Renormalons, R-evolution and Renormalon Sum Rule

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Motivation

Precise determination of the strong coupling α_s provided by investigations of the τ hadronic width

$$R_\tau = \frac{N_C}{2} S_{EW} |V_{ud}|^2 \left[1 + \delta^{(0)} + \delta_{EW} + \sum_{D \geq 2} \delta_{ud}^{(D)} \right].$$

Dominant theoretical uncertainty resides in

- Higher-order perturbative QCD corrections $\rightarrow$ Renormalons
- Different possibilities of resumming the perturbative series:

Fixed-order PT (FOPT) vs. Contour-improved PT (CIPT)

Fixed-order vs. Contour-improved PT

- Central quantity in the analysis of hadronic τ decays: Adler function $D(s) = -s\frac{\mathrm{d}}{\mathrm{d}s}\Pi(s)$,

$$i(p^2 g_{\mu\nu} - p_\mu p_\nu)\Pi(p^2) = \int \mathrm{d}x\, \mathrm{e}^{ipx} \langle 0|\, T\{j_\mu(x) j_\nu^\dagger(0)\}\, |0\rangle\, .$$

- R_τ expressed in terms of the Adler function: $(x = s/M_\tau^2)$

$$R_\tau = -6\pi i \oint_{|x|=1} \frac{\mathrm{d}x}{x}(1-x)^3(1+x)\, D(M_\tau^2 x).$$

- General structure of the Adler function in the chiral limit: $(a_\mu = \alpha_s(\mu)/\pi)$

$$D(s) = \frac{1}{4\pi^2}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k\, c_{n,k} \log^{k-1}\left(\frac{-s}{\mu^2}\right).$$

Fixed-order vs. Contour-improved PT

$$\Rightarrow \delta^{(0)} = \frac{1}{2\pi i} \sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k\, c_{n,k} \oint\limits_{|x|=1} \frac{\mathrm{d}x}{x} (1-x)^3 (1+x) \log^{k-1}\left(\frac{-M_\tau^2 x}{\mu^2}\right)$$

- FOPT: Set $\mu^2 = M_\tau^2$.

$$\delta_{\mathrm{FO}}^{(0)} = \sum_{n=0}^{\infty} a_{M_\tau}^n \sum_{k=1}^{n+1} k\, c_{n,k} J_{k-1}.$$

- CIPT: Set $\mu^2 = -M_\tau^2 x$.

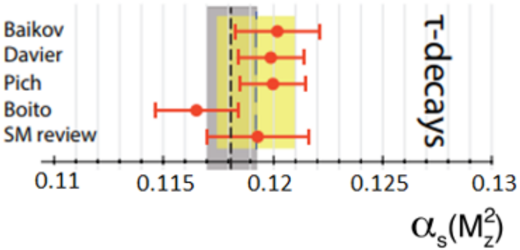
$$\delta_{\mathrm{CI}}^{(0)} = \sum_{n=0}^{\infty} c_{n,1} J_n^a(M_\tau^2)$$

$$J_n^a(M_\tau^2) = \frac{1}{2\pi i} \oint_{|x|=1} \frac{\mathrm{d}x}{x} (1-x)^3 (1+x) \left(\frac{\alpha_s(-M_\tau^2 x)}{\pi}\right)^n.$$

Fixed-order vs. Contour-improved PT

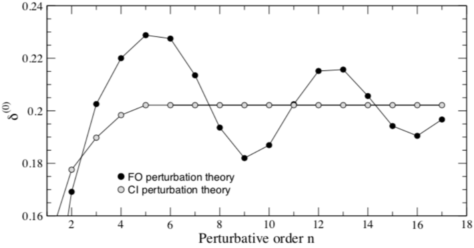
- Current status using corrections up to $\mathcal{O}(\alpha_s^4)$: $(\alpha_s(M_\tau) = 0.32)$

$$\delta_{FO}^{(0)} = 0.1959 \pm 0.006,$$
$$\delta_{CI}^{(0)} = 0.1814 \pm 0.003.$$
$$\Rightarrow \delta_{FO}^{(0)} - \delta_{CI}^{(0)} = 0.0145 >> \Delta\delta^{(0)}$$



