Bottom and charm quark masses from quarkonium at \( N^{3}\)LO

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Based on JHEP 01 (2018) 122

Vienna 18-05-2018
Outline

- Motivation: Heavy quark masses
- Quarkonium, the Cornell and static QCD potential
- Master formula and the MSR mass
- Massive lighter quarks in quarkonium and MSR mass
- Determination of charm and bottom mass (and $\alpha_s$)
- Calibration of the Cornell model
- Conclusions
Motivation
Heavy quark masses

Fundamental parameters of the Standard Model, need to be known with high precision

In this talk we focus only on **Bottom and Charm**

Play a fundamental role in flavor physics:

- Unitarity triangle
- Rare kaon decays
- Test the Standard Model at the precision frontier

Also play a role in Higgs physics (branching ratios)
Heavy quark masses

Fundamental parameters of the Standard Model, need to be known with high precision

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Play a fundamental role in flavor physics:

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Also play a role in Higgs physics (branching ratios)

But… quarks are confined particles, therefore their mass is not observable!

The mass of a heavy quark needs to be defined within perturbation theory…

… as any other parameter in the QCD Lagrangian (renormalization, \( \mu \)-dependence)

Only indirect measurements of quark masses possible.
Quarkonium

Charm and Bottom quarks discovered as $Q\bar{Q}$ bound states

SLAC and BNL (1974), $J/\Psi$ bound state

Fermilab (1977), $\Upsilon$ bound state
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Quarkonium

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SLAC and BNL (1974), $J/\Psi$ bound state

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Theoretical description in early days in terms of a very simple non-relativistic description, the Cornell Model, with only three parameters:

- Quark mass: $m_Q$
- Coulomb-type interaction: $\alpha_s$
- Linear raising potential: “string tension” $\sigma$

Interplay between perturbative and non-perturbative is crucial

[Eichten et. al. PRL 34:369–372 (1975)]
Cornell model and the static QCD potential
Cornell Model

\[ V_{\text{Cornell}}(r) = -C_F \frac{\alpha_s}{r} + \sigma r \]

“static potential”
Cornell Model

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"static potential"

these terms yield dependence on (n,l) quantum numbers

Solved numerically (Numerov)
Cornell Model

\[ V_{\text{Cornell}}(r) = -C_F \frac{\alpha_s}{r} + \sigma r + V_{LS} + V_{SS} + V_T + \text{nothing} \]

"static potential" spin-dependent \(1/m^2\) corrections

these terms yield dependence on \((n,l)\) quantum numbers
these terms give dependence on \((s,j)\) quantum numbers

Solved numerically (Numerov) Use perturbation theory

![Graph of Cornell potential](image)
Cornell Model

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“static potential”  \hspace{1cm} spin-dependent \( 1/m^2 \) corrections

these terms yield dependence \hspace{1cm} these terms give dependence
on (n,l) quantum numbers \hspace{1cm} on (s,j) quantum numbers

Solved numerically (Numerov)  \hspace{1cm} Use perturbation theory

\[ \text{spin-dependent } 1/m^2 \text{ corrections} \]

These terms yield dependence on (n,l) quantum numbers.

These terms give dependence on (s,j) quantum numbers.

Solved numerically (Numerov).

Use perturbation theory.
EFT treatment

Modern method to deal with problems widely separated scales

For quarkonium such theory is called NRQCD:

RGE-improved versions of NRQCD are called

pNRQCD  [Pineda, Soto; Brambilla, Pineda, Soto, Vairo NPB 566 (2000) 275]


Can be also used for other processes, such as t-\bar{t} production at threshold
Static QCD potential

\[ V_{\text{QCD}}(r) = V_{\text{static}}(r) + \frac{1}{m^n} \text{corrections} \]

\[ V_{\text{static}}(r) = -C_F \frac{\alpha_s(\mu)}{r} \left[ 1 + \sum_{i=1}^{\infty} \left( \frac{\alpha_s(\mu)}{4\pi} \right)^i a_{ij} \log^i(r \mu e^{\gamma_E}) \right] \text{ known to } \mathcal{O}(\alpha_s^4) \]

\[ V_{\text{static}}(r) [\text{GeV}] \]

-3.5 \rightarrow -3, 0.0 \rightarrow 0.20 \text{ fm}

\[ \text{r-independent } u = 1/2 \text{ renormalon} \]

[Peter, PRL 78 (1997) 602-605], [Lee et al, PRD 94 (2016) 054029]
Static QCD potential

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But a potential is not an observable! Energy is an observable

\[ E = 2 m_Q^{\text{pole}} + V_{\text{static}}(r) \]

has a \( u = 1/2 \) mass-independent renormalon

[Pineda, PhD thesis]
[Hoang et al, PRD 59 (1999) 114014]
**Static QCD potential**

\[ V_{\text{QCD}}(r) = V_{\text{static}}(r) + \frac{1}{m^n} \text{corrections} \]

many more terms known, also quantum corrections

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But a potential is not an observable!

Energy is an observable

\[ E = 2m_{\text{pole}}^Q + V_{\text{static}}(r) \]

Same renormalon! cancels in the difference

\[ E = 2m_{\text{MSR}}^Q(R) + \delta_M(R, \mu) + V_{\text{static}}(r) \]
Static QCD potential

\[ V_{\text{QCD}}(r) = V_{\text{static}}(r) + \frac{1}{m^n} \text{corrections} \]

many more terms known, also quantum corrections

\[ V_{\text{static}}(r) = -C_F \frac{\alpha_s(\mu)}{r} \left[ 1 + \sum_{i=1}^n \left[ \frac{\alpha_s(\mu)}{4\pi} \right]^i a_{ij} \log^i(r \mu e^{\gamma_E}) \right] \]

known to \( O(\alpha_s^4) \)

But a potential is not an observable! Energy is an observable

\[ E = 2 m_Q^{\text{pole}} + V_{\text{static}}(r) \]

Same renormalon! cancels in the difference

\[ E = 2 m_Q^{\text{MSR}}(R) + \delta_M(R, \mu) + V_{\text{static}}(r) \]

Important to use MSR mass because neither \( \log(r \mu) \) nor \( \log(R/\mu) \) should be large

It also makes qualitative agreement with Cornell model better
Master formula for quarkonia masses
Master formula \cite{Kiyo, Sumino NP B889 (2014) 156-191}

\[
E_X(\mu, n_\ell) = 2m_Q^{\text{pole}} \left[ 1 - \frac{C_F^2 \alpha_s(n_\ell)(\mu)^2}{8n^2} \sum_{i=0}^{\infty} \left( \frac{\alpha_s(n_\ell)(\mu)}{4\pi} \right)^i \varepsilon^{i+1} P_i(L_{n_\ell}) \right] + \text{non-perturbative}
\]

\cite{Penin, Steinhauser, PLB 538 (2002) 335-345}
\cite{Beneke, Kiyo Schuller, PLB 714 (2005) 67-90}
$E_X(\mu, n_\ell) = 2 m_Q^{\text{pole}} \left[ 1 - \frac{C_F^2 \alpha_s^{(n_\ell)}(\mu)^2}{8 n^2} \sum_{i=0}^{\infty} \left( \frac{\alpha_s^{(n_\ell)}(\mu)}{4\pi} \right)^i \varepsilon^{i+1} P_i(L_{n_\ell}) \right] + \text{non-perturbative}$

We assume $\mu_s = m v \gg \Lambda_{QCD}$ certainly true for $n < 4$

And either $\mu_{us} = m v^2 \gg \Lambda_{QCD}$ (true for $n = 1$) local condensates

or $\mu_{us} \sim \Lambda_{QCD}$ (possibly true for $n = 2$) non-local condensates

For $n = 3$ one seems to have $\mu_{us} < \Lambda_{QCD}$ perturbative and non-perturbative of the same order

We will estimate those by comparing fits with different datasets and by studies of perturbative stability

For a recent study of non-perturbative effects, see [T. Rauh 1803.05477 (2018)]
Master formula \cite{Kiyo, Sumino NP B889 (2014) 156-191}

\[ E_X(\mu, n_\ell) = 2m_Q^{\text{pole}} \left[ 1 - \frac{C_F^2 \alpha_s^{(n_\ell)}(\mu)^2}{8n^2} \sum_{i=0}^{\infty} \left( \frac{\alpha_s^{(n_\ell)}(\mu)}{4\pi} \right)^i \varepsilon^{i+1} P_i(L_{n_\ell}) \right] + \text{non-perturbative} \]

\[ L_{n_\ell} = \log\left( \frac{n\mu}{C_F \alpha_s^{(n_\ell)}(\mu)m_Q^{\text{pole}}} \right) + H_{n+\ell} \]

\[ P_i(L) = \sum_{j=0}^{i} c_{i,j} L^j \]

First quantum correction is \( \mathcal{O}(\alpha_s^2) \), but static potential starts at \( \mathcal{O}(\alpha_s) \)
$E_X(\mu, n_\ell) = 2m_Q^{\text{pole}} \left[ 1 - \frac{C_F^2}{8n^2} \frac{\alpha_s^{(n_\ell)}(\mu)^2}{4\pi} \sum_{i=0}^{\infty} \left( \frac{\alpha_s^{(n_\ell)}(\mu)}{4\pi} \right)^i \varepsilon^{i+1} P_i(L_{n_\ell}) \right] + \text{non-perturbative}$

