Small-x Resummation from Effective Field Theory

In Collaboration with
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Based on JHEP08(2016)025, and an upcoming paper

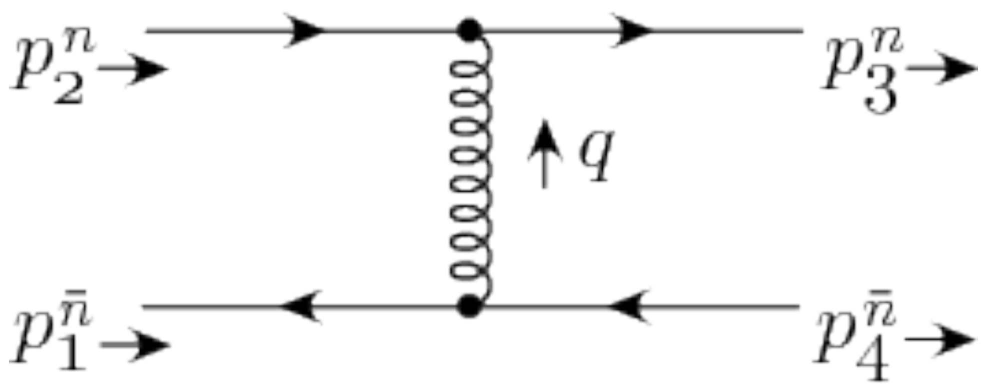
Introduction

Forward Scattering

Focus on a special region of kinematic phase space:
Forward Scattering limit.

Hard Scattering limit: $s \sim |t|$
Forward Scattering limit: $s \gg |t|$

$s = (p_1 + p_2)^2, \quad t = q^2 = (p_3 - p_2)^2$



$p_3\,, p_4$ separated by large rapidity.

$$s = 4k_\perp^2 \cosh^2 y, \quad t = -2k_\perp^2 \cosh y\, e^{-y}, \quad u = -2k_\perp^2 \cosh y\, e^{y}$$

$$s \simeq -u \simeq k_\perp^2 e^y\,, \quad t \simeq -k_\perp^2 \qquad\qquad y = \ln\left(\frac{s}{-t}\right)$$

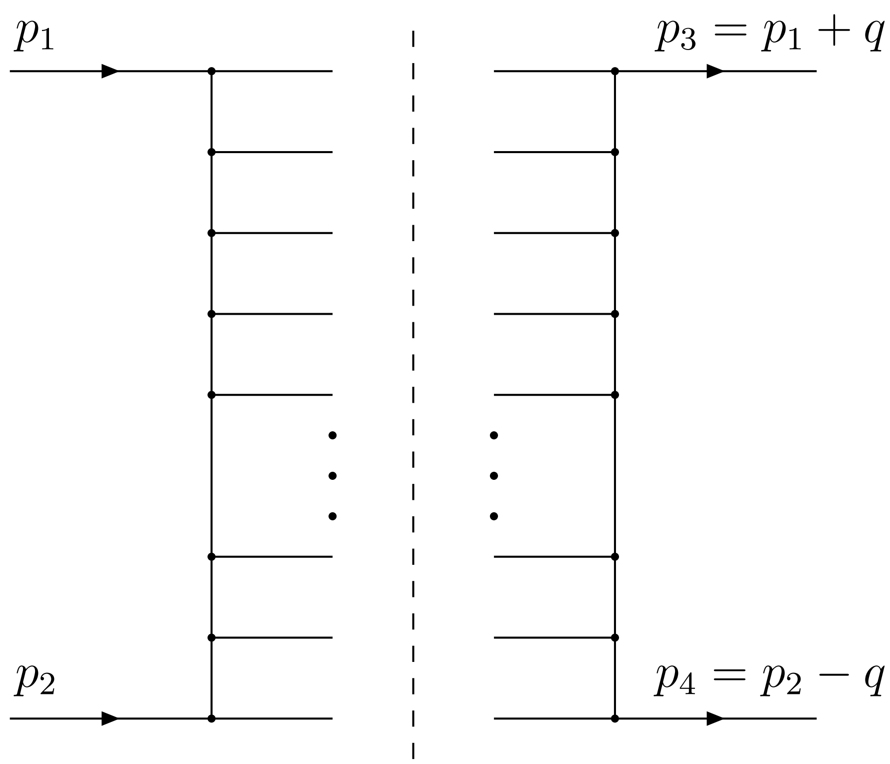
t-channel exchange dominant.

Forward Scattering in Scalar ϕ^3 Theory

$$\phi(p_1) + \phi(p_2) \to \phi(p_3) + \phi(p_4)$$

Regge behaviour observed for $s \to \infty$ and t fixed. $\quad \mathcal{M} \sim s^{\alpha(t)}$

Sum infinite set of ladder diagrams.



Rungs of the ladder on-shell, legs off-shell by $\sqrt{-t}$, particles strongly ordered in rapidity.

$$y_0 \gg y_1 \gg \ldots \gg y_n ; k_{i\perp} \simeq k_\perp$$

$$[x_1 x_2 \ldots x_n s + D(x_i, y_i, t)]^n$$

$$\frac{g^2}{s\, n!}[\beta(t)\ln(s)]^n$$

$$\mathcal{M} \sim \frac{1}{s}\sum_{n=0}^{\infty}\frac{\beta^n(t)}{n!}\ln^n s \to s^{-1+\beta(t)}$$

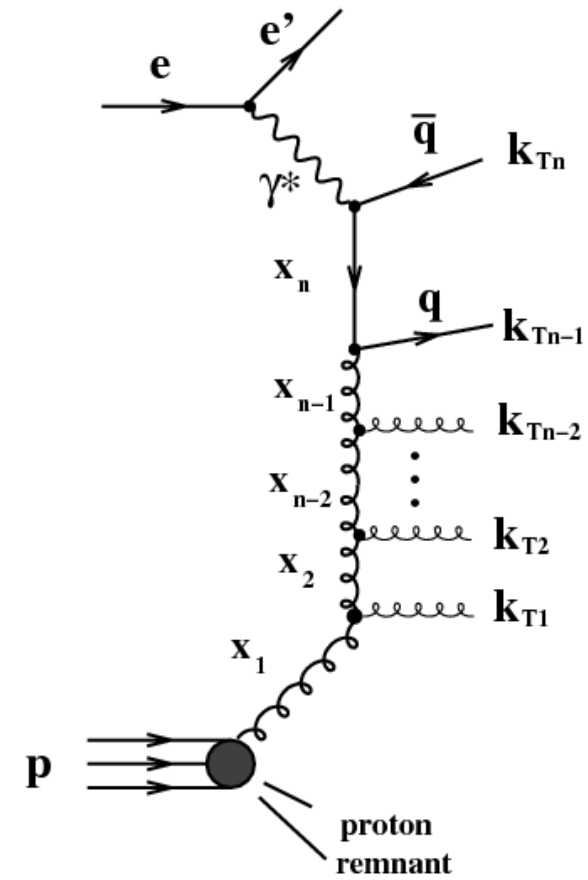
Also include cross-box diagrams.

[Polkinghorne; J. Math. Phys. 4(1963) 503, 1393, 1396]

Forward Scattering in QCD

This is a concern in QCD
$\rightarrow$ Breakdown of perturbation theory.

$$\alpha_s \ll 1, \quad \alpha_s \ln\left(\frac{s}{-t}\right) \simeq 1$$



Regge behavior in QCD is described by
Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation.

$$A(s) \sim \frac{1}{2\pi}\left(\frac{s}{-t}\right)^{4C_A \frac{\alpha_S}{\pi} \ln 2}$$

This corresponds to leading log resummation of terms $(\alpha_s \ln x)^n$

Other small-x resummation formalisms: BK, BJIMWLK,
the dipole approximation, multi-wilson lines EFT, etc.

[Balitsy, Lipatov; Sov. J. Nucl. Phys. 28, 822 (1978)]

[Del Duca; hep-ph/9503226]

Looking for BFKL in Experiments

Fit for total photoproduction cross section:

$$\sigma_{\gamma p}^{\mathrm{Tot}} = X s^{\epsilon} + Y s^{-\eta}$$

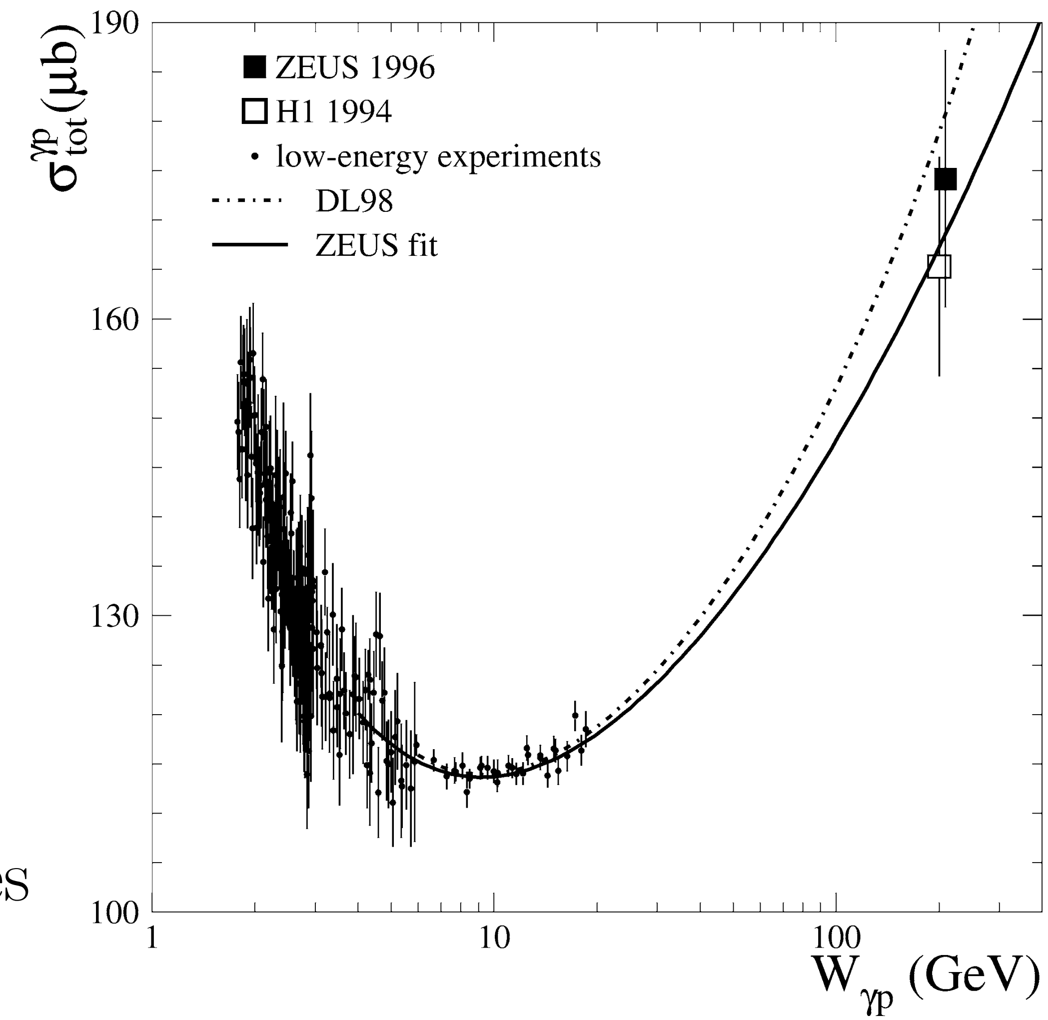
[ZEUS Collaboration; Nucl. Phys. B. 627 (2002) 3-28]

Regge behavior observed in the data:
$\epsilon \sim 0.08$.

BFKL Leading log solution gives
$\epsilon = 4 C_A \alpha_s / \pi \ln 2 \sim 0.5$ for $\alpha_s \sim 0.2$.

Naively including NLO corrections to BFKL gives
$\epsilon = 4 C_A \alpha_s / \pi \ln 2 (1 - 6.5 C_A \alpha_s / \pi) < 0$.

ZEUS Collaboration / Nuclear Physics B 627 (2002) 3–28



Looking for BFKL in experiments
is a very challenging problem!

Other signatures of BFKL: $F_2(x, Q^2)$ structure function measurement at HERA, Mueller-Navelet jets.

[Ducloué et al.; arXiv:1407.5106]

Soft Collinear Effective Theory

Sudakov decomposition (light cone coordinates):

$$n^\mu = (1,0,0,1) \quad \bar{n}^\mu = (1,0,0,-1)$$

$$p^\mu = \frac{n^\mu}{2}\,\bar{n}.p + \frac{\bar{n}^\mu}{2}\,n.p + p_\perp^\mu$$

$$p^+ = n.p = p^0 - p^3$$

$$p^\mu = (p^+, p^-, \vec{p}_\perp)$$

$$p^- = \bar{n}.p = p^0 + p^3$$

$$p^2 = p^+ p^- - \vec{p}_\perp{}^2$$

$$p_\perp^\mu = (0, p_1, p_2, 0)$$

$$\mathcal{L}_{\mathrm{QCD}}(\psi, A) \to \mathcal{L}_{\mathrm{SCET}} = \mathcal{L}_{\mathrm{S}}^{(0)}(\psi_S, A_S) + \sum_{n_i} \mathcal{L}_{n_i}^{(0)}(\xi_{n_i}, A_{n_i})$$

Power counting parameter: λ.

Symmetries of SCET: Gauge invariance for each sector, Lorentz invariance gets partially broken, discrete CPT symmetries of QCD.

$$p_n \sim \sqrt{s}(\lambda^2, 1, \lambda)$$

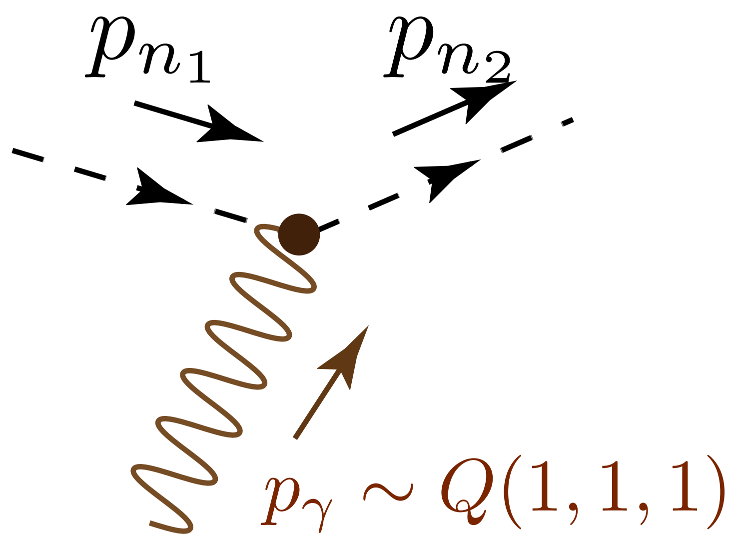
$$p_s \sim \sqrt{s}(1, \lambda^2, \lambda)$$

$$p_{\bar{n}} \sim \sqrt{s}(1, \lambda^2, \lambda)$$

Modes: Soft and collinear quarks and gluons $\psi_S, A_S, \xi_{n_i}, A_{n_i}$

SCET lectures by Iain Stewart:
http://courses.edx.org/c4x/MITx/8.EFTx/asset/notes scetnotes.pdf

Modes in SCET

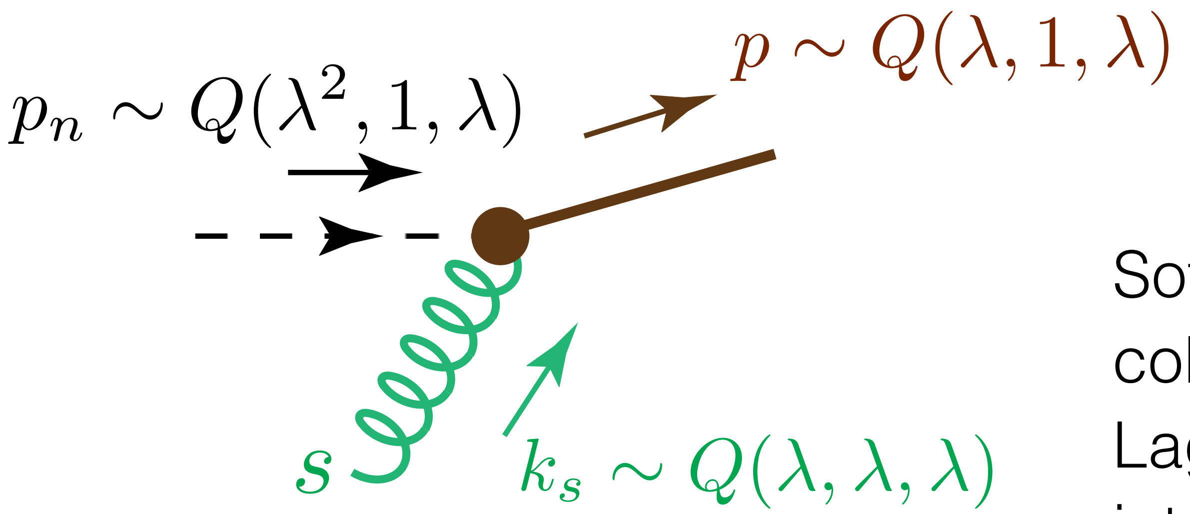


Hard interactions are integrated out in the effective theory. All the hard physics is then encoded in the matching coefficients

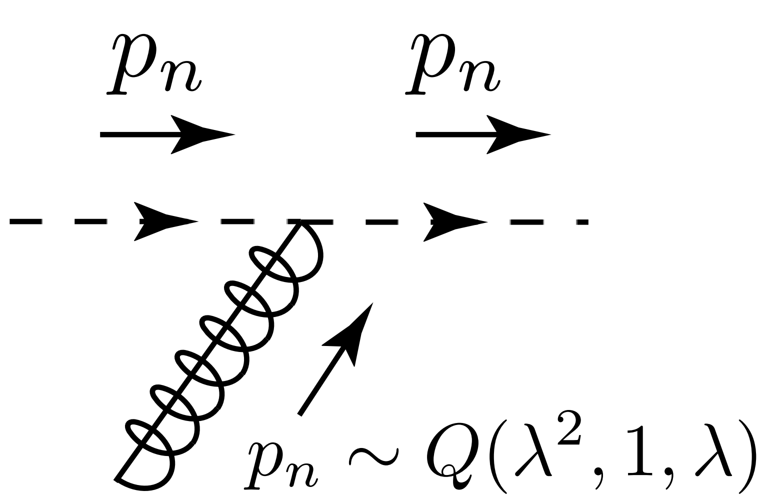
$$J^\mu_{\rm QCD}(x) = \bar{q}(x)\,\Gamma^\mu q(x)$$

$$\mathcal{O}^{\alpha\beta}_{q\bar{q}}(\tilde{p}_1\tilde{p}_2;x) = \bar{\chi}^{\alpha j}_{n_1,\tilde{p}_1}(x)\chi^{\beta j}_{n_2,\tilde{p}_2}(x)$$

$$J^\mu_{\rm QCD}(x) = \sum_{n_1,n_2}\int d^3\tilde{p}_1 d^3\tilde{p}_2 e^{i(\tilde{p}_1-\tilde{p}_2)\cdot x)}\left[C^\mu_{q\bar{q}\alpha\beta}\mathcal{O}^{\alpha\beta}_{q\bar{q}}(\tilde{p}_1,\tilde{p}_2;x) + C^\mu_{gg\lambda\rho}\mathcal{O}^{\lambda\rho}_{gg}(\tilde{p}_1,\tilde{p}_2;x)\right]$$



Soft particles with O(Q λ) momenta can knock collinear particles off shell. Not allowed in the SCET Lagrangian (without operators for Glauber and hard interactions).



