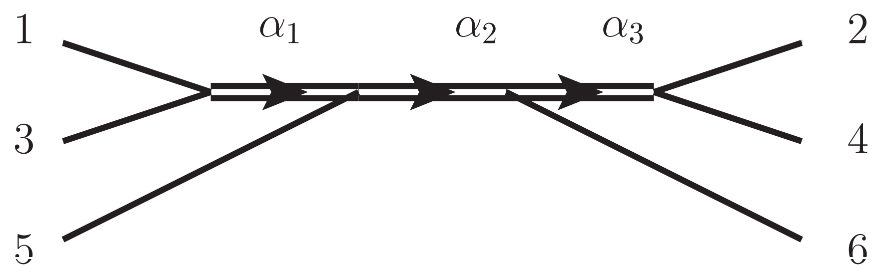


- Knowing the 3j and 6j Wigner coefficients we can immediately write down the scalar product with any basis vector!
- This only has to be done once for each Feynman diagram, and the scalar product with most basis vectors vanishes
- We only need to care about non-zero projections, we could list the non-zero 6j-coefficients
- Each sum over representations contains at most 8 terms for SU(3), at most $N_c^2 - 1$ for SU(N_c)

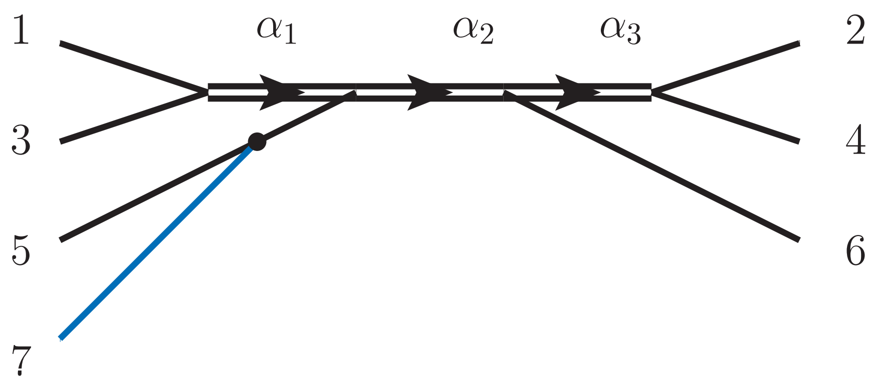
A parton shower perspective

- In a parton shower we start with some amplitude which we can assume that we have decomposed in the multiplet basis

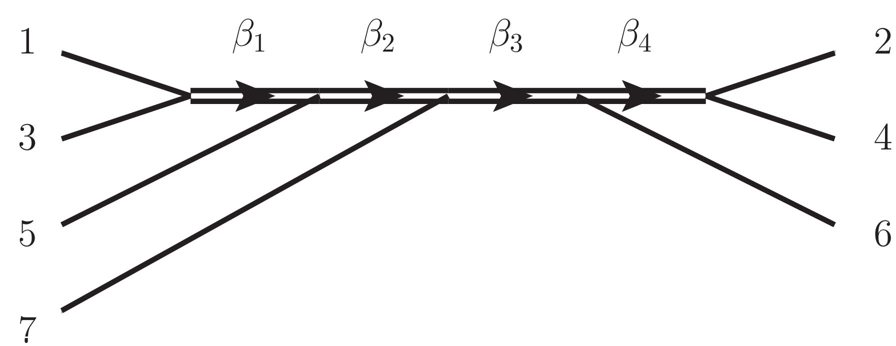
$$\text{Amp} = \sum_{\alpha_1,\alpha_2,\alpha_3} c_{\alpha_1,\alpha_2,\alpha_3}$$



- Knowing the decomposition for $N_g - 1$ gluons, how can we decompose the N_g gluon amplitude?