1) Truncated Adler Function

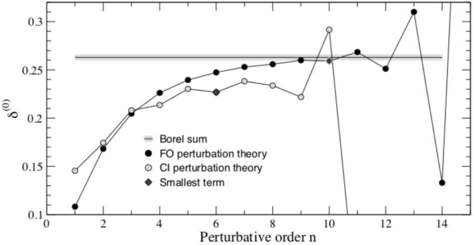
- All unknown coefficients $c_{n,1}$ are set to zero.
- CIPT becomes exact.
- FOPT oscillates around CIPT result.



2) Large-β_0 Approximation

$$\bullet B[D](u) = \frac{32}{3}\,\frac{\mathrm{e}^{-Cu}}{(2-u)} \times \sum_{k=2}^{\infty} \frac{(-1)^k\, k}{[k^2-(1-u)^2]^2}$$

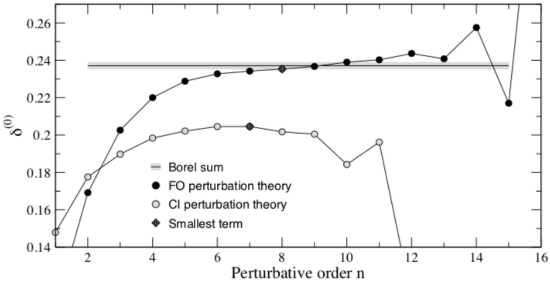
$$\bullet D(\alpha) = \int_0^{\infty} \mathrm{d}u\, \mathrm{e}^{-\frac{4\pi u}{\alpha_s \beta_0}}\, B[D](t)$$



3) Ansatz: Physical Model for the Adler Function

$$B[D](u) = \frac{d_{-1}^{\mathrm{UV}}}{(u+1)^{1+\gamma_1}} + \frac{d_2^{\mathrm{IR}}}{(u-2)^{1+\gamma_2}} + \frac{d_3^{\mathrm{IR}}}{(u-3)^{1+\gamma_3}} + d_0 + d_1 u$$

- Ansatz uses connection between renormalon singularities and operators in the OPE to reproduce the pole structure.
- Not possible to predict the residues.
- FOPT prevails.



Motivation

We want to address the following questions:

1. What can we tell about the ambiguities related to renormalon singularities?
   - How well does the Borel integral quantify this ambiguity?
   - Are there alternatives ways to estimate the size of the ambiguity?

2. Is it possible to improve the form of the Borel transform used in physical models for the Adler function if one does not rely on the large-β_0 approximation?
   - Can we gain more information on single renormalon singularities?

Outline

1. Introduction: Renormalons and Borel Summation
2. *R*-evolution and Renormalon Sum Rule
3. Application: Large-β_0 Approximation
4. Summary & Outlook

- As long as interactions are "weak" in QFTs, perturbation theory allows us to express observables F as series

$$F = \sum_n r_n(\mu)\,\alpha_s^n(\mu).$$

- BUT: These series are usually divergent with $r_n \overset{n\to\infty}{\sim} a^n n!\, n^b Z(\mu)$.
- Working assumption: Perturbative series is asymptotic.
  - Best approximation given when truncating at smallest term.
  - Truncation error typically of the order of this minimal term.
- Important source of divergence → Renormalons.
  - Related to small and large momentum behaviour in loop integrals (IR and UV renormalons).

How can we quantify the large order behaviour of perturbative series?

- In order to sum factorially divergent series Borel summation is especially useful:

  The Borel transform of a series $F = \sum_{n=0}^{\infty} r_n \, \alpha_s^{n+1}$ is defined as

  $$B[F](u) = \sum_{n=0}^{\infty} r_n \frac{u^n}{n!}.$$

  If $B[F](u)$ has no singularities for $u > 0$ and does not increase to rapidly for $u \to \infty$, the Borel integral

  $$\widetilde{F}(\alpha_s) = \int_0^{\infty} \mathrm{d}u \, \mathrm{e}^{-\frac{4\pi u}{\alpha_s \beta_0}} \, B[F](u)$$

  exists and F is called Borel-summable.

