Non-Abelian Discrete Groups and Neutrino Flavor Symmetry

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Content

1 Introduction

The discrete transformations (*e.g.*, rotation of a regular polygon) give rise to corresponding symmetries:

Discrete Symmetry

The well known fundamental symmetry in particle physics is,

C, P, T : Abelian

Non-Abelian Discrete Symmetry may be important for flavor physics of quarks and leptons.

The discrete symmetries are described by finite groups.

The classification of the finite groups has been completed in 2004,
(Gorenstein announced in 1981 that the finite simple groups had all been classified.)
about 100 years later than the case of the continuous groups.

Thompson, Gorenstein, Aschbacher ……

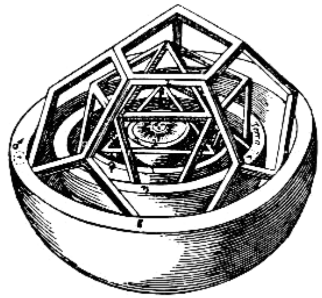
The classification of finite simple group

Theorem —
Every finite **simple group** is **isomorphic** to one of the following groups:

- a member of one of three infinite classes of such:
  - the **cyclic groups** of prime order, **Z_n (n: prime)**
  - the **alternating groups** of degree at least 5, **A_n (n>4)**
  - the **groups of Lie type** **$E_6(q)$, $E_7(q)$, $E_8(q)$, ……**
- one of 26 groups called the "**sporadic groups**" **Mathieu groups, Monster group …**
- the **Tits group** (which is sometimes considered a 27_{th} sporadic group).

See Web: http://brauer.maths.qmul.ac.uk/Atlas/v3/

Johannes Kepler

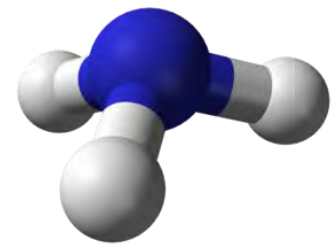


The Cosmographic Mystery

More than 400 years ago,
Kepler tried to understand cosmological structure by five Platonic solids.

Scientists like symmetries !

molecular symmetry



Finite groups are used to classify crystal structures, regular polyhedra, and the symmetries of molecules.
The assigned point groups can then be used to determine physical properties, spectroscopic properties and to construct molecular orbitals.

Finite groups also possibly control fundamental particle physics as well as chemistry and materials science.

Symmetry is an advantageous approach if the dynamics is unknown.

2 Examples of finite groups

Ishimori, Kobayashi, Ohki, Shimizu, Okada, M.T, PTP supprement, 183,2010,arXiv1003.3552,
Lect. Notes Physics (Springer) 858,2012

Finite group G

consists of a finite number of element of G.

- The number of elements in G is called **order**.
- The group G is called Abelian
  if all elements are commutable each other,i.e. $ab = ba$.
- The group G is called non-Abelian
  if all elements do not satisfy the commutativity.

Subgroup

If a subset H of the group G is also a group, H is called subgroup of G. The order of the subgroup H is a divisor of the order of G.

(Lagrange's theorem)

If a subgroup N of G satisfies $g^{-1}Ng = N$ for any element $g \in G$, the subgroup N is called a normal subgroup or an invariant subgroup.

The subgroup H and normal subgroup N of G satisfy $HN = NH$ and it is a subgroup of G, where HN denotes $\{h_i n_j \mid h_i \in H, n_j \in N\}$

Simple group

It is a nontrivial group whose only normal subgroups are the trivial group and the group itself.

A group that is not simple can be broken into two smaller groups, a normal subgroup and the quotient group (factor group), and the process can be repeat
If the group is finite, eventually one arrives at uniquely determined simple groups.

G is classified by Conjugacy Class

The number of irreducible representations is equal to the number of conjugacy classes. Schur's lemma

The elements $g^{-1}ag$ for $g \in G$ are called elements conjugate to the element a.

The set including all elements to conjugate to an element a of G, $\{g^{-1}ag, \forall g \in G\}$, is called a conjugacy class.

When $a^h = e$ for an element $a \in G$, the number h is called the order of a.
The conjugacy class including the identity e consists of the single element e.
All of elements in a conjugacy class have the same order

A pedagogical example, S_3

smallest non-Abelian finite group

S_3 consists of all permutations among three objects, (x_1, x_2, x_3) and its order is equal to 3! = 6.
All of six elements correspond to the following transformations,

$$e : (x_1, x_2, x_3) \to (x_1, x_2, x_3) \qquad a_1 : (x_1, x_2, x_3) \to (x_2, x_1, x_3)$$
$$a_2 : (x_1, x_2, x_3) \to (x_3, x_2, x_1) \qquad a_3 : (x_1, x_2, x_3) \to (x_1, x_3, x_2)$$
$$a_4 : (x_1, x_2, x_3) \to (x_3, x_1, x_2) \qquad a_5 : (x_1, x_2, x_3) \to (x_2, x_3, x_1)$$

Their multiplication forms a closed algebra, e.g.

$$a_1 a_2 = a_5 , \quad a_2 a_1 = a_4 , \quad a_4 a_2 = a_2 a_1 a_2 = a_3$$

By defining $a_1 = a$, $a_2 = b$, all of elements are written as $\{e, a, b, ab, ba, bab\}$.

These elements are classified to three conjugacy classes,

$C_1 : \{e\}$, $C_2 : \{ab, ba\}$, $C_3 : \{a, b, bab\}$. **$(ab)^3=(ba)^3=e$, $a^2=b^2=(bab)^2=e$**

The subscript of C_n, n, denotes the number of elements in the conjugacy class C_n .

Let us study irreducible representations of S_3.

The number of irreducible representations must be equal to **3**, because there are **3 conjugacy classes**.

A representation of G is a homomorphic map of elements of G onto matrices, D(g) for $g \in G$. D(g) are (n × n) matrices

Character

$$\chi_D(g) = \mathrm{tr}\ D(g) = \sum_{i=1}^{d_\alpha} D(g)_{ii}.$$

Orthogonality relations

$$\sum_{g \in G} \chi_{D_\alpha}(g)^* \chi_{D_\beta}(g) = N_G \delta_{\alpha\beta}, \qquad \sum_\alpha \chi_{D_\alpha}(g_i)^* \chi_{D_\alpha}(g_j) = \frac{N_G}{n_i} \delta_{C_i C_j},$$

Since $C_1 = \{e\}$ ($n_1=1$), the orthgonality relation is

$$\sum_\alpha [\chi_\alpha(C_1)]^2 = \sum_n m_n n^2 = m_1 + 4m_2 + 9m_3 + \cdots = N_G$$

m_n is number of n-dimensional irreducible representations

Irreducible representations of S_3 are two singlets 1 and 1', one doublet 2.

2 + 4×1 = 6

Since $(\chi_{1'}(C_2))^3 = 1, \qquad (\chi_{1'}(C_3))^2 = 1$ are satisfied,

Orthogonarity conditions determine the Character Table

	h	χ_1	$\chi_{1'}$	χ_2
C_1	1	1	1	2
C_2	3	1	1	-1
C_3	2	1	-1	0

C_1 : {e}, C_2 : {ab, ba}, C_3 : {a, b, bab}.

By using this table, we can construct the representation matrix for 2.

Because of $\chi_2(C_3) = 0$ we choose $a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **$a^2=e$**

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$

$$ab = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad ba = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad bab = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

We can change the representation through the unitary transformation, $U^\dagger gU$.

A lager group

is constructed from more than two groups *by a certain product.*

Consider two groups G_1 and G_2

Direct product

The direct product is denoted as $G_1 \times G_2$.

