QCD Correlators at Higher Orders

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Begin with the conventional Adler function $D(Q^2)$:

$$4\pi^2 D(Q^2) \equiv 1 + \hat{D}(Q^2) = 1 + a_Q + O(a_Q^2)$$

where $a_Q \equiv \alpha_s(Q^2)/\pi$.

$\hat{D}(Q^2)$ can be expressed through a Borel-integral:

$$\hat{D}(Q^2) \equiv \frac{2}{\beta_1} \int_0^\infty du e^{-2u/(\beta_1 a_Q^C)} B[\hat{D}](u)$$

with the large-$\beta_0$ Borel-transform: ($\beta_1 = 2\pi \beta_0 = 11/2 - N_f/3$) (Beneke 1993; Broadhurst 1993)

$$B[\hat{D}](u) = 8C_F \frac{e^{-Cu}}{(2-u)} \sum_{k=2}^\infty \frac{(-1)^k k}{[k^2 - (1-u)^2]^2}$$
Introducing the scheme-invariant coupling $A_Q$ in large-$\beta_0$:

$$\frac{1}{A_Q} \equiv \frac{1}{a^C_Q} + \frac{\beta_1}{2} C = \frac{1}{a^\text{MS}_Q} + \frac{\beta_1}{2} \hat{C}$$

with $\hat{C} = -\frac{5}{3}$ as well as $\hat{\Lambda} \equiv \Lambda^\text{MS} e^{-\hat{C}/2} \approx 2.3 \Lambda^\text{MS}$.

The Borel-integral can be expressed in scheme-invariant form:

$$\hat{D}(Q^2) \equiv \frac{2}{\beta_1} \int_0^{\infty} du e^{-2u/(\beta_1 A_Q)} \hat{B}[\hat{D}](u)$$

with

$$\hat{B}[\hat{D}](u) = \frac{8C_F}{(2-u)} \sum_{k=2}^{\infty} \frac{(-1)^k k}{[k^2 - (1-u)^2]^2}$$

Renormalon poles at all integer $u = 2, 3, 4, \ldots$ (IR poles)

and $u = -1, -2, -3, \ldots$ (UV poles).
The scalar correlator $\psi(Q^2)$ is defined as:

$$
\psi(Q^2 = -q^2) \equiv i \int dx e^{iqx} \langle \Omega | T \{j(x) j^\dagger(0)\} | \Omega \rangle
$$

where, for example,

$$j(x) = m : \bar{u}(x)s(x) :$$

and $m$ is a generic quark mass.

In large-$\beta_0$, $\psi''(Q^2)$ can be expressed as: (MJ, Miravitllas 2016)

$$
\psi''(Q^2) = \frac{N_c}{8\pi^2} \frac{\hat{m}^2}{Q^2} \left( \frac{\pi A_Q}{2} \right)^2 \left( \gamma_m^{(1)} / \beta_1 \right) \times
$$

$$
\left\{ 1 + \frac{2}{\beta_1} \int_0^\infty du e^{-2u/(\beta_1 A_Q)} \hat{B}[\psi''](u) \right\}
$$
The RGI quark mass $\hat{m}$ in full QCD is defined as:

$$m(\mu) \equiv \hat{m}[\alpha_s(\mu)]\gamma_m^{(1)} / \beta_1 \exp \left\{ \int_0^{a_\mu} da \left[ \frac{\gamma_m(a)}{\beta(a)} - \frac{\gamma_m^{(1)}}{\beta_1 a} \right] \right\}$$

The scheme-invariant Borel-transform is found to be:

(Broadhurst, Kataev, Maxwell 2001)

$$\hat{B}[\Psi''](u) = \frac{3}{2} C_F \left[ (1 - u) G_D(u) - 1 \right]$$

with

$$G_D(u) = \frac{2}{1-u} - \frac{1}{2-u} + \frac{2}{3} \sum_{p=3}^{\infty} \frac{(-1)^p}{(p-u)^2} - \frac{2}{3} \sum_{p=1}^{\infty} \frac{(-1)^p}{(p+u)^2}$$

which explicitly displays the renormalon structure.
Coupling evolution

Scale evolution of $\alpha_s$ is given by the $\beta$-function:

$$-Q \frac{da_Q}{dQ} \equiv \beta(a_Q) = \beta_1 a_Q^2 + \beta_2 a_Q^3 + \beta_3 a_Q^4 + \ldots.$$ 

with $a_Q = \alpha_s / \pi$.

The scale invariant parameter $\Lambda$ can be defined by:

$$\frac{\Lambda}{Q} \equiv e^{-\frac{1}{\beta_1} a_Q} \left[ a_Q \right]^{-\frac{\beta_2}{\beta_1^2}} \exp\left\{ \int_0^{a_Q} \frac{da}{\tilde{\beta}(a)} \right\},$$

where

$$\frac{1}{\tilde{\beta}(a)} \equiv \frac{1}{\beta(a)} - \frac{1}{\beta_1 a^2} + \frac{\beta_2}{\beta_1^2 a}$$

is free of singularities as $a \to 0$. 
However, $\Lambda$ depends on the renormalisation scheme.

$$a' \equiv a + c_1 a^2 + c_2 a^3 + c_3 a^4 + \ldots .$$

Then, $\Lambda$ transforms as: \cite{Celmaster, Gonsalves 1979}

$$\Lambda' = \Lambda e^{c_1/\beta_1}.$$ 

This suggests to define a “novel” “C-scheme” coupling $\hat{a}^C_Q$:

$$\frac{1}{\hat{a}^C_Q} + \frac{\beta_2}{\beta_1} \ln \hat{a}^C_Q - \frac{\beta_1}{2} C \equiv \beta_1 \ln \frac{Q}{\Lambda}$$

$$= \frac{1}{a_Q} + \frac{\beta_2}{\beta_1} \ln a_Q - \beta_1 \int_0^{a_Q} \frac{da}{\beta(a)}$$

\cite{Boito, MJ, Miravitllas 2016}
The $\beta$-function of $\hat{a}_Q^C$ reads simply:  
(Brown, Yaffe, Zhai 1992)

$$
- Q \frac{d\hat{a}_Q^C}{dQ} \equiv \beta(\hat{a}_Q^C) = \frac{\beta_1 (\hat{a}_Q^C)^2}{1 - \frac{\beta_2}{\beta_1} \hat{a}_Q^C} = -2 \frac{d\hat{a}_Q^C}{dC}
$$

Relating a general coupling $a_Q$ and $\bar{a}_Q \equiv \hat{a}_Q^{C=0}$ reads:

$$
a_Q = \bar{a}_Q + \left( \frac{\beta_3}{\beta_1} - \frac{\beta_2^2}{\beta_1^2} \right) \bar{a}_Q^3 + \left( \frac{\beta_4}{2\beta_1} - \frac{\beta_2^3}{2\beta_1^3} \right) \bar{a}_Q^4
$$

$$
+ \left( \frac{\beta_5}{3\beta_1} - \frac{\beta_2\beta_4}{6\beta_1^2} + \frac{5\beta_2^2}{3\beta_1^2} - \frac{3\beta_2^2\beta_3}{\beta_1^3} + \frac{7\beta_2^4}{6\beta_1^4} \right) \bar{a}_Q^5 + O(\bar{a}_Q^6)
$$