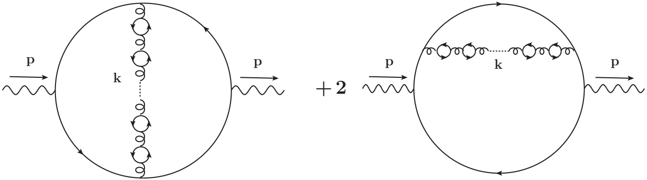
Example

- Consider the Adler function:

$$D(p^2) = 4\pi^2 p^2 \frac{\mathrm{d}\Pi(p^2)}{\mathrm{d}p^2}$$

$$i(p^2 g_{\mu\nu} - p_\mu p_\nu)\Pi(p^2) = \int \mathrm{d}x\, \mathrm{e}^{ipx} \langle 0|\, T\{j_\mu(x) j_\nu^\dagger(0)\}\, |0\rangle\,.$$

- Only exactly computable all order contributions: Bubble chain diagrams of massless quarks.



Example

- Calculation: $(Q^2 = -p^2,\ \mu^2 = Q^2,\ \alpha_s = \alpha_s(Q),\ \hat{k}^2 = -k^2/Q^2)$

$$D \sim \sum_{n=0}^{\infty} \alpha_s \int_0^\infty \frac{\mathrm{d}\hat{k}^2}{\hat{k}^2}\,\omega(\hat{k}^2)\left[\frac{n_f \alpha_s}{6\pi} \ln\left(\hat{k}^2 \mathrm{e}^{-5/3}\right)\right]^n$$

- Large logarithmic enhancement for $\hat{k}^2 >> 1$ and $\hat{k}^2 << 1$.
- Split integral at $\hat{k}^2 = \mathrm{e}^{5/3}$ and perform integration for small and large momenta:

$$\omega(\hat{k}^2) = \frac{3C_F}{2\pi}\hat{k}^4 + \mathcal{O}(\hat{k}^6 \ln \hat{k}^2),$$

$$\omega(\hat{k}^2) = \frac{C_F}{3\pi}\frac{1}{\hat{k}^2}\left(\ln \hat{k}^2 + \frac{5}{6}\right) + \mathcal{O}\left(\frac{\ln \hat{k}^2}{\hat{k}^4}\right)$$

Example

- Result: $\left(\int_0^1 \mathrm{d}x \ln^n(x) = (-1)^n\ \mathrm{n!}\right)$

$$D \sim \sum_{n=0}^{\infty} \alpha_s^{n+1} \left[\mathrm{e}^{10/3} \left(-\frac{n_f}{12\pi} \right)^n n! + \mathrm{e}^{-5/3} \left(\frac{n_f}{6\pi} \right)^n n! \left(n + \frac{11}{6} \right) + \ldots \right]$$

- Leading IR renormalon and leading UV renormalon behaviour.
- Drawback of bubble chain diagrams: n_f-terms only give small contribution of the complete perturbative coefficients.
- Take step beyond the quark bubble diagrams → Naive Non-Abelianization (NNA)

$$n_f \to -\frac{3}{2} \left(\frac{11}{3} C_A - \frac{2}{3} n_f \right) = -\frac{3}{2} \beta_0$$

⇒ Large-β_0 approximation.

- Takes some non-Abelian contributions into account.

Example

$$D \sim \sum_{n=0}^{\infty} \left(\frac{\alpha_s \beta_0}{4\pi} \right)^{n+1} \left[\mathrm{e}^{10/3} \left(\frac{1}{2} \right)^n n! + \mathrm{e}^{-5/3} (-1)^n n! \left(n + \frac{11}{6} \right) + \dots \right]$$

- Perform Borel transformation in variable u:

$$B[D](u) \sim \frac{2\,\mathrm{e}^{10/3}}{2-u} + \mathrm{e}^{-5/3} \left(\frac{1}{(1+u)^2} + \frac{5}{6} \frac{1}{(1+u)} \right) + \dots$$

- Information on divergent behaviour of $D(\alpha_s)$ encoded in the singularities of its Borel transform $\rightarrow$ "Renormalon poles".
  - UV-poles $\rightarrow$ sign-alternating factorial divergence.
  - IR-poles $\rightarrow$ fixed-sign factorial divergence.
- Poles closest to the origin $u = 0$ of the Borel plane dominate high order behaviour.