Multiplication rule

$(a_1,a_2)\,(b_1,b_2) = (a_1b_1,\ a_2b_2)$ for $a_1,b_1 \in G_1$ and $a_2,b_2 \in G_2$

(outer) semi-direct product

The semi-direct product is denoted as $G_1 \rtimes_f G_2$.

Multiplication rule

$(a_1,a_2)\,(b_1,b_2) = (a_1\, f_{a2}\,(b_1),\ a_2b_2)$ for $a_1,b_1 \in G_1$ and $a_2,b_2 \in G_2$

where $f_{a2}\,(b_1)$ *denotes a homomorphic map from* G_2 *to* G_1.

Consider the group G and its **subgroup H** and **normal subgroup N**.
When *$G = NH = HN$ and $N \cap H = \{e\}$, the semi-direct* product $N \rtimes_f H$ *is isomorphic to G,*
where we use the map ***f*** *as* $f_{hi}\,(n_j) = h_i\, n_j\, (h_i)^{-1}$.

Example of semi-direct product

semi-direct product, $Z_3 \rtimes Z_2$.

Here we denote the Z_3 and Z_2 generators by c and h, i.e., $c^3 = e$ and $h^2 = e$.
In this case, can be written by $h\, c\, h^{-1} = c^m$

only the case with m = 2 is non-trivial, $h\, c\, h^{-1} = c^2$

This algebra is **isomorphic** to S_3, *and h and c are identified as a and ab.*

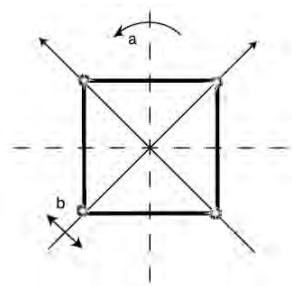
$$N=(e,\ ab,\ ba),\quad H=(e,\ a) \Rightarrow NH=HN \simeq S_3$$

Z_3 Z_2

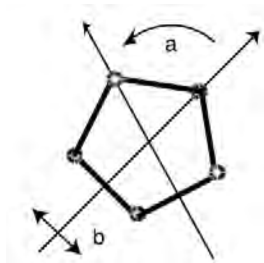
Semi-direct products generates a larger non-Abelian groups

Dihedral group $Z_N \rtimes Z_2 \simeq D_N$, $\Delta(2N)$; $a^N = e$, $b^2 = e$, $bab = a^{-1}$ **order : 2N**

D_4
square



D_5



Regular pentagon

$\Delta(3N^2) \simeq (Z_N \times Z'_N) \rtimes Z_3$, $a^N = a'^N = b^3 = e$, $a a' = a' a$, $bab^{-1} = a^{-1}(a')^{-1}$; $ba'b^{-1} = a$

$\Delta(27)$

$\Delta(6N^2) \simeq (Z_N \times Z'_N) \rtimes S_3$, **$\Delta(6N^2)$ group includes the subgroup, $\Delta(3N^2)$**

$a^N = a'^N = b^3 = c^2 = (bc)^2 = e$, $aa' = a'a$, $bab^{-1} = a^{-1}(a')^{-1}$, $b a'b^{-1} = a$,$cac^{-1} = (a')^{-1}$, $ca'c^{-1} = a^{-1}$

$\Delta(6)=S_3$ $\Delta(24) \simeq S_4$ $\Delta(54)$ .....

Familiar non-Abelian finite groups

			order
S_n :	$S_2 = Z_2$, S_3, S_4 …	Symmetric group	N !
A_n:	$A_3 = Z_3$, $A_4 = T$, A_5 …	Alternating group	(N !)/2
D_n:	$D_3 = S_3$, D_4, D_5 …	Dihedral group	2N
$Q_{N(\text{even})}$:	Q_4, Q_6 ….	Binary dihedral group	2N
$\Sigma(2N^2)$:	$\Sigma(2) = Z_2$, $\Sigma(18)$, $\Sigma(32)$, $\Sigma(50)$ …		$2N^2$
$\Delta(3N^2)$:	$\Delta(12) = A_4$, $\Delta(27)$ …		$3N^2$
$T_{N(\text{prime number})} \simeq Z_N \rtimes Z_3$:	T_7, T_{13}, T_{19}, T_{31}, T_{43}, T_{49}		3N
$\Sigma(3N^3)$:	$\Sigma(24)=Z_2 \times (12)$, $\Sigma(81)$ …		$3N^3$
$\Delta(6N^2)$:	$\Delta(6)=S_3$, $\Delta(24)=S_4$, $\Delta(54)$ …		$6N^2$
T' : double covering group of $A_4 = T$			24

Subgroups are important for particle physics because symmetry breaks down to them.

Ludwig Sylow in 1872:

Theorem 1:
For every prime factor *p* with multiplicity *n* of the order of a finite group *G*, there exists a Sylow *p*-subgroup of *G*, of order p^n.

A_4 has subgroups with order 4 and 3, respectively.

$$12 = 2^2 \times 3$$

Actually, $(Z_2 \times Z_2)$ (klein group) and Z_3 are the subgroup of A_4 .

For flavour physics, we are interested in finite groups with **triplet representations**.

S_3 has two singlets and one doublet: **1, 1', 2,** **no triplet representation.**

Some examples of non-Abelian Finite groups with triplet representation, which are often used in Flavor symmetry

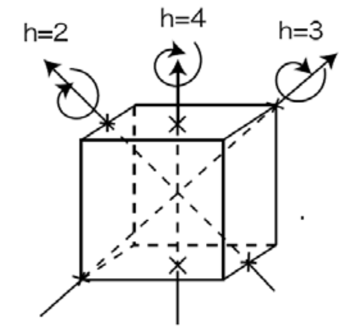
$$S_4, \quad A_4, \quad A_5$$

S_4 group

All permutations among four objects, 4 ! =24 elements

24 elements are generated by **S, T and U**:

$S^2=T^3=U^2=1, \quad ST^3 = (SU)^2 = (TU)^2 = (STU)^4 = 1$



Symmetry of a cube

5 conjugacy classes

C1: 1 h=1

C3: S, T^2ST, TST^2 h=2

C6: U, TU, SU, T^2U, STSU, ST^2SU h=2

C6': STU, TSU, T^2SU, ST^2U, TST^2U, T^2STU h=4

C8: T, ST, TS, STS, T^2, ST^2, T^2S, ST^2S h=3

	h	χ_1	$\chi_{1'}$	χ_2	χ_3	$\chi_{3'}$
C_1	1	1	1	2	3	3
C_3	2	1	1	2	-1	-1
C_6	2	1	-1	0	1	-1
$C_{6'}$	4	1	-1	0	-1	1
C_8	3	1	1	-1	0	0

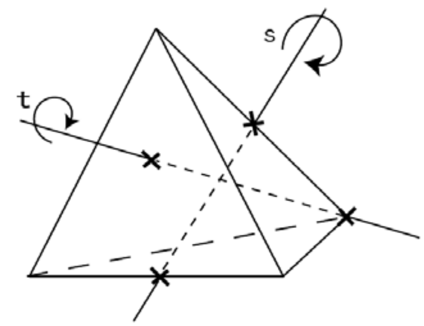
Irreducible representations:

1, 1', 2, 3, 3'

For triplet 3 and 3'

$$U = \mp \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}; \quad \omega = e^{2\pi i/3}$$

A_4 group



Symmetry of tetrahedron

Even permutation group of four objects (1234)
12 elements (order 12) are generated by
S and T: $S^2=T^3=(ST)^3=1$: S=(14)(23), T=(123)

4 conjugacy classes

C1: 1 h=1
C3: S, T^2ST, TST^2 h=2
C4: T, ST, TS, STS h=3
C4': T^2, ST^2, T^2S, ST^2S h=3

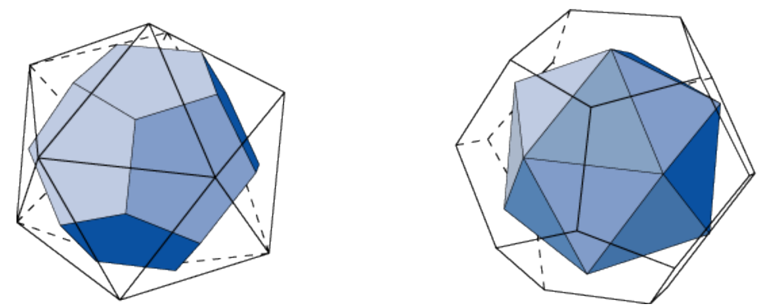
	h	χ_1	$\chi_{1'}$	$\chi_{1''}$	χ_3
C_1	1	1	1	1	3
C_3	2	1	1	1	-1
C_4	3	1	ω	ω^2	0
$C_{4'}$	3	1	ω^2	ω	0

Irreducible representations: 1, 1', 1", 3
The minimum group containing triplet without doublet.