The coupling $\hat{a}_Q^C$ at arbitrary $C$ is obtained from $\bar{a}_Q$ via:

$$
\bar{a}_Q = \hat{a}_Q^C + \frac{\beta_1}{2} C (\hat{a}_Q^C)^2 + \left( \frac{\beta_2}{2} C + \frac{\beta_1^2}{4} C^2 \right) (\hat{a}_Q^C)^3
$$

$$
+ \left( \frac{\beta_2^2}{2\beta_1} C + \frac{5\beta_1\beta_2}{8} C^2 + \frac{\beta_1^3}{8} C^3 \right) (\hat{a}_Q^C)^4 + O((\hat{a}_Q^C)^5)
$$
\[ \hat{a}(M_\tau) \] as a function of \( C \) for \( \alpha_s(M_\tau) = 0.316(10) \).
C-scheme quark mass

Scheme dependence of the C-scheme quark mass:

\[
\frac{1}{m_Q^C} \frac{d m_Q^C}{d C} = \frac{d \hat{a}_Q^C}{d C} \frac{d Q}{\hat{a}_Q^C} \frac{1}{m_Q^C} \frac{d m_Q^C}{d Q} = -\frac{1}{2} \hat{\gamma}_m(\hat{a}_Q^C)
\]

RGI quark mass \( \hat{m} \) also scheme invariant:

\[
\hat{m} \equiv m_Q^C \left[ \hat{\alpha}_s(Q) \right]^{-\gamma_m(1)/\beta_1} \exp \left\{ \int_0^{\hat{a}_Q^C} d\hat{a} \left[ \frac{\gamma_m(1)}{\beta_1 \hat{a}} - \hat{\gamma}_m(\hat{a}) \right] \right\}
\]

Normalisation condition for \( m_Q^C \):

\[
m_{\overline{\text{MS}}} = m_{Q}^{C=0} \equiv \bar{m}_Q
\]
Adler function (in full QCD)

\[ 4\pi^2 D(a_Q) - 1 \equiv \hat{D}(a_Q) = \sum_{n=1}^{\infty} c_{n,1} a_Q^n \]

\[ = a_Q + 1.640 a_Q^2 + 6.371 a_Q^3 + 49.08 a_Q^4 + \ldots \]

Expressed in terms of the coupling \( \bar{a}_Q \):

\[ \hat{D}(a_Q) = \sum_{n=1}^{\infty} \bar{c}_{n,1} \bar{a}_Q^n \]

\[ = \bar{a}_Q + 1.640 \bar{a}_Q^2 + 7.682 \bar{a}_Q^3 + 61.06 \bar{a}_Q^4 + \ldots \]

Analytically, the coefficient \( \bar{c}_{4,1} \) is given by:

(Baikov, Chetyrkin, Kühn 2008)

\[ \bar{c}_{4,1} = \frac{357259199}{93312} - \frac{1713103}{432} \zeta_3 + \frac{4185}{8} \zeta_3^2 + \frac{34165}{96} \zeta_5 - \frac{1995}{16} \zeta_7 \]
Scalar correlator

(in full QCD)

\[ \psi''(Q^2) = \frac{N_c}{8\pi^2} \frac{m_Q^2}{Q^2} \left\{ 1 + \sum_{n=1}^{\infty} d''_{n,1} a_Q^n \right\} \]

Expressed in terms of the RGI quark mass \( \hat{m} \):

\[ \psi''(Q^2) = \frac{N_c}{8\pi^2} \frac{\hat{m}^2}{Q^2} [\alpha_s(Q)]^{2\gamma_m^{(1)}}/\beta_1 \left\{ 1 + \sum_{n=1}^{\infty} r_n a_Q^n \right\} \]

Expressed in terms of the coupling \( \bar{a}_Q \):

\[ \psi''(Q^2) = \frac{N_c}{8\pi^2} \frac{\hat{m}^2}{Q^2} [\bar{\alpha}_s(Q)]^{2\gamma_m^{(1)}}/\beta_1 \left\{ 1 + \sum_{n=1}^{\infty} \bar{r}_n \bar{a}_Q^n \right\} \]