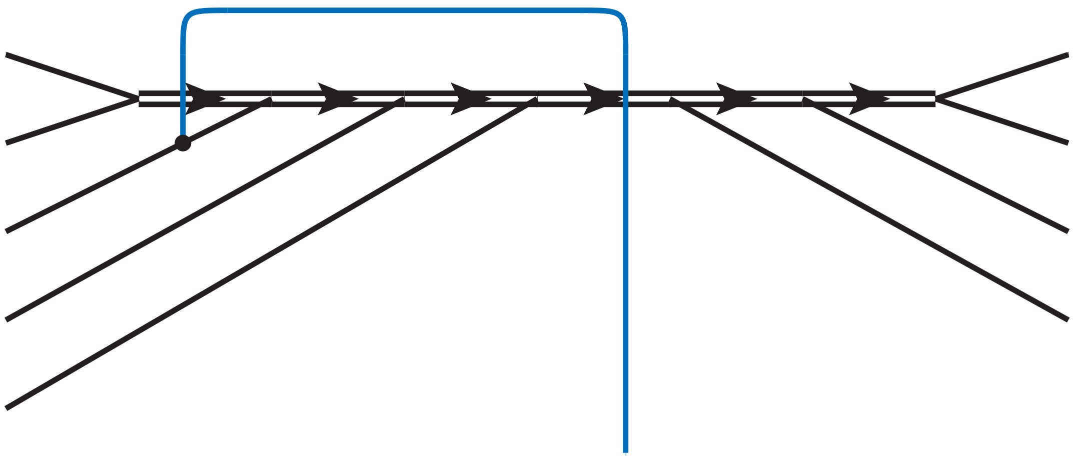


$$= \sum_{\beta_1, \beta_2, \ldots} \tilde{c}_{\beta_1, \beta_2, \ldots}$$

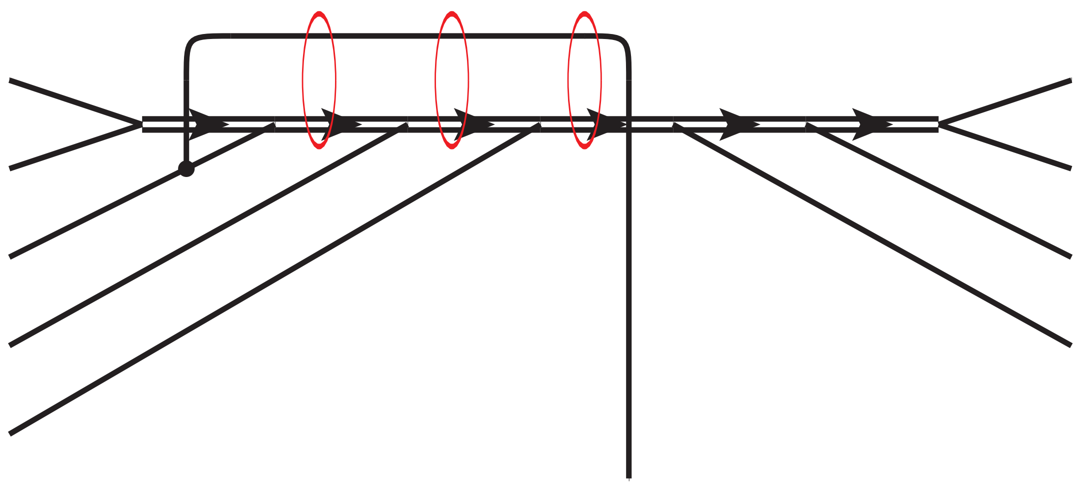


- Scalar products? Too slow!

Let one of the gluons emit a new gluon:

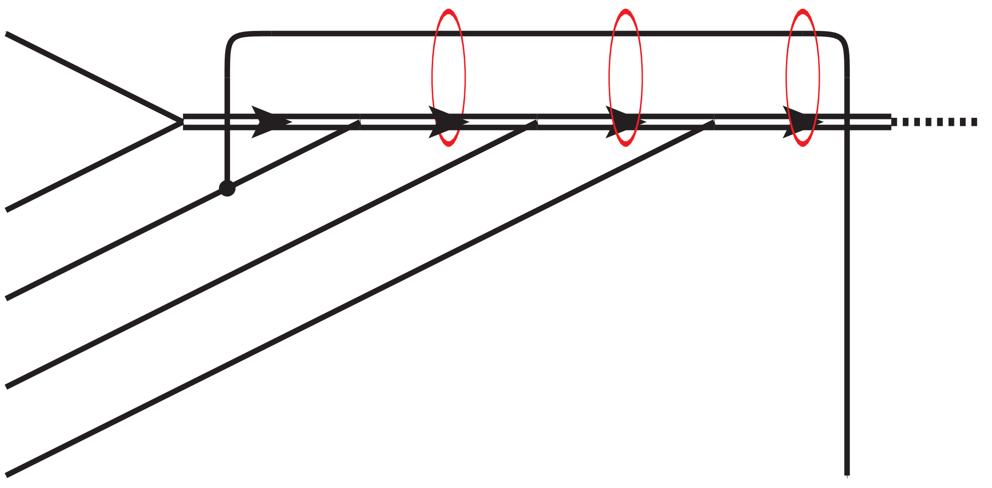


To decompose the affected side, we may insert the completeness relation repeatedly:

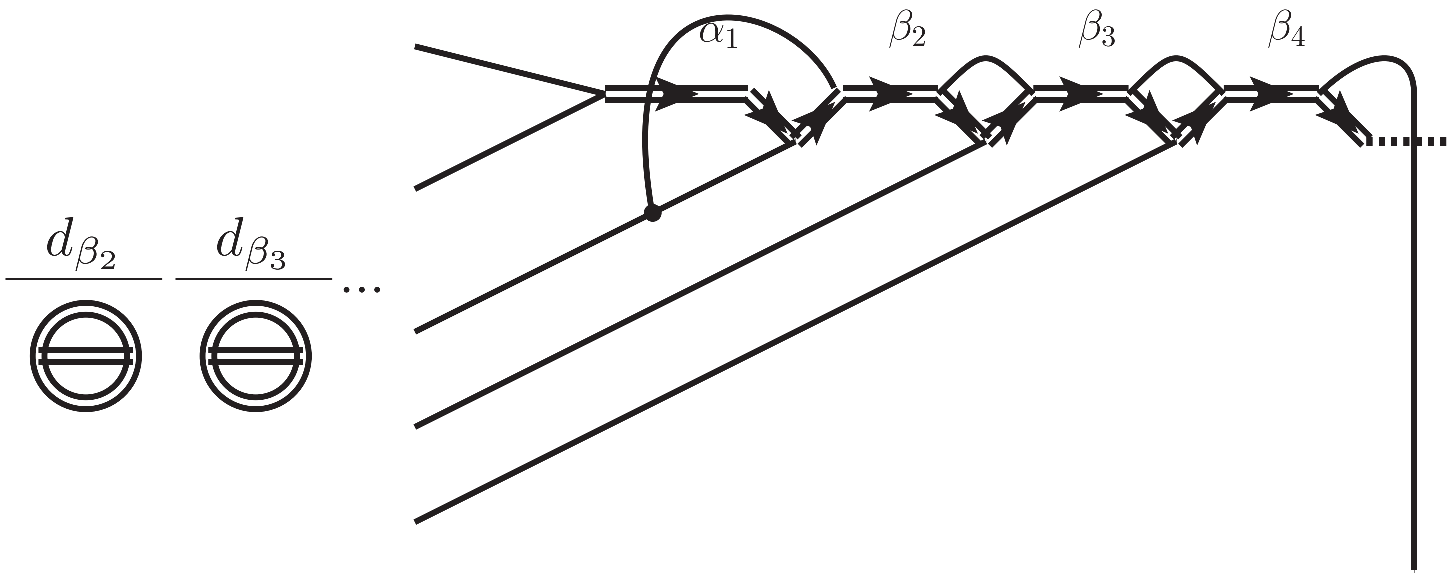


The representations on the other side (here right) don't change

Consider the affected side:

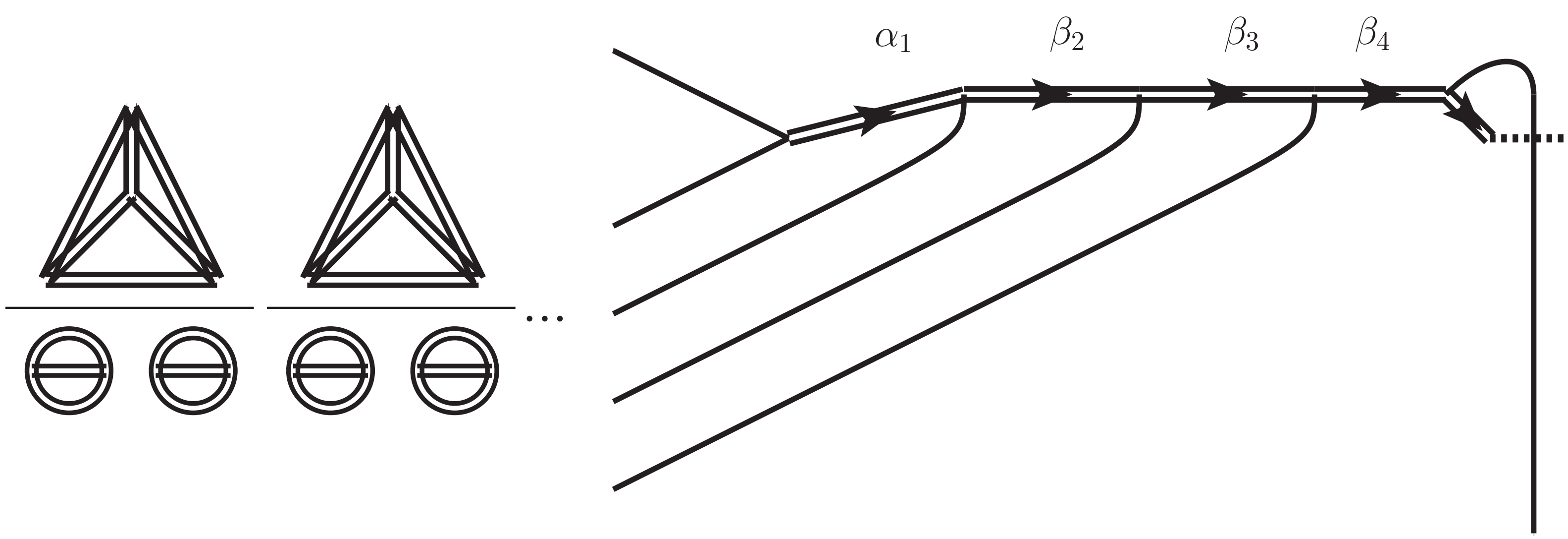


Inserting completeness relations we get a sum of terms of form:



What we have here are just vertex corrections which can be rewritten in terms of 3j and 6j coefficients

Giving us a sum of terms of form:



i.e., knowing the 3j and 6j symbols we can write down the resulting vectors

- By inserting the new gluon "in the middle" in the basis we guarantee that the emitted gluon need never "be transported" across more than ∼ half of the reps
- Typically we get only a small fraction of all basis vectors in the larger basis:

N_g	5→6	6→7	7→8	8→9	9→10
$N_c = 3$	0.094	0.027	0.012	0.0032	0.0014
$N_c \geq N_g$	0.071	0.014	0.0054	0.00092	0.00032

(Y. Du, M.S. & J. Thorén, JHEP 1505 (2015) 119, 1503.00530)

Consider the sum of all terms from all emissions (all emitters and all vectors) and compare to the number encountered when squaring a tree-level amplitude

N_g	Fraction ($N_c = 3$)	All terms ($N_c = 3$)	(# tree vectors)^2 (any N_c)
5→6	0.094	2 184	$(120)^2$
6→7	0.027	16 372	$(720)^2$
7→8	0.012	212 914	$(5\ 040)^2$
8→9	0.0032	1 758 620	$(40\ 320)^2 \sim 10^9$
9→10	0.0014	25 407 328	$(362\ 880)^2 \sim 10^{11}$

Numbers will be somewhat reduced by clever vertex choices, and non-general linear combinations

Loops?

- Tree level color treatment can be treated as in the shower case above: we have some color structure and add a parton
- What about loops?
- Well: Taking a color structure and exchanging a gluon between two legs, corresponds to a linear map in color space from the color basis in question for that number of partons to itself. This can be described by a matrix Color correlator, soft anomalous dimension matrix
- This matrix can be calculated in a way similar to the gluon emission case
- The scaling is not quite as good, but rather comparable to the case of having one more parton, but this is a thumb rule for LO vs. NLO in general

Conclusion

- QCD color structure can — due to confinement — always be dealt with in a purely diagrammatic way, using group invariant quantities
- In this presentation, I have argued that multiplet bases can be used and I have described how to color structure can be treated using group invariants, Wigner 3j and 6j coefficients, which can be calculated once and for all
- In multiplet bases the decomposition step – not the squaring step – is the hard step, but overall, for example in parton showers or recursion, there are fewer terms to keep track of