Example

- General structure of Borel transform for a renormalon pole at $u = p/2$:

$$B[D](u) \sim \sum_{l=0}^{\infty} g_l^{(p)} \frac{\Gamma(1+\hat{b}_1 p+\delta-l)}{(u-p/2)^{1+\hat{b}_1 p+\delta-l}}$$

$$\frac{1}{(u-p/2)^{1+\delta}} \sim \left(\frac{\mu}{Q}\right)^p \sum_n \alpha_s^n(\mu) \left[\Gamma(1+n+\delta)\left(\frac{2\beta_0}{p}\right)^{n+\delta} + \dots\right]$$

$$\hat{b}_1 = \beta_1/(2\beta_0), \qquad g_l^{(p)} = g_l^{(p)}(p, \{\beta_n\})$$

- Deficiencies of the large-β_0 approximation:
  - Analytic structure of Borel transform and strength of its singularities not reproduced correctly.
  - Does not yield all singularities $\rightarrow$ e.g. instantons.
- We still use large-β_0 approximation to study main features related to renormalon poles.

Ambiguity of the Borel Integral

- Evaluate the Borel integral

$$\widetilde{D}(\alpha_s) = \int_0^\infty \mathrm{d}u \; \mathrm{e}^{-\frac{4\pi u}{\beta_0 \alpha_s}} \, B[D](u).$$

- $B[D](u)$ has singularities for $u > 0$ (IR renormalons) $\rightarrow$ To regulate the integral we can move the contour above or below the singularities in the complex plane.
- Ambiguity of the Borel integral!
- How well does the Borel integral quantify the size of the ambiguity?
  - Traditional ansatz: $\Delta D(\alpha_s) = \mathrm{Im}\left[\int_0^\infty \mathrm{d}u \; \mathrm{e}^{-\frac{4\pi u}{\beta_0 \alpha_s}} \, B[D](u)\right]$
  - Alternative ways to quantify the ambiguity?

Ambiguity of the Borel Integral

- Estimate ambiguity of the leading IR renormalon: $B[D](u) \sim \frac{1}{u-2}$

$$\Delta D(Q^2) \sim \oint \mathrm{d}u \; \mathrm{e}^{-\frac{4\pi u}{\beta_0 \alpha_s(Q)}} \frac{1}{u-2} \propto \left(\mathrm{e}^{-\frac{2\pi}{\beta_0 \alpha_s(Q)}} \right)^4 \sim \frac{\Lambda_{\mathrm{QCD}}^4}{Q^4}$$

- Ambiguity is related to non-perturbative power corrections.
- Important connection:

  IR pole at $u > 0 \leftrightarrow$ Addition of higher dimensional terms in OPE
- In general: IR renormalon singularity at $u = p/2$ is related to non-pert. matrix element $\langle 0 | \mathcal{O} | 0 \rangle \sim \Lambda_{\mathrm{QCD}}^p$.

Operator Product Expansion

- Consider OPE for the Adler function:

$$D(Q) = C_0(Q,\mu) + C_{GG}(Q,\mu)\frac{\langle 0|\, G^A_{\mu\nu} G^{A,\mu\nu}\,|0\rangle}{Q^4} + \mathcal{O}(1/Q^6)$$

- Gluon condensate $\langle 0|\, G^A_{\mu\nu} G^{A,\mu\nu}\,|0\rangle \sim \Lambda^4_{\mathrm{QCD}}$ cancels ambiguity caused by IR renormalon at $u = 2$.
  $\rightarrow$ Scaling behaviour of power corrections can be inferred from IR renormalon poles
- Strength of renormalon poles related to anomalous dimension of operators in OPE.

