QCD Correlators at Higher Orders

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Adler function in large-β_0

Begin with the conventional Adler function $D(Q^2)$:

$$4\pi^2 D(Q^2) \equiv 1 + \hat{D}(Q^2) = 1 + a_Q + \mathscr{O}(a_Q^2)$$

where $a_Q \equiv \alpha_s(Q^2)/\pi$.

$\hat{D}(Q^2)$ can be expressed through a Borel-integral:

$$\hat{D}(Q^2) \equiv \frac{2}{\beta_1} \int\limits_0^\infty \mathrm{d}u\, \mathrm{e}^{-2u/(\beta_1 a_Q^C)} B[\hat{D}](u)$$

with the large-β_0 Borel-transform: $(\beta_1 = 2\pi\beta_0 = 11/2 - N_f/3)$
(Beneke 1993; Broadhurst 1993)

$$B[\hat{D}](u) = 8C_F \frac{\mathrm{e}^{-Cu}}{(2-u)} \sum_{k=2}^\infty \frac{(-1)^k k}{[k^2-(1-u)^2]^2}$$

Introducing the scheme-invariant coupling A_Q in large-β_0:

$$\frac{1}{A_Q} \equiv \frac{1}{a_Q^C} + \frac{\beta_1}{2}\, C = \frac{1}{a_Q^{\overline{\rm MS}}} + \frac{\beta_1}{2}\, \widehat{C}$$

with $\widehat{C} = -5/3$ as well as $\widehat{\Lambda} \equiv \Lambda^{\overline{\rm MS}}\, {\rm e}^{-\widehat{C}/2} \approx 2.3\, \Lambda^{\overline{\rm MS}}$.

The Borel-integral can be expressed in scheme-invariant form:

$$\hat{D}(Q^2) \equiv \frac{2}{\beta_1} \int\limits_0^\infty {\rm d}u\, {\rm e}^{-2u/(\beta_1 A_Q)}\, \hat{B}[\hat{D}](u)$$

with

$$\hat{B}[\hat{D}](u) = \frac{8 C_F}{(2-u)} \sum_{k=2}^{\infty} \frac{(-1)^k k}{[k^2 - (1-u)^2]^2}$$

Renormalon poles at all integer $u = 2, 3, 4, \ldots$ (IR poles)
and $u = -1, -2, -3, \ldots$ (UV poles).

Scalar correlator in large-β_0

The scalar correlator $\Psi(Q^2)$ is defined as:

$$\Psi(Q^2=-q^2) \equiv i\int \mathrm{d}x\, \mathrm{e}^{iqx}\langle\Omega|T\{j(x)j^\dagger(0)\}|\Omega\rangle$$

where, for example,

$$j(x) = m :\bar{u}(x)s(x):$$

and m is a generic quark mass.

In large-β_0, $\Psi''(Q^2)$ can be expressed as: (MJ, Miravitllas 2016)

$$\Psi''(Q^2) = \frac{N_c}{8\pi^2}\frac{\hat{m}^2}{Q^2}\left(\pi A_Q\right)^{2\gamma_m^{(1)}/\beta_1} \times$$

$$\left\{1+\frac{2}{\beta_1}\int\limits_0^\infty \mathrm{d}u\, \mathrm{e}^{-2u/(\beta_1 A_Q)}\,\hat{B}[\Psi''](u)\right\}$$

The RGI quark mass $\hat{m}$ in full QCD is defined as:

$$m(\mu) \equiv \hat{m}\,[\alpha_s(\mu)]^{\gamma_m^{(1)}/\beta_1} \exp\left\{ \int_0^{a_\mu} \mathrm{d}a \left[\frac{\gamma_m(a)}{\beta(a)} - \frac{\gamma_m^{(1)}}{\beta_1 a} \right] \right\}$$

The scheme-invariant Borel-transform is found to be:

(Broadhurst, Kataev, Maxwell 2001)

$$\hat{B}[\Psi''](u) = \frac{3}{2}\,C_F \left[(1-u)\,G_D(u) - 1 \right]$$

with

$$G_D(u) = \frac{2}{1-u} - \frac{1}{2-u} + \frac{2}{3}\sum_{p=3}^{\infty} \frac{(-1)^p}{(p-u)^2} - \frac{2}{3}\sum_{p=1}^{\infty} \frac{(-1)^p}{(p+u)^2}$$

which explicitly displays the renormalon structure.