$L_{n_\ell} = \log \left( \frac{n\mu}{C_F\alpha_s^{(n_\ell)}(\mu)m_Q^{\text{pole}}} \right) + H_{n+\ell}$

$P_i(L) = \sum_{j=0}^{i} c_{i,j} L^j$

First quantum correction is $\mathcal{O}(\alpha_s^2)$, but static potential starts at $\mathcal{O}(\alpha_s)$

bookkeeping parameter that labels the various orders in the $\Upsilon$-expansion


Crucial when cancelling the static potential renormalon in the quarkonium mass

Important when figuring out alternative perturbative expansions

Sets up a counting for the MSR-mass parameter $R$
Master formula [Kiyo, Sumino NP B889 (2014) 156-191]

\[ E_X(\mu, n_\ell) = 2m_Q^{\text{pole}} \left[ 1 - \frac{C_F^2 \alpha_s^{(n_\ell)}(\mu)^2}{8n^2} \sum_{i=0}^{\infty} \left( \frac{\alpha_s^{(n_\ell)}(\mu)}{4\pi} \right)^i \varepsilon^{i+1} P_i(L_{n_\ell}) \right] + \text{non-perturbative} \]

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Master formula

\[ E_X(\mu, n_\ell) = 2 m_Q^{\text{pole}} \left[ 1 - \frac{C_F^2 \alpha_s(n_\ell)(\mu)^2}{8 n^2} \sum_{i=0}^{\infty} \left( \frac{\alpha_s(n_\ell)(\mu)}{4\pi} \right)^i \varepsilon^{i+1} P_i(L_{n_\ell}) \right] + \text{non-perturbative} \]

\begin{align*}
L_{n_\ell} &= \log \left( \frac{n\mu}{C_F \alpha_s(n_\ell)(\mu)m_Q^{\text{pole}}} \right) + H_{n+\ell} \\

P_i(L) &= \sum_{j=0}^{i} c_{i,j} L^j
\end{align*}

[Kiyo, Sumino NP B889 (2014) 156-191]
Master formula [Kiy, Sumino NP B889 (2014) 156-191]

\[ E_X(\mu, n_\ell) = 2 m_Q^{\text{pole}} \left[ 1 - \frac{C_F^2}{8n^2} \alpha_s(n_\ell)(\mu)^2 \sum_{i=0}^{\infty} \left( \frac{\alpha_s(n_\ell)(\mu)}{4\pi} \right)^i \varepsilon^{i+1} P_i(L_{n_\ell}) \right] + \text{non-perturbative} \]

\[ L_{n_\ell} = \log \left( \frac{n\mu}{C_F \alpha_s(n_\ell)(\mu)m_Q^{\text{pole}}} \right) + H_{n+\ell} \]

\[ P_i(L) = \sum_{j=0}^{i} c_{i,j} L^j \]

Argument in logs non-trivial: explicit \( \mu \) dependence as well as through \( \alpha_s(n_\ell)(\mu) \)

Dependence gets even more complex when switching to the MSR mass
Master formula

\[ E_X(\mu, n_\ell) = 2 m_Q^{\text{pole}} \left[ 1 - \frac{C_F^2 \alpha_s^{(n_\ell)}(\mu)^2}{8n^2} \sum_{i=0}^{\infty} \left( \frac{\alpha_s^{(n_\ell)}(\mu)}{4\pi} \right)^i \varepsilon^{i+1} P_i(L_{n_\ell}) \right] + \text{non-perturbative} \]

\[ L_{n_\ell} = \log \left( \frac{n\mu}{C_F \alpha_s^{(n_\ell)}(\mu)m_Q^{\text{pole}}} \right) + H_{n+\ell} \]

Argument in logs non-trivial: explicit \( \mu \) dependence as well as through \( \alpha_s^{(n_\ell)}(\mu) \)

Dependence gets even more complex when switching to the MSR mass

Harmonic number \( H_n \equiv \sum_{i=1}^{n} \frac{1}{i} \) grouped with the log for convenience
Master formula [Kiyu, Sumino NP B889 (2014) 156-191]

\[ EX(\mu, n_\ell) = 2m_Q^{\text{pole}} \left[ 1 - \frac{C_F^2 \alpha_s(n_\ell)(\mu)^2}{8n^2} \sum_{i=0}^{\infty} \left( \frac{\alpha_s(n_\ell)(\mu)}{4\pi} \right)^i \varepsilon^{i+1} P_i(L_{n_\ell}) \right] + \text{non-perturbative} \]

\[ L_{n_\ell} = \log \left( \frac{n\mu}{C_F \alpha_s(n_\ell)(\mu)m_Q^{\text{pole}}} \right) + H_{n+\ell} \]

\[ P_i(L) = \sum_{j=0}^{i} c_{i,j} L^j \]

In this formula corrections in \( \frac{1}{m_Q} \) and \( \alpha_s \) are of the same order: \( m_Q \) only scale involved
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\[ L_{n_\ell} = \log \left( \frac{n\mu}{C_F \alpha_s(n_\ell)(\mu)m_Q^{\text{pole}}} \right) + H_{n+\ell} \quad P_i(L) = \sum_{j=0}^{i} c_{i,j} L^j \]

In this formula corrections in \( \frac{1}{m_Q} \) and \( \alpha_s \) are of the same order: \( m_Q \) only scale involved

\( c_{i,0} \) are known to up to \( i = 3 \), \( c_{i,j} > 0 \) can be computed demanding \( \mu \) independence

\[ c_{k,j+1} = \frac{2}{j+1} \left\{ (j+2) \beta_{k-1-j} c_{j,j} + \sum_{i=j+1}^{k-1} \beta_{k-1-i} \left[ (i+2) c_{i,j} - (j+1) c_{i,j+1} \right] \right\} \]
\[ E_X(\mu, n_{\ell}) = 2 m_Q^{\text{pole}} \left[ 1 - \frac{C_F^2 \alpha_s^{(n_{\ell})}(\mu)^2}{8 n^2} \sum_{i=0}^{\infty} \left( \frac{\alpha_s^{(n_{\ell})}(\mu)}{4\pi} \right)^i \varepsilon^{i+1} P_i(L_{n_{\ell}}) \right] + \text{non-perturbative} \]

\[ L_{n_{\ell}} = \log \left( \frac{n \mu}{C_F \alpha_s^{(n_{\ell})}(\mu) m_Q^{\text{pole}}} \right) + H_{n+\ell} \]

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\( c_{i,j} \) depend on the bound-state quantum numbers: \( (n, l, j, s) \)

\( c_{3,0} \) depends on \( \log(\alpha_s) \): first hint of ultra-soft effects. Could be resumed within EFTs.
Master formula [Kiyo, Sumino NP B889 (2014) 156-191]

\[ E_X(\mu, n_\ell) = 2 m_Q^{\text{pole}} \left[ 1 - \frac{C_F^2 \alpha_s(n_\ell)(\mu)^2}{8n^2} \sum_{i=0}^{\infty} \left( \frac{\alpha_s(n_\ell)(\mu)}{4\pi} \right)^i \varepsilon^{i+1} P_i(L_{n_\ell}) \right] + \text{non-perturbative} \]

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\[ c_{k,j+1} = \frac{2}{j+1} \left\{ (j + 2) \beta_{k-1-j} c_{j,j} + \sum_{i=j+1}^{k-1} \beta_{k-1-i} [(i + 2) c_{i,j} - (j + 1) c_{i,j+1}] \right\} \]

\( c_{i,j} \) depend on the bound-state quantum numbers: (\( n, l, j, s \))

\( c_{3,0} \) depends on \( \log(\alpha_s) \): first hint of ultra-soft effects. Could be resumed within EFTs.

This formula has an \( m_Q \)-independent renormalon equal to that of \(-2 m_Q^{\text{pole}}\) inherited from the QCD static potential.
Master formula in short-distance scheme and finite charm quark mass effects
Master formula in short-distance scheme

\[ m_Q^{\text{pole}} = m_Q^{\text{SD}} \left[ 1 + \sum_{n=1}^{\infty} \varepsilon^n \delta_n^{\text{SD}} \left( \frac{\alpha_s^{(n_F)}(\mu)}{4\pi} \right)^n \right] \]
Master formula in short-distance scheme

\[ m^\text{pole}_Q = m^\text{SD}_Q \left[ 1 + \sum_{n=1}^{\infty} \varepsilon^n \delta_n^\text{SD} \left( \frac{\alpha_s(n, \mu)}{4\pi} \right)^n \right] \]

Depends on some scale \( R \)
Master formula in short-distance scheme

\[ m_Q^{\text{pole}} = m_Q^{\text{SD}} \left[ 1 + \sum_{n=1}^{\infty} \varepsilon^n \delta_n^{\text{SD}} \left( \frac{\alpha_s(n_{\ell})(\mu)}{4\pi} \right)^n \right] \]

- Depends on some scale \( R \)
- \( \gamma \) - counting scheme parameter
Master formula in short-distance scheme

\[ m_{Q}^{\text{pole}} = m_{Q}^{\text{SD}} \left[ 1 + \sum_{n=1}^{\infty} \varepsilon^n \delta_n^{\text{SD}} \left( \frac{\alpha_s^{(n_{\ell})}(\mu)}{4\pi} \right)^n \right] \]