For triplet $S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}; \quad \omega = e^{2\pi i/3}$

A_5 group (simple group)

The A_5 group is isomorphic to the symmetry of a regular icosahedron and a regular dodecahedron.



60 elements are generated S and T .

$$S^2 = (ST)^3 = 1 \text{ and } T^5 = 1$$

5 conjugacy classes

	h	1	3	$3'$	4	5
C_1	1	1	3	3	4	5
C_{15}	2	1	-1	-1	0	1
C_{20}	3	1	0	0	1	-1
C_{12}	5	1	ϕ	$1-\phi$	-1	0
$C_{12'}$	5	1	$1-\phi$	ϕ	-1	0

Irreducible representations:

1, 3, 3', 4, 5

For triplet 3

$$S = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\phi & \frac{1}{\phi} \\ \sqrt{2} & \frac{1}{\phi} & -\phi \end{pmatrix} \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{5}} & 0 \\ 0 & 0 & e^{\frac{8\pi i}{5}} \end{pmatrix}$$

$$\phi = \frac{1+\sqrt{5}}{2}$$ Golden Ratio

3 Flavor symmetry with non-Abelian Discrete group

3.1 Towards non-Abelian Discrete flavor symmetry

In Quark sector

There was no information of lepton flavor mixing before 1998.

Discrete Symmetry and Cabibbo Angle,
Phys. Lett. 73B (1978) 61, S.Pakvasa and H.Sugawara

S_3 symmetry is assumed for the Higgs interaction with the quarks and the leptons for the self-coupling of the Higgs bosons.

S_3 doublet **S_3 singlets** **S_3 doublet**

$$\left\{\begin{pmatrix} p_1 \\ n_1 \end{pmatrix}_L, \begin{pmatrix} p_2 \\ n_2 \end{pmatrix}_L\right\} \Bigg| \ \{p_{1R}\}, \{p_{2R}\}, \{n_{1R}, n_{2R}\} \quad \Rightarrow \quad \tan\theta_c = m_d/m_s$$

one S_3 singlet $\{\phi_0\}$ and one S_3 doublet $\{\phi_1, \phi_2\}$

A Geometry of the generations, 3 generations
Phys. Rev. Lett. 75 (1995) 3985, L.J.Hall and H.Murayama

$(S(3))^3$ flavor symmetry for quarks Q, U, D

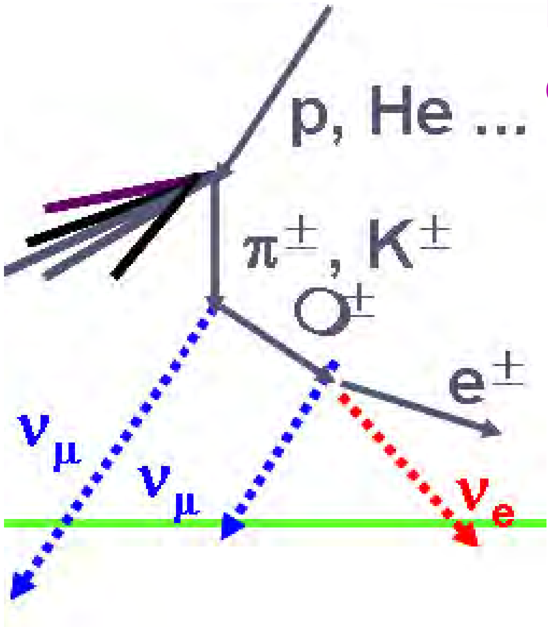
$(S(3))^3$ flavor symmetry and p ---> K^0 e^+ , (SUSY version)
Phys. Rev.D 53 (1996) 6282, C.D.Carone, L.J.Hall and H.Murayama

fundamental sources of flavor symmetry breaking are gauge singlet fields ϕ :flavons
Incorporating the lepton flavor based on the discrete flavor group $(S_3)^3$.

1998 Revolution in Neutrinos !

Atmospheric neutrinos brought us informations of neutrino masses and flavor mixing.

$$P_{\nu_\mu \to \nu_\mu} = 1 - 4\left|U_{\mu 3}\right|^2 \left(1 - \left|U_{\mu 3}\right|^2\right) \sin^2 \frac{\Delta_{13}}{2} + 2\left|U_{\mu 2}\right|^2 \left|U_{\mu 3}\right|^2 \Delta_{12} \sin \Delta_{13} + \mathcal{O}(\Delta_{12}^2)$$



First clear evidence of neutrino oscillation was discovered in 1998

$$R = \frac{(\nu_\mu + \overline{\nu}_\mu)/(\nu_e + \overline{\nu}_e)|_{DATA}}{(\nu_\mu + \overline{\nu}_\mu)/(\nu_e + \overline{\nu}_e)|_{MC}} = 0.65 \pm 0.05 \pm 0.08$$

Multi-GeV

MC $\quad (\nu_\mu + \overline{\nu}_\mu)/(\nu_e + \overline{\nu}_e)|_{MC} \approx 2$

Before 2012 (no data for θ_{13})

Neutrino Data presented **$\sin^2\theta_{12} \sim 1/3$, $\sin^2\theta_{23} \sim 1/2$**

Harrison, Perkins, Scott (2002) proposed

Tri-bimaximal Mixing of Neutrino flavors.

$$\sin^2\theta_{12} = 1/3,\ \sin^2\theta_{23} = 1/2,\ \sin^2\theta_{13} = 0,$$

$$U_{\rm tri-bimaximal} = \begin{pmatrix} \sqrt{2/3} & \sqrt{1/3} & 0 \\ -\sqrt{1/6} & \sqrt{1/3} & -\sqrt{1/2} \\ -\sqrt{1/6} & \sqrt{1/3} & \sqrt{1/2} \end{pmatrix}$$

PDG

$$U_{\rm PMNS} \equiv \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{CP}} & c_{23}c_{13} \end{pmatrix}$$

Tri-bimaximal Mixing of Neutrinos motivates to consider
Non-Abelian Discrete Flavor Symmetry.

Tri-bimaximal Mixing (TBM) is realized by the mass matrix

$$m_{TBM} = \frac{m_1+m_3}{2}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} + \frac{m_2-m_1}{3}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix} + \frac{m_1-m_3}{2}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

in the diagonal basis of charged leptons.

Mixing angles are independent of neutrino masses.

Integer (inter-family related) matrix elements suggest Non-Abelian Discrete Flavor Symmetry.