Coupling evolution

Scale evolution of α_s is given by the β-function:

$$-Q\,\frac{\mathrm{d}a_Q}{\mathrm{d}Q} \equiv \beta(a_Q) = \beta_1\,a_Q^2 + \beta_2\,a_Q^3 + \beta_3\,a_Q^4 + \ldots .$$

with $a_Q = \alpha_s/\pi$.

The scale invariant parameter Λ can be defined by:

$$\frac{\Lambda}{Q} \equiv \mathrm{e}^{-\frac{1}{\beta_1 a_Q}}\,[a_Q]^{-\frac{\beta_2}{\beta_1^2}}\exp\left\{\int\limits_0^{a_Q}\frac{\mathrm{d}a}{\tilde\beta(a)}\right\},$$

where

$$\frac{1}{\tilde\beta(a)} \equiv \frac{1}{\beta(a)} - \frac{1}{\beta_1 a^2} + \frac{\beta_2}{\beta_1^2 a}$$

is free of singularites as $a \to 0$.

C-scheme coupling

However, Λ depends on the renormalisation scheme.

$$a' \equiv a + c_1\, a^2 + c_2\, a^3 + c_3\, a^4 + \ldots .$$

Then, Λ transforms as: (Celmaster, Gonsalves 1979)

$$\Lambda' = \Lambda\, \mathrm{e}^{c_1/\beta_1} .$$

This suggests to define a "novel" "C-scheme" coupling $\hat{a}_Q^C$:

$$\frac{1}{\hat{a}_Q^C} + \frac{\beta_2}{\beta_1} \ln \hat{a}_Q^C - \frac{\beta_1}{2}\, C \equiv \beta_1 \ln \frac{Q}{\Lambda}$$

$$= \frac{1}{a_Q} + \frac{\beta_2}{\beta_1} \ln a_Q - \beta_1 \int\limits_0^{a_Q} \frac{\mathrm{d}a}{\tilde{\beta}(a)}$$

(Boito, MJ, Miravitllas 2016)

The β-function of $\hat{a}_Q^C$ reads simply: (Brown, Yaffe, Zhai 1992)

$$-Q\,\frac{\mathrm{d}\hat{a}_Q^C}{\mathrm{d}Q} \equiv \hat{\beta}(\hat{a}_Q^C) = \frac{\beta_1\,(\hat{a}_Q^C)^2}{\left(1-\frac{\beta_2}{\beta_1}\,\hat{a}_Q^C\right)} = -2\,\frac{\mathrm{d}\hat{a}_Q^C}{\mathrm{d}C}$$

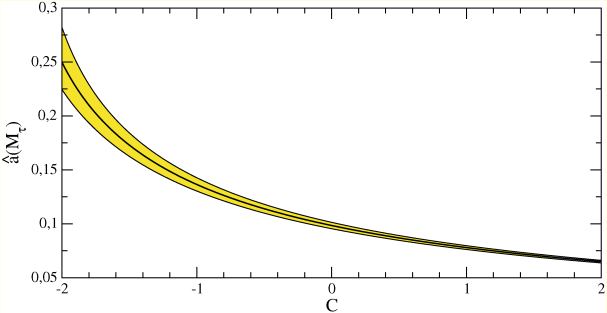
Relating a general coupling a_Q and $\bar{a}_Q \equiv \hat{a}_Q^{C=0}$ reads:

$$a_Q = \bar{a}_Q + \left(\frac{\beta_3}{\beta_1} - \frac{\beta_2^2}{\beta_1^2}\right)\bar{a}_Q^3 + \left(\frac{\beta_4}{2\beta_1} - \frac{\beta_2^3}{2\beta_1^3}\right)\bar{a}_Q^4$$
$$+ \left(\frac{\beta_5}{3\beta_1} - \frac{\beta_2\beta_4}{6\beta_1^2} + \frac{5\beta_3^2}{3\beta_1^2} - \frac{3\beta_2^2\beta_3}{\beta_1^3} + \frac{7\beta_2^4}{6\beta_1^4}\right)\bar{a}_Q^5 + \mathcal{O}(\bar{a}_Q^6)$$