- Depends on some scale \( R \)
- \( \gamma \)-counting scheme parameter

Depends on powers of \( \log \left( \frac{\mu}{R} \right) \) and proportional to \( \frac{R}{m_{Q}^{\text{SD}}} \)
Master formula in short-distance scheme

\[ m^\text{pole}_Q = m^\text{SD}_Q \left[ 1 + \sum_{n=1}^{\infty} \varepsilon^n \delta^n \left( \frac{\alpha_s^{(n_e)}(\mu)}{4\pi} \right)^n \right] \]

- Depends on some scale \( R \)
- \( \Upsilon \)- counting scheme parameter

- Depends on powers of \( \log \left( \frac{\mu}{R} \right) \) and proportional to \( \frac{R}{m^\text{SD}_Q} \)

\[ E_X (\mu, n_e) = 2 m^\text{SD}_Q \left\{ 1 + \sum_{i=1}^\infty \varepsilon^i \left( \frac{\alpha_s^{(n_e)}(\mu)}{4\pi} \right)^i \left[ \delta^\text{SD}_i - F P^\text{SD}_{i-1} - (1 - \delta_{i,1}) F \sum_{j=1}^{i-1} \delta^\text{SD}_j P^\text{SD}_{i-j-1} \right] \right\} \]

\[ F = \frac{\pi C_F^2 \alpha_s^{(n_e)}(\mu)}{2n^2} \]
Master formula in short-distance scheme

\[ m_Q^{\text{pole}} = m_Q^{\text{SD}} \left[ 1 + \sum_{n=1} \varepsilon^n \delta_n^{\text{SD}} \left( \frac{\alpha_s^{(n_{\ell})}(\mu)}{4\pi} \right)^n \right] \]

Depends on some scale \( R \)

\( \gamma \) - counting scheme parameter

Depends on powers of \( \log \left( \frac{\mu}{R} \right) \) and proportional to \( \frac{R}{m_Q^{\text{SD}}} \)

\[ E_X(\mu, n_{\ell}) = 2 m_Q^{\text{SD}} \left\{ 1 + \sum_{i=1} \varepsilon^i \left( \frac{\alpha_s^{(n_{\ell})}(\mu)}{4\pi} \right)^i \left[ \delta_i^{\text{SD}} - F P_i^{\text{SD}} - (1 - \delta_{i,1}) F \sum_{j=1}^{i-1} \delta_j^{\text{SD}} P_{i-j-1}^{\text{SD}} \right] \right\} \]

\[ F = \frac{\pi C_F^2 \alpha_s^{(n_{\ell})}(\mu)}{2n^2} \]

Different powers of \( \alpha_s \) in the same \( \varepsilon \) order
Master formula in short-distance scheme

\[ m_Q^{\text{pole}} = m_Q^{\text{SD}} \left[ 1 + \sum_{n=1}^{\infty} \varepsilon^n \delta_n^{\text{SD}} \left( \frac{\alpha_s^{(n_\ell)}(\mu)}{4\pi} \right)^n \right] \]

Depends on some scale \( R \)

\( \Upsilon \)- counting scheme parameter

Depends on powers of \( \log \left( \frac{\mu}{R} \right) \) and proportional to \( \frac{R}{m_Q^{\text{SD}}} \)

\[ E_X(\mu, n_\ell) = 2 m_Q^{\text{SD}} \left\{ 1 + \sum_{i=1}^{i-1} \varepsilon^i \left( \frac{\alpha_s^{(n_\ell)}(\mu)}{4\pi} \right)^i \left[ \delta_i^{\text{SD}} - FP_{i-1}^{\text{SD}} - (1 - \delta_{i,1}) F \sum_{j=1}^{i-1} \delta_j^{\text{SD}} P_{i-j-1}^{\text{SD}} \right] \right\} \]

\[ F = \frac{\pi C_F^2 \alpha_s^{(n_\ell)}(\mu)}{2n^2} \]

\( F \) depend on \( L_{SD} = \log \left( \frac{n \mu}{C_F \alpha_s^{(n_\ell)}(\mu) m_Q^{\text{SD}}} \right) + H_{n+\ell} \)

and \( c_{i,j}, \delta_i^{\text{SD}} \)

Different powers of \( \alpha_s \) in the same \( \varepsilon \) order
Master formula in short-distance scheme

\[ m_Q^{\text{pole}} = m_Q^{\text{SD}} \left[ 1 + \sum_{n=1}^{\infty} \varepsilon^n \delta_n^{\text{SD}} \left( \frac{\alpha_s^{(n, \ell)}(\mu)}{4\pi} \right)^n \right] \]

 Depends on some scale \( R \)

 Depends on powers of \( \log \left( \frac{\mu}{R} \right) \) and proportional to \( \frac{R}{m_Q^{\text{SD}}} \)

\[ E_X(\mu, n_\ell) = 2 m_Q^{\text{SD}} \left\{ 1 + \sum_{i=1}^{\infty} \varepsilon^i \left( \frac{\alpha_s^{(n, \ell)}(\mu)}{4\pi} \right)^i \left[ \delta_i^{\text{SD}} - F P_i^{\text{SD}} - (1 - \delta_{i,1}) F \sum_{j=1}^{i-1} \delta_j^{\text{SD}} P_{i-j-1}^{\text{SD}} \right] \right\} \]

\[ F = \frac{\pi C_F^2 \alpha_s^{(n, \ell)}(\mu)}{2n^2} \]

depend on \( L_{\text{SD}} = \log \left( \frac{n\mu}{C_F \alpha_s^{(n, \ell)}(\mu) m_Q^{\text{SD}}} \right) + H_{n+\ell} \)

and \( c_{i,j}, \delta_i^{\text{SD}} \)

The scale that minimizes these logs will be denoted generically \( \mu_S \)
Master formula in short-distance scheme

\[ m_Q^{\text{pole}} = m_Q^{\text{SD}} \left[ 1 + \sum_{n=1} \varepsilon^n \delta_n^{\text{SD}} \left( \frac{\alpha_s(n^\ell)(\mu)}{4\pi} \right)^n \right] \]

Depends on some scale \( R \)

Depends on powers of \( \log \left( \frac{\mu}{R} \right) \) and proportional to \( \frac{R}{m_Q^{\text{SD}}} \)

\[ E_X(\mu, n^\ell) = 2 m_Q^{\text{SD}} \left\{ 1 + \sum_{i=1} \varepsilon^i \left( \frac{\alpha_s(n^\ell)(\mu)}{4\pi} \right)^i \left[ \delta_i^{\text{SD}} - F P_{i-1}^{\text{SD}} - (1 - \delta_{i,1}) F \sum_{j=1}^{i-1} \delta_j^{\text{SD}} P_{i-j-1}^{\text{SD}} \right] \right\} \]

\[ F = \frac{\pi C_F^2 \alpha_s(n^\ell)(\mu)}{2n^2} \]

depend on \( L_{\text{SD}} = \log \left( \frac{n\mu}{C_F \alpha_s(n^\ell)(\mu)m_Q^{\text{SD}}} \right) + H_{n+\ell} \)

and \( c_{i,j} \), \( \delta_i^{\text{SD}} \)

The scale that minimizes these logs will be denoted generically \( \mu_S \)

Two kinds of logs if short-distance mass is used. To minimize both of them simultaneously one either has a tunable scale \( R \), or a fixed scale \( R \sim \mu_S \)
Master formula in short-distance scheme

\[ m_{Q_{\text{pole}}} = m_{Q_{\text{SD}}} \left[ 1 + \sum_{n=1}^{\infty} \varepsilon^n \delta_{n}^{\text{SD}} \left( \frac{\alpha_s^{(n_{\ell})}(\mu)}{4\pi} \right)^n \right] \]

Depends on some scale \( R \)

Depends on powers of \( \log\left(\frac{\mu}{R}\right) \) and proportional to \( \frac{R}{m_{Q_{\text{SD}}}} \)

\[ E_X(\mu, n_{\ell}) = 2 m_{Q_{\text{SD}}} \left\{ 1 + \sum_{i=1}^{\infty} \varepsilon^i \left( \frac{\alpha_s^{(n_{\ell})}(\mu)}{4\pi} \right)^i \left[ \delta_{i}^{\text{SD}} - F P_{i-1}^{\text{SD}} - (1 - \delta_{i,1}) F \sum_{j=1}^{i-1} \delta_{j}^{\text{SD}} P_{i-j-1}^{\text{SD}} \right] \right\} \]

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The scale that minimizes these logs will be denoted generically \( \mu_{S} \)

Two kinds of logs if short-distance mass is used. To minimize both of the them simultaneously one either has a tunable scale \( R \), or a fixed scale \( R \sim \mu_{S} \)

MSR mass

I-S mass
Effects from massive lighter quarks

\[ E_X(\mu, n_\ell, m_Q^{\text{pole}}, m_q^{\text{pole}}) = E_X(\mu, n_\ell, m_Q^{\text{pole}}) + \varepsilon^2 \delta E_X^{(1)} + \varepsilon^3 \delta E_X^{(2)} + \cdots \]
Effects from massive lighter quarks

\[ E_X(\mu, n_\ell, m_Q^\text{pole}, m_q^\text{pole}) = E_X(\mu, n_\ell, m_Q^\text{pole}) + \varepsilon^2 \delta E_X^{(1)} + \varepsilon^3 \delta E_X^{(2)} + \ldots \]