Hint for the symmetry in TBM

$$m_{TBM} = \frac{m_1+m_3}{2}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + \frac{m_2-m_1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} + \frac{m_1-m_3}{2}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$$

A_4 symmetric

Assign A_4 triplet $\underline{3}$ for $(\nu_e,\nu_\mu,\nu_\tau)_L$

E. Ma and G. Rajasekaran, PRD64(2001)113012

$$\mathbf{3}\times\mathbf{3}\Rightarrow\mathbf{3}+\mathbf{3}+\mathbf{1}+\mathbf{1}'+\mathbf{1}''$$

$$\mathbf{3}\times\mathbf{3}\Rightarrow\mathbf{1}=a_1b_1+a_2b_3+a_3b_2$$

The third matrix is A_4 symmetric !

The first and second matrices are Unit matrix and Democratic matrix, respectively, which could be derived from S_3 symmetry.

In 2012

θ_{13} was measured by Daya Bay, RENO, T2K, MINOS, Double Chooz

Tri-bimaximal mixing was ruled out !

$$\theta_{13} \simeq 9^\circ \simeq \theta_c/\sqrt{2}$$

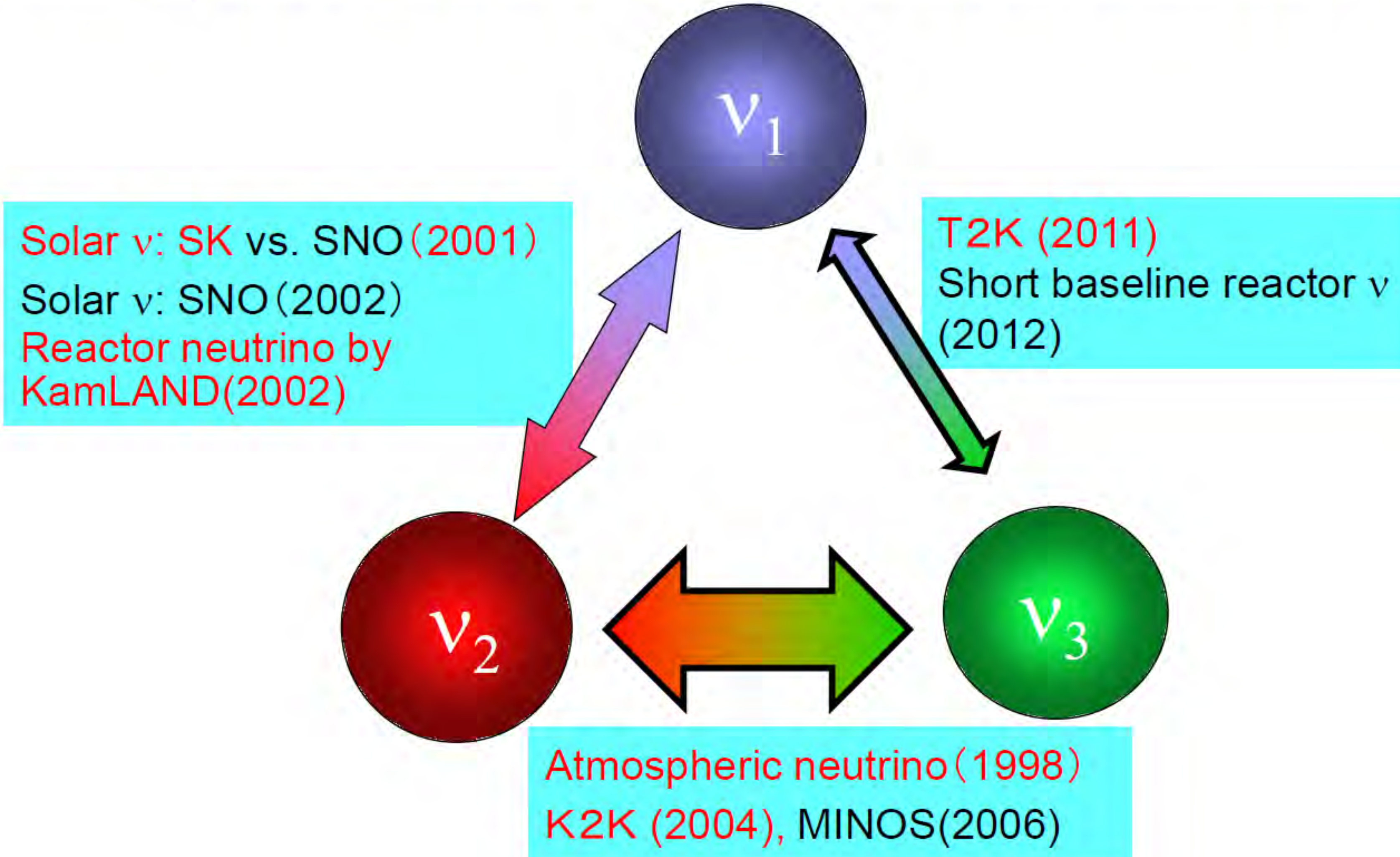
Rather large θ_{13} suggests to search for CP violation !

$$J_{CP} = s_{23}c_{23}s_{12}c_{12}s_{13}c_{13}^2 \sin\delta_{CP} \simeq 0.0327 \sin\delta$$

J_{CP}(quark)$\sim 3\times10^{-5}$

Challenge for flavor and CP symmetries for leptons

Summary of discoveries of neutrino oscillations



Neutrino mixing vs. quark mixing

Neutrino mixing

(3σ C.L. range)

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

$$\begin{pmatrix} 0.800\text{-}0.844 & 0.515\text{-}0.581 & 0.139\text{-}0.155 \\ 0.229\text{-}0.516 & 0.438\text{-}0.699 & 0.614\text{-}0.790 \\ 0.249\text{-}0.528 & 0.462\text{-}0.715 & 0.595\text{-}0.776 \end{pmatrix}$$

I. Esteban et al., JHEP 01 (2017) 087

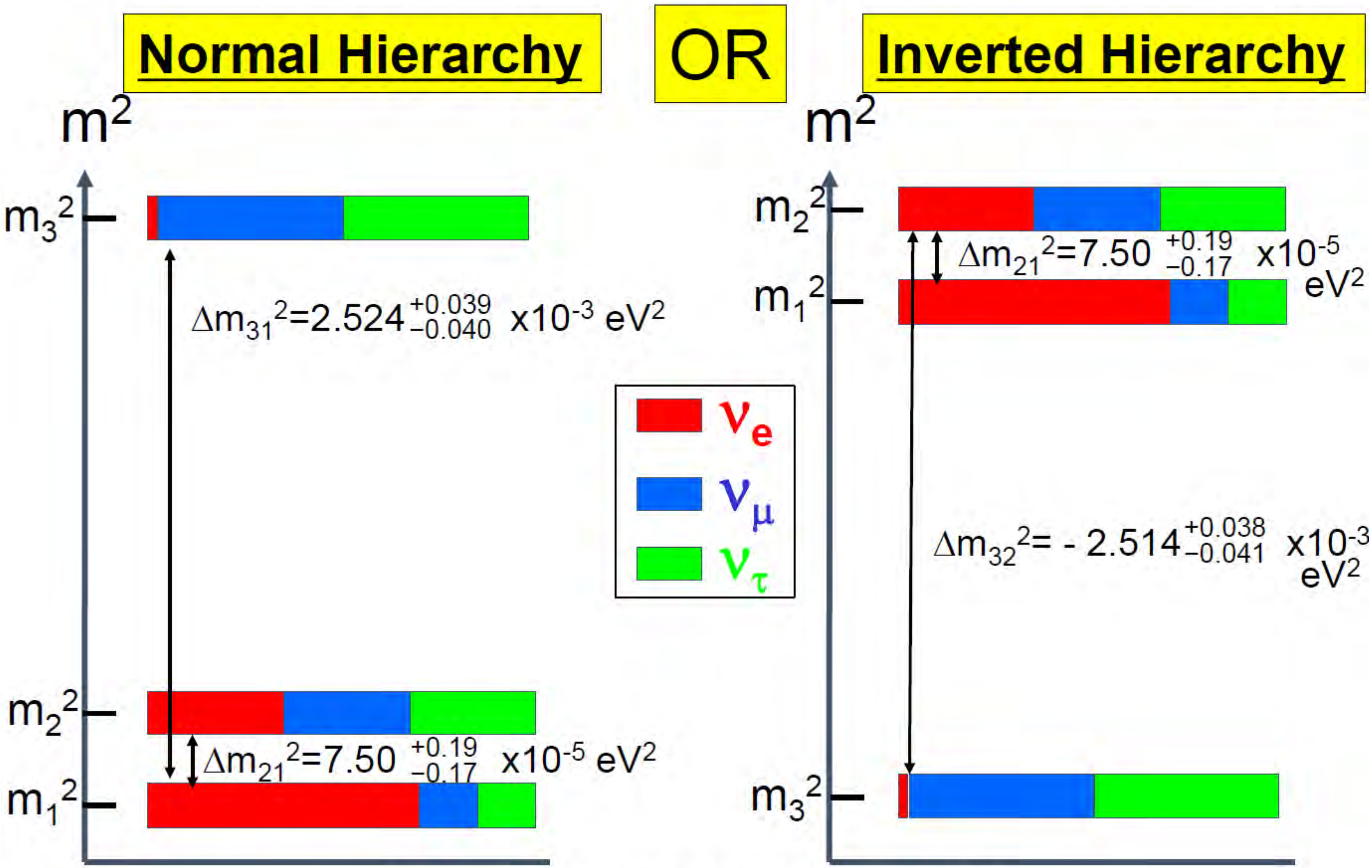
Quark mixing (CKM matrix)