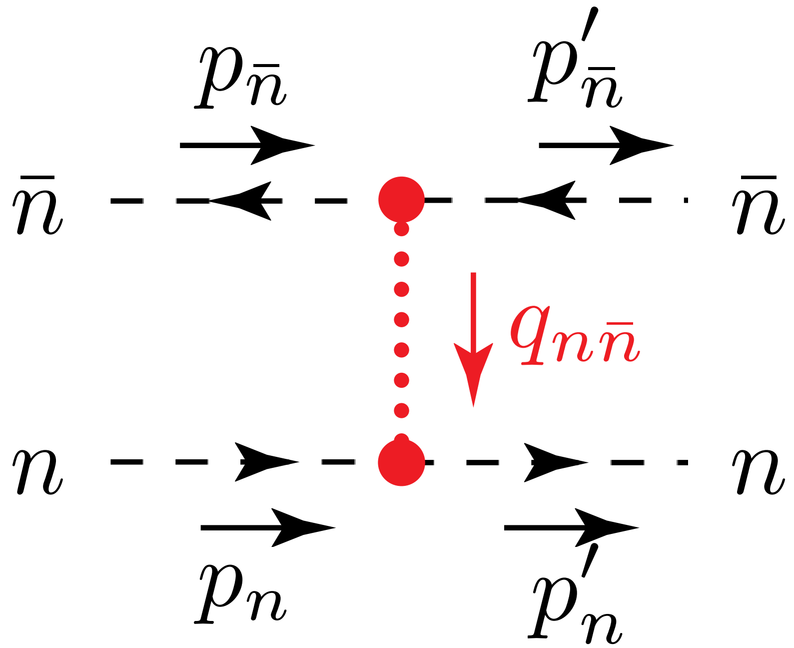
Interactions with collinear gluons are allowed

Formalism

$q\bar{q}$ Forward Scattering Example

Kinematics of Forward Scattering

$$q(p_n) + \bar{q}(p_{\bar{n}}) \to q(p'_n) + \bar{q}(p'_{\bar{n}})$$

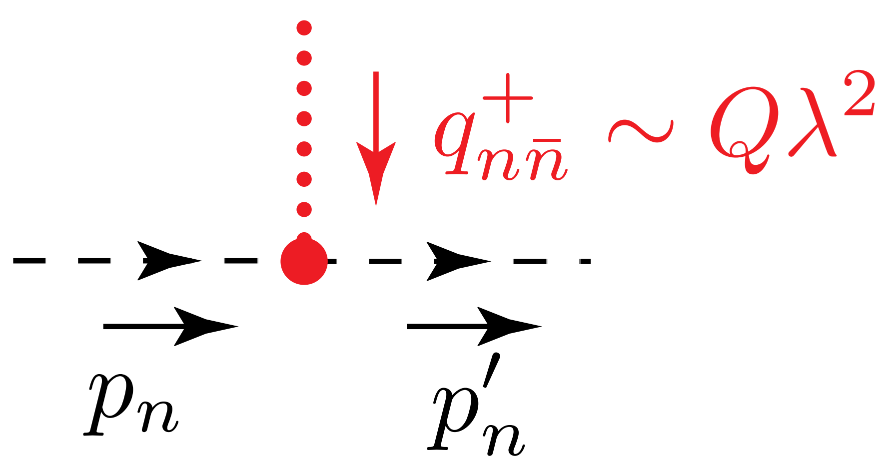


The momentum configuration we want for forward scattering:

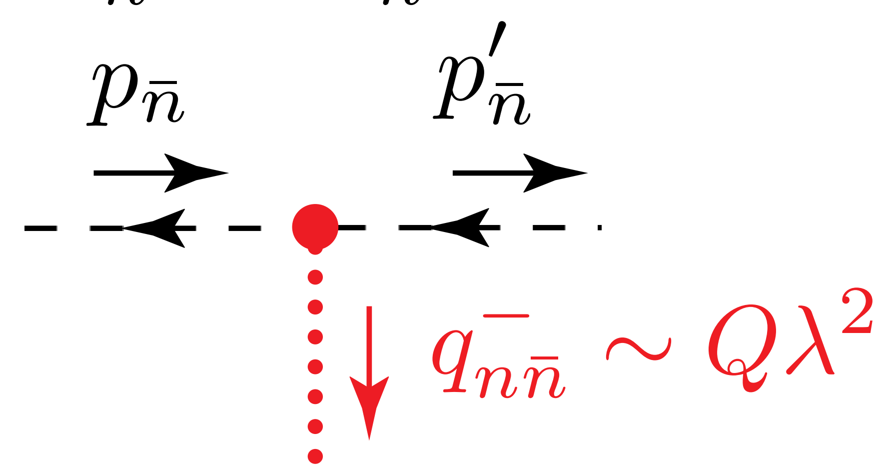
$$p_n \sim p'_n \sim (\underbrace{Q\lambda^2}_{p^+}, \underbrace{Q}_{p^-}, \underbrace{Q\lambda}_{p_\perp})$$

$$p_{\bar{n}} \sim p'_{\bar{n}} \sim (\underbrace{Q}_{p^+}, \underbrace{Q\lambda^2}_{p^-}, \underbrace{Q\lambda}_{p_\perp})$$

$$p'^+_n = p^+_n + q^+$$



$$p'^-_{\bar{n}} = p^-_{\bar{n}} - q^-$$

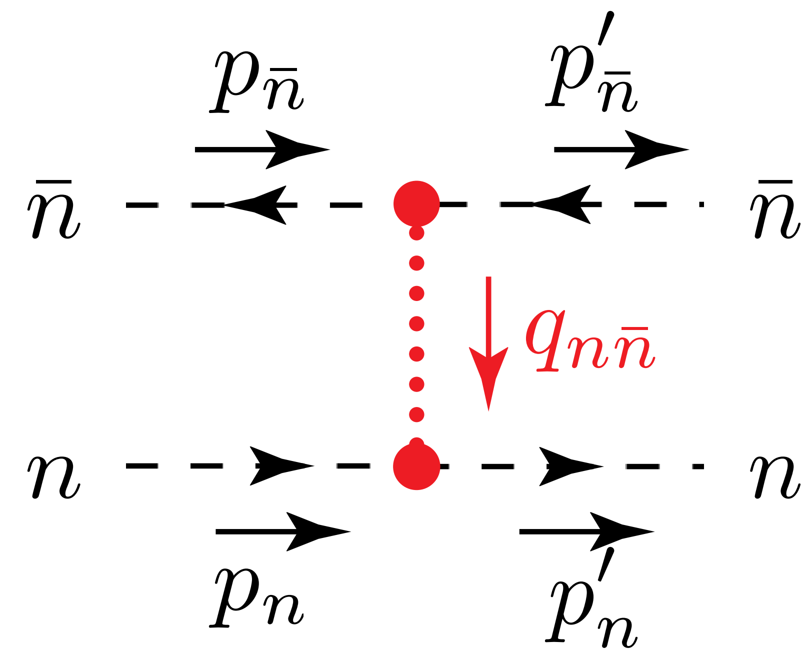


$$q^\mu_{n\bar{n}} \sim (Q\lambda^2, Q\lambda^2, Q\lambda)$$

$$p'^-_n \sim p^-_n \qquad p'^+_{\bar{n}} \sim p^+_{\bar{n}}$$

Forward Scattering Operators

$$q^{\mu}_{n\bar{n}} \sim (Q\lambda^2\,, Q\lambda^2\,, Q\lambda) \qquad q_\perp^2 \gg q^+\,q^-$$



The QCD matrix element is given by:

$$\mathcal{M}_{\rm QCD} = i\left[\bar{u}_n \frac{\bar{n}\!\!\!/}{2} T^B u_n\right]\left[\frac{-8\pi\alpha_s(\mu)\delta^{BC}}{\vec{q}_\perp^{\,2}}\right]\left[\bar{v}_{\bar{n}} \frac{n\!\!\!/}{2} \bar{T}^C v_{\bar{n}}\right]$$

Suggests the following form for the SCET scattering amplitude:

$$\mathcal{M}_{\rm SCET} = \langle\, q(p'_n)\,\bar{q}(p'_{\bar{n}})\,|\,O^{qq}_{ns\bar{n}}\,|\,q(p_n)\,\bar{q}(p_{\bar{n}})\,\rangle$$

$$O^{qq}_{ns\bar{n}} = \mathcal{O}^{qB}_n \frac{1}{\mathcal{P}_\perp^2} \mathcal{O}^{BC}_s \frac{1}{\mathcal{P}_\perp^2} \mathcal{O}^{qC}_n$$

$$\mathcal{O}^{qB}_n = \overline{\chi}_n T^B \frac{\bar{n}\!\!\!/}{2}\chi_n \qquad \mathcal{O}^{qB}_{\bar{n}} = \overline{\chi}_{\bar{n}} T^B \frac{n\!\!\!/}{2}\chi_{\bar{n}} \qquad \mathcal{O}^{BC}_s \to 8\pi\alpha_s\delta^{BC}\mathcal{P}_\perp^2$$

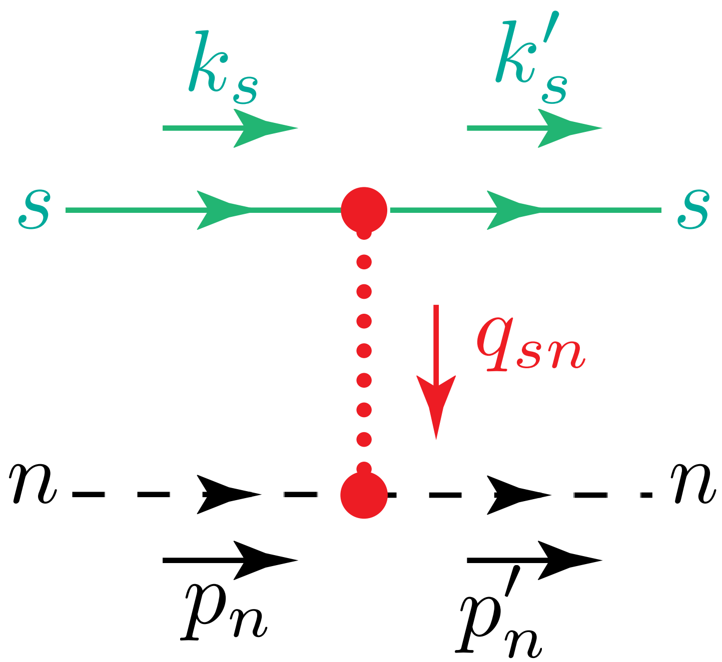
(for no soft fields)

$$\mathcal{P}^\mu_\perp \phi_p(x) = p^\mu_\perp \phi_p(x)$$

picks out the $\mathcal{O}(\lambda)$ perp label momentum

n-s forward scattering

$$q(p_n) + \bar{q}(k_s) \to q(p_n') + \bar{q}(k_s')$$



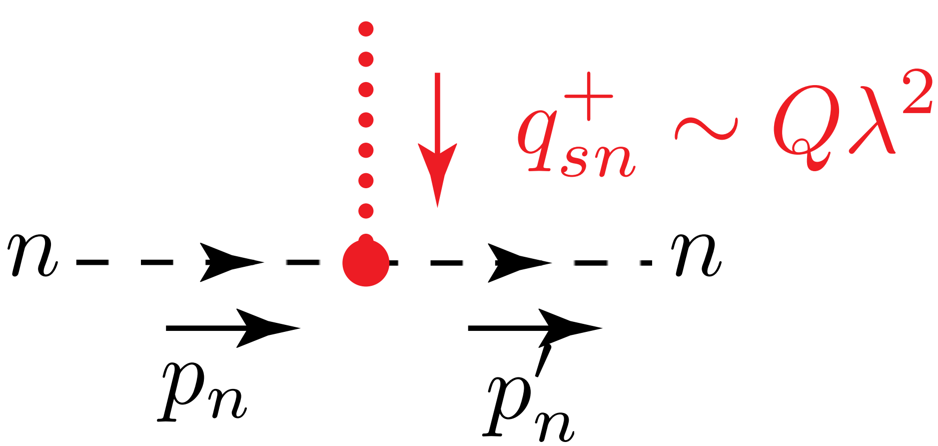
Glauber exchanges can also couple soft and collinear fields

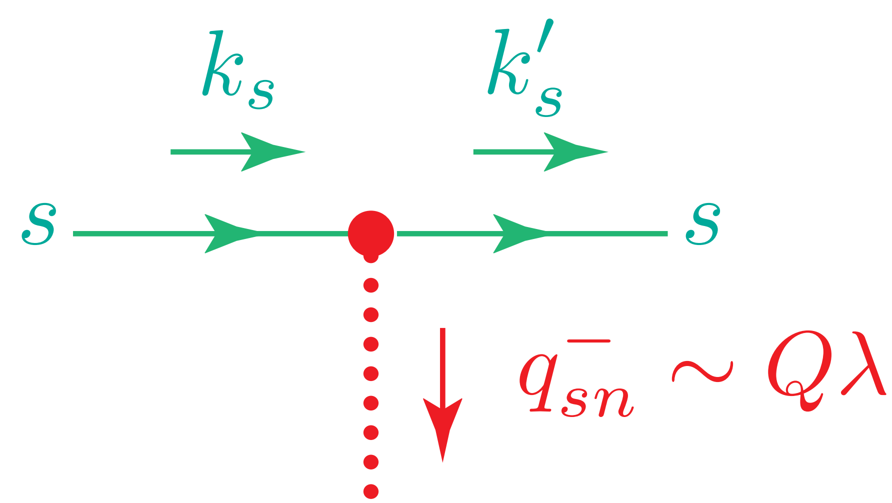
$$p_n \sim p_n' \sim (\underbrace{Q\lambda^2}_{p^+}, \underbrace{Q}_{p^-}, \underbrace{Q\lambda}_{p_\perp})$$

$$k_s \sim k_s' \sim (\underbrace{Q\lambda}_{p^+}, \underbrace{Q\lambda}_{p^-}, \underbrace{Q\lambda}_{p_\perp})$$

$$p_n'^+ = p_n^+ + q^+$$

$$k_s'^- = k_s^- - q^-$$





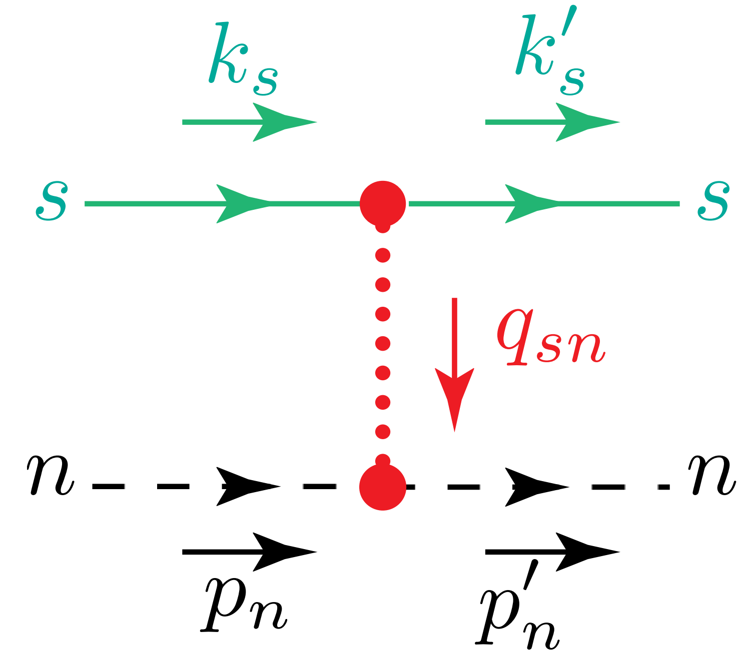
$$q_{sn}^\mu \sim (Q\lambda^2, Q\lambda, Q\lambda)$$

$$p_n'^- \sim p_n^- \qquad k_s'^+ \sim k_s^+$$

Forward Scattering Operators

n-s forward scattering is then described by the following gauge invariant operator:

$$O^{qg}_{ns} = \mathcal{O}^{qB}_{n}\frac{1}{\mathcal{P}^2_\perp}\mathcal{O}^{g_n B}_{s}$$



$$\mathcal{O}^{qB}_{n} = \overline{\chi}_n T^B \frac{\slashed{\bar{n}}}{2}\chi_n \qquad \mathcal{O}^{q_n B}_{s} = 8\pi\alpha_s\left(\bar{\psi}^n_S T^B \frac{\slashed{n}}{2}\psi^n_S\right)$$