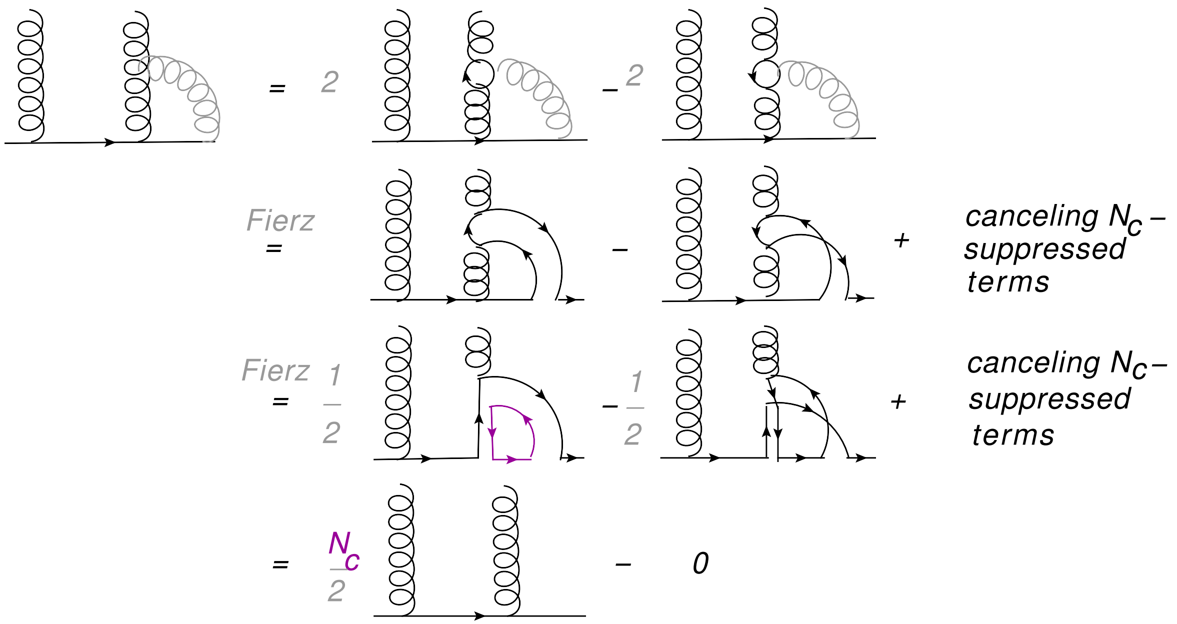
Outlook

- What is needed is the 6js for many partons
- I am confident that high enough multiplicity for the method to be beneficial can be reached
- With present strategies, I am confident that we could go to 7 gluons plus $q\bar{q}$-paris, perhaps to 8 and possibly to 9
- For example, the parton shower that me and Simon worked on would be speeded up by this method
- This could remove the color squaring step from the list of bottle necks
- What is needed is also a general and accessible implementation

Thank you for your attention!