$$\begin{pmatrix} 0.97434 & 0.22506 & 0.00357 \\ 0.22492 & 0.97351 & 0.0414 \\ 0.00875 & 0.0403 & 0.99915 \end{pmatrix}$$

Particle Data Group (2016)

They are so much different!

Neutrino mass and mixing (what we know now)



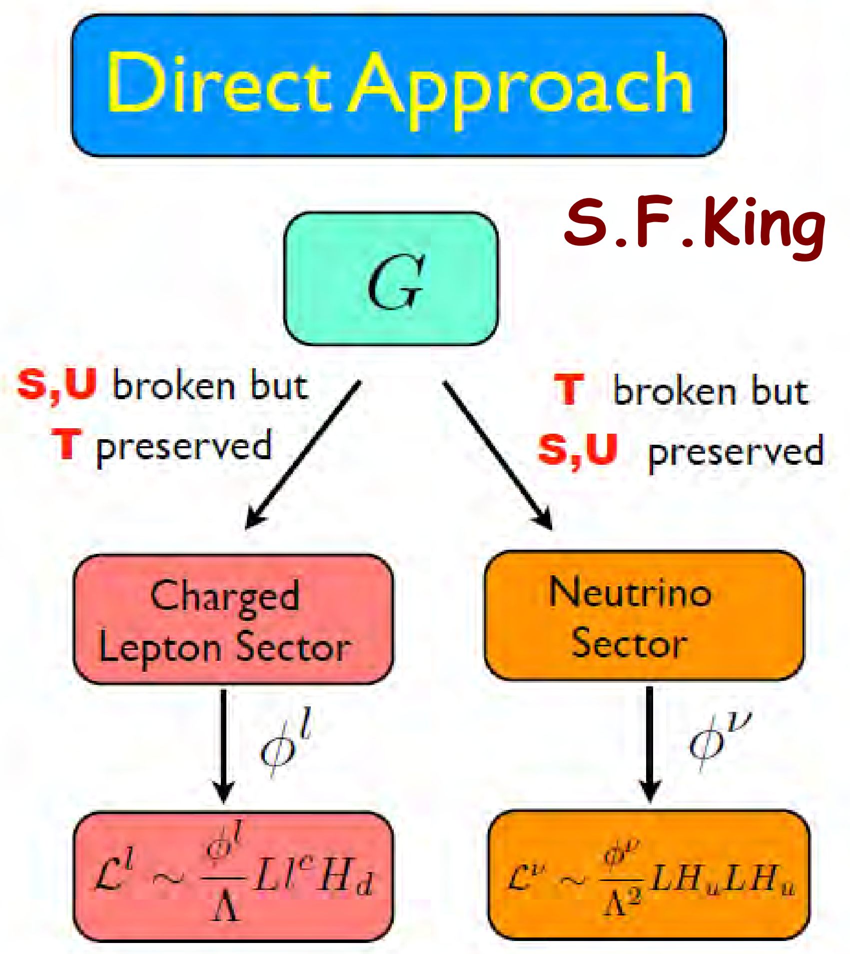
Numbers from I. Esteban et al., JHEP 01 (2017) 087

3.2 Direct approach of Flavor Symmetry

Suppose Flavor symmetry group G
Consider only Mass matrices !

Different subgroups of G are preserved in Yukawa sectors of Neutrinos and Charged leptons, respectively.

S, T, U are generators of Finite groups



arXiv: 1402.4271 King, Merle, Morisi, Simizu, M.T

Consider S_4 flavor symmetry:

24 elements are generated by S, T and U:

$S^2=T^3=U^2=1, \quad ST^3=(SU)^2=(TU)^2=(STU)^4=1$

Irreducible representations: 1, 1', 2, 3, 3'

It has subgroups, nine Z_2, four Z_3, three Z_4, four $Z_2 \times Z_2$ (K_4)

Suppose S_4 is spontaneously broken to one of subgroups:

Neutrino sector preserves (1, S, U, SU) (K_4)

Charged lepton sector preserves (1, T, T^2) (Z_3)

For 3 and 3'

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}; \quad \omega = e^{2\pi i/3}$$

$$U = \mp \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Neutrino and charged lepton mass matrices respect S, U and T, respectively:

$$S^T m^{\nu}_{LL} S = m^{\nu}_{LL}, \quad U^T m^{\nu}_{LL} U = m^{\nu}_{LL}, \quad T^{\dagger} Y_e Y_e^{\dagger} T = Y_e Y_e^{\dagger}$$

⬇

$$[S, m^{\nu}_{LL}] = 0, \quad [U, m^{\nu}_{LL}] = 0, \quad [T, Y_e Y_e^{\dagger}] = 0$$

Mixing matrices diagonalize mass matrices also diagonalize S,U, and T, respectively !

The charged lepton mass matrix is diagonal because T is diagonal matrix.

$$V_{\nu} = \begin{pmatrix} 2/\sqrt{6} & /1\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$$

Tri-bimaximal mixing $\theta_{13}=0$

C.S.Lam, PRD98(2008) arXiv:0809.1185

which digonalizes both S and U.

Independent of mass eigenvalues !

Klein Symmetry can reproduce Tri-bimaximal mixing.

If S_4 is spontaneously broken to another subgroups,

Neutrino sector preserves SU (Z_2)

Charged lepton sector preserves T (Z_3),

mixing matrix is changed !

$$(SU)^T m_{LL}^{\nu} SU = m_{LL}^{\nu}, \quad T^\dagger Y_e Y_e^\dagger T = Y_e Y_e^\dagger$$

⬇

$$[SU, m_{LL}^{\nu}] = 0, \quad [T, Y_e Y_e^\dagger] = 0$$

Tri-maximal mixing

TM$_1$

$$V_\nu = \begin{pmatrix} 2/\sqrt{6} & c/\sqrt{3} & s/\sqrt{3} \\ -1/\sqrt{6} & c/\sqrt{3} - s/\sqrt{2} & -s/\sqrt{3} - c/\sqrt{2} \\ -1/\sqrt{6} & c/\sqrt{3} + s/\sqrt{2} & -s/\sqrt{3} + c/\sqrt{2} \end{pmatrix}$$

$c = \cos\theta, \quad s = \sin\theta$ includes CP phase.