\( m_Q \) heavy quark

\( m_q \) massive lighter quark
Effects from massive lighter quarks

\[ E_X(\mu, n_\ell, m_Q^{\text{pole}}, m_q^{\text{pole}}) = E_X(\mu, n_\ell, m_Q^{\text{pole}}) + \epsilon^2 \delta E_X^{(1)} + \epsilon^3 \delta E_X^{(2)} + \cdots \]

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\( m_Q \) heavy quark  \hspace{1cm} \text{massless result}  \hspace{1cm} \text{corrections from massive lighter quarks}  \hspace{1cm} \text{higher order terms currently unknown, but …}

\( m_q \) massive lighter quark


Computed in [Hoang hep-ph/0008102] for the ground state

Effects from massive lighter quarks

\[
E_X(\mu, n_\ell, m_Q^{\text{pole}}, m_q^{\text{pole}}) = E_X(\mu, n_\ell, m_Q^{\text{pole}}) + \varepsilon^2 \delta E_X^{(1)} + \varepsilon^3 \delta E_X^{(2)} + \ldots
\]

- \( m_Q \): heavy quark
- \( m_q \): massive lighter quark

massless result + corrections from massive lighter quarks


Computed in [Hoang hep-ph/0008102] for the ground state


\( n_\ell \) scheme: massless limit manifest, decoupling limit not well defined

\( n_\ell - 1 \) scheme: decoupling limit manifest, massless limit not well defined
Effects from massive lighter quarks

\[ E_X(\mu, n_\ell, m_Q^{\text{pole}}, m_q^{\text{pole}}) = E_X(\mu, n_\ell, m_Q^{\text{pole}}) + \varepsilon^2 \delta E_X^{(1)} + \varepsilon^3 \delta E_X^{(2)} + \cdots \]

\( m_Q \) heavy quark

massless result

corrections from massive lighter quarks

\( m_q \) massive lighter quark

higher order terms currently unknown, but …


Computed in [Hoang hep-ph/0008102] for the ground state


\( n_\ell \) scheme: massless limit manifest, decoupling limit not well defined

\( n_\ell - 1 \) scheme: decoupling limit manifest, massless limit not well defined

Observation made in [Brambilla, Sumino, Vairo PRD65 (2002) 043001]: true answer very very very close to decoupling limit:

Use decoupling limit plus trick to parametrize

Use \( n_\ell - 1 \) scheme plus corrections to incorporate \( O(\varepsilon^3) \) charm quark mass effects
Effects from massive lighter quarks

\[ 10^{-2} \delta E_X^{(2)} [\text{GeV}] \]

\[ m^*_c(n) = \frac{(1 \text{ GeV})}{\sqrt{n}} \]

below \( m^*_c \) use fit function of the form
\[ f(m_c) = m_c \left[ a + b \log(m_c) \right] \]

above use decoupling limit

demand smooth junction

It appears clear we will be in need of massive lighter quarks effect on the short-distance mass as well
Schemes for quarks masses
The pole mass

\[ \Sigma(p, m_0) + \cdots = \frac{1}{\not{p} - m_0 - } \]

\( m_0 = \text{bare mass} \)

\text{quark mass defined in context of perturbation theory}
The pole mass

\[ \Sigma(p, m_0) \]

\( \mu \)-independent has divergences

\[ \frac{1}{\not{p} - m_0} \]

\( m_0 = \text{bare mass} \)

The pole mass \( \Sigma(p, m_0) \) is \( \mu \)-independent.

The whole diagram at \( p^2 = m^2 \) is absorbed into the mass definition.

**Pole scheme**: propagator has a pole for \( \not{p} \to m_p \)

\[ m_p = m_0 + \Sigma(m_p, m_0) \]

pole mass is \( \mu \)-independent
The pole mass

\[ \Sigma(p, m_0) \]

\( \mu \)-independent has divergences

\[ \frac{1}{p - m_0} \]

\( m_0 = \text{bare mass} \)

Quark mass defined in context of perturbation theory

**Pole scheme:** Propagator has a pole for \( \not{p} \to m_p \)

\[ m_p = m_0 + \Sigma(m_p, m_0) \]

Pole mass is \( \mu \)-independent

Absorbs into mass parameter UV fluctuations from scales \( > 0 \)

Linear sensitivity to infrared momenta: factorially growing coefficients in perturbation theory

Sensitivity to non-perturbative regime

\[ \mathcal{O}(\Lambda_{\text{QCD}}) \]

Renormalon
The $\overline{\text{MS}}$ mass

**$\overline{\text{MS}}$ scheme**: propagator is finite, subtract only $\frac{1}{\epsilon}$ in dimensional regularization

$$\overline{m}(\mu) = m_0 + \Sigma(m_p, m_0) \frac{1}{\epsilon} \quad \text{$\overline{\text{MS}}$ mass is $\mu$-dependent}$$
The \( \overline{\text{MS}} \) mass

**\( \overline{\text{MS}} \) scheme:** propagator is finite, subtract only \( \frac{1}{\epsilon} \) in dimensional regularization

\[
\overline{m}(\mu) = m_0 + \Sigma(m_p, m_0)|\frac{1}{\epsilon} \quad \text{\( \overline{\text{MS}} \) mass is \( \mu \)-dependent}
\]

\[
m_p - \overline{m}(\mu) = \Sigma(m_p, m_0)|_{\text{finite}} \equiv \delta m_{\overline{\text{MS}}} (\mu) \quad \text{no renormalon problem}
\]

\( \mu \)-dependent

Absorbs into mass parameter UV fluctuations from scales \( > \overline{m}(\mu) \)
The $\overline{\text{MS}}$ mass

$\overline{\text{MS}}$ scheme: propagator is finite, subtract only $\frac{1}{\epsilon}$ in dimensional regularization

$$\bar{m}(\mu) = m_0 + \sum (m_p, m_0) \frac{1}{\epsilon}$$  \(\overline{\text{MS}}\) mass is $\mu$-dependent

$$m_p - \bar{m}(\mu) = \sum (m_p, m_0) \big|_{\text{finite}} \equiv \delta m_{\overline{\text{MS}}}(\mu)$$  no renormalon problem

$\mu$ - dependent

Absorbs into mass parameter UV fluctuations from scales $> \bar{m}(\bar{m})$

$$\delta m_{\overline{\text{MS}}}(\mu) = \bar{m}(\mu) \sum_{n=1}^{\infty} \left[ \frac{\alpha_s^{(n_\ell + n_h)}(\mu)}{4\pi} \right]^n \sum_{m=0}^{n} a_{n,m}(n_\ell, n_h) \log^m \left( \frac{\bar{m}(\mu)}{\mu} \right)$$

This equations encodes the $\mu$-anomalous dimension of the $\overline{\text{MS}}$ mass
The $\overline{\text{MS}}$ mass

**$\overline{\text{MS}}$ scheme:** propagator is finite, subtract only $\frac{1}{\epsilon}$ in dimensional regularization

$$\overline{m}(\mu) = m_0 + \Sigma(m_p, m_0)\frac{1}{\epsilon} \quad \text{$\overline{\text{MS}}$ mass is $\mu$-dependent}$$

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This series contains the renormalon

Absorbs into mass parameter UV fluctuations from scales $> \overline{m}(\overline{m})$

$$\delta m_{\overline{\text{MS}}} (\mu) = \overline{m}(\mu) \sum_{n=1}^{\infty} \left[ \frac{\alpha_s(n_{\ell}+n_h)(\mu)}{4\pi} \right]^n \sum_{m=0}^n a_{n,m}(n_{\ell}, n_h) \log^m \left( \frac{\overline{m}(\mu)}{\mu} \right)$$

This equations encodes the $\mu$-anomalous dimension of the $\overline{\text{MS}}$ mass

Let us define $\overline{m} \equiv \overline{m}(\overline{m})$:

$$m_p - \overline{m} = \overline{m} \sum_{n=1}^{\infty} \left[ \frac{\alpha_s(n_{\ell}+n_h)(\mu)}{4\pi} \right]^n a_{n,0}(n_{\ell}, n_h)$$

This series contains the renormalon

The $\overline{\text{MS}}$ mass is very far from a kinetic or threshold mass, resembles a coupling constant

Cannot be used in processes for which the quark mass is no longer a dynamical scale
Based on the observation that the B-meson mass $M_B = m_b^{\text{pole}} + \Lambda$ is renormalon free, ambiguity cancels in the sum of these two terms.
Based on the observation that the B-meson mass $M_B = m_b^{\text{pole}} + \Lambda$ is renormalon free $m_b$-independent, therefore the ambiguity in the pole mass is mass-independent.
Based on the observation that the B-meson mass $M_B = m^\text{pole}_b + \bar{\Lambda}$ is renormalon free,

Wilson coefficient is 1 to all orders,

No anomalous dimension,

therefore the ambiguity in the pole mass is scheme and scale independent.
Status of perturbative coefficients

Considering all lighter quarks massless, coefficients known up to $\mathcal{O}(\alpha_s^4)$