Operator Product Expansion - General Case

- OPE of a dimensionless observable σ in the $\overline{\text{MS}}$ scheme:

$$\sigma(Q) = \hat{C}_0(Q)\,\mathbb{1} + \overline{C}_1(Q,\mu)\frac{\overline{\theta}_1(\mu)}{Q^p} + \dots$$

$$= \hat{C}_0(Q)\mathbb{1} + \hat{C}_1(Q)\frac{\hat{\theta}_1}{Q^p} + \dots \text{(RGI-OPE)}$$

- Perturbative coefficients in $\hat{C}_0$ factorially enhanced,
  $\rightarrow$ Recall: $\hat{C}_0 \sim (\mu/Q)^p \sum_n \alpha_s^n(\mu)\, n!\, (2\beta_0/p)^n$ for large n.
- Cancellations between $\hat{C}_0(Q)$ and $\hat{\theta}_1$ at large orders.
- Cancellation between $\hat{C}_0$ and $\hat{\theta}_1$ improved by switching to the MSR-scheme that subtracts renormalon contributions at new scale R:

$$C_0(Q,R) = \hat{C}_0(Q) - \left(\frac{R}{Q}\right)^p \delta C_0(R)$$

$$\delta C_0(R) = \sum_n a_n(R)\alpha_s(R) \sim \left(\frac{\mu}{R}\right)^p \sum_n \alpha_s^n(\mu)\, n!\left(\frac{2\beta_0}{p}\right)^n$$

- OPE in the MSR-scheme becomes:

$$\sigma(Q) = C_0(Q,R)\mathbb{1} + \hat{C}_1(Q)\frac{\theta_1(R)}{Q^p} + \ldots,$$
$$\theta_1(R) = \hat{\theta}_1 + R^p \delta C_0(R)$$

- What are appropriate values for R in the OPE?
  - $\theta_1(R)$ requires $R \sim \Lambda_{\mathrm{QCD}}$.
  - $C_0(Q,R)$ requires $R \sim Q$.
- No choice for R avoids large logs in both, $C_0(Q,R)$ and $\theta_1(R)$.
  $\to$ Solution: RGE for the scale R.

$$\Rightarrow C_0(Q, \Lambda \gtrsim \Lambda_{\mathrm{QCD}}) = C_0(Q,Q) + \int_\Lambda^Q \mathrm{d}\ln R \frac{\mathrm{d}C_0(Q,R)}{\mathrm{d}\ln R}$$

$\to$ R-evolution.

R-evolution

- R-evolution equation:

$$\frac{\mathrm{d}\, C_0(Q,R)}{\mathrm{d}\ln R} = -\frac{1}{Q^p}\frac{\mathrm{d}}{\mathrm{d}\ln R}\left(R^P \delta C_0(R)\right) = -\left(\frac{R}{Q}\right)^p \sum_{n=0}^{\infty} \gamma_n^R \left(\frac{\alpha_s(R)}{4\pi}\right)^{n+1}$$

$$\Rightarrow C_0(Q,R_1) - C_0(Q,R_0) = -\frac{1}{Q^p}\sum_{n=0}^{\infty}\gamma_n^R \int_{R_0}^{R_1} \mathrm{d}\ln R\; R^p \left(\frac{\alpha_s(R)}{4\pi}\right)^{n+1}$$

- Sums systematically asymptotic renormalon series and large logs to all orders.
- Free of renormalon ambiguities.
  (Recall: Ambiguity $\sim \mathcal{O}(\Lambda_{\mathrm{QCD}}^p)$, independent of R)
- R-evolution equation takes higher power IR sensitivities into account.
  (Common RGEs only have logarithmic scale dependence.)