The coupling $\hat{a}_Q^C$ at arbitrary C is obtained from $\bar{a}_Q$ via:

$$\bar{a}_Q = \hat{a}_Q^C + \frac{\beta_1}{2}\,C\,(\hat{a}_Q^C)^2 + \left(\frac{\beta_2}{2}\,C + \frac{\beta_1^2}{4}\,C^2\right)(\hat{a}_Q^C)^3$$
$$+ \left(\frac{\beta_2^2}{2\beta_1}\,C + \frac{5\beta_1\beta_2}{8}\,C^2 + \frac{\beta_1^3}{8}\,C^3\right)(\hat{a}_Q^C)^4 + \mathcal{O}((\hat{a}_Q^C)^5)$$

C-scheme coupling



$\hat{a}(M_\tau)$ as a function of C for $\alpha_s(M_\tau) = 0.316(10)$.

C-scheme quark mass

Scheme dependence of the C-scheme quark mass:

$$\frac{1}{m_Q^C}\frac{\mathrm{d}m_Q^C}{\mathrm{d}C} = \frac{\mathrm{d}\hat{a}_Q^C}{\mathrm{d}C}\frac{\mathrm{d}Q}{\mathrm{d}\hat{a}_Q^C}\frac{1}{m_Q^C}\frac{\mathrm{d}m_Q^C}{\mathrm{d}Q} = -\frac{1}{2}\,\hat{\gamma}_m(\hat{a}_Q^C)$$

RGI quark mass $\hat{m}$ also scheme invariant:

$$\widehat{m} \equiv m_Q^C\,[\hat{\alpha}_s^C(Q)]^{-\gamma_m^{(1)}/\beta_1}\exp\left\{\int\limits_0^{\hat{a}_Q^C}\mathrm{d}\hat{a}\left[\frac{\gamma_m^{(1)}}{\beta_1\,\hat{a}} - \frac{\hat{\gamma}_m(\hat{a})}{\hat{\beta}(\hat{a})}\right]\right\}$$

Normalisation condition for m_Q^C:

$$m_Q^{\overline{\mathrm{MS}}} \equiv m_Q^{C=0} \equiv \bar{m}_Q$$

Adler function (in full QCD)

$$4\pi^2 D(a_Q) - 1 \;\equiv\; \hat{D}(a_Q) \;=\; \sum_{n=1}^{\infty} c_{n,1}\, a_Q^n$$

$$= a_Q + 1.640\, a_Q^2 + 6.371\, a_Q^3 + 49.08\, a_Q^4 + \ldots$$

Expressed in terms of the coupling $\bar{a}_Q$:

$$\hat{D}(a_Q) \;=\; \sum_{n=1}^{\infty} \bar{c}_{n,1}\, \bar{a}_Q^n$$

$$= \bar{a}_Q + 1.640\, \bar{a}_Q^2 + 7.682\, \bar{a}_Q^3 + 61.06\, \bar{a}_Q^4 + \ldots$$

Analytically, the coefficient $\bar{c}_{4,1}$ is given by:

(Baikov, Chetyrkin, Kühn 2008)

$$\bar{c}_{4,1} = \frac{357259199}{93312} - \frac{1713103}{432}\zeta_3 + \frac{4185}{8}\zeta_3^2 + \frac{34165}{96}\zeta_5 - \frac{1995}{16}\zeta_7$$

Scalar correlator (in full QCD)

$$\Psi''(Q^2) = \frac{N_c}{8\pi^2}\,\frac{m_Q^2}{Q^2}\left\{1 + \sum_{n=1}^{\infty} d''_{n,1}\, a_Q^n\right\}$$

Expressed in terms of the RGI quark mass $\hat{m}$:

$$\Psi''(Q^2) = \frac{N_c}{8\pi^2}\,\frac{\hat{m}^2}{Q^2}\,[\alpha_s(Q)]^{2\gamma_m^{(1)}/\beta_1}\left\{1 + \sum_{n=1}^{\infty} r_n\, a_Q^n\right\}$$