Numerically, at \( N_f = 3 \):

\[ \bar{r}_1 = 5.457, \quad \bar{r}_2 = 25.45, \quad \bar{r}_3 = 142.4, \quad \bar{r}_4 = 932.7. \]
Analytically, the coefficient $\bar{r}_4$ is given by:

$\bar{r}_4 = \frac{49275071521973}{8264970432} - \frac{10679302931}{1889568} \zeta_3 + \frac{601705}{648} \zeta_3^2 + \frac{117947335}{209952} \zeta_5 - \frac{3285415}{20736} \zeta_7$

(Baikov, Chetyrkin, Kühn 2006)

The even-integer $\zeta$-function terms ($\zeta_4$ and $\zeta_6$) present in both $r_3$, $r_4$ and $\gamma_m$, $\beta_5$ cancel each other.

Feature of the C-scheme conjectured on the basis of the scalar quark and gluonium correlators. (MJ, Miravitllas 2018)

Since then demonstrated for several more physical quantities. (Davies, Vogt 2018)

And proven for massless correlators up to six loops. (Baikov, Chetyrkin 2018)
**Borel transforms**

**Conventional Borel transform for the Adler function:**

\[ \hat{D}(Q^2) = \int_0^\infty dt e^{-t/\hat{\alpha}_Q} B[\hat{D}](t) \quad (t = 2u/\beta_1) \]

**Modified Borel transform for the Adler function:**

(Brown, Yaffe, Zhai 1992)

\[ \hat{D}(Q^2) = \int_0^\infty dt e^{-t/\hat{\alpha}_Q} \left( \frac{t}{\hat{\alpha}_Q} \right)^{\beta_2/\beta_1} t^{\beta_1/2} e^{\beta_1/2 C t} B[\hat{D}](t) \]

**Conventional Borel transform for the scalar correlator:**

\[ \psi''(Q^2) = \frac{N_c}{8\pi^2} \frac{\hat{m}^2}{Q^2} \left( \pi \hat{\alpha}_Q \right)^{2\gamma_m^{(1)}/\beta_1} \left\{ 1 + \int_0^\infty dt e^{-t/\hat{\alpha}_Q} B[\psi''](t) \right\} \]
Structure of IR renormalon poles (Beneke 1999)

General term in the Operator Product Expansion:

\[
\hat{C}_{O_d}(\hat{a}_Q) \frac{\langle \hat{O}_d \rangle}{Q^d} = \hat{C}_{O_d}^{(0)} [\hat{a}_Q] \delta \left[ 1 + \hat{C}_{O_d}^{(1)} \hat{a}_Q + \hat{C}_{O_d}^{(2)} \hat{a}_Q^2 + \ldots \right] \frac{\langle \hat{O}_d \rangle}{Q^d}
\]

Express \(Q\)-dependence in terms of \(\bar{a}_Q\):

\[
\hat{C}_{O_d}(\bar{a}_Q) \frac{\langle \hat{O}_d \rangle}{Q^d} = \hat{C}_{O_d}^{(0)} e^{-\beta_1 \bar{a}_Q^d} \left[ \delta - d \frac{\beta_2}{\beta_1} \hat{a}_Q \right] \left[ 1 + \hat{C}_{O_d}^{(1)} \bar{a}_Q + \hat{C}_{O_d}^{(2)} \bar{a}_Q^2 + \ldots \right]
\]

Take Ansatz for Borel transform of IR renormalon pole:

\[
B[\hat{D}_p^{IR}](u) \equiv \frac{d_p^{IR}}{(p-u)^\gamma} \left[ 1 + b_1 (p-u) + b_2 (p-u)^2 + \ldots \right]
\]
The imaginary ambiguity takes the form:

\[ \text{Im} \left[ \hat{D}_p^{\text{IR}}(\bar{a}_Q) \right] = \pm \frac{2\pi^2}{\beta_1} d_p^{\text{IR}} e^{-\frac{2p}{\beta_1 \bar{a}_Q}} (\bar{a}_Q)^{1-\gamma} \left[ 1 + b_1 \frac{\beta_1}{2} (\gamma - 1) \bar{a}_Q 
\right.
\]

\[ + b_2 \frac{\beta_1^2}{4} (\gamma - 1)(\gamma - 2) \bar{a}_Q^2 + \ldots \right] \]