Thus the Glauber Lagrangian is given by:

$$\mathcal{L}^{\mathrm{II}(0)}_G = e^{-ix\cdot\mathcal{P}}\sum_{n,\bar{n}}\sum_{i,j=q,g} O^{ij}_{ns\bar{n}} + e^{-ix\cdot\mathcal{P}}\sum_{n}\sum_{i,j=q,g} O^{ij}_{ns}$$

$$\equiv e^{-ix\cdot\mathcal{P}}\sum_{n,\bar{n}}\sum_{i,j=q,g} \mathcal{O}^{iB}_{n}\frac{1}{\mathcal{P}^2_\perp}\mathcal{O}^{BC}_{s}\frac{1}{\mathcal{P}^2_\perp}\mathcal{O}^{jC}_{\bar{n}} + e^{-ix\cdot\mathcal{P}}\sum_{n}\sum_{i,j=q,g}\mathcal{O}^{iB}_{n}\frac{1}{\mathcal{P}^2_\perp}\mathcal{O}^{j_n B}_{s}$$

Summary of all components of Glauber operators:

$$\mathcal{O}^{qB}_{n} = \overline{\chi}_n T^B \frac{\slashed{\bar{n}}}{2}\,\chi_n \qquad \mathcal{O}^{gB}_{n} = \frac{i}{2} f^{BCD}\mathcal{B}^{C}_{n\perp\mu}\,\frac{\bar{n}}{2}\cdot(\mathcal{P}+\mathcal{P}^\dagger)\mathcal{B}^{D\mu}_{n\perp}$$

$$\mathcal{O}^{qB}_{\bar{n}} = \overline{\chi}_{\bar{n}} T^B \frac{\slashed{n}}{2}\,\chi_{\bar{n}} \qquad \mathcal{O}^{gB}_{\bar{n}} = \frac{i}{2} f^{BCD}\mathcal{B}^{C}_{\bar{n}\perp\mu}\,\frac{n}{2}\cdot(\mathcal{P}+\mathcal{P}^\dagger)\mathcal{B}^{D\mu}_{\bar{n}\perp}$$

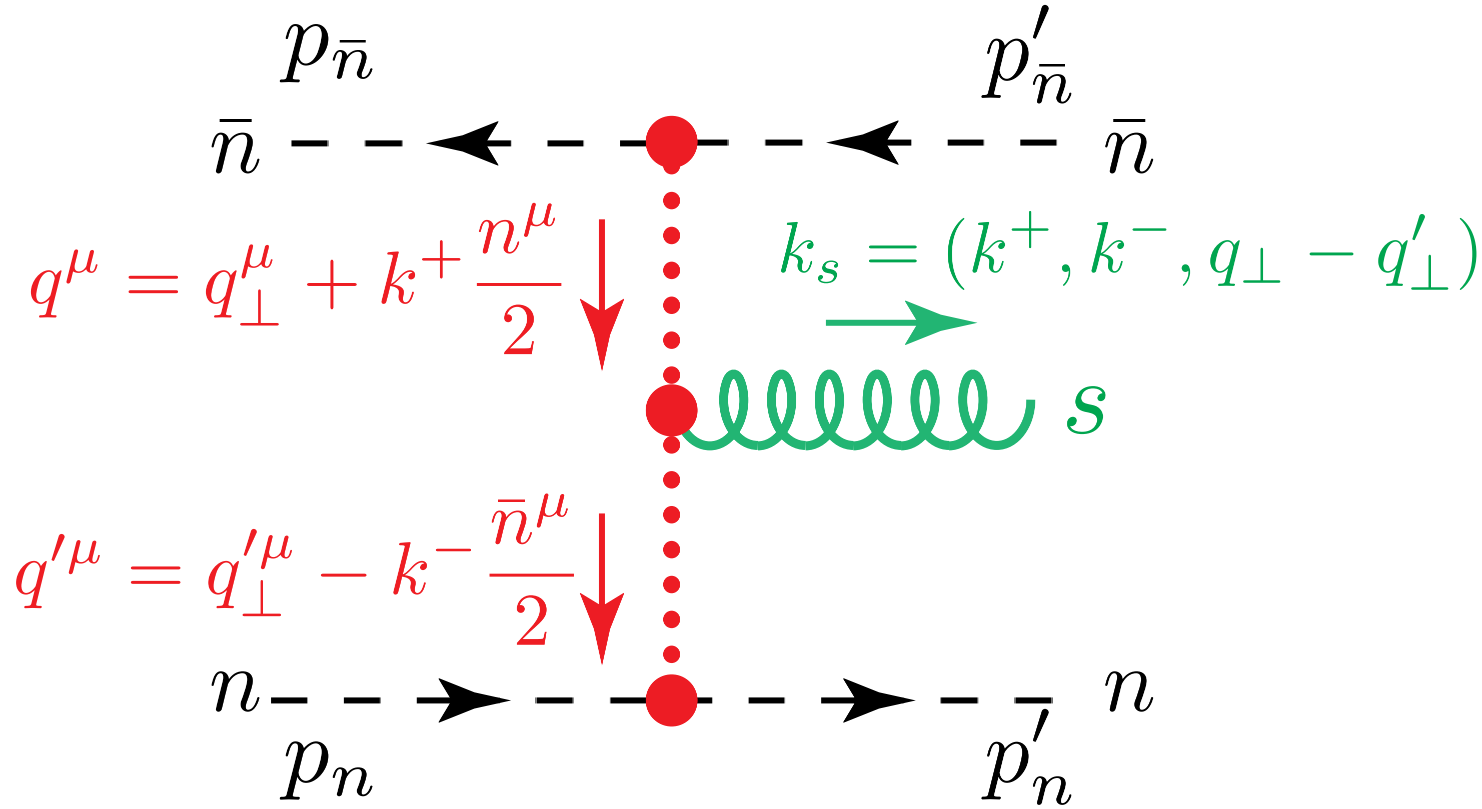
$$\mathcal{O}^{BC}_{s} = 8\pi\alpha_s\Big\{\mathcal{P}^\mu_\perp\mathcal{S}^T_n\mathcal{S}_{\bar{n}}\mathcal{P}_{\perp\mu} - \mathcal{P}^\perp_\mu g\widetilde{\mathcal{B}}^{n\mu}_{S\perp}\mathcal{S}^T_n\mathcal{S}_{\bar{n}} - \mathcal{S}^T_n\mathcal{S}_{\bar{n}}g\widetilde{\mathcal{B}}^{\bar{n}\mu}_{S\perp}\mathcal{P}^\perp_\mu - g\widetilde{\mathcal{B}}^{n\mu}_{S\perp}\mathcal{S}^T_n\mathcal{S}_{\bar{n}}g\widetilde{\mathcal{B}}^{\bar{n}}_{S\perp\mu} - \frac{n_\mu\bar{n}_\nu}{2}\mathcal{S}^T_n ig\widetilde{G}^{\mu\nu}_s\mathcal{S}_{\bar{n}}\Big\}^{BC}$$

$$\mathcal{O}^{q_n B}_{s} = 8\pi\alpha_s\left(\bar{\psi}^n_S\,T^B\frac{\slashed{\bar{n}}}{2}\psi^n_S\right) \qquad \mathcal{O}^{g_n B}_{s} = 8\pi\alpha_s\left(\frac{i}{2} f^{BCD}\mathcal{B}^{nC}_{S\perp\mu}\,\frac{n}{2}\cdot(\mathcal{P}+\mathcal{P}^\dagger)\mathcal{B}^{nD\mu}_{S\perp}\right)$$

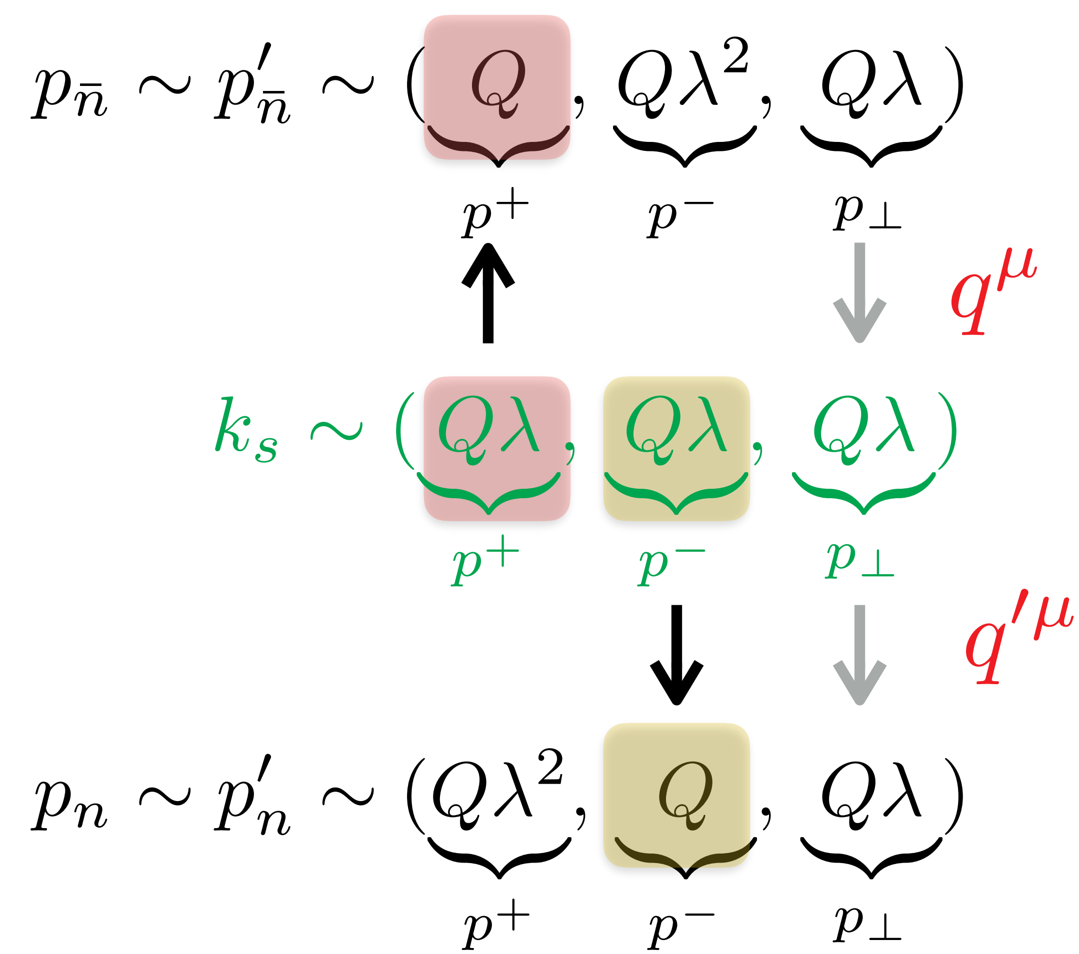
$$\mathcal{O}^{q_{\bar{n}} B}_{s} = 8\pi\alpha_s\left(\bar{\psi}^{\bar{n}}_S\,T^B\frac{\slashed{n}}{2}\psi^{\bar{n}}_S\right) \qquad \mathcal{O}^{g_{\bar{n}} B}_{s} = 8\pi\alpha_s\left(\frac{i}{2} f^{BCD}\mathcal{B}^{\bar{n}C}_{S\perp\mu}\,\frac{\bar{n}}{2}\cdot(\mathcal{P}+\mathcal{P}^\dagger)\mathcal{B}^{\bar{n}D\mu}_{S\perp}\right)$$

Sublabel Momenta

Consider the case of gluon emission from the mid rapidity operator:



The routing of soft momenta should preserve power counting



Introduce labels for routing O(λ) soft momenta on collinear and soft operators

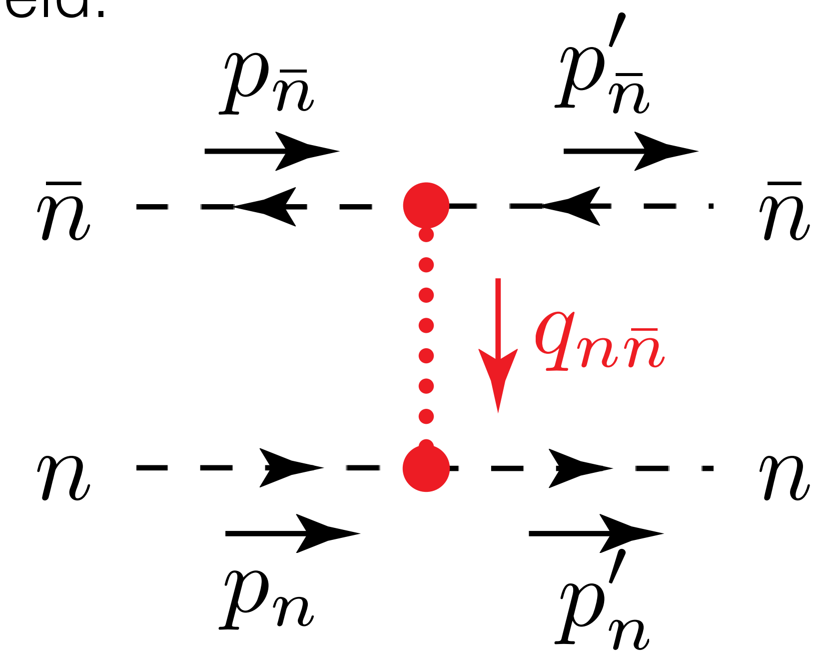
$$\int d^4x\, \mathcal{L}_G^{\mathrm{II}(0)} = \sum_{n,\bar n} \sum_{i,j=q,g} \int [dx^\pm] \sum_{k^+,k^-} \int \frac{d^2q_\perp}{q_\perp^2} \frac{d^2q'_\perp}{q'^2_\perp}\, \mathcal{O}^{iA}_{n,k^-}(q_\perp)\, \mathcal{O}^{AB}_{s,-k^\pm}(q_\perp, q'_\perp)\, \mathcal{O}^{jB}_{\bar n,k^+}(-q'_\perp)$$

$$+ \sum_{n} \sum_{i,j=q,g} \int [dx^\pm] \sum_{k^-} \int \frac{d^2q_\perp}{q_\perp^2}\, \mathcal{O}^{iA}_{n,k^-}(q_\perp)\, \mathcal{O}^{j_n A}_{s,-k^-}(-q_\perp)\,.$$

Features of Glauber Potential

We describe the forward scattering regime through a non-local Glauber potential instead of trying to introduce a mode for the off shell Glauber field.

$$V_{\mathrm{G}}(q_\perp) = -\frac{8\pi\alpha_s(\mu)}{\vec{q}_\perp^{\,2}}$$

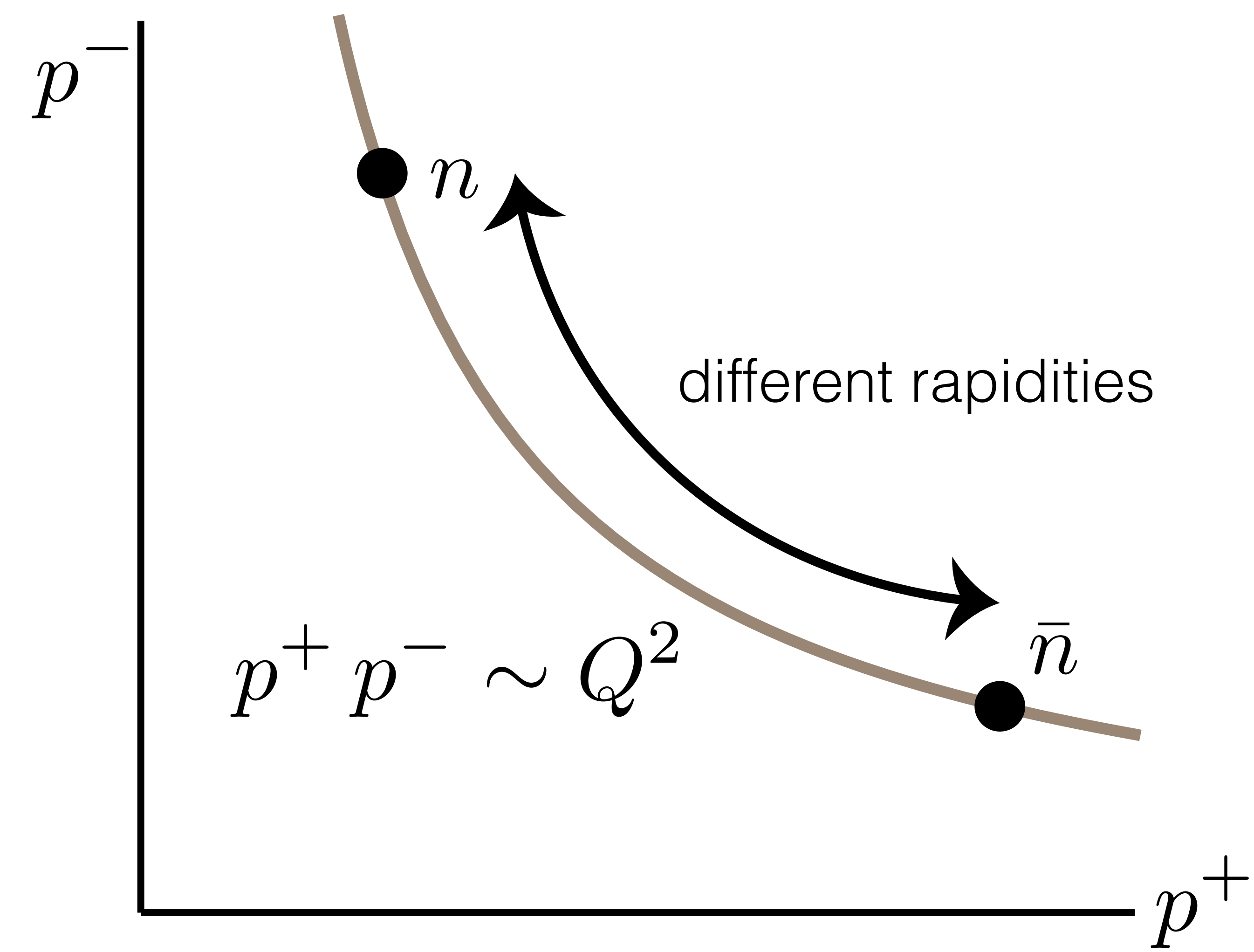


$$q^\mu_{n\bar n} \sim (Q\lambda^2, Q\lambda^2, Q\lambda) \qquad q_\perp^2 \gg q^+ q^-$$

This suggests that the Glauber exchange does NOT resolve the longitudinal x^+ and x^- directions, hence instantaneous in these directions.