Expressed in terms of the coupling $\bar{a}_Q$:

$$\Psi''(Q^2) = \frac{N_c}{8\pi^2}\,\frac{\hat{m}^2}{Q^2}\,[\bar{\alpha}_s(Q)]^{2\gamma_m^{(1)}/\beta_1}\left\{1 + \sum_{n=1}^{\infty} \bar{r}_n\, \bar{a}_Q^n\right\}$$

Numerically, at $N_f = 3$:

$$\bar{r}_1 = 5.457\,, \quad \bar{r}_2 = 25.45\,, \quad \bar{r}_3 = 142.4\,, \quad \bar{r}_4 = 932.7\,.$$

Analytically, the coefficient $\bar{r}_4$ is given by:

(Baikov, Chetyrkin, Kühn 2006)

$$\bar{r}_4 = \frac{49275071521973}{8264970432} - \frac{10679302931}{1889568}\zeta_3 + \frac{601705}{648}\zeta_3^2 + \frac{117947335}{209952}\zeta_5 - \frac{3285415}{20736}\zeta_7$$

The even-integer ζ-function terms (ζ_4 and ζ_6) present in both r_3, r_4 and γ_m, β_5 cancel each other.

Feature of the C-scheme conjectured on the basis of the scalar quark and gluonium correlators. (MJ, Miravitllas 2018)

Since then demonstrated for several more physical quantities.

(Davies, Vogt 2018)

And proven for massless correlators up to six loops.

(Baikov, Chetyrkin 2018)

Borel transforms

Conventional Borel transform for the Adler function:

$$\hat{D}(Q^2) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t/\hat{a}_Q^C}\, B[\hat{D}](t) \qquad (t = 2u/\beta_1)$$

Modified Borel transform for the Adler function:

(Brown, Yaffe, Zhai 1992)

$$\hat{D}(Q^2) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t/\hat{a}_Q^C} \left(\frac{t}{\hat{a}_Q^C}\right)^{\beta_2/\beta_1 t} \mathrm{e}^{\beta_1/2\, C\, t}\, \hat{B}[\hat{D}](t)$$

Conventional Borel transform for the scalar correlator:

$$\Psi''(Q^2) = \frac{N_c}{8\pi^2} \frac{\hat{m}^2}{Q^2} \left(\pi \hat{\alpha}_Q^C\right)^{2\gamma_m^{(1)}/\beta_1} \left\{ 1 + \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t/\hat{a}_Q^C} B[\Psi''](t) \right\}$$

Structure of IR renormalon poles

(Beneke 1999)

General term in the Operator Product Expansion:

$$\widehat{C}_{O_d}(\hat{a}_Q)\,\frac{\langle \widehat{O}_d\rangle}{Q^d} = \widehat{C}^{(0)}_{O_d}\,[\hat{a}_Q]^{\delta}\left[1+\widetilde{C}^{(1)}_{O_d}\,\hat{a}_Q+\widetilde{C}^{(2)}_{O_d}\,\hat{a}_Q^2+\ldots\right]\frac{\langle \widehat{O}_d\rangle}{Q^d}$$

Express Q-dependence in terms of $\bar{a}_Q$:

$$\frac{\widehat{C}_{O_d}(\bar{a}_Q)}{Q^d} = \frac{\widehat{C}^{(0)}_{O_d}}{\Lambda^d}\,\mathrm{e}^{-\frac{d}{\beta_1\bar{a}_Q}}\,[\bar{a}_Q]^{\delta-d\frac{\beta_2}{\beta_1^2}}\left[1+\widetilde{C}^{(1)}_{O_d}\,\bar{a}_Q+\widetilde{C}^{(2)}_{O_d}\,\bar{a}_Q^2+\ldots\right]$$

Take Ansatz for Borel transform of IR renormalon pole:

$$B[\widehat{D}^{\mathrm{IR}}_p](u) \equiv \frac{d^{\mathrm{IR}}_p}{(p-u)^{\gamma}}\left[1+b_1(p-u)+b_2(p-u)^2+\ldots\right]$$