- 1-loop: [Tarrach (1981)]
- 2-loop: [Gray, Broadhurst, Grafe, Schilcher (1990)]
- 4-loop: [Marquard, Smirnov, Smirnov, Steinhauser (2015), Marquard, Smirnov, Smirnov, Steinhauser, Wellmann (2016)]
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Asymptotic form known for all orders from renormalon behavior

$$a_n^{\text{asy}} = 4\pi N_{1/2} (2\beta_0)^{n-1} \sum_{\ell=0}^{\infty} g_\ell \left( 1 + \hat{b}_1 \right)_{n-1-\ell}$$
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Corrections from massive lighter quarks known up to $O(\alpha_s^3)$

- 2-loop: [Gray, Broadhurst, Grafe, Schilcher (1990)]
- 3-loop: [Bekavac, Grozin, Seidel, Steinhauser (2007)]
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Corrections from massive lighter quarks known up to $\mathcal{O}(\alpha_s^3)$

- 2-loop: [Gray, Broadhurst, Grafe, Schilcher (1990)]
- 3-loop: [Bekavac, Grozin, Seidel, Steinhauser (2007)]

4-loop and higher can be estimated within a few percent

[Lepeãnk, Hoang, Preisser (2017)], [VM, P.G. Ortega (2017), this talk]
The MSR mass

We exploit the fact that the ambiguity is mass-independent since the renormalon only sees light flavors, express the series in terms of $\alpha_s^{(n_\ell)}$

Either by setting $n_h = 0$ (Natural MSR mass or MSRn)
Or expressing $\alpha_s^{(n_\ell+1)}(\bar{m}_Q)$ in terms of $\alpha_s^{(n_\ell)}(\bar{m}_Q)$ (Practical MSR mass or MSRp)
The MSR mass

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Or expressing $\alpha_s^{(n_{\ell}+1)}(\overline{m}_Q)$ in terms of $\alpha_s^{(n_{\ell})}(\overline{m}_Q)$ (Practical MSR mass or MSRp)

$$m_Q^{\text{pole}} - m_Q^{\text{MSR}}(R) = R \sum_{n=1}^{\infty} a_n^{\text{MSR}}(n_{\ell}) \left( \frac{\alpha_s^{(n_{\ell})}(R)}{4\pi} \right)^n$$

same ambiguity as MS to pole relation

The MSRn mass can be easily matched to the MS mass at $R = \overline{m}_Q$

By construction $m_Q^{\text{MSRp}}(\overline{m}_Q) = \overline{m}_Q(\overline{m}_Q)$ to all orders
The MSR mass

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By construction $m_Q^{\text{MSRp}}(\bar{m}_Q) = \bar{m}_Q(\bar{m}_Q)$ to all orders

Both realizations coincide at $O(\alpha_s)$

Difference of masses is renormalon-free as long as series expressed in terms of $\alpha_s$ at the same scale

Last statement true for any series
In a given physical situation one has a perturbative expansion in terms of $\alpha_s(\mu)$.

Therefore we have to choose $\mu \sim R$.

The value of $\mu$ is in general much smaller than $\overline{m}_Q$ for the cases we care.

Therefore there are large logs of $\overline{m}_Q/R$ that need to be summed up:

$$R \frac{d}{dR} m_Q^{\text{MSR}}(R) = - R \gamma^R[\alpha_s(R)] = - R \sum_{n=0}^{\infty} \gamma_n^R \left( \frac{\alpha_s(R)}{4\pi} \right)^{n+1}$$
In a given physical situation one has a perturbative expansion in terms of $\alpha_s(\mu)$.

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The value of $\mu$ is in general much smaller than $\overline{m}_Q$ for the cases we care.

Therefore there are large logs of $\overline{m}_Q/R$ that need to be summed up:

$$R \frac{d}{dR} m_{\text{MSR}}^Q(R) = - R \gamma^R [\alpha_s(R)] = - R \sum_{n=0}^{\infty} \gamma_n^R \left( \frac{\alpha_s(R)}{4\pi} \right)^{n+1}$$

Pole mass is $R$-independent $\rightarrow$ $R$-anomalous dimension from MSR definition

Ambiguity $R$-independent $\rightarrow$ $R$-anomalous dimension renormalon-free

General formula

$$\gamma_n^R = a_{n+1}^{\text{MSR}} - 2 \sum_{j=0}^{n-1} (n - j) \beta_j a_{n-j}^{\text{MSR}}$$
In a given physical situation one has a perturbative expansion in terms of $\alpha_s(\mu)$.

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$$R \frac{d}{dR} m_{\text{MSR}}(R) = -R \gamma^R [\alpha_s(R)] = -R \sum_{n=0}^{\infty} \gamma^R_n \left( \frac{\alpha_s(R)}{4\pi} \right)^{n+1}$$

pole mass is R-independent $\quad \rightarrow \quad$ R-anomalous dimension from MSR definition

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general formula $\quad \gamma^R_n = a_{n+1}^{\text{MSR}} - 2 \sum_{j=0}^{n-1} (n - j) \beta_j a_{n-j}^{\text{MSR}}$

renormalon cancels between these two terms

[Hoang, Jain, Scimemi, Stewart (2008)]
In a given physical situation one has a perturbative expansion in terms of $\alpha_s(\mu)$. Therefore we have to choose $\mu \sim R$.

The value of $\mu$ is in general much smaller than $\overline{m}_Q$ for the cases we care. Therefore there are large logs of $\overline{m}_Q/R$ that need to be summed up:

$$R \frac{d}{dR} m^{\text{MSR}}_Q(R) = -R \gamma^R [\alpha_s(R)] = -R \sum_{n=0}^{\infty} \gamma_n^R \left( \frac{\alpha_s(R)}{4\pi} \right)^{n+1}$$

pole mass is R-independent $\quad \rightarrow \quad$ R-anomalous dimension from MSR definition

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general formula $\gamma_n^R = a_{n+1}^{\text{MSR}} - 2 \sum_{j=0}^{n-1} (n - j) \beta_j a_{n-j}^{\text{MSR}}$ renormalon cancels between these two terms

Solution to RGE equation:

$$m^{\text{MSR}}_Q(R_2) - m^{\text{MSR}}_Q(R_1) = \int_{R_1}^{R_2} dR \gamma_n^R [\alpha_s^{(n\epsilon)}(R)]$$

Sums up large logs of $R_2/R_1$ associated to the renormalon to all orders in pert. theory

These logs are also summed up e.g. in [Pineda (2001)] for the Renormalon Subtracted mass...
Massive lighter quarks

\[ m_Q^{\text{pole}} - \overline{m}_Q = \delta \overline{m}_Q(\overline{m}_Q) + \overline{m}_Q \Delta_{m_\alpha}^{\overline{\text{MS}}}(\overline{m}_Q, \xi) \]

\[ O(\alpha_s^2) \]

\[ O(\alpha_s^3) \]
Massive lighter quarks

\[ m_Q^{\text{pole}} - \overline{m}_Q = \delta m_Q(\overline{m}_Q) + \overline{m}_Q \Delta^{\text{MS}}(\overline{m}_Q, \xi) \]

\[ \Delta^{\text{MS}}(\overline{m}_Q, \xi) = \sum_{n=2} \left( \frac{\alpha_s(n_\ell + n_h)(\overline{m}_Q)}{4\pi} \right)^n \Delta^{\text{MS}}(n_\ell, n_h, \xi) \]

\[ \xi = \frac{\overline{m}_q}{\overline{m}_Q} \]

where for our calculation only four-loop corrections \([24–26]\) are needed.
Massive lighter quarks

\[ m^\text{pole}_Q - \overline{m}_Q = \delta \overline{m}_Q(\overline{m}_Q) + \overline{m}_Q \Delta_{\overline{m}_q}^{\text{MS}}(\overline{m}_Q, \xi) \]

massless

\[ \Delta_{\overline{m}_c}^{\text{MS}}(\overline{m}_Q, \xi) = \sum_{n=2} \left( \frac{\alpha_s(n_{\ell} + n_h)(\overline{m}_Q)}{4\pi} \right)^n \Delta_n^{\text{MS}}(n_{\ell}, n_h; \xi) \]

\[ \xi = \frac{\overline{m}_q}{\overline{m}_Q} \]

Two obvious constraints

\[ \Delta_n^{\text{MS}}(n_{\ell}, n_h, 0) = 0 \]
\[ \Delta_n^{\text{MS}}(n_{\ell}, n_h, 1) = a_n^{\text{MS}}(n_{\ell} - 1, n_h + 1) - a_n^{\text{MS}}(n_{\ell}, n_h) \]
Massive lighter quarks

\[ m_Q^{\text{pole}} - \bar{m}_Q = \delta \bar{m}_Q(\bar{m}_Q) + \bar{m}_Q \Delta_{\MS}(\bar{m}_Q, \xi) \]

\begin{align*}
\Delta_{\MS}(\bar{m}_Q, \xi) &= \sum_{n=2} \left( \frac{\alpha_s^{n_L+n_h}}{4\pi} \right)^n \Delta_{\MS}(n_L, n_h, \xi) \\
\xi &= \frac{\bar{m}_q}{\bar{m}_Q}
\end{align*}

\[ \mathcal{O}(\alpha_s^2) \]

\[ \mathcal{O}(\alpha_s^3) \]