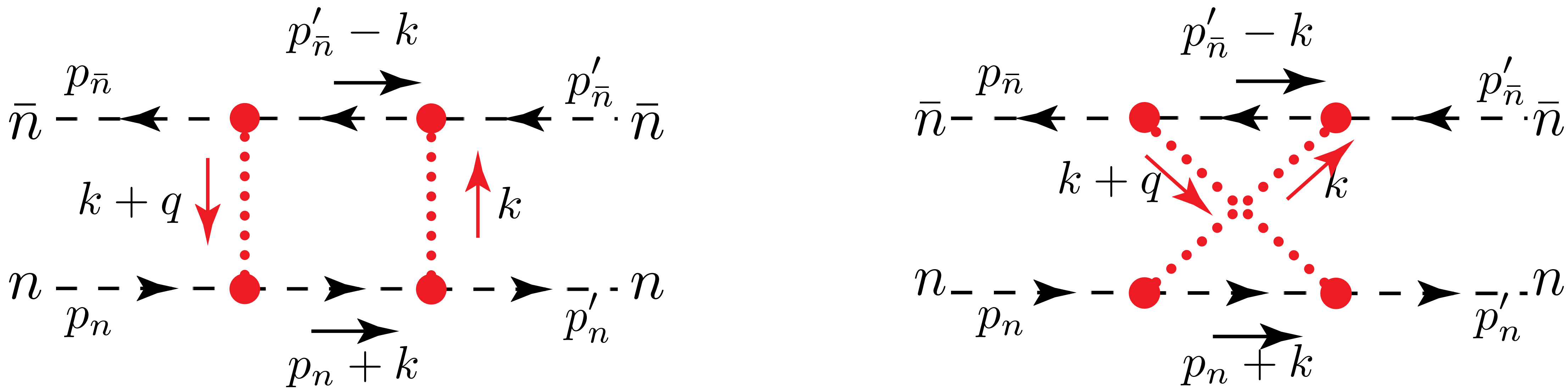
Connects fields living at two different rapidities, but same invariant mass.

What are then consequences of such nature of the Glauber potential?



Rapidity Divergences

Consider the box and cross box Glauber exchange:



$$k^-, q^- \ll p_n^-$$

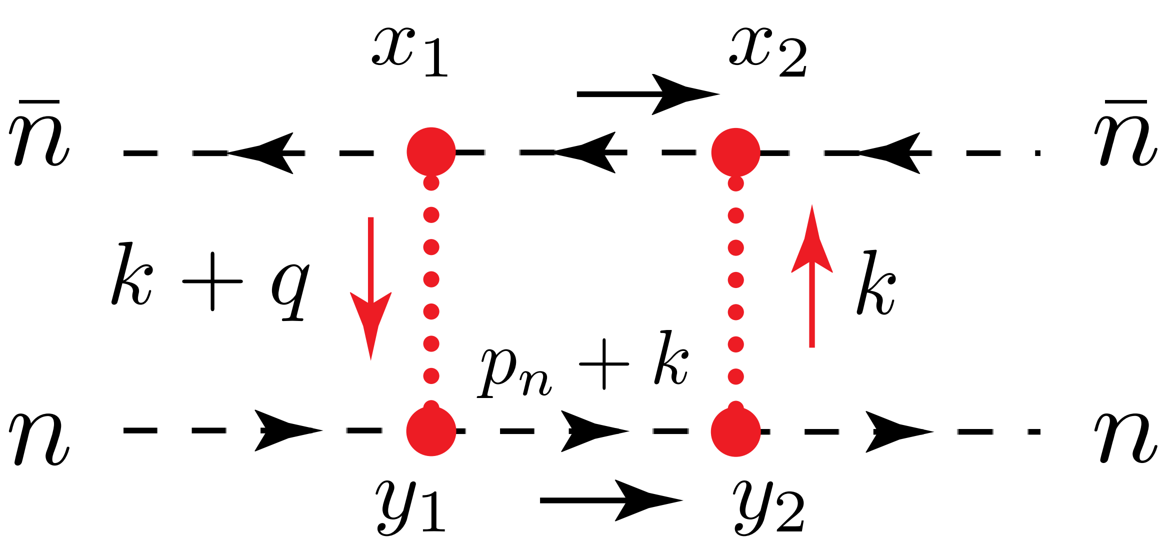
$$k^+, q^+ \ll p_{\bar n}^+$$

$$I_{\rm Gbox} = \int \frac{d^{d-2}k_\perp}{(2\pi)^d\, 2(\vec k_\perp^2)(\vec k_\perp + \vec q_\perp)^2} \times \frac{1}{2} \int \frac{dk^+}{k^+ + p_n^+ + \Delta_n(p_n^-, \vec k_\perp) + i0} \int \frac{-dk^-}{k^- - p_{\bar n}'^- + \Delta_{\bar n}(p_{\bar n}^+, \vec k_\perp + \vec q_\perp) - i0}$$

$$I_{\rm cGbox} = \int \frac{d^{d-2}k_\perp}{(2\pi)^d\, 2(\vec k_\perp^2)(\vec k_\perp + \vec q_\perp)^2} \times \frac{1}{2} \int \frac{dk^+}{k^+ + p_n^+ + \Delta_n(p_n^-, \vec k_\perp) + i0} \int \frac{+dk^-}{k^- + p_{\bar n}^- + \Delta_{\bar n}(p_{\bar n}^+, \vec k_\perp + \vec q_\perp) + i0}$$

These integrals cannot be regulated by dimensional regularization. We need to introduce a regulator that distinguishes modes at different rapidities.

Rapidity Divergences



Potential static in the longitudinal distance:

$$x_1^- = y_1^-\,,\; x_1^+ = y_1^+$$

$$x_2^- = y_2^-\,,\; x_2^+ = y_2^+$$

n-collinear propagator: $$i\frac{\not{n}}{2}\,\frac{\overbrace{p_n^-}^{\mathcal{O}(\lambda^0)} + \overbrace{\boxed{k^-}}^{\mathcal{O}(\lambda^2)}}{(p_n^+ + k^+)(\underbrace{p_n^-}_{\mathcal{O}(\lambda^0)} + \underbrace{\boxed{k^-}}_{\mathcal{O}(\lambda^2)}) - \vec{k}_\perp^{\,2} + i0}$$

drop these terms

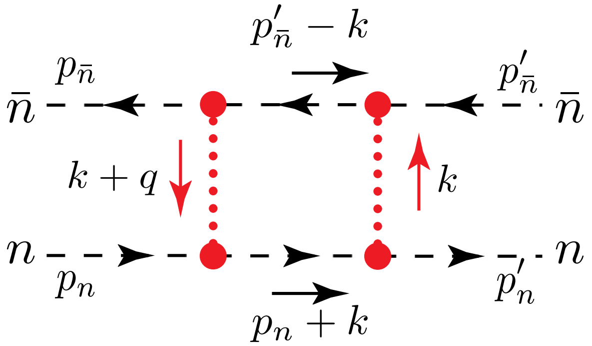
The O(λ²) Glauber momenta gets dropped. To regulate the k± divergences introduce a slight dependence on the longitudinal momenta, so that the Glauber exchange is no longer instantaneous in these directions

For every Glauber operator insertion insert this factor to make the Glauber exchange no longer static in the longitudinal distance, but only in the limit η ->0:

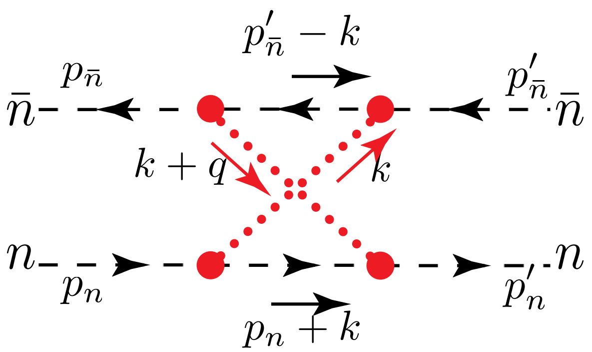
$$w^2\frac{\nu^\eta}{|2k_z|^\eta} = w^2\frac{\nu^\eta}{|k^- - k^+|^\eta}$$

Rapidity Divergences

The cross box diagram is zero with this regulator



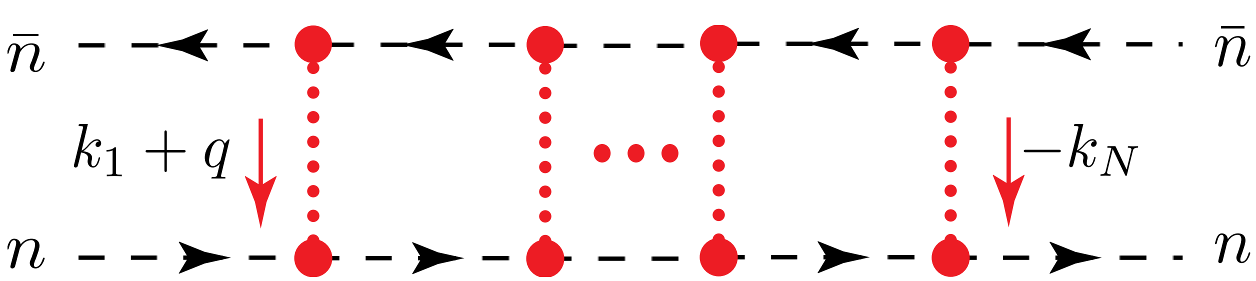
$$I_{\rm Gbox} = \frac{-i}{4\pi}\int \frac{d^{d-2}k_\perp}{(2\pi)^d\, 2(\vec k_\perp^{\,2})(\vec k_\perp + \vec q_\perp)^2}\Big[-i\pi + \mathcal{O}(\eta)\Big]$$



$$I_{\rm cGbox} = 0$$

Same as the result stated earlier from Polkinghorne's calculation.

This allows one to just consider a series of ladder diagrams. Diagrams with one or more crossed Glauber propagators are zero



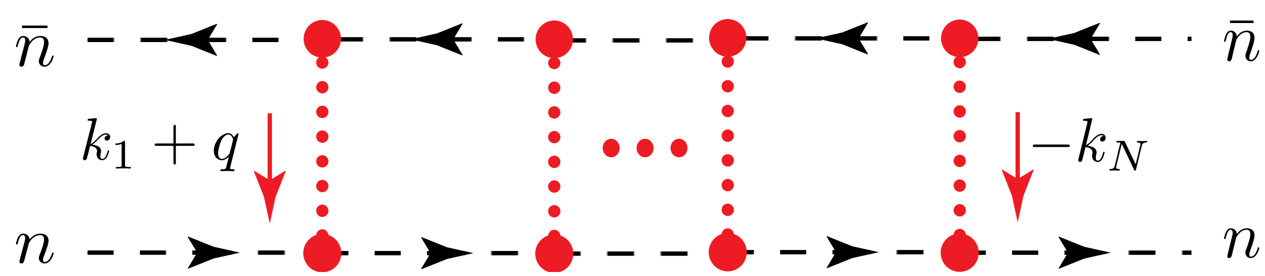
$$= 2(-ig^2)^{N+1}\mathcal{S}^{n\bar n}_{(N+1)} I_\perp^{(N)}(q_\perp)\ \frac{1}{(N+1)!}\Big[1+\mathcal{O}(\eta)\Big]$$

$$I_\perp^{(N)}(q_\perp) = \int \frac{{d\!\!{}^-}^{d-2}k_{1\perp}\cdots {d\!\!{}^-}^{d-2}k_{N\perp}\,(\iota^\epsilon\mu^{2\epsilon})^{N+1}}{(\vec k_{1\perp}+\vec q_\perp)^2(\vec k_{2\perp}-\vec k_{1\perp})^2\cdots(\vec k_{N\perp}-\vec k_{(N-1)\perp})^2\,\vec k_{N\perp}^{\,2}} \qquad \mathcal{S}^{n\bar n} = \Big[\bar u_n \frac{\bar{n}\!\!\!/}{2} u_n\Big]\Big[\bar v_{\bar n}\frac{n\!\!\!/}{2} v_{\bar n}\Big]$$

Rapidity Divergences

One can see the exponentiation by doing a Fourier transform to the impact parameter space:

$\mathrm{F.T.}_\perp$



$$= \frac{1}{(N+1)!}\left[i\phi(b_\perp)\right]^{N+1} 2\mathcal{S}^{n\bar{n}}$$

$$\phi(b_\perp) = -\mathbf{T}_1^A \otimes \mathbf{T}_2^A\, g^2(\mu) \int \frac{\slashed{d}^{d-2}q_\perp\,(\iota^\epsilon \mu^{2\epsilon})}{\vec{q}_\perp^{\,2}}\, e^{i\vec{q}_\perp\cdot\vec{b}_\perp}$$

The sum of all the ladder diagrams give

$$\int \slashed{d}^{d-2}q_\perp\, e^{i\vec{q}_\perp\cdot\vec{b}_\perp} \sum_{N=0}^{\infty} \mathrm{G.Box}_N^{2\to2}(q_\perp) = \left(\tilde{G}(b_\perp)-1\right)2\mathcal{S}^{n\bar{n}}$$

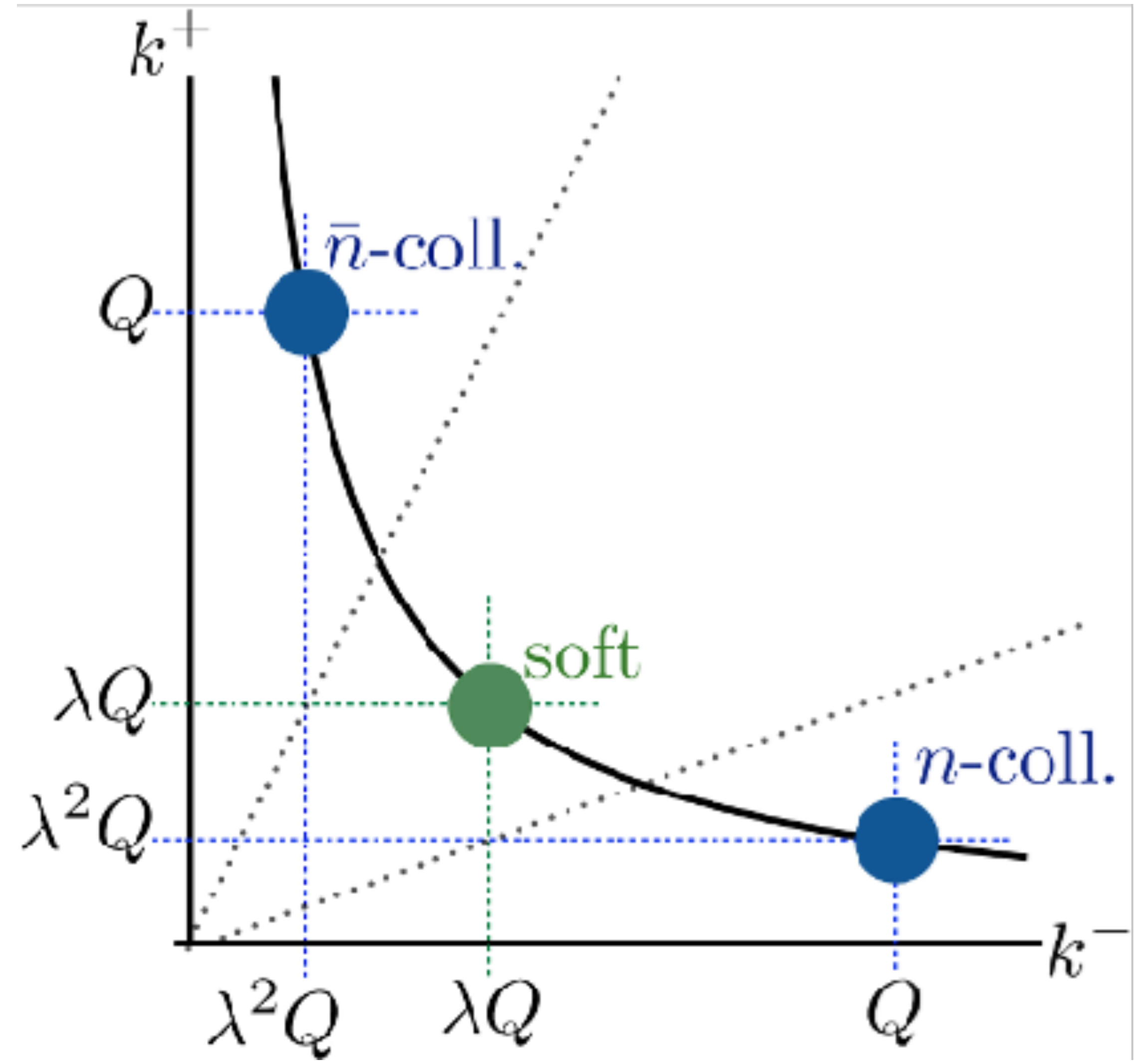
$$\tilde{G}(b_\perp) = e^{i\phi(b_\perp)}$$

$$G(q_\perp) = \int d^2b_\perp\, e^{-i\vec{q}_\perp\cdot\vec{b}_\perp}\, e^{i\phi(b_\perp)}$$