The imaginary ambiguity takes the form:

$$\mathrm{Im}\Big[\widehat{D}_p^{\mathrm{IR}}(\bar{a}_Q)\Big] = \pm\frac{2\pi^2}{\beta_1}\,d_p^{\mathrm{IR}}\,\mathrm{e}^{-\frac{2p}{\beta_1\bar{a}_Q}}\,(\bar{a}_Q)^{1-\gamma}\Big[1+b_1\frac{\beta_1}{2}(\gamma-1)\,\bar{a}_Q + b_2\frac{\beta_1^2}{4}(\gamma-1)(\gamma-2)\,\bar{a}_Q^2+\ldots\Big]$$

One can identify:

$$p=\frac{d}{2}\,,\quad \gamma = 1-\delta+2p\,\frac{\beta_2}{\beta_1^2}\,,$$

$$b_1 = \frac{2\widetilde{C}_{O_d}^{(1)}}{\beta_1(\gamma-1)}\,,\quad b_2 = \frac{4\widetilde{C}_{O_d}^{(2)}}{\beta_1^2(\gamma-1)(\gamma-2)}\,.$$

Assume ambiguity $\pm i\Delta_p^{\rm IR}\Lambda^d$ for matrix element $\langle \widehat{O}_d \rangle$.

Cancellation of ambiguities with PT entails:

$$\widehat{C}_{O_d}^{(0)} \Delta_p^{\rm IR} = \frac{2\pi^2}{\beta_1}\, C_0^{(0)} d_p^{\rm IR}$$

Universality of ambiguity for correlators A and B leads to:

$$\frac{C_0^{(0)}(A)}{\widehat{C}_{O_d}^{(0)}(A)}\, d_p^{\rm IR}(A) = \frac{C_0^{(0)}(B)}{\widehat{C}_{O_d}^{(0)}(B)}\, d_p^{\rm IR}(B)$$

Example: Gluon condensate renormalon in large-β_0:

$$C_0^{(0)} = \frac{N_c}{12\pi^2}\,, \quad \widehat{C}_{GG}^{(0)} = \frac{1}{6}\,, \quad d_2^{\rm IR} = \frac{3C_F}{2}\, {\rm e}^{-2C}$$

The invariant combination reads:

$$\frac{C_0^{(0)}}{\widehat{C}_{GG}^{(0)}}\, d_2^{\rm IR} = \frac{3}{8\pi^2}(N_c^2 - 1)\, {\rm e}^{-2C}$$

Borel models

(Beneke, MJ 2008)

To incorporate known renormalon structure, use an Ansatz for the Adler function:

$$B[\widehat{D}](u) = B[\widehat{D}_2^{\rm IR}](u) + B[\widehat{D}_3^{\rm IR}](u) + B[\widehat{D}_1^{\rm UV}](u) + d_0^{\rm PO}$$

Fitting $\bar{c}_{1,1}$ to $\bar{c}_{4,1}$, the parameters are found to be:

$$d_2^{\rm IR} = 2.74\,, \qquad d_3^{\rm IR} = -7.72\,,$$
$$d_1^{\rm UV} = -2.12\cdot 10^{-2}\,, \qquad d_0^{\rm PO} = 0.289\,.$$

The Borel model predicts: $\bar{c}_{5,1} \approx 329 \;\Rightarrow\; c_{5,1} \approx 264$.

(BJ08: ≈ 280)

Imposing $d_2^{\rm IR}$ in the scalar correlator model yields $C \approx -1.6$.

(Boito, MJ, Miravitllas: in preparation)

Summary

- The C-scheme coupling $\hat{a}_Q^C$ was introduced to study scheme dependence in Borel models for QCD correlators.
- Its corresponding β-function $\hat{\beta}(\hat{a})$ is found to be manifestly scheme invariant.
- In the C-scheme, the ζ_4 term in $\bar{r}_4$ of the scalar correlator cancels against the corresponding ζ_4 term in β_5.
- Expressing the coupling prefactor in terms of $\hat{a}_Q^C$ resums dominant corrections in the scalar correlator.
  The remaining corrections are more "Adler function like".

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Thank You!