One can identify:

\[ p = \frac{d}{2}, \quad \gamma = 1 - \delta + 2\rho \frac{\beta_2}{\beta_1^2}, \]

\[ b_1 = \frac{2\tilde{C}_{Od}^{(1)}}{\beta_1 (\gamma - 1)}, \quad b_2 = \frac{4\tilde{C}_{Od}^{(2)}}{\beta_1^2 (\gamma - 1)(\gamma - 2)}. \]
Assume ambiguity $\pm i\Delta_{p}^{\text{IR}} \wedge^{d}$ for matrix element $\langle \hat{O}_{d} \rangle$.

Cancellation of ambiguities with PT entails:

$$\hat{C}_{O_{d}}^{(0)} \Delta_{p}^{\text{IR}} = \frac{2\pi^{2}}{\beta_{1}} C_{0}^{(0)} d_{p}^{\text{IR}}$$

Universality of ambiguity for correlators $A$ and $B$ leads to:

$$\frac{C_{0}^{(0)}(A)}{\hat{C}_{O_{d}}^{(0)}(A)} d_{p}^{\text{IR}}(A) = \frac{C_{0}^{(0)}(B)}{\hat{C}_{O_{d}}^{(0)}(B)} d_{p}^{\text{IR}}(B)$$

Example: Gluon condensate renormalon in large-$\beta_{0}$:

$$C_{0}^{(0)} = \frac{N_{c}}{12\pi^{2}} , \quad \hat{C}_{GG}^{(0)} = \frac{1}{6} , \quad d_{2}^{\text{IR}} = \frac{3C_{F}}{2} e^{-2C}$$

The invariant combination reads:

$$\frac{C_{0}^{(0)}}{\hat{C}_{GG}^{(0)}} d_{2}^{\text{IR}} = \frac{3}{8\pi^{2}} (N_{c}^{2} - 1) e^{-2C}$$
To incorporate known renormalon structure, use an Ansatz for the Adler function:

\[ B[\hat{D}](u) = B[\hat{D}_2^{\text{IR}}](u) + B[\hat{D}_3^{\text{IR}}](u) + B[\hat{D}_1^{\text{UV}}](u) + d_0^{\text{PO}} \]

Fitting \( \bar{c}_{1,1} \) to \( \bar{c}_{4,1} \), the parameters are found to be:

\[ d_2^{\text{IR}} = 2.74, \quad d_3^{\text{IR}} = -7.72, \]
\[ d_1^{\text{UV}} = -2.12 \cdot 10^{-2}, \quad d_0^{\text{PO}} = 0.289. \]

The Borel model predicts: \( \bar{c}_{5,1} \approx 329 \Rightarrow c_{5,1} \approx 264 \).

(BJ08: \( \approx 280 \))

Imposing \( d_2^{\text{IR}} \) in the scalar correlator model yields \( C \approx -1.6 \).

(Boito, MJ, Miravitllas: in preparation)
Summary

- The $C$-scheme coupling $\hat{a}_Q^C$ was introduced to study scheme dependence in Borel models for QCD correlators.

- Its corresponding $\beta$-function $\hat{\beta}(\hat{a})$ is found to be manifestly scheme invariant.

- In the $C$-scheme, the $\zeta_4$ term in $\bar{r}_4$ of the scalar correlator cancels against the corresponding $\zeta_4$ term in $\beta_5$.

- Expressing the coupling prefactor in terms of $\hat{a}_Q^C$ resums dominant corrections in the scalar correlator. The remaining corrections are more “Adler function like”.
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Thank You!