Rapidity Divergences

Rapidity divergences also occur in loops other than Glauber exchanges: A general feature of the $\mathrm{SCET_{II}}$ theory

On factorizing a matrix element into soft and collinear pieces one finds rapidity divergences that only cancel in the sum:



$$I = \int_{\mu_L}^{Q} \frac{dk^+}{k^+} \to \int_{\mu_L}^{\Lambda} \frac{dk^+}{k^+} + \int_{\Lambda}^{Q} \frac{dk^+}{k^+}$$

$$\to \int_{\mu_L}^{\infty} \frac{dk^+}{k^+} + \int_{\infty}^{Q} \frac{dk^+}{k^+}$$

To treat these divergences we include modify the momentum space Wilson lines:

$$W_n = \sum_{\text{perms}} \exp\left[-\frac{gw^2}{\bar{n}\cdot\mathcal{P}} \frac{|\,\bar{n}\cdot\mathcal{P}_g\,|^{-\eta}}{\nu^{-\eta}} \bar{n}\cdot A_n\right]$$

$$S_n = \sum_{\text{perms}} \exp\left[-\frac{gw}{n\cdot\mathcal{P}} \frac{|\,2\mathcal{P}_{g3}\,|^{-\eta/2}}{\nu^{-\eta/2}} n\cdot A_s\right]$$

v acts as a rapidity cutoff. w is a bookkeeping parameter

$$\nu \frac{\partial}{\partial\nu} w^2(\nu) = -\eta w^2(\nu)\,, \qquad \lim_{\eta\to 0} w(\nu) = 1$$

Rapidity Divergences

Incorporate rapidity regulators in the Wilson lines and in the Glauber Lagrangian

$$S_n = \sum_{\text{perms}} \exp\left\{ \frac{-g}{n\cdot\mathcal{P}} \left[\frac{w|2\mathcal{P}^z|^{-\eta/2}}{\nu^{-\eta/2}} n\cdot A_s \right] \right\}, \qquad S_{\bar n} = \sum_{\text{perms}} \exp\left\{ \frac{-g}{\bar n\cdot\mathcal{P}} \left[\frac{w|2\mathcal{P}^z|^{-\eta/2}}{\nu^{-\eta/2}} \bar n\cdot A_s \right] \right\},$$

$$W_n = \sum_{\text{perms}} \exp\left\{ \frac{-g}{\bar n\cdot\mathcal{P}} \left[\frac{w^2|\bar n\cdot\mathcal{P}|^{-\eta}}{\nu^{-\eta}} \bar n\cdot A_n \right] \right\}, \quad W_{\bar n} = \sum_{\text{perms}} \exp\left\{ \frac{-g}{n\cdot\mathcal{P}} \left[\frac{w^2|n\cdot\mathcal{P}|^{-\eta}}{\nu^{-\eta}} n\cdot A_{\bar n} \right] \right\}.$$

$$\begin{aligned}\int d^4x\, \mathcal{L}_G^{\mathrm{II}(0)} &= \sum_{n,\bar n}\sum_{i,j=q,g}\int [dx^\pm] \sum_{k_r^+,k_r^-} \int \frac{d^2q_\perp}{q_\perp^2}\frac{d^2q'_\perp}{q'^2_\perp}\, \mathcal{O}_{s,-k_r^\pm}^{AB}(q_\perp,q'_\perp) \\ &\times \left[\mathcal{O}_{n,k_r^-}^{iA}(q_\perp)\, w^2 \left| \frac{in\cdot\overleftarrow{\partial} + i\bar n\cdot\vec{\partial}}{\nu} \right|^{-\eta} \mathcal{O}_{\bar n,k_r^+}^{jB}(-q'_\perp) \right] \\ &+ \sum_n \sum_{i,j=q,g} \int [dx^\pm] \sum_{k_r^-} \int \frac{d^2q_\perp}{q_\perp^2}\, \mathcal{O}_{n,-k_r^-}^{iA}(q_\perp)\, w^2 \left| \frac{-\beta_{ns}\, k_r^- + i\bar n\cdot\vec{\partial}}{\nu} \right|^{-\eta} \mathcal{O}_{s,k_r^-}^{j_n A}(-q_\perp)\end{aligned}$$

$$\beta_{ns} \sim \mathcal{O}(\lambda)$$

BFKL from Glauber Lagrangian

We are in now position to write down cross section for forward scattering

$$\sigma_{pp'\to X} \sim \sum_X \langle pp' | U^\dagger_{(1,1)} | X \rangle \langle X | U_{(1,1)} | pp \rangle$$

$U_{(k,k')}$ refers to the term in Glauber Lagrangian with k n-collinear operator insertions and k' n-bar-collinear operators.

$$U_{(1,1)} = i \int [dx^\pm][dx'^\pm] \sum_{k^\pm} \int \frac{d^2 q_\perp}{q_\perp^2} \frac{d^2 q'_\perp}{q'^2_\perp} \big[\mathcal{O}^{qA}_{n,k^-}(q_\perp) + \mathcal{O}^{gA}_{n,k^-}(q_\perp)\big](\tilde{x}) \big[\mathcal{O}^{qB}_{\bar{n},k^+}(q'_\perp) + \mathcal{O}^{gB}_{\bar{n},k^+}(q'_\perp)\big](\tilde{x}') \\ \times O^{AB}_{s(1,1),-k^\pm}(q_\perp, q'_\perp)(\tilde{x}, \tilde{x}') \,.$$

$O_{s(1,1)}$ contains a single insertion of the 3-rapidity and a T product of two 2-rapidity operators.

For this operator we can factorize the cross section by splitting the final state into different rapidity sectors:

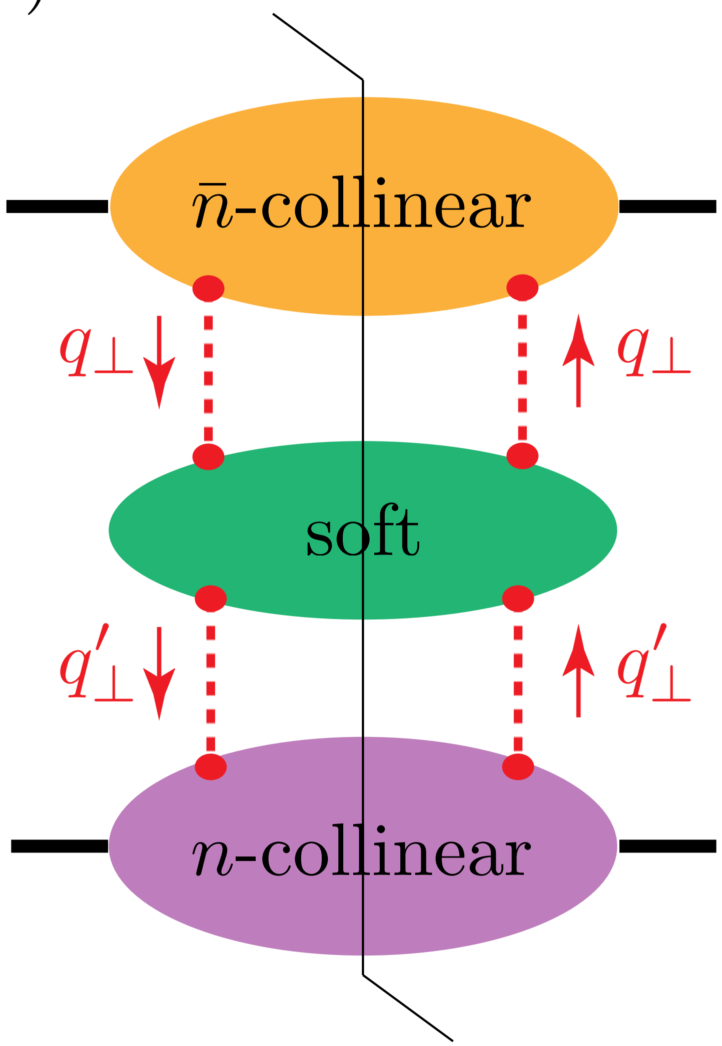
$$|X\rangle \to |X_n\rangle \, |X_{\bar{n}}\rangle \, |X_s\rangle$$

BFKL from Glauber Lagrangian

The cross section can then be manipulated to give

$$\sigma_{pp'\to X} = \int d^2q_\perp d^2q'_\perp C_{\bar{n}}(q'_\perp, p'^+) S_G(q_\perp, q'_\perp) C_n(q'_\perp, p^-)$$

We have factorized the cross section by separating the collinear and soft matrix elements of Glauber operators. This factorization holds for lowest order graphs that give Leading Log BFKL.



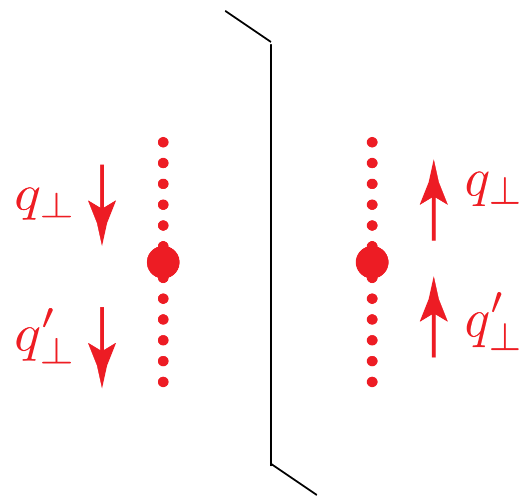
$$C_n(q'_\perp, p^-) \equiv \sum_{i,i'=q,g} \sum_{X_n} (2\pi)^3 \delta^2(P_{X_n} - q'_\perp)\delta(P^-_{X_n} - p^-)$$
$$\times \langle p|\mathcal{O}_n^{i'A\dagger}(0)|X_n\rangle\langle X_n|\mathcal{O}_n^{iA}(0)|p\rangle$$

$$S_G(q_\perp, q'_\perp) = \frac{1}{\vec{q}_\perp^{\,2}\vec{q}_\perp^{\,\prime 2}} \sum_{X_s} (2\pi)^4 \delta^2(q_\perp - q'_\perp - P^\perp_{X_s})\delta(q^+ - P^+_{X_s})\delta(q'^-_\perp + P^-_{X_s})$$
$$\times \langle 0|O^{AB}_{s(1,1)}(q_\perp, q'_\perp)^\dagger|X_s\rangle\langle X_s|O^{AB}_{s(1,1)}(q_\perp, q'_\perp)|0\rangle$$

BFKL for the Soft Function

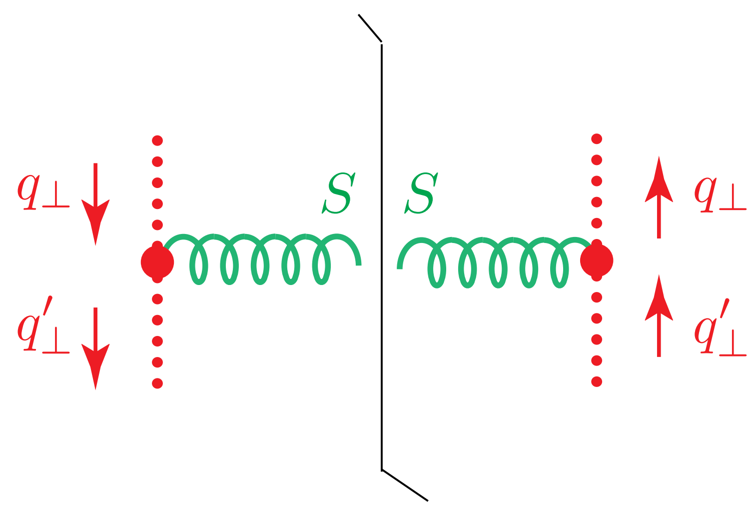
We can derive the LL BFKL equation for soft function by collecting all the rapidity divergent pieces for either purely soft or collinear diagrams. We choose soft here:

Tree level bare graph:



$$= S_G^{(0)}(q_\perp, q'_\perp) = (8\pi\alpha_s)(N_c^2-1)(2\pi)^2\delta^2(\vec{q}_\perp - \vec{q}\,'_\perp)$$

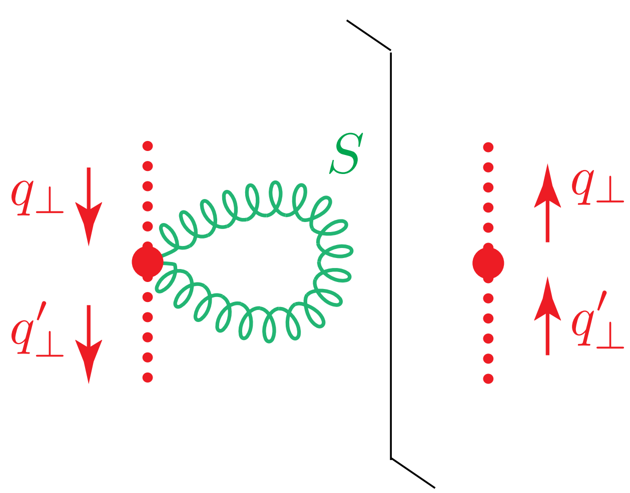
Real radiation:



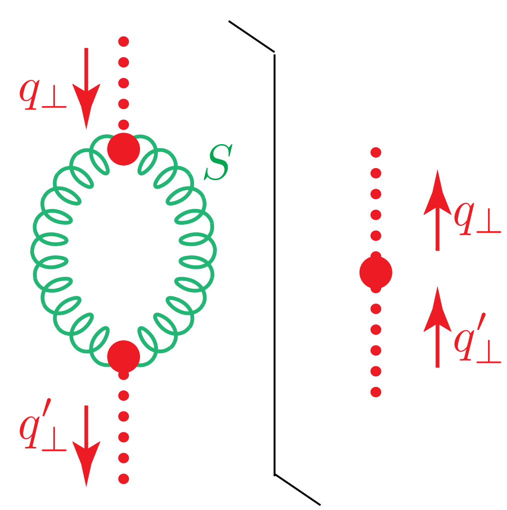
$$= \frac{C_A\alpha_s}{\pi^2} w^2 \Gamma\left(\frac{\eta}{2}\right) \int \frac{d^2k_\perp}{(\vec{k}_\perp - \vec{q}_\perp)^2} S_G^{(0)}(k_\perp, q'_\perp)$$

Virtual corrections:

2



$+2$



$$= -\frac{C_A\alpha_s}{2\pi^2} w^2 \Gamma\left(\frac{\eta}{2}\right) \int d^2k_\perp \frac{\vec{q}_\perp^{\,2}}{\vec{k}_\perp^{\,2}(\vec{k}_\perp - \vec{q}_\perp)^2} S_G^{(0)}(q_\perp, q'_\perp)$$

BFKL for the Soft Function

Putting the pieces together we get:

$$S_G^{\mathrm{bare}}(q_\perp, q'_\perp) = S_G^{(0)}(q_\perp, q'_\perp) + \frac{\alpha_s C_A}{\pi^2} w^2 \Gamma\left(\frac{\eta}{2}\right) \int \frac{d^2 k_\perp}{(\vec{k}_\perp - \vec{q}_\perp)^2} \left[S_G^{(0)}(k_\perp, q'_\perp) - \frac{\vec{q}_\perp^{\,2}}{2\vec{k}_\perp^{\,2}} S_G^{(0)}(q_\perp, q'_\perp) \right]$$

The bare soft function is then renormalized:

$$S_G(\vec{q}_\perp, \vec{q}'_\perp, \nu) = \int d^2 k_\perp \; Z_{S_G}(q_\perp, k_\perp) \; S_G^{\mathrm{bare}}(k_\perp, q'_\perp)$$

$$Z_{S_G}(q_\perp, k_\perp) = \delta^2(\vec{q}_\perp - \vec{k}_\perp) - \frac{2 C_A \alpha_s(\mu) w^2(\nu)}{\pi^2 \, \eta} \left[\frac{1}{(\vec{k}_\perp - \vec{q}_\perp)^2} - \delta^2(\vec{q}_\perp - \vec{k}_\perp) \int \frac{d^2 k'_\perp \; \vec{q}_\perp^{\,2}}{2 \vec{k}_\perp^{\prime 2} (\vec{k}'_\perp - \vec{q}_\perp)^2} \right]$$

This now allows us to derive an RG equation for the soft function in rapidity space …

$$0 = \nu \frac{d}{d\nu} S_G^{\mathrm{bare}}(q_\perp, q'_\perp) = \nu \frac{d}{d\nu} \int d^2 k_\perp \; Z_{S_G}^{-1}(q_\perp, k_\perp) \; S_G(k_\perp, q'_\perp, \nu)$$

$$\nu \frac{d}{d\nu} S_G(q_\perp, q'_\perp, \nu) = \int d^2 k_\perp \; \gamma_{S_G}(q_\perp, k_\perp) \; S_G(k_\perp, q'_\perp, \nu)$$

BFKL for the Collinear Function from consistency

… and the RG equation is precisely the Leading Log BFKL equation:

$$\nu \frac{d}{d\nu} S_G(q_\perp, q'_\perp, \nu) = \frac{2C_A \alpha_s(\mu)}{\pi^2} \int d^2 k_\perp \left[\frac{S_G(k_\perp, q'_\perp, \nu)}{(\vec{k}_\perp - \vec{q}_\perp)^2} - \frac{\vec{q}_\perp^{\,2}\, S_G(q_\perp, q'_\perp, \nu)}{2\vec{k}_\perp^2 (\vec{k}_\perp - \vec{q}_\perp)^2} \right]$$

Through the consistency relation one can determine the RG for the collinear sectors:

$$\nu \frac{d}{d\nu} \int d^2 q_\perp d^2 q'_\perp C_{\bar{n}}(q'_\perp, p'^+) S_G(q_\perp, q'_\perp) C_n(q'_\perp, p^-) = 0$$

$$\gamma_C(q_\perp, q'_\perp) = -\frac{1}{2}\, \gamma_{S_G}(q_\perp, q'_\perp)$$

$$\nu \frac{d}{d\nu} C_n(q_\perp, p^-, \nu) = -\frac{C_A \alpha_s}{\pi^2} \int d^2 k_\perp \left[\frac{C_n(k_\perp, p^-, \nu)}{(\vec{k}_\perp - \vec{q}_\perp)^2} - \frac{\vec{q}_\perp^{\,2}\, C_n(q_\perp, p^-, \nu)}{2\vec{k}_\perp^2 (\vec{k}_\perp - \vec{q}_\perp)^2} \right]$$

This is an important result since the same collinear function appears in the DIS cross section in the small-x limit.