Two obvious constraints

\[ \Delta_{\MS}(n_L, n_h, 0) = 0 \]

\[ \Delta_{\MS}(n_L, n_h, 1) = a_{\MS}(n_L-1, n_h+1) - a_{\MS}(n_L, n_h) \]
**Massive lighter quarks**

\[ m_Q^{\text{pole}} - \bar{m}_Q = \delta m_Q(\bar{m}_Q) + \frac{\bar{m}_Q \Delta \overline{\text{MS}}(\bar{m}_Q, \xi)}{m_q} \]  

**mass corrections**

\[ \Delta \overline{\text{MS}}(\bar{m}_Q, \xi) = \sum_{n=2} \left( \frac{\alpha_s(n_{\ell} + n_h \bar{m}_Q)}{4\pi} \right)^n \Delta \overline{\text{MS}}(n_{\ell}, n_h, \xi) \]

\[ \xi = \frac{m_q}{\bar{m}_Q} \]

**Two obvious constraints**

\[ \Delta \overline{\text{MS}}(n_{\ell}, n_h, 0) = 0 \]

\[ \Delta \overline{\text{MS}}(n_{\ell}, n_h, 1) = a_n \Delta \overline{\text{MS}}(n_{\ell} - 1, n_h + 1) - a_n \Delta \overline{\text{MS}}(n_{\ell}, n_h) \]

**Massive lighter quarks effects on the MSR mass**

\[ \delta m_Q^{\text{MSR}}(R, \bar{m}_q) = \delta m_Q^{\text{MSR}}(R) + R \Delta \overline{m}_q(R, \xi_R) ; \]

\[ \Delta \overline{m}_q(R, \xi_R) = \sum_{k=2}^\infty \Delta^{(k)}(\xi_R) \left( \frac{\alpha_s(n_{\ell} \bar{m}_q)}{4\pi} \right)^k, \quad \xi_R = \frac{m_q}{R} \]

**differently implemented in** [Lepenik, Hoang, Preisser (2017)]
\[
\delta m_Q^{\text{MSR}}(R, \bar{m}_q) = \delta m_Q^{\text{MSR}}(R) + R \Delta \bar{m}_q(R, \xi_R), \\
\Delta \bar{m}_q(R, \xi_R) = \sum_{k=2} \Delta^{(k)}_{\bar{m}_q}(\xi_R) \left( \frac{\alpha_s^{(n})}{4\pi} \right)^k, \\
- \frac{d}{dR} m_Q^{\text{MSR}}(R) = \gamma^R[\alpha_s^{(n})] + \delta\gamma^R[\xi_R, \alpha_s^{(n})].
\]

exact Heavy Quark symmetry for MSRn

\[m_Q^{\text{pole}} - m_Q^{\text{MSRn}}(\bar{m}_q) = m_q^{\text{pole}} - \bar{m}_q\]
MSR with Massive lighter quarks

\[
\delta m_Q^{\text{MSR}}(R, \bar{m}_q) = \delta m_Q^{\text{MSR}}(R) + R \Delta \bar{m}_q(R, \xi_R),
\]

\[
\Delta \bar{m}_q(R, \xi_R) = \sum_{k=2}^{\infty} \Delta^{(k)}(\xi_R) \left( \frac{\alpha_s^{(n_e)}(R)}{4\pi} \right)^k,
\]

\[
- \frac{d}{dR} m_Q^{\text{MSR}}(R) = \gamma^R[\alpha_s^{(n_e)}(R)] + \delta \gamma^R[\xi_R, \alpha_s^{(n_e)}(R)]
\]

Exact Heavy Quark symmetry for MSRn

\[
m_Q^{\text{pole}} - m_Q^{\text{MSRn}}(\bar{m}_q) = m_q^{\text{pole}} - \bar{m}_q
\]

Mass-dependent R-anomalous dimension

Massless R-anomalous dimension

\[
\delta \gamma^R[\xi_R, \alpha_s^{(n_e)}(R)] = \sum_{n=1}^{\infty} \delta \gamma^n(\xi_R) \left( \frac{\alpha_s^{(n_e)}(R)}{4\pi} \right)^{n+1}
\]
\[ \delta m_{Q}^{\text{MSR}}(R, \bar{m}_q) = \delta m_{Q}^{\text{MSR}}(R) + R \Delta \bar{m}_q(R, \xi_R), \]

\[ \Delta \bar{m}_q(R, \xi_R) = \sum_{k=2} \Delta^{(k)} \bar{m}_q(\xi_R) \left( \frac{\alpha_s^{(n\ell)}(R)}{4\pi} \right)^k, \]

\[ -\frac{d}{dR} m_{Q}^{\text{MSR}}(R) = \gamma^R [\alpha_s^{(n\ell)}(R)] + \delta \gamma^R [\xi_R, \alpha_s^{(n\ell)}(R)] \]

**mass-dependent R-anomalous dimension**

\[ \delta \gamma^R [\xi_R, \alpha_s^{(n\ell)}(R)] = \sum_{n=1} \delta \gamma_n^R(\xi_R) \left( \frac{\alpha_s^{(n\ell)}(R)}{4\pi} \right)^{n+1} \]

\[ \Delta_n(\xi_R) = \xi_R \frac{d\Delta_n(\xi_R)}{d\xi_R} - 2 \sum_{j=0}^{n-2} (n-j) \beta_j \Delta_{n-j}(\xi_R) \]

**massless R-anomalous dimension**

**exact Heavy Quark symmetry for MSRn**

\[ m_{Q}^{\text{pole}} - m_{Q}^{\text{MSRn}(\bar{m}_q)} = m_{q}^{\text{pole}} - \bar{m}_q \]
\[ \delta m_Q^{\text{MSR}}(R, \bar{m}_q) = \delta m_Q^{\text{MSR}}(R) + R \Delta \bar{m}_q(R, \xi_R), \]

\[ \Delta \bar{m}_q(R, \xi_R) = \sum_{k=2} \Delta^{(k)} \bar{m}_q(\xi_R) \left( \frac{\alpha_s^{(n_\ell)}(R)}{4\pi} \right)^k, \]

exact Heavy Quark symmetry for MSRn

\[ m_Q^{\text{pole}} - m_Q^{\text{MSRn}}(\bar{m}_q) = m_q^{\text{pole}} - \bar{m}_q \]

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mass-dependent R-anomalous dimension

\[ \delta \gamma_n^{\text{R-anom}}(\xi_R) = \Delta_n(\xi_R) - \xi_R \frac{d\Delta_n(\xi_R)}{d\xi_R} - 2 \sum_{j=0}^{n-2} (n - j) \beta_j \Delta_{n-j}(\xi_R) \]

renormalon cancels among these terms

Various contributions to \( \delta y^R_2 \)

- \( \delta y^R_2 = \Delta_2(\xi) - \xi \frac{d\Delta_2(\xi)}{d\xi} \Delta_2(\xi) \)
- \( \xi \frac{d\Delta_2(\xi)}{d\xi} \)
- \( \Delta_2(\xi) \)

Various contributions to \( \delta y^R_3 \)

- \( \Delta_3(\xi) - \xi \frac{d\Delta_3(\xi)}{d\xi} \Delta_3(\xi) \)
- \( \xi \frac{d\Delta_3(\xi)}{d\xi} \Delta_3(\xi) \)
- \( \beta_0 \Delta_2(\xi) \)
- \( \Delta_3(\xi) \)
- \( \delta y^R_3 \)

huge cancellations among the various contributions!
MSR with Massive lighter quarks

\[ \delta m_{Q}^{\text{MSR}}(R, \bar{m}_q) = \delta m_{Q}^{\text{MSR}}(R) + R \Delta \bar{m}_q(R, \xi_R), \]

\[ \Delta \bar{m}_q(R, \xi_R) = \sum_{k=2} \Delta^{(k)}(\xi_R) \left( \frac{\alpha_s^{(n\ell)}(R)}{4\pi} \right)^k, \]

\[ -\frac{d}{dR} m_{Q}^{\text{MSR}}(R) = \gamma^R[\alpha_s^{(n\ell)}(R)] + \delta \gamma^R[\xi_R, \alpha_s^{(n\ell)}(R)] \]

mass-dependent R-anomalous dimension

\[ \delta \gamma^R[\xi_R, \alpha_s^{(n\ell)}(R)] = \sum_{n=1} \delta \gamma_n^R(\xi_R) \left( \frac{\alpha_s^{(n\ell)}(R)}{4\pi} \right)^{n+1} \]

massless R-anomalous dimension

\[ \delta \gamma_n^R(\xi_R) = \Delta_n(\xi_R) - \xi_R \frac{d\Delta_n(\xi_R)}{d\xi_R} - 2 \sum_{j=0}^{n-2} (n-j) \beta_j \Delta_{n-j}(\xi_R) \]

renormalon cancels among these terms

Various contributions to \( \delta \gamma_2^R \)

Various contributions to \( \delta \gamma_3^R \)

huge cancelations among the various contributions!

requiring exactly \( \delta \gamma_n^R(\xi) = 0 \)

prediction for higher orders \( \Delta_n(\xi) \approx \xi \Delta_n(1) + 2 \xi \sum_{j=0}^{n-2} (n-j) \beta_j \int_{\xi}^1 dx \frac{\Delta_{n-j}(x)}{x^2} \)

satisfies \( \xi = 0 \) and \( \xi = 1 \) constraints
MSR with $n_f - 1$ active flavors

Physical situations in which one runs to scales $R < \bar{m}_q$ and $\bar{m}_q$ is integrated out

Therefore we must integrate the quark $q$ in the MSR mass as well.
MSR with $n_{\ell} - 1$ active flavors

Physical situations in which one runs to scales $R < \overline{m}_q$ and $\overline{m}_q$ is integrated out.