Backup: Gluon exchange

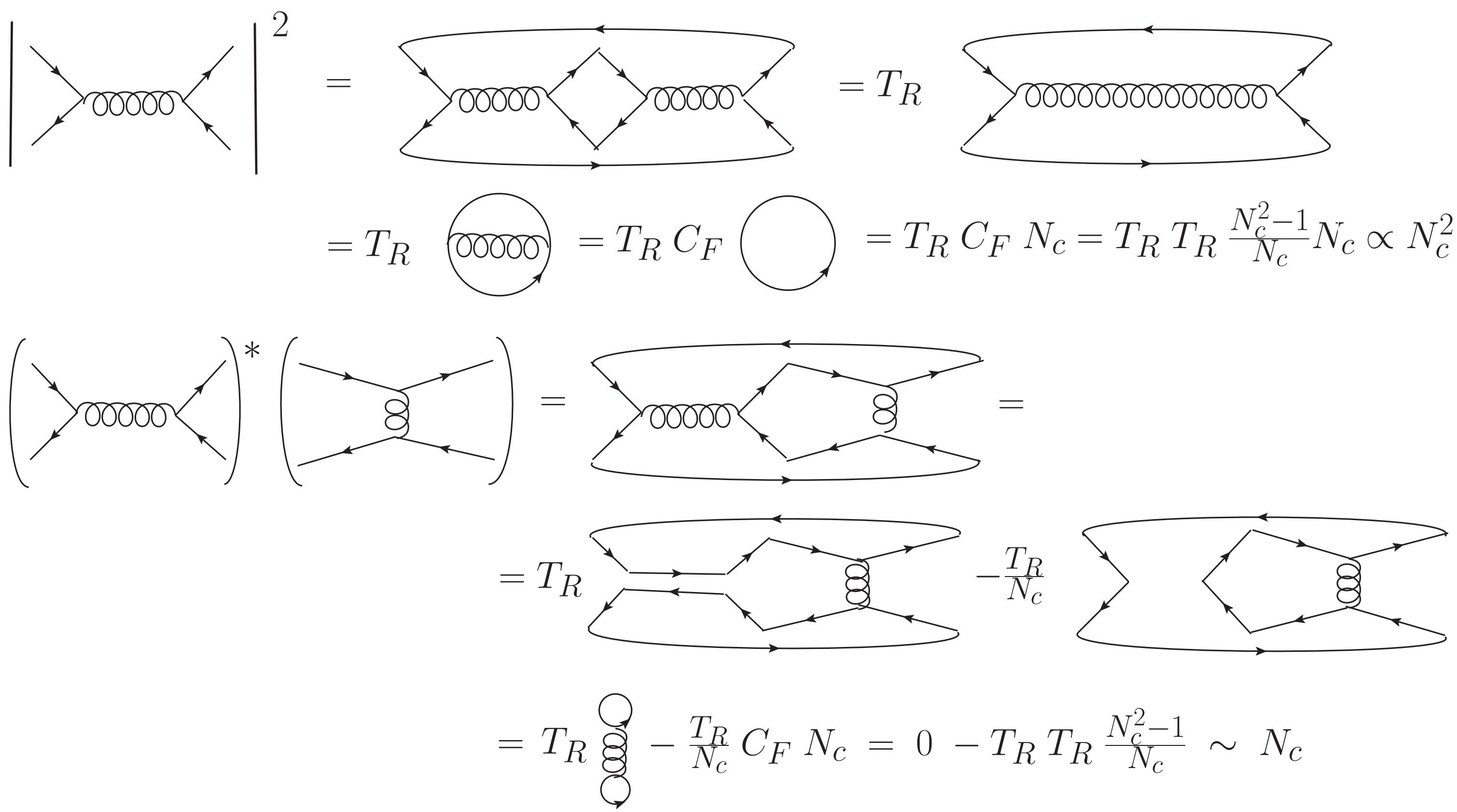
A gluon exchange in this basis "directly" i.e. without using scalar products gives back a linear combination of (at most 4) basis tensors



- N_c-enhancement possible only for near by partons
  $\to$ only "color neighbors" radiate in the $N_c \to \infty$ limit

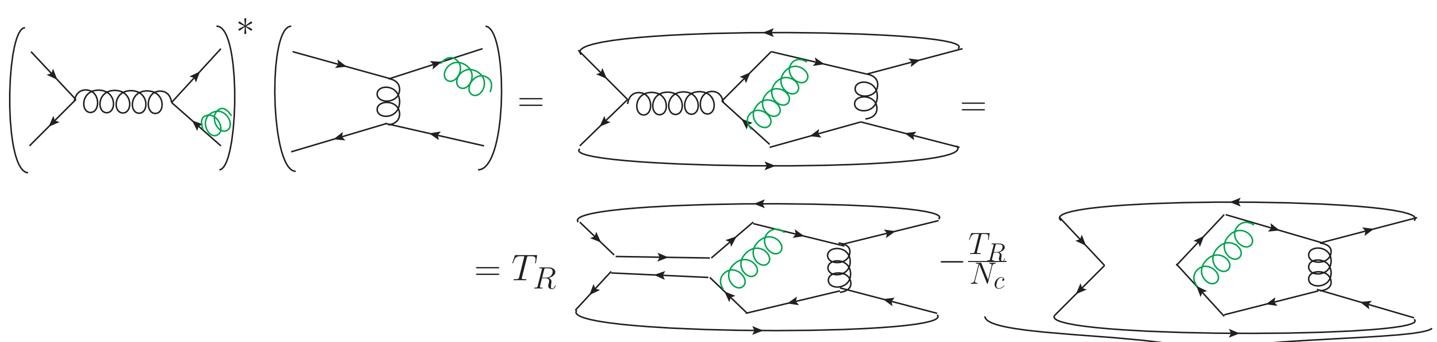
Backup: N_c-suppressed terms

That non-leading color terms are suppressed by $1/N_c^2$, is guaranteed only for same order α_s diagrams with only gluons ('t Hooft 1973)



Backup: N_c-suppressed terms

For a parton shower there may also be terms which only are suppressed by one power of N_c



Is 0 without emission, with $\sim N_c^2$ did not enter in any form, genuine "shower" contribution

Is $\sim N_c$ without emission, with $\sim N_c^2$ "included" in shower, contribution from hard process

The leading N_c contribution scales as N_c^2 before emission and N_c^3 after