Application

Small-x Deep Inelastic Scattering

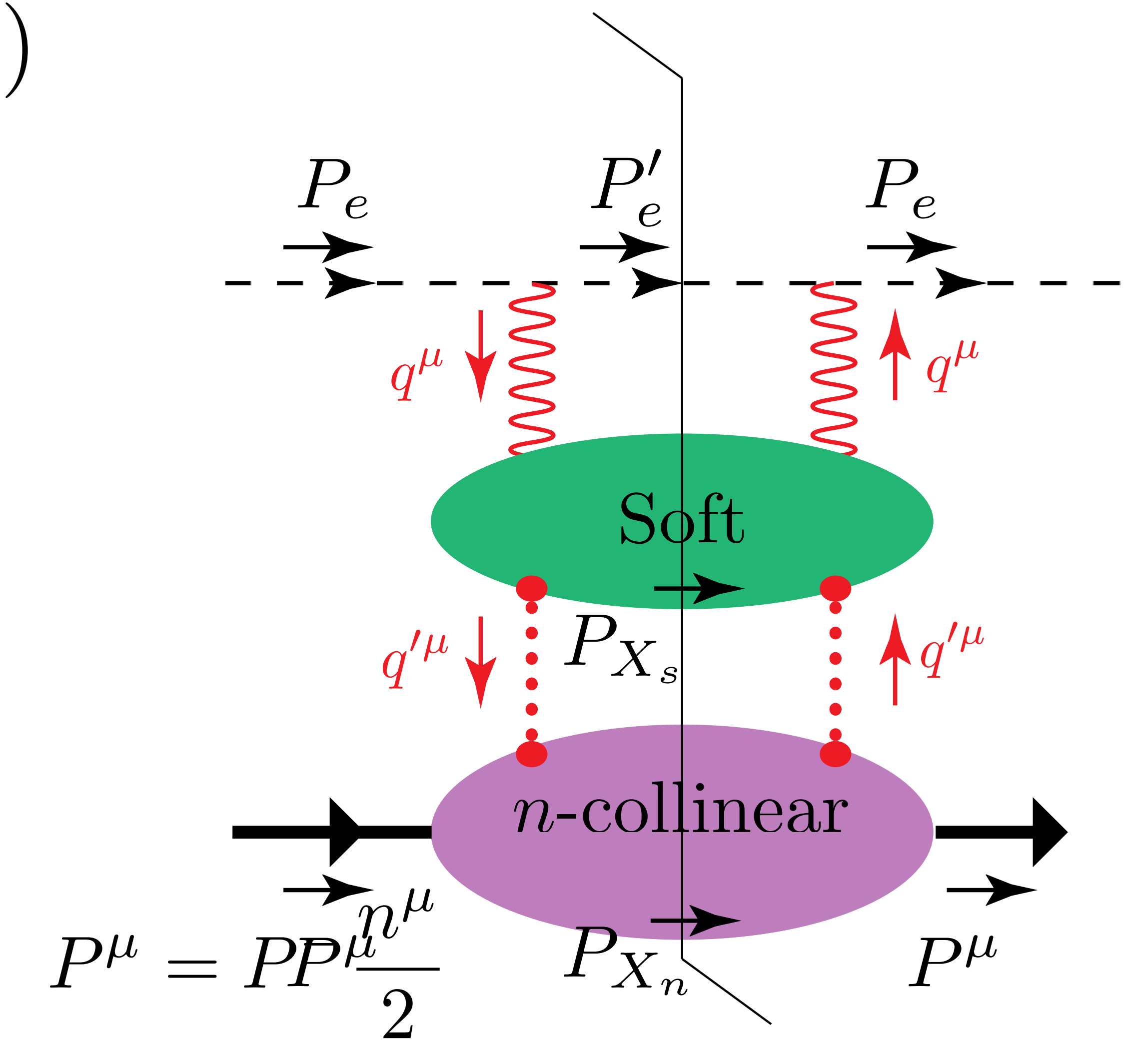
DIS Kinematics

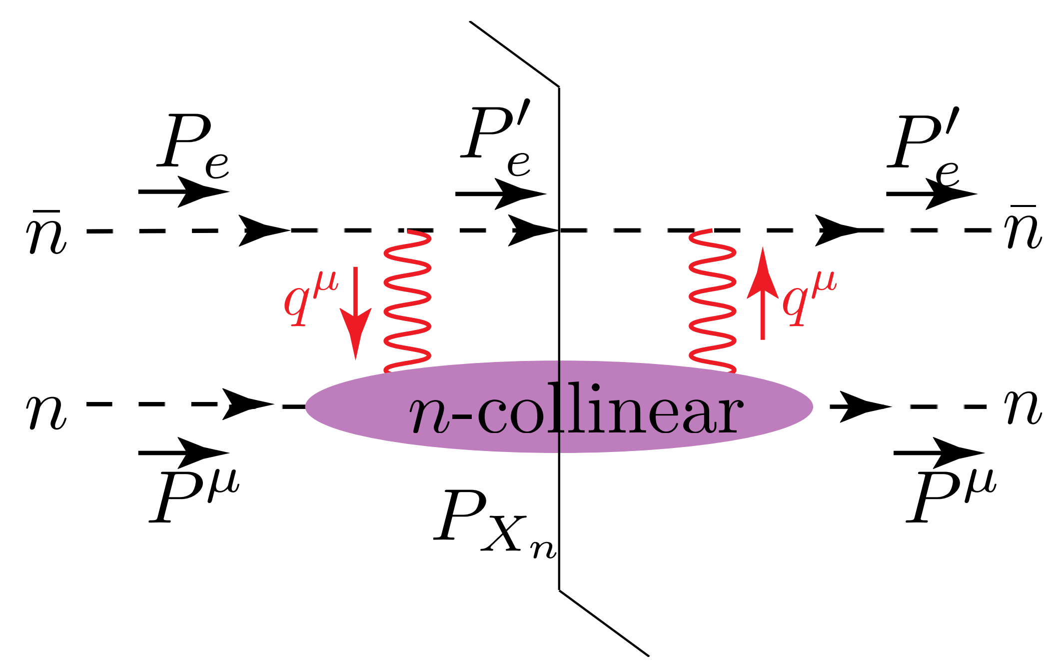
$$s = (P_e + P)^2$$

$$q^2 = -Q^2 = (P_e' - P_e)^2$$

$$e(P_e) + p(P) \to e(P_e') + p(P_X)$$

- We have the same scaling for the momenta to give us the forward scattering configuration $Q^2 << s$.
- The Photon is treated without expanding. We have two QCD rapidity sectors now.
    - The more interesting piece has a soft sector at the scale Q.
- BFKL arises between the Soft and collinear sector.
    - Even the direct piece has BFKL between soft and collinear sectors, but it is at the scale $\sim\Lambda_{QCD}$, not at Q





DIS Kinematics

Fixing x and Q^2 implies:

$$q_\perp^2 = -Q^2 \qquad q^+ = k_s^+ = -\frac{Q^2}{xP^-} = -\frac{Q^2}{x\sqrt{s}}$$

$$Q^2 = x\,y\,s$$

$$q^\mu = \Big(\underbrace{y\sqrt{s}}_{k_s^+ \sim \mathcal{O}(\lambda)} \,, \underbrace{-xy\sqrt{s}}_{\mathcal{O}(\lambda^2)}, \underbrace{q_\perp^\mu}_{\mathcal{O}(\lambda)} \Big)$$

x fixes this

Q^2 fixes this

Power counting:

$$\boxed{\begin{aligned} \lambda &\sim \frac{Q}{\sqrt{s}} \\ x &\sim \lambda \\ y &\sim \lambda \end{aligned}}$$

recall the flow of momenta:

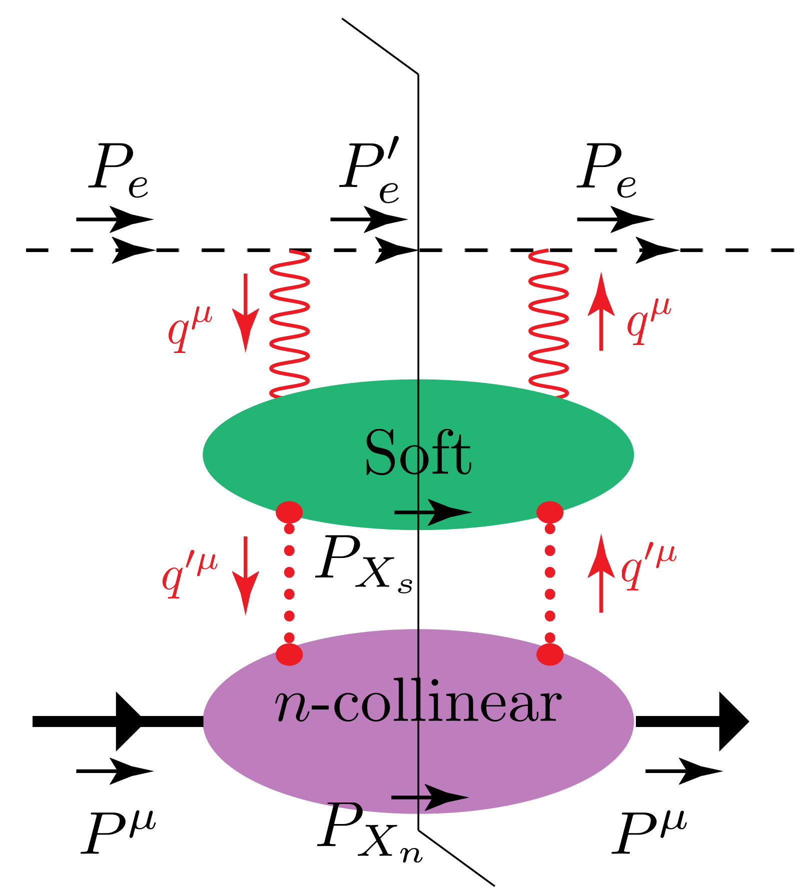
$$p_{\bar{n}} \sim p'_{\bar{n}} \sim (\underbrace{Q}_{p^+}, \underbrace{Q\lambda^2}_{p^-}, \underbrace{Q\lambda}_{p_\perp})$$

$$\uparrow \qquad \downarrow q^\mu$$

$$k_s \sim (\underbrace{Q\lambda}_{p^+}, \underbrace{Q\lambda}_{p^-}, \underbrace{Q\lambda}_{p_\perp})$$

$$\downarrow \qquad \downarrow q'^\mu$$

$$p_n \sim p'_n \sim (\underbrace{Q\lambda^2}_{p^+}, \underbrace{Q}_{p^-}, \underbrace{Q\lambda}_{p_\perp})$$



Small-x Factorization of Hadronic Tensor

Here we sketch a quick derivation of the factorization of the Hadronic tensor:

$$\frac{d\sigma}{dx dQ^2} = \sum_{I,I'=V,A} L^{II'}_{\mu\nu}(x,Q^2) W^{II'\mu\nu}(x,Q^2)$$

Focus on the hadronic part:

$$W^{\mu\nu}(x,Q^2) = \sum_{\text{avg. spin}} \sum_X \int d^4x'' e^{x''\cdot q} \langle P | J^\mu_{\text{QCD}}(0) | X \rangle \langle X | J^\nu_{\text{QCD}}(x'') | P \rangle$$

$$\langle X | J^{\mu f}_{\text{QCD}}(x'') | P \rangle \to \langle X_n X_s | \mathrm{T} J^{\mu f}_{\text{QCD}}(x'') i \int d^4x \mathcal{L}^{(0)}_G(x) | P \rangle$$

$$= \sum_{ij=q,g} \int \frac{dx^+ dx^-}{2} (2\pi)^2 \sum_{k^-} \int \frac{d^2 q'_\perp}{\vec{q}'^2_\perp}$$

$$\langle X_n X_s | \mathrm{T} \mathcal{O}^{iA}_{n,k^-}(q'_\perp)(\tilde{x}) \mathcal{O}^{j_n A}_{s,-k^-}(-q'_\perp)(\tilde{x}) J^{\mu f}_{\text{QCD}}(x'') | P \rangle$$

To be able to factorize the matrix element strip out the O(λ^0) momentum from J_{QCD} so that it only has O(λ) soft momenta left:

$$J^{\mu f}_{\text{QCD}}(x'') = e^{\frac{i}{2} x''\cdot \mathcal{P}^-} J^{\mu f}_{\text{soft}}(x'') e^{-\frac{i}{2} x''\cdot \mathcal{P}^-}$$

$$= e^{\frac{i}{2} x''\cdot \mathcal{P}^-} \bar{\psi}^f_S \gamma^\mu \psi^f_S(x'') e^{-\frac{i}{2} x''\cdot \mathcal{P}^-}$$

Small-x Factorization of Hadronic Tensor

Steps that follow:

- Power counting allows us to drop the soft momenta label on the collinear operators.
- O(λ²) residual momenta can be dropped everywhere
- The k⁻ label on the soft momenta can then be made continuous
- Apply momentum conservation

$$W^{\mu\nu}(x,Q^2) = \int \frac{d^2q'_\perp}{\vec{q}'^2_\perp} C_n(q'_\perp, P^-, \nu, \Lambda_{\rm QCD})\, P^- S^{\mu\nu}(q_\perp, q'_\perp, xP^-, \nu)\Big[1+\mathcal{O}(x)\Big]$$

$$S^{\mu\nu}(q_\perp, q'_\perp, xP^-, \nu) \equiv \frac{1}{\vec{q}'^2_\perp}\sum_{X_s}(2\pi)^3\delta^2(q_\perp - q'_\perp - P^\perp_{X_s})\,\delta(q^+ - P^+_{X_s})$$
$$\times \sum_{j,j'=qg}\langle 0|\overline{\mathrm{T}}\int d^2\tilde{x}'\,\mathcal{O}_s^{j'_n A}(\tilde{x}')J^{\mu f}_{\rm soft}(0,-q_\perp)|X_s\rangle\langle X_s|\mathrm{T}\int d^2\tilde{x}\,\mathcal{O}_s^{j_n A}(\tilde{x})J^{\nu f}_{\rm soft}(0,q_\perp)|0\rangle$$

$$C_n(q'_\perp, P^-, \nu, \Lambda_{\rm QCD})P^-\delta^{AA'} \equiv \sum_{\rm spins}\sum_{X_n}(2\pi)^3\delta^2(P^\perp_{X_n} - \vec{q}'_\perp)\delta(P^-_{X_n} - P^-)$$
$$\times \sum_{i,i'=qg}\langle \mathcal{O}_n^{i'A'}(0)|X_n\rangle\langle X_n|\mathcal{O}_n^{iA}(0)|P\rangle$$

Sum logs of x

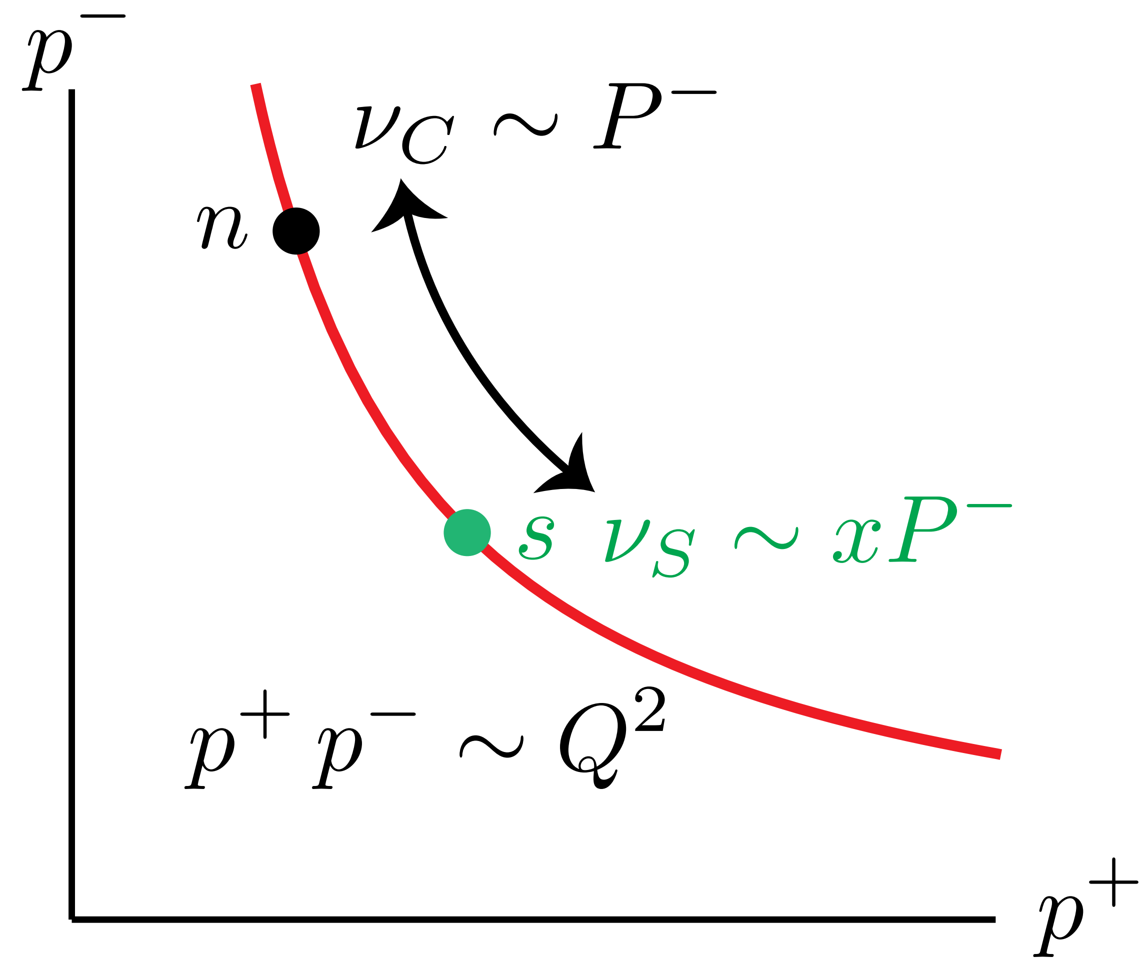
$$\frac{d\sigma}{dx dQ^2} = \frac{-\alpha_{\rm em}^2}{2x^2 s^2}\left(\frac{-s}{Q^2}\right) L_{\mu\nu}(Q^2) W^{\mu\nu}(x,Q^2)$$

$$\frac{\bar{n}_\mu \bar{n}_\nu}{(P^-)^2} W_{\mu\nu} \xrightarrow[x\to 0]{} \underbrace{\frac{xy^2}{2Q^2} F_L}_{\mathcal{O}(\lambda)} + \underbrace{\frac{2x}{Q^2}(1-y) F_2}_{\mathcal{O}(\lambda^{-1})}$$

$$F_2 = \frac{Q^2 \bar{n}_\mu \bar{n}_\nu W^{\mu\nu}}{2xs} + \ldots\,, \qquad F_L = \frac{2x^3 s}{Q^2} n_\mu n_\nu W^{\mu\nu}$$

$F_{2,L}$ correspond to taking appropriate projections on the soft function $S^{\mu\nu}$.