R-evolution: Connection to Borel integral

- Consider large-β_0 approximation $\rightarrow$ R-evolution series collapses exactly to a single term:

$$\left[C_0(Q,R_1) - C_0(Q,R_0)\right]_{LL} = -\gamma_0^R \int_{R_0}^{R_1} \mathrm{d}R \, \frac{R^{p-1}}{Q^p} \left(\frac{\alpha_s(R)}{4\pi}\right)$$

- Change of variables: $t_R = -2\pi/(\beta_0 \alpha_s(R))$ and $u = -p\,(t/t_R - 1)/2$

$$\left[C_0(Q,R_1) - C_0(Q,R_0)\right]_{LL} = \int_0^\infty \mathrm{d}u \,[B(R_1,u) - B(R_0,u)]\, \mathrm{e}^{-\frac{4\pi u}{\beta_0 \alpha_s}}$$

$$B(R,u) \sim \left(\frac{R}{Q}\right)^p \frac{a_0}{u - p/2}$$

- Renormalon ambiguities cancel in difference of the Borel functions.
- 1st Application of R-evolution: Estimation of renormalon ambiguities.

2^{nd} Application of R-evolution - Renormalon Sum Rule

- Solution of R-evolution equation yields analytic expression for the normalization of singular terms in the Borel transform.
- Derivation:

$$R\,\frac{\mathrm{d}}{\mathrm{d}R}\,C_0(Q,R) = -\left(\frac{R}{Q}\right)^p \gamma^R[\alpha_s(R)] = -\left(\frac{R}{Q}\right)^p \sum_{n=0}^{\infty} \gamma_n^R \left(\frac{\alpha_s(R)}{4\pi}\right)^{n+1+\delta}$$

$$\Rightarrow C_0(Q,R) - \hat{C}_0(Q) = -\int_0^R \mathrm{d}\ln\bar{R}\,\frac{\bar{R}^p}{Q^p}\,\gamma^{\bar{R}}[\alpha_s(\bar{R})]$$

$$= \left(\frac{\Lambda_{\mathrm{QCD}}}{Q}\right)^p \sum_{k=0}^{\infty} S_k^{(p)}\,\frac{\mathrm{e}^{i\pi(\hat{b}_1 p + k)}}{p^{-\hat{b}_1 p - \delta}}\,\Gamma(-\hat{b}_1 p - k - \delta, p\,t_R)$$

$$t_R = -2\pi/(\beta_0\alpha_s(R)), \qquad \hat{b}_1 = \beta_1/(2\beta_0), \qquad S_k^{(p)} = S_k^{(p)}(a_n, \beta_n)$$

Renormalon Sum Rule

- Algebraic manipulations:
  - Asymptotic expansion of incomplete Gamma functions in $\alpha_s(R)$.
  - Perform Borel transform.
  - Use identities for hypergeometric functions.
- Leads to:

$$B[C_0(Q,R) - \hat{C}_0(Q)](u) = -2\left(\frac{R}{Q}\right)^p \left[P_{p/2} \sum_{l=0}^{\infty} g_l^{(p)} \frac{\Gamma(1+\hat{b}_1 p+\delta-l)}{(p-2u)^{1+\hat{b}_1 p+\delta-l}} + \ldots\right]$$

$$P_{p/2} = \sum_{k=0}^{\infty} \frac{S_k^{(p)}\, p^{k+\hat{b}_1 p+\delta}}{\Gamma(1+k+\hat{b}_1 p+\delta)}.$$

- Analytic expression for normalization $P_{p/2}$ of singular contributions that quantify the $\mathcal{O}(\Lambda_{\rm QCD}^p)$ renormalon ambiguity → Renormalon Sum Rule.

Renormalon Sum Rule

Sum Rule as a probe for renormalon ambiguities:

$$P_{p/2} = \sum_{k=0}^{\infty} \frac{S_k^{(p)}\, p^{k+\hat{b}_1 p+\delta}}{\Gamma(1+k+\hat{b}_1 p+\delta)}$$

- Apply sum rule $P_{p/2}$ to any perturbative series as a probe for $\mathcal{O}(\Lambda_{\text{QCD}}^p)$ renormalon ambiguities.
  - $P_{p/2} \approx 0$ or $P_{p/2} \neq 0$.
- Application: Use Sum Rule to gain more information on the coefficients of renormalon poles in physical models for the Adler function.

  $\rightarrow$ Study Adler function in the large-β_0 approximation to illustrate the applications of the R-evolution.