Θ is not fixed by the flavor symmetry.

Mixing sum rules

$$\sin^2\theta_{23} = \frac{1}{2}\frac{1}{\cos^2\theta_{13}} \geq \frac{1}{2}, \quad \sin^2\theta_{12} \simeq \frac{1}{\sqrt{3}} - \frac{2\sqrt{2}}{3}\sin\theta_{13}\cos\delta_{CP} + \frac{1}{3}\sin^2\theta_{13}\cos 2\delta_{CP}$$

Mixing pattern in A_5 flavor symmetry

It has subgroups, ten Z_3, six Z_5, five $Z_2 \times Z_2$ (K_4) .

Suppose A_5 is spontaneously broken to one of subgroups:
Neutrino sector preserves S and U (K_4)
Charged lepton sector preserves T (Z_5)

$$S^T m^\nu_{LL} S = m^\nu_{LL}, \quad U^T m^\nu_{LL} U = m^\nu_{LL}, \quad T^\dagger Y_e Y_e^\dagger T = Y_e Y_e^\dagger$$

$$[S, m^\nu_{LL}] = 0, \quad [U, m^\nu_{LL}] = 0, \quad [T, Y_e Y_e^\dagger] = 0$$

$$S= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\phi & \frac{1}{\phi} \\ \sqrt{2} & \frac{1}{\phi} & -\phi \end{pmatrix} \qquad T= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{5}} & 0 \\ 0 & 0 & e^{\frac{8\pi i}{5}} \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

F. Feruglio and Paris, JHEP 1103(2011) 101 arXiv:1101.0393

$$U_{GR} = \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ \frac{\sin\theta_{12}}{\sqrt{2}} & -\frac{\cos\theta_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{\sin\theta_{12}}{\sqrt{2}} & -\frac{\cos\theta_{12}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$\theta_{13}=0$

$\tan\theta_{12} = 1/\phi$: $\phi = \frac{1+\sqrt{5}}{2}$ **Golden ratio**

Neutrino mass matrix has μ-τ symmetry.

$$m_\nu = \begin{pmatrix} x & y & y \\ y & z & w \\ y & w & z \end{pmatrix}$$ with $z + w = x - \sqrt{2}y$

$\sin^2\theta_{12} = 2/(5+\sqrt{5}) = 0.2763\ldots$

which is rather smaller than the experimental data.

$$\sin^2\theta_{12} = 0.306 \pm 0.012$$

3.3 CP symmetry in neutrinos

Possibility of predicting CP phase δ_{CP} in FLASY

A hint : under μ-τ symmetry

$$|U_{\mu i}| = |U_{\tau i}| \; i = 1, 2, 3$$

$$\cos\theta_{23} = \sin\theta_{23} = \frac{1}{\sqrt{2}}$$

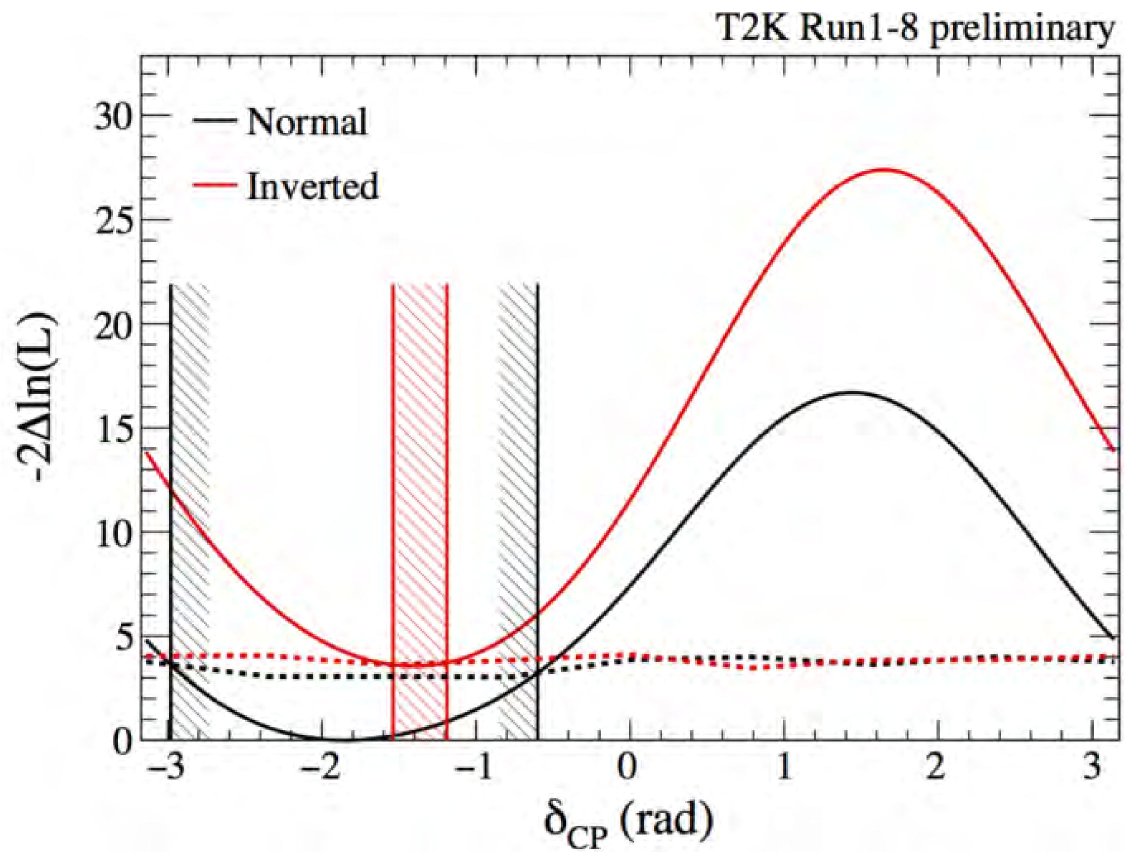
$$\sin\theta_{13}\cos\delta = 0$$

$\delta = \pm\frac{\pi}{2}$ is predicted since we know $\theta_{13} \neq 0$

Ferreira, Grimus, Lavoura, Ludl, JHEP2012,arXiv: 1206.7072

Exciting Era of Observation of CP violating phase @T2K and NOvA

T2K reported the constraint on δ_{CP} **August 4, 2017**



Feldman-Cousins method

The 2σ CL confidence interval:

Normal hierarchy: [-2.98, -0.60] radians

Inverted hierarchy: [-1.54, -1.19] radians

CP conserving values (0,π) fall outside of 2σ intervals

G.Ecker, W.Grimus and W.Konetschny, Nucl. Phys. B 191 (1981) 465
G.Ecker, W.Grimus and H.Neufeld, Nucl.Phys.B 229(1983) 421