F_2 also has a contribution from a direct piece that is shown later.



$$F_{2,L}\left(x, \frac{Q^2}{\Lambda_{\rm QCD}^2}\right) = \int \frac{d^2 q'_\perp}{\vec{q}_\perp^{\,\prime 2}} S_{2,L}\left(q_\perp, q'_\perp, \frac{\nu_S}{xP^-}, \nu\right) C_n\left(\frac{q'_\perp}{\Lambda_{\rm QCD}}, \frac{\nu_S}{P^-}\right)$$

$$= \int \frac{d^2 q'_\perp}{\vec{q}_\perp^{\,\prime 2}} \int \frac{d^2 q''_\perp}{\vec{q}_\perp^{\,\prime\prime 2}} S_{2,L}\left(q_\perp, q'_\perp, \frac{\nu_S}{xP^-}, \nu\right) U_\nu(\nu_S, \nu_C, q'_\perp, q''_\perp)\, C_n\left(\frac{q''_\perp}{\Lambda_{\rm QCD}}, \frac{\nu_C}{P^-}\right)$$

Here the logs of x are related to rapidity divergence and are resummed by BFKL.

$$\boxed{\ln x = \ln\left(\frac{\nu_S}{\nu_C}\right)}$$

μ-Factorization of the Collinear Function

We assume that l ~ Q is a perturbative scale and hence we need to resum the large logs in the collinear function

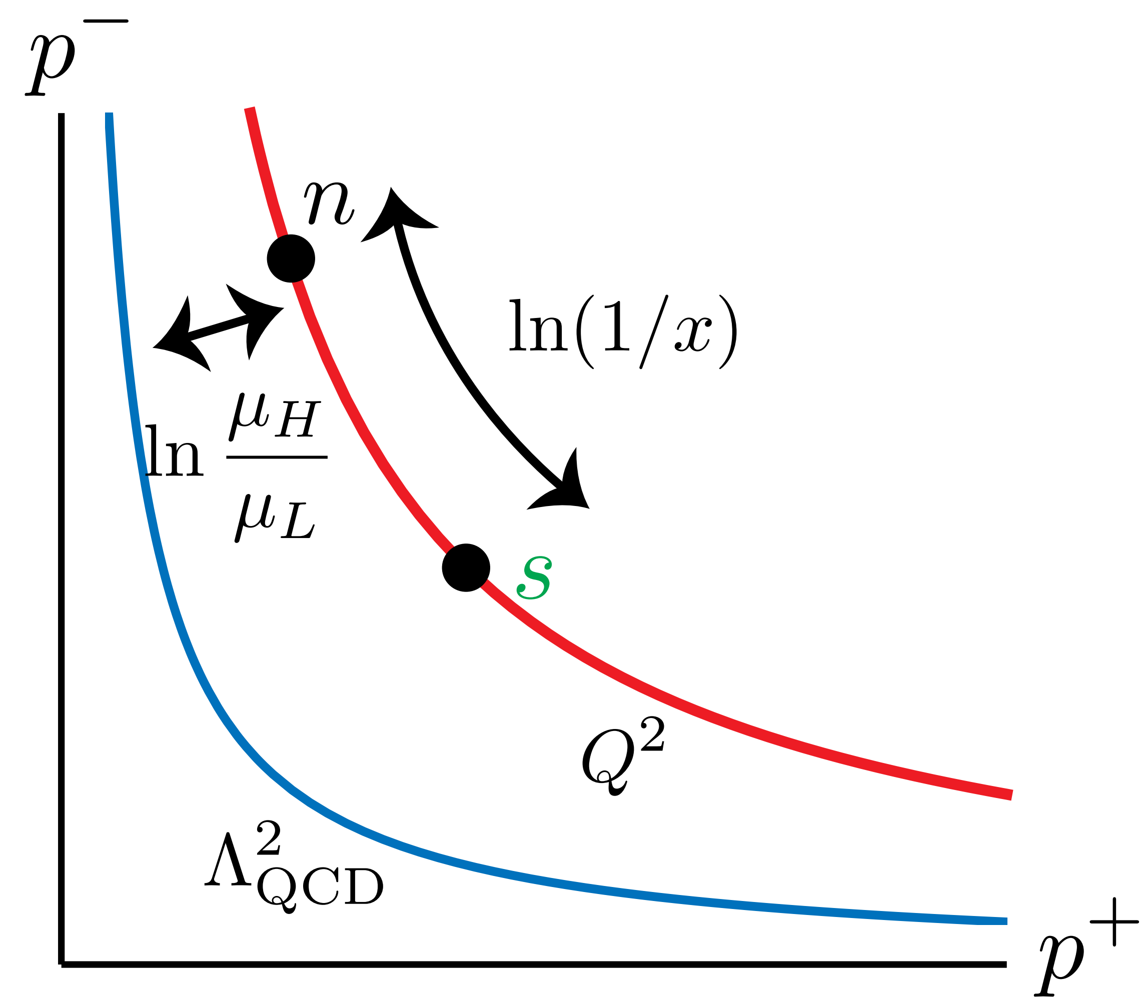
$$C_n\left(\underbrace{\frac{l_\perp}{\Lambda_{\rm QCD}}}_{\gg 1}, \underbrace{\frac{\nu_C}{P^-}}_{\sim 1}\right)$$

$$C_n\left(\frac{l_\perp}{\Lambda_{\rm QCD}}, \frac{\nu_C}{P^-}\right) = \int_0^1 d\xi H_j\left(\frac{l_\perp}{\mu}, \frac{\nu_C}{P^-\xi}\right) f_j\left(\xi, \frac{\mu}{\Lambda_{\rm QCD}}\right)\left[1+\mathcal{O}\left(\frac{\Lambda_{\rm QCD}}{Q}\right)\right]$$

$$= \int_0^1 d\xi \int_0^1 \frac{d\xi'}{\xi'} H_j\left(\frac{l_\perp}{\mu_H}, \frac{\nu_C}{P^-\xi\xi'}\right) U_H^{kj}(\mu_H, \mu_\Lambda, \xi') f_j\left(\xi, \frac{\mu_\Lambda}{\Lambda_{\rm QCD}}\right)$$

Here U does the DGLAP μ resummation between the matching coefficient and the pdf.

The arguments are fixed by noting that all the objects appearing in this equation are dimensionless, and hence can depend only on ratios of various energy scales and dimensionless parameters



BFKL at the scale of the pdf

$$C_n\left(\frac{l_\perp}{\Lambda_{\rm QCD}}, \frac{\nu_C}{P^-}\right) = \int_0^1 d\xi\, H_j\left(\frac{l_\perp}{\mu}, \frac{\nu_C}{P^-\xi}\right) f_j\left(\xi, \frac{\mu}{\Lambda_{\rm QCD}}\right)$$

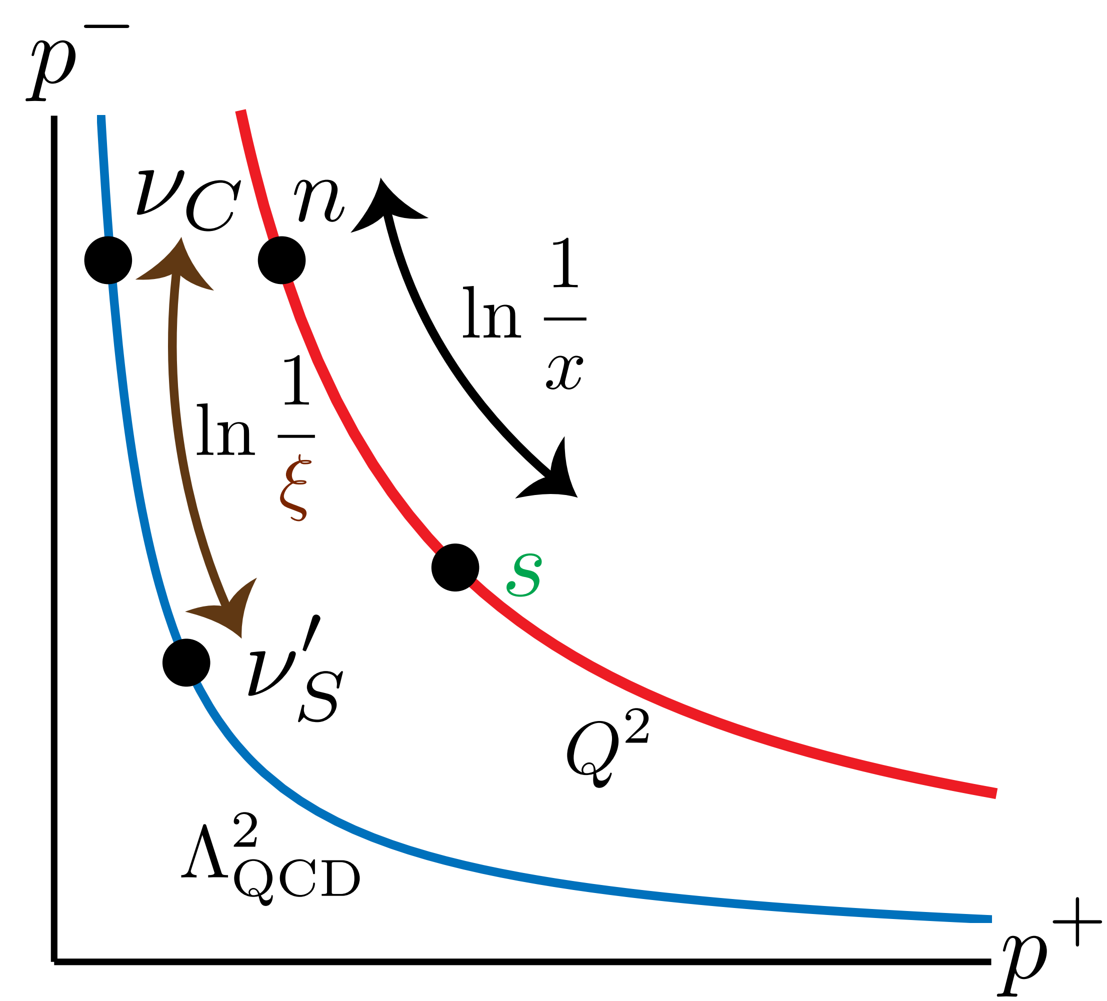
$$= \int_0^1 d\xi \int_0^1 \frac{d\xi'}{\xi'} H_j\left(\frac{l_\perp}{\mu_H}, \frac{\nu_C}{P^-\xi\xi'}\right) U_H^{kj}(\mu_H, \mu_\Lambda, \xi')\, f_j\left(\xi, \frac{\mu_\Lambda}{\Lambda_{\rm QCD}}\right)$$

We note that for the choice of $\nu_C = P^-$, the logs of x on the left hand side are resummed, but we are still left with logs of ξ and ξ'.

This requires a further factorization in rapidity for the H and the pdf.

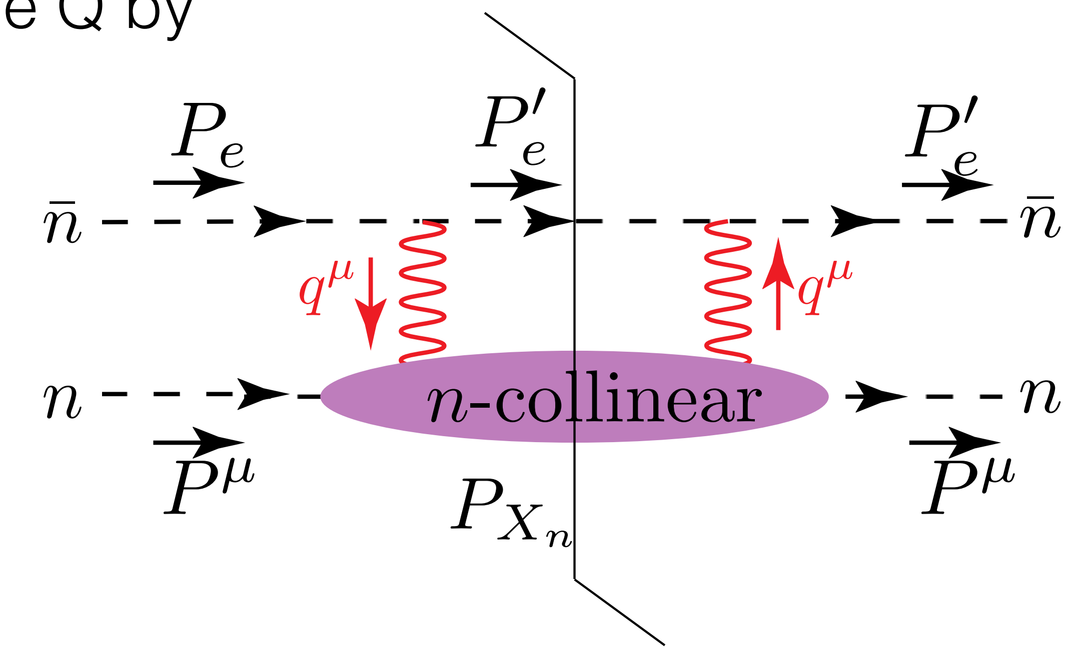
$$\ln\frac{1}{\xi} = \ln\frac{\nu_C}{\nu'_S}\,, \qquad \nu'_S = \xi P^-$$

$$\ln\frac{1}{\xi'\xi} = \ln\frac{\nu_C}{\nu''_S}\,, \qquad \nu''_S = \xi\xi' P^-$$



The Direct Piece for F_2

F_2 also gets contribution from a direct photon exchange with Glauber scaling. This piece has no soft modes at scale Q by definition. We denote this piece as $\Delta F(x,Q^2)$

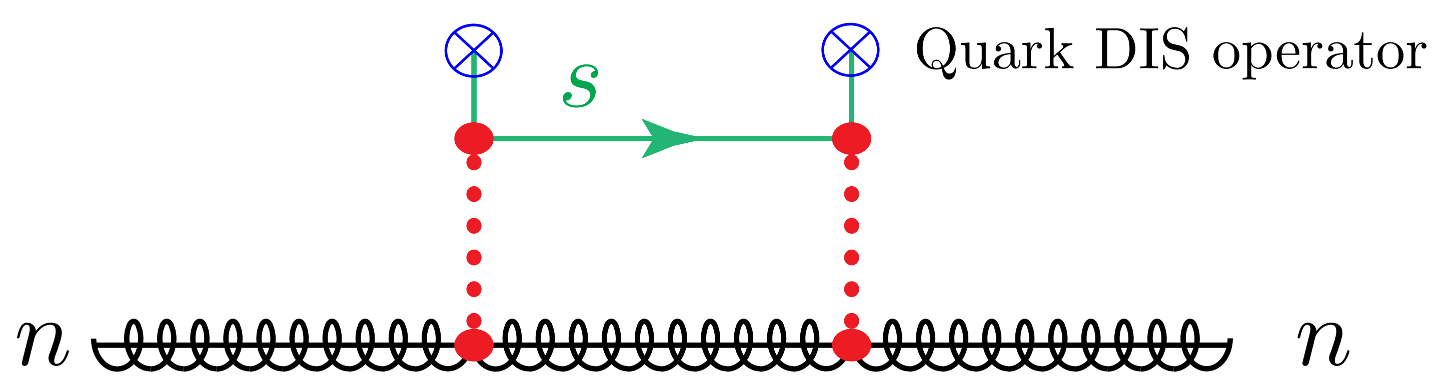


$$\Delta F(x, Q^2) = \sum_{i=q,g} \int_x^1 \frac{dz}{z} H'_i\left(\frac{x}{z}, \frac{\mu^2}{Q^2}\right) f_i(z,\mu)$$

The logs of z in the pdf are resummed by a rapidity factorization at the Λ_{QCD} scale:

$$f_i(z,\mu) = \int \frac{d^2k_\perp}{\vec{k}_\perp^{\,2}} S_i^f\left(\frac{k_\perp}{\mu}, \frac{\nu}{zP^-}\right) C\left(\frac{k_\perp}{\Lambda_{\rm QCD}}, \frac{\nu}{P^-}\right)$$

One can see that the bottom part of the diagram is same as before - hence get the same C, but this time it is non-perturbative. We are here mainly interested in the RG running for μ and ν of the pdf.