- Borel transform of the Adler function in large-β_0: [Broadhurst '93]

$$B[D](u)\Big|_{\mu^2=-p^2} = \frac{4}{\beta_0}\,\frac{32}{3}\,\mathrm{e}^{-Cu}\,\frac{1}{2-u}\sum_{k=2}^{\infty}\frac{(-1)^k\,k}{(k^2-(1-u)^2)^2}$$

- C: Scheme-dependent constant ($C=-5/3$ in $\overline{\mathrm{MS}}$).
- Pole structure:
  - Simple pole at $u=2$.
  - Simple and double poles at integer u (except for $u=0,1$).
- Taylor expanding $B[D](u)$ in u and performing the Borel integral term by term gives:

$$D(Q)=\sum_{n=0}^{\infty} a_n\,\alpha_s^{n+1}(Q),$$

$$a_n = a_n^{2,(1)} + \sum_{k_{\mathrm{IR}}=3}^{\infty}\left[a_n^{k_{\mathrm{IR}},(1)} + a_n^{k_{\mathrm{IR}},(2)}\right] + \sum_{k_{\mathrm{UV}}=-\infty}^{-1}\left[a_n^{k_{\mathrm{UV}},(1)} + a_n^{k_{\mathrm{UV}},(2)}\right]$$

$$a_n^{k,(1)} \sim n!/k^{(n+1)}, \qquad a_n^{k,(2)} \sim (n+1)!/k^{(n+2)}$$

Application: Large-β_0 Approximation

R-evolution for the Adler function:

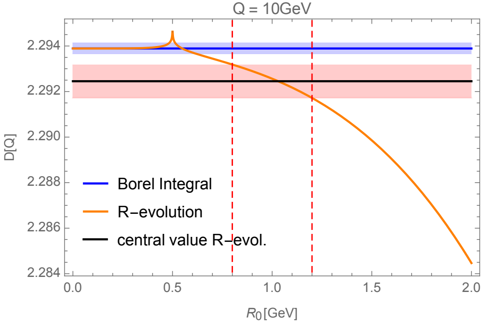
- Consider contribution of a simple IR renormalon pole at $u = p/2 > 0$:

$$D^{(p/2)}(Q, R_0) = \frac{1}{Q^p} \int_{R_0 \gtrsim \Lambda_{\rm QCD}}^{Q} \mathrm{d} \ln R \, R^p \, \gamma^R_{p/2}[\alpha_s(R)]$$

- In large-β_0: $\gamma^R[\alpha_s(R)]$ reduces exactly to a single term.

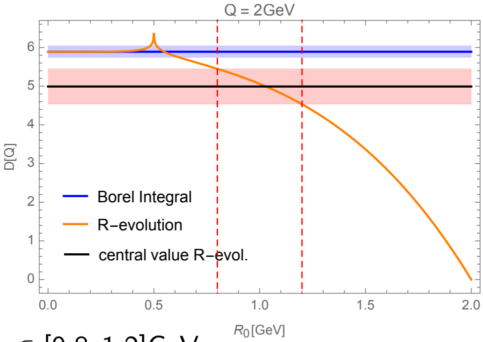
$$\Rightarrow D^{(p/2)}(Q, R_0) \sim \mathrm{e}^{-\frac{2\pi p}{\beta_0 \alpha_s(Q)}} \left[\Gamma\left(0, -\frac{2\pi p}{\beta_0 \alpha_s(R_0)}\right) - \Gamma\left(0, -\frac{2\pi p}{\beta_0 \alpha_s(Q)}\right) \right]$$

Example: $u = 2$ **renormalon** $(Q = 10\,\text{GeV})$



- Variation: $R_0 \in [0.8, 1.2]$GeV.
- Central values:
  - $\sigma_B = 2.2939 \pm 0.0002$.
  - $\sigma_R = 2.2925 \pm 0.0007$.
- Relative deviation of the R-evolution central value compared to the Borel integral: $\Delta_{\frac{R-B}{B}} \approx 0.06\%$.