Therefore we must integrate the quark q in the MSR mass as well.

We define the $\text{MSR}^{(n_{\ell} - 1)}$ mass as

$$m_Q^{\text{pole}} - m_Q^{\text{MSR}^{(n_{\ell} - 1)}}(R) = m_q^{\text{pole}} - m_q^{\text{MSR}^{(R)}}$$

It is smoothly matched with the $\text{MSR}^{(n_{\ell})}$ mass at $R = \overline{m}_q$.

Essential to study the ambiguity of the pole mass. [Hoang, Lepenik, Preisser, ‘17]
MSR with $n_\ell - 1$ active flavors

Physical situations in which one runs to scales $R < \bar{m}_q$ and $\bar{m}_q$ is integrated out

Therefore we must integrate the quark $q$ in the MSR mass as well

We define the MSR$(n_\ell - 1)$ mass as

$$m_Q^{\text{pole}} - m_Q^{\text{MSR} (n_\ell - 1)} (R) = m_Q^{\text{pole}} - m_Q^{\text{MSR} (R)}$$

It is smoothly matched with the MSR$(n_\ell)$ mass at $R = \bar{m}_q$

Essential to study the ambiguity of the pole mass

VFNS-like sequence of running and matching
While we worked on our MSR mass with massive lighter quarks the article by Hoang, Lepenik and Preissner appeared

\[ m_Q^{\text{pole}} - m_Q^{\text{MSR}}(R, \overline{m}_q) = \delta m_Q^{\text{MSR}}(R) + \overline{m}_Q \Delta \overline{m}_q(\overline{m}_Q, \overline{m}_q/\overline{m}_Q) \]

massless term R-independent
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massless term

R-independent

Therefore R-evolutions is the same as for massless quarks

Identical matching to \( \overline{MS} \) mass at \( R = \overline{m}_Q \)

Different matching for MSR\(^{(n_\ell)}\) and MSR\(^{(n_\ell - 1)}\) masses at \( R = \overline{m}_q \)
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Different matching for \( \text{MSR}^{(n\ell)} \) and \( \text{MSR}^{(n\ell-1)} \) masses at \( R = \bar{m}_q \)

Prediction for higher order corrections from imposing exact Heavy Quark Symmetry

almost identical to our prediction
Comment on “Vienna implementation” [Hoang, Lepenik, Preisser, ’17]

While we worked on our MSR mass with massive lighter quarks the article by Hoang, Lepenik and Preisser appeared

$$m_Q^{\text{pole}} - m_Q^{\text{MSR}} (R, \bar{m}_q) = \delta m_Q^{\text{MSR}} (R) + \bar{m}_Q \Delta \bar{m}_q (\bar{m}_Q, \bar{m}_q / \bar{m}_Q)$$

\(m_{\text{pole}}\) massless term \(\Delta \bar{m}_{\text{q}}\) R-independent

Therefore R-evolutions is the same as for massless quarks

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Different matching for \(\text{MSR}^{(n_\ell)}\) and \(\text{MSR}^{(n_\ell - 1)}\) masses at \(R = \bar{m}_q\)

Prediction for higher order corrections from imposing exact Heavy Quark Symmetry

Different aims lead to slightly different versions of the MSR mass

For all practical purposes can be considered identical
Analysis
Scale variation and charm mass dependence
## Scale dependence investigation

<table>
<thead>
<tr>
<th>“Popular” scheme choices in the literature</th>
<th>[\overline{\text{MS}}]: large logs of (\frac{\overline{m}_b}{\mu}) in subtractions</th>
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## Scale dependence investigation

### “Popular” scheme choices in the literature

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### Our choice: MSR mass (either version)

- RS mass (Pineda): no smooth transition to $(n_l - 1)$ scheme  
  [Ayala, Czvetic, Pineda (2016)]

### “Popular” scale variations in the literature: Independent scale variation (one at a time)

- [Ayala, Czvetic, Pineda (2016)]
**Scale dependence investigation**

"Popular" **scheme choices** in the literature

$\overline{\text{MS}}$: large logs of $\frac{m_b}{\mu}$ in subtractions

[Brambilla, Vairo, Sumino], [Kiyo, Mishima, Sumino]

Our choice: MSR mass (either version)

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"Popular" **scale variations** in the literature: Independent scale variation (one at a time)

[Ayala, Czvetic, Pineda (2016)]

**Principle of minimal sensitivity** [Brambilla, Vairo, Sumino]

Take the scale at maximum or minimum, double that scale to estimate uncertainties
Not defined at all orders. Large dependence on order and quantum numbers other than $n$.
Results in ranges that cover relativistic scales. Renders small perturbative uncertainties.

---

![Graphs showing scale dependence](image-url)

- MRSp scheme with $R = \overline{m}_b(\overline{m}_b)$
- MRSp scheme with $R = \overline{m}_b(\overline{m}_c)$
- MRSp scheme with $R = \overline{m}_c(\overline{m}_c)$
Scale dependence investigation

Scale variation should (only) depend on the principal quantum number n, since the argument of perturbative logs depends on n (but not on other numbers)

Should not depend on the perturbative order

It should also depend on bottomonium vs charmonium
Scale dependence investigation

Scale variation should (only) depend on the principal quantum number $n$, since the argument of perturbative logs depends on $n$ (but not on other numbers).

Should not depend on the perturbative order.

It should also depend on bottomonium vs charmonium.

Our criterion: argument of logs should vary between $2^{\pm \phi}$

$$\left( \mu_{\text{nat}} \pm \Delta \mu \right) \sim \frac{2^{\pm \phi} C_F \alpha_s^{(n \ell)}(\mu) m_Q}{n}$$

We take $\phi = 0.5$ but extend the upper limit to 4 GeV (similar to scale variation in relativistic sum rules [Dehnadi, Hoang, Mateu (2013, 2015)]).
Scale dependence investigation

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$$\mu_{n=1} \sim 1.9^{+1.6}_{-0.4} \text{ GeV} \quad \mu_{n=2} \sim 1.25 \pm 0.25 \text{ GeV}$$
Scale dependence investigation

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Our criterion: argument of logs should vary between \( 2^{\pm \phi} \)

\[
(\mu_{\text{nat}} \pm \Delta \mu) \sim \frac{2^{\pm \phi} C_F \alpha_s^{(n \ell)}(\mu) m_Q}{n}
\]

We take \( \phi = 0.5 \) but extend the upper limit to 4 GeV (similar to scale variation in relativistic sum rules [Dehnadi, Hoang, Mateu (2013, 2015)])

For \( n = 3 \) one gets a lower scale below 1 GeV. It seems no scale choice can make the perturbative series both convergent and compatible.
Scale dependence investigation

Scale variation should (only) depend on the principal quantum number \( n \), since the argument of perturbative logs depends on \( n \) (but not on other numbers).

Should not depend on the perturbative order

It should also depend on bottomonium vs charmonium

Our criterion: argument of logs should vary between \( 2^{\pm \phi} \)

\[
\left( \mu_{\text{nat}} \pm \Delta \mu \right) \sim \frac{2^{\pm \phi} C_F \alpha_s(n_F)(\mu)m_Q}{n}
\]

We take \( \phi = 0.5 \) but extend the upper limit to 4 GeV (similar to scale variation in relativistic sum rules [Dehnadi, Hoang, Mateu (2013, 2015)])

For charm this criterion renders the lower scale below 1 GeV. However the following choice \( 1.2 \text{ GeV} \geq \mu_{\text{charm}} \geq 4 \text{ GeV} \) makes for a convergent and compatible perturbative series
Charm mass dependence

This confirms that the \((n_l - 1)\) scheme is the most accurate to describe finite charm quark mass effect.
Perturbative correlations
Perturbative correlations

Perturbative uncertainties highly correlate quarkonium masses, since:

1. All masses are determined from the same static potential ($\mu$ dependence)
2. Same quark mass for all bound states (R dependence)
Perturbative correlations

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1. All masses are determined from the same static potential ($\mu$ dependence)
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But for different values of $n$ we use different scale variation $\longrightarrow$ linear rescaling

$$\mu_2(\mu) = \mu, \quad R_2(R) = R \quad 1 \text{ GeV} \leq \mu, R \leq 4 \text{ GeV}$$

$$\mu_{1,3}(\mu) = 1.5 \text{ GeV} + 2.5 (\mu - 1 \text{ GeV})/3, \quad R_{1,3}(\mu) = 1.5 \text{ GeV} + 2.5 (R - 1 \text{ GeV})/3$$
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$$
\begin{align*}
\mu_2(\mu) &= \mu, & \quad R_2(R) &= R & \quad 1 \text{ GeV} \leq \mu, R \leq 4 \text{ GeV} \\
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\end{align*}
$$

Perturbative covariance matrix approach: severely affected by D’Agostini bias

We make our $\chi^2$ function depend on ($\mu, R$) and scan on the range shown above

$$
\chi^2(\overline{m}_Q, \mu, R) = \sum_i \left( \frac{M_i^{\text{exp}} - M_i^{\text{pert}}(\mu, R, \overline{m}_Q)}{\sigma_i^{\text{exp}}} \right)^2
$$
Perturbative correlations