Generalized CP Symmetry

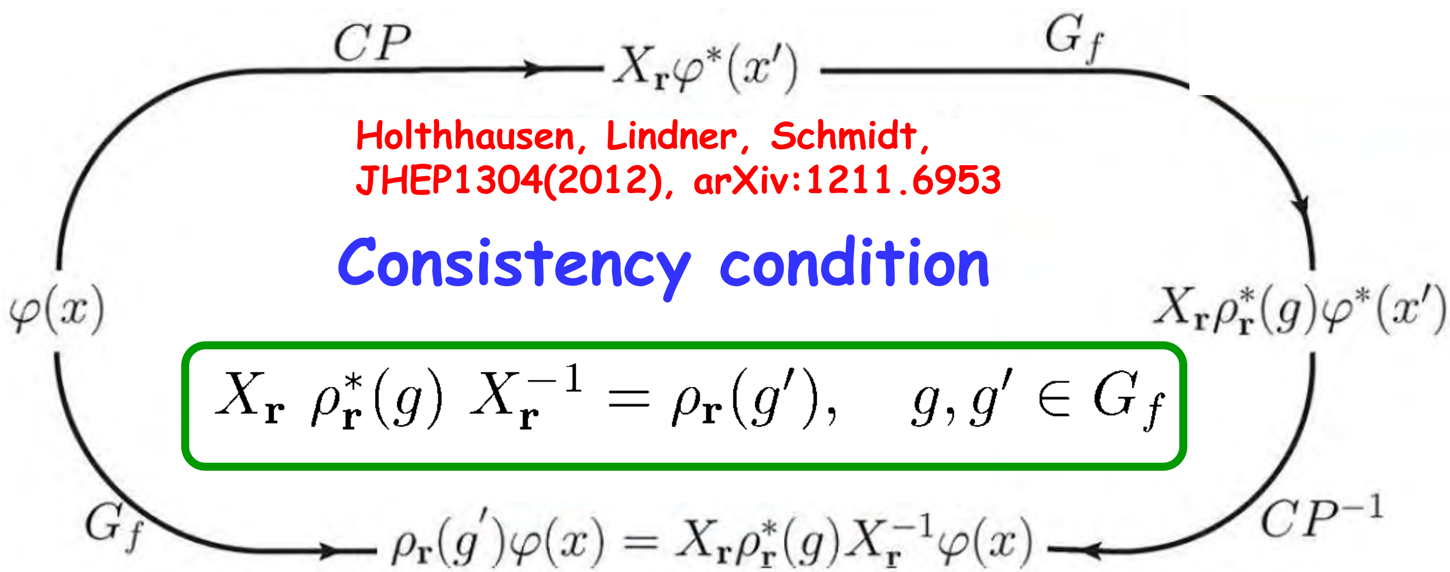
CP Symmetry $\varphi(x) \xrightarrow{\text{CP}} X_{\mathbf{r}} \varphi^*(x'), \quad x' = (t, -\mathbf{x})$

Flavour Symmetry $\varphi(x) \xrightarrow{g} \rho_{\mathbf{r}}(g) \varphi^*(x), \quad g \in G_f$

$$X_{\mathbf{r}}^{\nu T} m_{\nu LL} X_{\mathbf{r}}^{\nu} = m_{\nu LL}^*$$

$$X_{\mathbf{r}}^{\ell \dagger} (m_\ell^\dagger m_\ell) X_{\mathbf{r}}^{\ell} = (m_\ell^\dagger m_\ell)^*$$

X_r must be consistent with Flavor Symmetry $\rho_{\mathbf{r}}(g)$



Holthhausen, Lindner, Schmidt, JHEP1304(2012), arXiv:1211.6953

Consistency condition

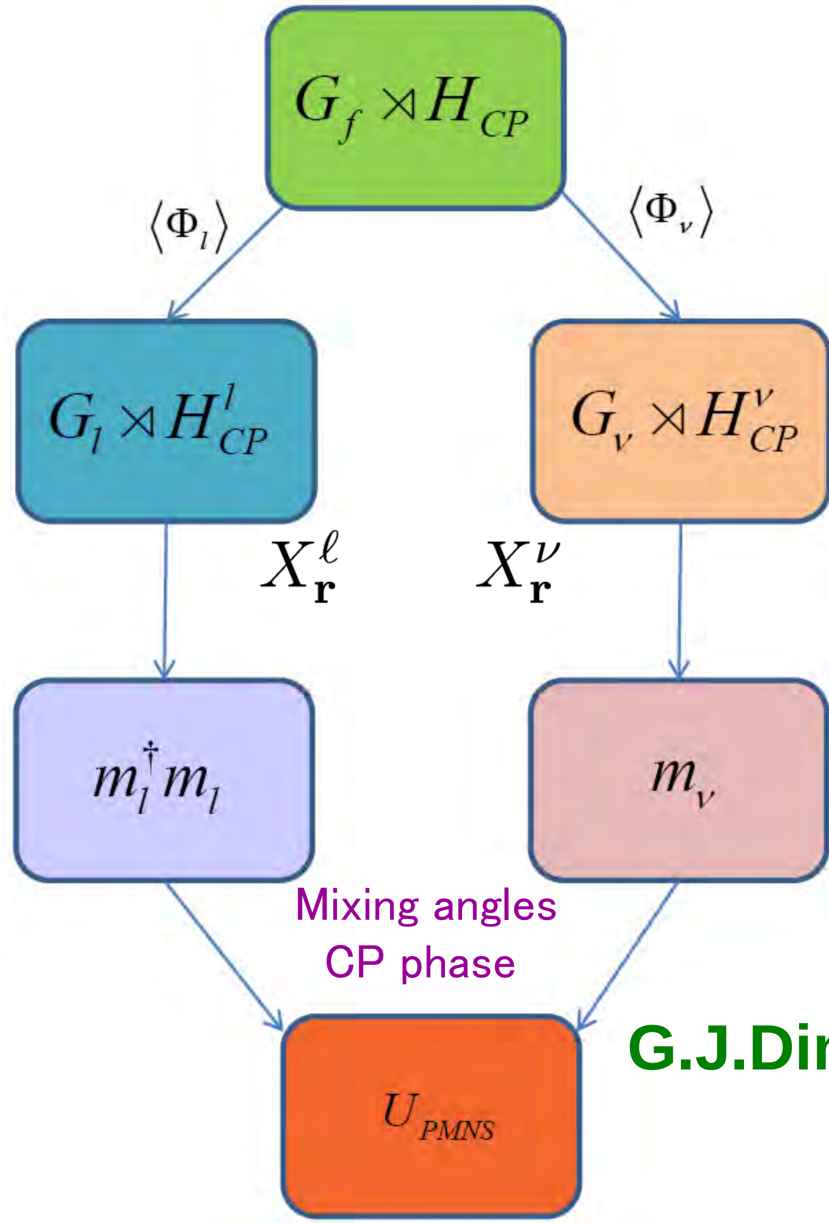
$$X_{\mathbf{r}} \; \rho_{\mathbf{r}}^*(g) \; X_{\mathbf{r}}^{-1} = \rho_{\mathbf{r}}(g'), \quad g, g' \in G_f$$

Mu-Chun Chen, Fallbacher, Mahanthappa, Ratz, Trautner, Nucl.Phys. B883 (2014) 267-305

Suppose a symmetry including FLASY and CP symmetry:

$$G_{CP} = G_f \rtimes H_{CP}$$

is broken to the subgroups in neutrino sector and charged lepton sector.