Example: $u = 2$ **renormalon** ($Q = 2\,$**GeV**)



- Variation: $R_0 \in [0.8, 1.2]$GeV.
- Central values:
  - $\sigma_B = 5.89 \pm 0.15$.
  - $\sigma_R = 4.99 \pm 0.46$.
- Relative deviation of the R-evolution central value compared to the Borel integral: $\Delta_{\frac{R-B}{B}} \approx 15\%$.

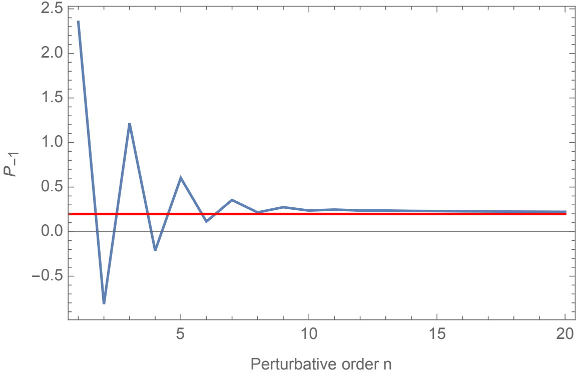
Application: Large-β_0 Approximation

Renormalon Sum Rule in large-β_0 approximation:

$$
\begin{aligned}
B[C_0(Q,R)-\hat{C}_0(Q)](u) &= (-2)^{-\delta}\left(\frac{R}{Q}\right)^p P_{p/2}\,\frac{1}{(u-p/2)^{1+\delta}},\\
P_{p/2} &= \sum_{k=0}^{\infty}\frac{S_k^{(p)}\,p^{k+\delta}}{\Gamma(1+k+\delta)},\\
S_k^{(p)} &= \frac{\gamma_k^R}{(2\beta_0)^{k+1}},\\
\gamma_k^R &= a_k\cdot p - 2\,(n+\delta)\,\beta_0\,a_{k-1}.
\end{aligned}
$$

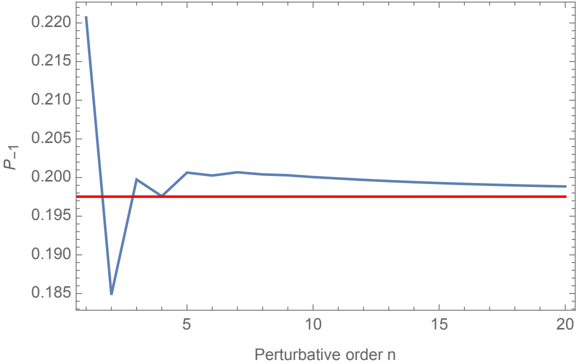
- Different possibilities:
  - $P_{p/2}$ converges to non-zero value $\to \mathcal{O}(\Lambda_{\mathrm{QCD}}^p)$ renormalon.
  - $P_{p/2}\approx 0 \to \mathcal{O}(\Lambda_{\mathrm{QCD}}^p)$ renormalon not present.
  - $P_{p/2}$ diverges $\to \mathcal{O}(\Lambda_{\mathrm{QCD}}^{p'})$ renormalon with $p'<p$ exists.

Sum Rule as a probe for the leading double pole at $u = -1$. (C=-5/3)



- Sum rule converges to the residue of the leading UV double pole at $u = -1$ (red line).

Sum Rule as a probe for the leading double pole at $u = -1$. (C=7/10)



- Convergence of the sum rule can be improved with an appropriate choice of C.

Summary & Outlook

- We discussed FOPT and CIPT.
- We addressed the following issues:
  - Quantification of ambiguities related to IR renormalon poles.
  - Possibilities to improve the form of the Borel transform used in physical models for the Adler function.
- Concept of R-evolution:
  - Provides alternative way to quantify the size of renormalon ambiguities.
  - Renormalon Sum Rule: Estimation of the normalization of singular terms in the Borel transform for a given perturbative series.

Summary & Outlook

Outlook:

- Use Renormalon Sum Rule to gain more information about the pole structure of $B[D](u)$ in physical models for the Adler function.
  - Apply Sum Rule to exactly known coefficients up to $\mathcal{O}(\alpha_s^4)$.
- α_s from τ decays:
  - Impact of our results on the value of α_s determined from hadronic τ decays?