Perturbative uncertainties highly correlate quarkonium masses, since

1. All masses are determined from the same static potential ($\mu$ dependence)
2. Same quark mass for all bound states ($R$ dependence)

But for different values of $n$ we use different scale variation $\mu^2(\mu) = \mu$, $R^2(R) = R$

$\mu_{1,3}(\mu) = 1.5 \text{ GeV} + 2.5 (\mu - 1 \text{ GeV})/3$, $R_{1,3}(\mu) = 1.5 \text{ GeV} + 2.5 (R - 1 \text{ GeV})/3$

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$$\chi^2(\bar{m}_Q, \mu, R) = \sum_i \left( \frac{M_i^{\text{exp}} - M_i^{\text{pert}}(\mu, R, \bar{m}_Q)}{\sigma_i^{\text{exp}}} \right)^2$$

This approach correctly propagates the theoretical correlations and avoids de bias

We also vary the strong coupling constant and the charm (bottom) mass for bottomonium (charmonium)
Fits to data
Different data sets

Bottomonium

1. Set$_{n=1}$ = \{ $\eta_b(1S)$, $\Upsilon(1S)$ \}.

2. Set$_{n=2}$ = \{ $\chi_{b0}(1P)$, $\chi_{b1}(1P)$, $h_b(1P)$, $\chi_{b2}(1P)$, $\eta_b(2S)$, $\Upsilon(2S)$ \}.

3. Set$_{n=3}$ = \{ $\Upsilon(1D)$, $\chi_{b0}(2P)$, $\chi_{b1}(2P)$, $h_b(2P)$, $\chi_{b2}(2P)$, $\Upsilon(3S)$ \}.

4. Set$_{L=P}$ = \{ $\chi_{b0}(1P)$, $\chi_{b1}(1P)$, $h_b(1P)$, $\chi_{b2}(1P)$ \}.

5. Set$_{n\leq2}$ = Set$_{n=1} \cup$ Set$_{n=2}$.

6. Set$_{n\leq3}$ = Set$_{n=1} \cup$ Set$_{n=2} \cup$ Set$_{n=3}$.

+ determinations from individual states
Different data sets

**Bottomonium**

1. Set$_{n=1} = \{ \eta_b(1S), \Upsilon(1S) \}$.

2. Set$_{n=2} = \{ \chi_{b0}(1P), \chi_{b1}(1P), h_b(1P), \chi_{b2}(1P), \eta_b(2S), \Upsilon(2S) \}$.

3. Set$_{n=3} = \{ \Upsilon(1D), \chi_{b0}(2P), \chi_{b1}(2P), h_b(2P), \chi_{b2}(2P), \Upsilon(3S) \}$.

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5. Set$_{n\leq2} = \text{Set}_{n=1} \cup \text{Set}_{n=2}$.

6. Set$_{n\leq3} = \text{Set}_{n=1} \cup \text{Set}_{n=2} \cup \text{Set}_{n=3}$.

+ determinations from individual states

**Charmonium**

$\eta_c(1S), J/\psi(1S)$  + determinations from individual states
Results for bottom and charm

$\alpha_s(m_2) = 0.1181 \pm 0.0011$

$\overline{m}_b(\overline{m}_b)$ [GeV] $N^3$LO in the MSRn scheme with $\alpha_s^{(n=3)}$

$\overline{m}_c(\overline{m}_c) = 1.28 \pm 0.03$ GeV

$\eta_c(1S)$ $J/\psi(1S)$
Results for bottom and charm

![Graphs showing results for bottom and charm](image)

Dependence with $\alpha_s$

![Graphs showing dependence with $\alpha_s$](image)
Convergence

\[ \alpha_s(m_Z) = 0.1181 \pm 0.0011 \]

\[ \bar{m}_c(m_c) = 1.28 \pm 0.03 \text{ GeV} \]

\[ n = 1, 2 \]

\[ \alpha_s(n=3) \]

\[ MSRn \text{ scheme with } n = 1, 2 \]

\[ n = 3 \]

\[ \eta_c(1S) \]

\[ J/\psi(1S) \]

\[ n = 1 \]
\[ m_b(m_b) [\text{GeV}] \quad \text{MSRn scheme with } \alpha_s^{(n=3)} \]

- \[ \alpha_s(m_Z) = 0.1181 \pm 0.0011 \]
- \[ \overline{m}_c(\overline{m}_c) = 1.28 \pm 0.03 \text{ GeV} \]

**World Average**

\[ n = 1 \]

\[ n = 2 \]

\[ n = 1, 2 \]

**Convergence**

\[ m_c(m_c) [\text{GeV}] \quad \text{MSRn scheme} \]

- \[ \eta_c(1S) \]
- \[ J/\psi(1S) \]

**Comparison to data**

\[ M_{\text{state}} [\text{GeV}]: \text{theory at N}^3\text{LO vs experiment} \]

- \[ \alpha_s(m_Z) = 0.1181 \pm 0.0011 \]
- \[ \overline{m}_c(\overline{m}_c) = 1.28 \pm 0.03 \text{ GeV} \]
- \[ \overline{m}_b(\overline{m}_b) = 4.216 \pm 0.039 \text{ GeV} \]
Final results

\[
\bar{m}_b(\bar{m}_b) = 4.216 \pm 0.009_{\text{exp}} \pm 0.034_{\text{pert}} \pm 0.017\alpha_s \pm 0.0008\bar{m}_c \text{ GeV} \\
= 4.216 \pm 0.039 \text{ GeV}
\]

\[
\bar{m}_c(\bar{m}_c) = 1.273 \pm 0.0005_{\text{exp}} \pm 0.054_{\text{pert}} \pm 0.006\alpha_s \pm 0.0001\bar{m}_b \text{ GeV} \\
= 1.273 \pm 0.054 \text{ GeV},
\]

Results in MSRn and MSRp schemes nearly identical

Estimate uncertainty from non-perturbative corrections comparing fits from various datasets

Estimate uncertainty from missing finite charm mass effects by comparing fits in \( n_l \) and \( n_l - 1 \) schemes
Final results

\[ \overline{m}_b(\overline{m}_b) = 4.216 \pm 0.009_{\text{exp}} \pm 0.034_{\text{pert}} \pm 0.017\alpha_s \pm 0.0008\overline{m}_c \text{ GeV} \]
\[ = 4.216 \pm 0.039 \text{ GeV} \]
\[ \overline{m}_c(\overline{m}_c) = 1.273 \pm 0.0005_{\text{exp}} \pm 0.054_{\text{pert}} \pm 0.006\alpha_s \pm 0.0001\overline{m}_b \text{ GeV} \]
\[ = 1.273 \pm 0.054 \text{ GeV} \]

Results in MSRn and MSRp schemes nearly identical

Estimate uncertainty from non-perturbative corrections comparing fits from various datasets

Estimate uncertainty from missing finite charm mass effects by comparing fits in \( n_f \) and \( n_f - 1 \) schemes

Simultaneous determination

\[ \overline{m}_b(\overline{m}_b) = 4.219 \pm 0.0002_{\text{exp}} \pm 0.062_{\text{pert}} \text{ GeV} \]
\[ \alpha_s^{(n_f=5)}(m_Z) = 0.1178 \pm 0.00001_{\text{exp}} \pm 0.0050_{\text{pert}} \]

Very strong correlation between these two parameters
If correlation broken, competitive \( \alpha_s \) possible
Comparison to previous determinations
Comparison to other determinations

<table>
<thead>
<tr>
<th>Method</th>
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<td>Kiyo 2016: N^3LL, MS</td>
<td>4.20</td>
</tr>
<tr>
<td>Ayala 2016: N^3LL, RS</td>
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<tr>
<td>Brambilla 2002: N^2LL, MS</td>
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</tr>
<tr>
<td>This work: N^3LL, MSR</td>
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It appears that non-relativistic analyses yield large values for the bottom mass as compared to the world average.
Comparison to other determinations

$m_b$ from Bottomonium

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$m_b$ from other methods

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<td>Hoang et al. '12</td>
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It appears that non-relativistic analyses yield large values for the bottom mass as compared to the world average.

Same statement does not hold true for charm mass.
Calibration of the Cornell model
Very preliminary!

Simple idea: can I relate the Cornell model parameters with QCD parameters?
Simple idea: can I relate the Cornell model parameters with QCD parameters?

Strategy: generate QCD predictions for bottomonium masses up to \( n = 2 \) and scan over the bottom mass for a wide range. We keep \( m_c = 0 \) and \( \alpha_s(1.3 \text{ GeV}) \) fixed.

We generate perturbative uncertainties adapting our scale variation to a variable bottom mass.
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If we choose $R = 1 \text{ GeV}$ the intersect is very close to zero, and fully compatible with zero within errors.
Conclusions
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- Quarkonia masses are a good place to determine the quark masses with high precision.
- Employing a low-scale short-distance mass as the MSR is crucial to cancel de renormalon and avoid large logs.
- Charm mass effects in bottomonium are close to the decoupling limit: integrate out charm and add power corrections.
- Effects from massive lighter quarks must then be incorporated into MSR mass, and the lighter quark can be integrated out.
- Very precise bottom mass determination, charm also good.
- This setup can be used to calibrate quark models such as Cornell.