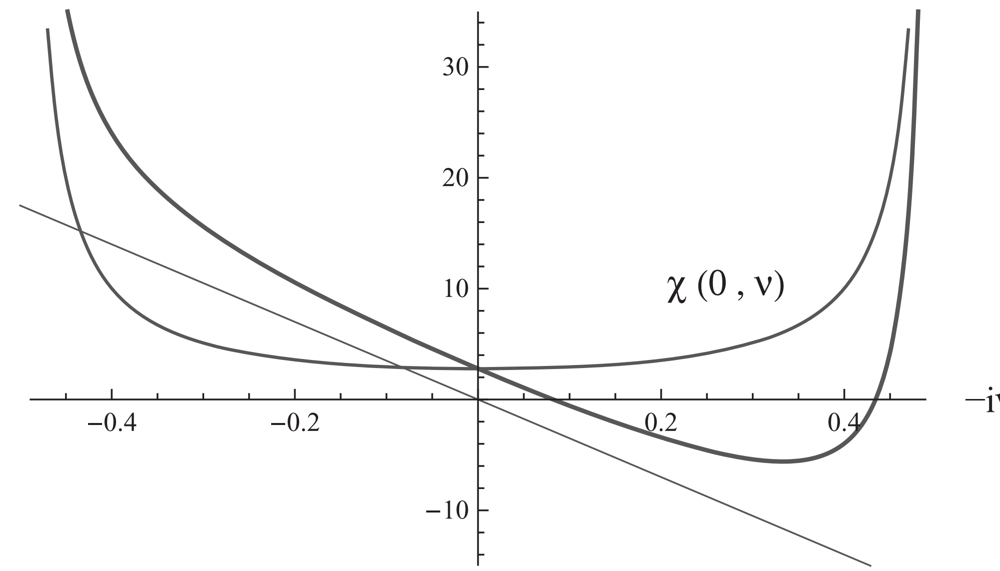
Solution to the BFKL equation

One solves BFKL by finding eigenfunctions of its integral kernel:

$$\int d^2q_\perp K_{BFKL}(l,q) f(\vec{q}_\perp) \equiv \frac{1}{\pi}\int \frac{d^2q_\perp}{(\vec{l}_\perp - \vec{q}_\perp)^2}\left[f(\vec{q}_\perp) - \frac{l_\perp^2}{2q_\perp^2} f(\vec{l}_\perp)\right]$$

Eigenfunctions: $f_{n,\gamma}(\vec{q}_\perp) = (\vec{q}_\perp^{\,2})^{\gamma-1} e^{in\phi_q}$

$$\int d^2q_\perp K_{BFKL}(l,q) q_\perp^{2(\gamma-1)} e^{in\phi_q} = \chi(n,\gamma) l_\perp^{2(\gamma-1)} e^{in\phi_l}$$



$$\chi(n,\gamma) = 2\psi(1) - \psi\left(\gamma + \frac{|n|}{2}\right) - \psi\left(1 - \gamma + \frac{|n|}{2}\right)$$

For our case n = 0, since there’s no angular dependence.

Using Mellin's inversion theorem we define γ transform as :

$$g(\gamma) = \int \frac{d^2 l_\perp}{\mu^2}\left(\frac{\vec{l}_\perp^{\,2}}{\mu^2}\right)^{\gamma-1} g\left(\frac{\vec{l}_\perp^{\,2}}{\mu^2}\right), \qquad g\left(\frac{\vec{l}_\perp^{\,2}}{\mu^2}\right) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+\infty} \frac{d\gamma}{2\pi i}\left(\frac{\vec{l}_\perp^{\,2}}{\mu^2}\right)^{-\gamma} g(\gamma)$$

Then solution for the BFKL Kernel is given by

$$g\left(\frac{\vec{l}_\perp^{\,2}}{\mu^2}, \frac{\vec{l}_\perp'^{\,2}}{\mu^2}, \frac{\nu}{\nu_0}\right) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\gamma}{2\pi i}\,\frac{1}{\pi} e^{\chi(\gamma)\ln\frac{\nu}{\nu_0}} f_\gamma\left(\frac{\vec{l}_\perp^{\,2}}{\mu^2}\right) f_{\gamma^\star}\left(\frac{\vec{l}_\perp'^{\,2}}{\mu^2}\right)$$

Factorization in Moment Space

Using the solution of the BFKL equation we now have

$$W_f^{\mu\nu} = P^- \sum_{j=q,g} \int_0^1 d\xi \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+\infty} \frac{d\gamma}{2\pi i} f_j\left(\xi, \frac{\mu}{\Lambda_{\rm QCD}}\right) U\left(\gamma, \frac{\nu_C}{\nu_S}\right) H_j\left(\gamma, \frac{\nu_C}{\xi P^-}\right)\left(\frac{\vec{q}_\perp^{\,2}}{\mu^2}\right)^{\gamma} S^{\mu\nu}\left(\gamma, q_\perp, \frac{\nu_S}{xP^-}\right)$$

Where we have defined:

$$U\left(\gamma, \frac{\nu_C}{\nu_S}\right) = \frac{1}{\pi} e^{\frac{\alpha_s C_A}{\pi}\chi(\gamma)\ln\frac{\nu_C}{\nu_S}}$$

$$H_j\left(\gamma, \frac{\nu_C}{\xi P^-}\right) \equiv \int \frac{d^2 l_\perp}{\mu^2}\left(\frac{l_\perp^2}{\mu^2}\right)^{-\gamma-1} H_j\left(\frac{l_\perp}{\mu}, \frac{\nu_C}{\xi P^-}\right)$$

$$H_j\left(\frac{l_\perp}{\mu}, \frac{\nu_C}{\xi P^-}\right) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+\infty} \frac{d\gamma}{2\pi i}\left(\frac{l_\perp^2}{\mu^2}\right)^{\gamma} H_j\left(\gamma, \frac{\nu_C}{\xi P^-}\right) \qquad S^{\mu\nu}\left(\gamma, q_\perp, \frac{\nu_S}{xP^-}\right) \equiv \int \frac{d^2 q'_\perp}{\vec{q}_\perp^{\,\prime 2}}\left(\frac{\vec{q}_\perp^{\,\prime 2}}{\vec{q}_\perp^{\,2}}\right)^{\gamma} S^{\mu\nu}\left(q'_\perp, q_\perp, \frac{\nu_S}{xP^-}\right)$$

Now define N-Mellin moments with respect to x as:

$$g(N) = \int_0^1 dx\, x^{N-1} g(x)\,, \qquad g(x) = \int_{c-i\infty}^{c+i\infty} \frac{dN}{2\pi i} x^{-N} g(N)$$

c is in the region on x axis where g(x) is analytic.

$$\begin{aligned}\bar{F}_2(N, Q^2) &= \int_0^1 dx\, x^{N-1} F_2(x, Q^2) \\ &= \int \frac{d\gamma}{2\pi i}\,\frac{1}{\pi}\left(\frac{Q^2}{\mu}\right)^{\gamma} \sum_j \frac{H_j(\gamma,1) S_2(\gamma,1)}{N - \frac{\alpha_s C_A}{\pi}\chi(\gamma)} \bar{f}_j\left(1 + \frac{\alpha_s C_A}{\pi}\chi(\gamma), \frac{\mu}{\Lambda_{\rm QCD}}\right) \\ &\quad + \bar{H}'_j\left(N, \frac{\mu}{Q}\right)\bar{f}_j\left(N+1, \frac{\mu}{\Lambda_{\rm QCD}}\right)\end{aligned}$$

Consistency between μ and ν RGE

BFKL for H gives

$$H_j\left(\frac{l_\perp}{\mu},\frac{\nu_C}{\xi P^-}\right)=\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+\infty}\frac{d\gamma}{2\pi i}U\left(\gamma,\frac{\nu_S'}{\nu_C}\right)\left(\frac{\vec{l}_\perp^{\,2}}{\mu^2}\right)^{\gamma}\int\frac{d^2 l_\perp'^2}{\mu^2}\left(\frac{l_\perp'^2}{\mu^2}\right)^{-\gamma-1}H_j\left(\frac{l_\perp'}{\mu},\frac{\nu_S'}{\xi P^-}\right)$$

$$=\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+\infty}\frac{d\gamma}{2\pi i}\xi^{-\frac{\alpha_s C_A}{\pi}\chi(\gamma)}\left(\frac{\vec{l}_\perp^{\,2}}{\mu^2}\right)^{\gamma}H_j\left(\gamma,\frac{\nu_S'}{\xi P^-}\right)$$

DGLAP for H gives

$$\mu^2\frac{d}{d\mu^2}H_j\left(\frac{l_\perp}{\mu},\frac{\nu_C}{\xi P^-}\right)=-\int_0^1\frac{d\xi'}{\xi'}H_k\left(\frac{l_\perp}{\mu},\frac{\nu_C}{\xi\xi' P^-}\right)P_{kj}(\xi')$$

Combining these two equations and taking N Mellin moment we get

$$\int\frac{d\gamma}{2\pi i}\,\frac{1}{\pi}\sum_k\frac{H_k(\gamma,1)\left[\gamma\delta_{kj}-\bar{P}_{kj}\left(\frac{\alpha_s C_A}{\pi}\chi(\gamma)\right)\right]}{N-\frac{\alpha_s C_A}{\pi}\chi(\gamma)}\left(\frac{\mu}{Q^2}\right)^{-\gamma}=0$$

This defines N as follows: $N\equiv\dfrac{\alpha_s C_A}{\pi}\chi(\gamma_N)$

μ-ν consistency equation: $\displaystyle\sum_k H_k(\gamma_N,1)\left[\gamma_N-\bar{P}_{kj}(N)\right]=0$

Consistency between μ and ν RGE

Hence at Leading Log accuracy we obtain resummed gluon anomalous dimension:

$$\boxed{H_g(\gamma_N,1)\big[\gamma_N - \bar{P}_{gg}(N)\big]} + \quad H_q(\gamma_N,1)\big[-\bar{P}_{qg}(N)\big] \quad = 0$$

$$H_g(\gamma_N,1)\big[-\bar{P}_{gq}(N)\big] \quad + \quad H_q(\gamma_N,1)\big[\gamma_N - \bar{P}_{qq}(N)\big] = 0$$

Following similar steps for the direct piece ΔF yields:

$$\frac{\overbrace{\bar{P}_{gg}(N)}^{\text{LL}}}{\gamma_N}\overbrace{S_g^f(\gamma_N,1)}^{\text{LL}} + \frac{\overbrace{\bar{P}_{gq}(N)}^{\text{LL}}}{\gamma_N}\overbrace{S_q^f(\gamma_N,1)}^{\text{NLL}} = \overbrace{S_g^f(\gamma_N,1)}^{\text{LL}}$$

$$\frac{\overbrace{\bar{P}_{qg}(N)}^{\text{NLL}}}{\gamma_N}\overbrace{S_g^f(\gamma_N,1)}^{\text{LL}} + \frac{\overbrace{\bar{P}_{qq}(N)}^{\alpha_s}}{\gamma_N}\overbrace{S_q^f(\gamma_N,1)}^{\text{NLL},\mathcal{O}(\alpha_s)} = \overbrace{S_q^f(\gamma_N,1)}^{\text{NLL}}$$

$$\bar{P}_{gg}(N) = \gamma_N\,, \qquad S_q^f(\gamma_N) = \frac{\bar{P}_{qg}}{\gamma_N}(N)S_g^f(\gamma_N)$$

The first is same as we derived for H. The second proves that at the lowest order in small-x the quark pdf is directly related to the gluon pdf.

$$\Delta F_2 = -\frac{\bar{P}_{gq}(N)}{\gamma_N}f_g(N)$$

Final results for structure functions

Hence having resummed the leading small-x logs we now have following relations for the coefficient functions:

$$\bar{F}_2(N) = C_2^g(N)\bar{f}_g(N)\,, \qquad \bar{F}_L(N) = C_L^g(N)\bar{f}_g(N)$$

Where the coefficient functions are given by:

$$C_L^g(N) = S_L(\gamma_N)R(\gamma_N)H(\gamma_N,1)$$

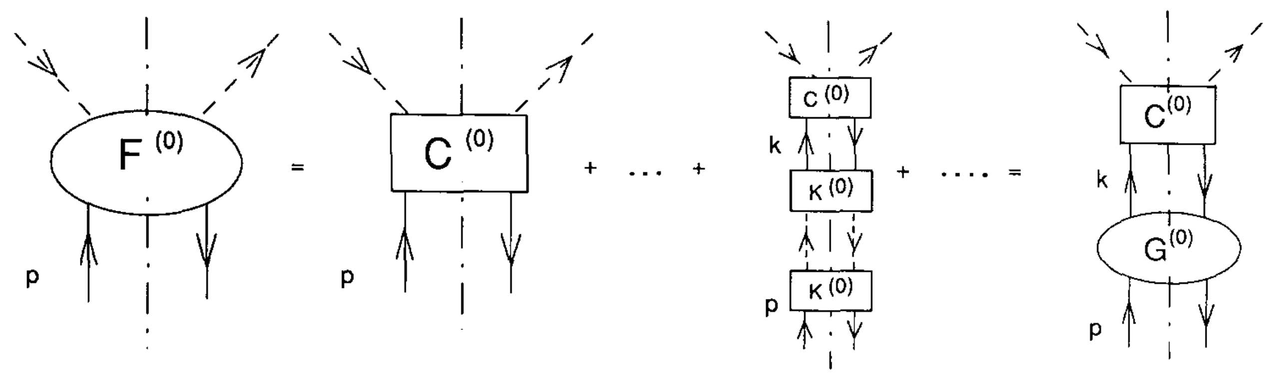
$$C_2^g(N) = S_2(\gamma_N)R(\gamma_N)H(\gamma_N,1) - \frac{2n_f\bar{P}_{qg}(N)}{\gamma_N}$$

$S_L(\gamma_N)$ and $S_2(\gamma_N)$ are the moments of our soft function. n_f is the number of quark flavors

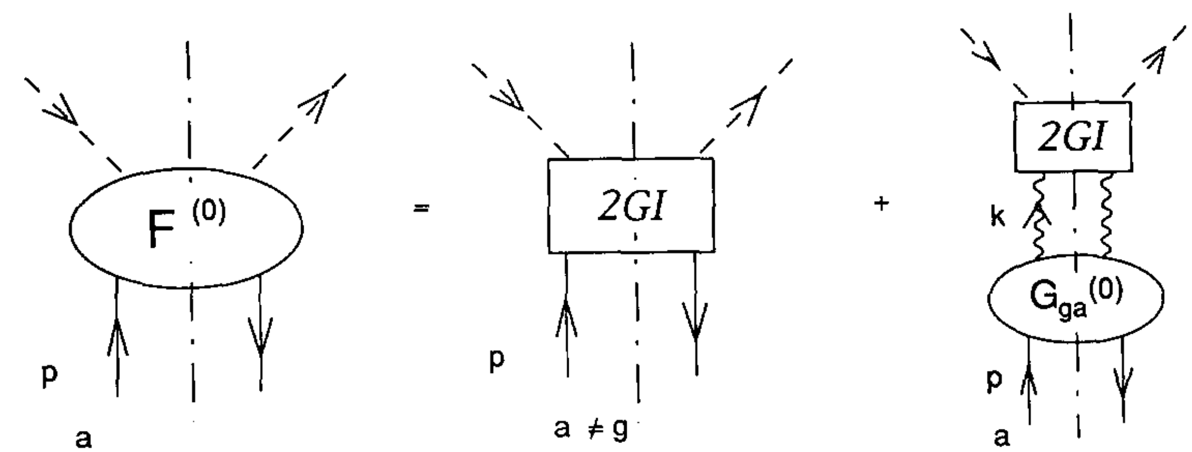
Comparison with previous work

- This problem has been long under investigation. First resummation for the coefficient function: Catani-Hautmann, 1994
- Diagrammatic approach.
- Their approach involved solving the n=4+2ε dimensional BFKL eqn. to keep track of the IR divergences, as they deal with the small-x resummation.
- Not gauge invariant (at least not clear a priori).
- Deals with bare quantities, unlike our factorization theorem where the functions in our factorization formulae have been renormalized.
- Our results for S_2 and S_L in the γ space agree with theirs. We get the same consistency equations.

2PI Kernels for collinear singularities:



2GI Kernels for high energy limit:



BFKL in 4+2ε dimensions:

$$\mathcal{F}_N^{(0)}(\vec{q}_\perp;\alpha_s,\mu,\epsilon) = \delta^{(2+2\epsilon)}(\vec{q}_\perp) + \frac{\alpha_s C_A}{\pi N}\int \frac{d^{2+2\epsilon}k_\perp}{(2\pi\mu)^{2\epsilon}}\frac{1}{\pi\vec{k}_\perp^2}\left[\mathcal{F}_N^{(0)}(\vec{q}_\perp-\vec{k}_\perp) - \frac{\vec{q}_\perp\cdot(\vec{q}_\perp-\vec{k}_\perp)}{(\vec{q}_\perp-\vec{k}_\perp)^2}\mathcal{F}_N^{(0)}(\vec{q}_\perp)\right]$$

Summary

- Forward Scattering formalism in Effective Field theory gives a very unique understanding of the underlying physics.

- Rich phenomenology of Glauber exchanges

- Use of μ-ν space RGEs for DGLAP and BFKL evolutions.

- Our goal is to get further understanding of NLL small-x resummation using this new forward scattering formalism in SCET.

Thank you