G.J.Ding

CP symmetry gives

$$X^{\nu T}_{\mathbf{r}} m_{\nu LL} X^{\nu}_{\mathbf{r}} = m^*_{\nu LL}$$

$$X^{\ell\dagger}_{\mathbf{r}} (m^\dagger_\ell m_\ell) X^{\ell}_{\mathbf{r}} = (m^\dagger_\ell m_\ell)^*$$

An example of S_4 model

Ding, King, Luhn, Stuart, JHEP1305, arXiv:1303.6180

One example of S_4: G_ν={S} and X_3^ν={U} , X_3^l ={1}

satisfy the consistency condition

$$X_{\mathbf{r}}\ \rho_{\mathbf{r}}^*(g)\ X_{\mathbf{r}}^{-1} = \rho_{\mathbf{r}}(g'), \quad g, g' \in G_f$$

$$m_{\nu LL} = \alpha \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

respects G_ν={S}

CP symmetry $X_{\mathbf{r}}^{\nu T} m_{\nu LL} X_{\mathbf{r}}^{\nu} = m_{\nu LL}^*$

⬇

α, β, γ are real, ε is imaginary.

$$V_\nu = \begin{pmatrix} 2c/\sqrt{6} & 1/\sqrt{3} & 2s/\sqrt{6} \\ -c/\sqrt{6}+is/\sqrt{2} & 1/\sqrt{3} & -s/\sqrt{6}-ic/\sqrt{2} \\ -c/\sqrt{6}+is/\sqrt{2} & 1/\sqrt{3} & -s/\sqrt{6}+ic/\sqrt{2} \end{pmatrix}$$

$$c = \cos\theta, \quad s = \sin\theta$$

⬇

$$\sin^2\theta_{13} = \frac{2}{3}\sin^2\theta, \quad \sin^2\theta_{12} = \frac{1}{2+\cos 2\theta}, \quad \sin^2\theta_{23} = \frac{1}{2}$$

$$|\sin\delta_{CP}| = 1, \quad \sin\alpha_{21} = \sin\alpha_{31} = 0$$

$$\delta_{CP} = \pm\pi/2$$

The predicton of CP phase depends on the respected Generators of FLASY and CP symmetry. Typically, it is simple value, 0, π, ±π/2 .

A_4, A_5, $\Delta(6N^2)$ …

3.4 Indirect approach of Flavor Symmetry

Model building by flavons

Flavor symmetry G is broken by flavon (SU_2 singlet scalors) VEV's.
Flavor symmetry controls Yukaw couplings
among leptons and flavons with special vacuum alignments.

Consider an example : A_4 model

	Leptons	flavons	
A_4 triplets	(L_e, L_μ, L_τ)	$\phi_\nu(\phi_{\nu 1}, \phi_{\nu 2}, \phi_{\nu 3})$	couple to neutrino sector
		$\phi_E(\phi_{E1}, \phi_{E2}, \phi_{E3})$	couple to charged lepton sector

A_4 singlets $\quad e_R : \mathbf{1} \quad \mu_R : \mathbf{1}" \quad \tau_R : \mathbf{1}'$

Mass matrices are given by A_4 invariant couplings with flavons

$$3_L \times 3_L \times 3_{flavon} \rightarrow 1 \,, \qquad 3_L \times 1_R^{(')(``)} \times 3_{flavon} \rightarrow 1$$

G. Altarelli, F. Feruglio, Nucl.Phys. B720 (2005) 64

Flavor symmetry G is broken by VEV of flavons

$3_L \times 3_L \times 3_{flavon} \to 1$

$$m_{\nu LL} \sim y \begin{pmatrix} 2\langle\phi_{\nu 1}\rangle & -\langle\phi_{\nu 3}\rangle & -\langle\phi_{\nu 2}\rangle \\ -\langle\phi_{\nu 3}\rangle & 2\langle\phi_{\nu 2}\rangle & -\langle\phi_{\nu 1}\rangle \\ -\langle\phi_{\nu 2}\rangle & -\langle\phi_{\nu 1}\rangle & 2\langle\phi_{\nu 3}\rangle \end{pmatrix}$$

$3_L \times 1_R (1_R', 1_R'') \times 3_{flavon} \to 1$

$$m_E \sim \begin{pmatrix} y_e\langle\phi_{E1}\rangle & y_e\langle\phi_{E3}\rangle & y_e\langle\phi_{E2}\rangle \\ y_\mu\langle\phi_{E2}\rangle & y_\mu\langle\phi_{E1}\rangle & y_\mu\langle\phi_{E3}\rangle \\ y_\tau\langle\phi_{E3}\rangle & y_\tau\langle\phi_{E2}\rangle & y_\tau\langle\phi_{E1}\rangle \end{pmatrix}$$

However, specific Vacuum Alingnments preserve S and T generator.

Take $\langle\phi_{\nu 1}\rangle = \langle\phi_{\nu 2}\rangle = \langle\phi_{\nu 3}\rangle$ and $\langle\phi_{E2}\rangle = \langle\phi_{E3}\rangle = 0$

$\Rightarrow \quad \langle\phi_\nu\rangle \sim (1,1,1)^T \, , \qquad \langle\phi_E\rangle \sim (1,0,0)^T$

$$S\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} , \quad T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

Then, $\langle\phi_\nu\rangle$ preserves S and $\langle\phi_E\rangle$ preserves T.

m_E is a diagonal matrix, on the other hand, $m_{\nu LL}$ is

$$m_{\nu LL} \sim 3y \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - y \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Rank 2

two generated masses and one massless neutrinos !

(0, 3y, 3y)

Flavor mixing is not fixed !

Adding A_4 singlet $\xi : \mathbf{1}$ in order to fix flavor mixing matrix.

$\mathbf{3_L} \times \mathbf{3_L} \times \mathbf{1_{flavon}} \rightarrow \mathbf{1}$

$$m_{\nu LL} \sim y_1 \begin{pmatrix} 2\langle\phi_{\nu 1}\rangle & -\langle\phi_{\nu 3}\rangle & -\langle\phi_{\nu 2}\rangle \\ -\langle\phi_{\nu 3}\rangle & 2\langle\phi_{\nu 2}\rangle & -\langle\phi_{\nu 1}\rangle \\ -\langle\phi_{\nu 2}\rangle & -\langle\phi_{\nu 1}\rangle & 2\langle\phi_{\nu 3}\rangle \end{pmatrix} + y_2\langle\xi\rangle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$\langle\phi_{\nu 1}\rangle = \langle\phi_{\nu 2}\rangle = \langle\phi_{\nu 3}\rangle$, which preserves S symmetry.

$$m_{\nu LL} = 3a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - a \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Flavor mixing is determined: Tri-bimaximal mixing. $\boldsymbol{\theta_{13}=0}$

$$m_\nu = 3a + b,\ b,\ 3a - b \ \Rightarrow m_{\nu_1} - m_{\nu_3} = 2m_{\nu_2}$$

There appears a Neutrino Mass Sum Rule.

This is a minimal framework of A_4 symmetry predicting mixing angles and masses.

A_4 model easily realizes non-vanishing θ_{13} .

Y. Simizu, M. Tanimoto, A. Watanabe, PTP 126, 81(2011)

$$\mathbf{3} \times \mathbf{3} \Rightarrow \mathbf{1} \quad = a_1 * b_1 + a_2 * b_3 + a_3 * b_2$$
$$\mathbf{3} \times \mathbf{3} \Rightarrow \mathbf{1}' \quad = a_1 * b_2 + a_2 * b_1 + a_3 * b_3$$
$$\mathbf{3} \times \mathbf{3} \Rightarrow \mathbf{1}'' \quad = a_1 * b_3 + a_2 * b_2 + a_3 * b_1$$

ξ $\qquad\qquad$ ξ'

$$\mathbf{1} \times \mathbf{1} \Rightarrow \mathbf{1} \quad , \qquad \mathbf{1}'' \times \mathbf{1}' \Rightarrow \mathbf{1}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Additional Matrix

$$M_\nu = a\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + b\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} + c\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} + d\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

$$a = \frac{y^\nu_{\phi_\nu}\alpha_\nu v_u^2}{\Lambda}, \quad b = -\frac{y^\nu_{\phi_\nu}\alpha_\nu v_u^2}{3\Lambda}, \quad c = \frac{y^\nu_{\xi}\alpha_\xi v_u^2}{\Lambda}, \quad d = \frac{y^\nu_{\xi'}\alpha_{\xi'} v_u^2}{\Lambda} \qquad a = -3b$$

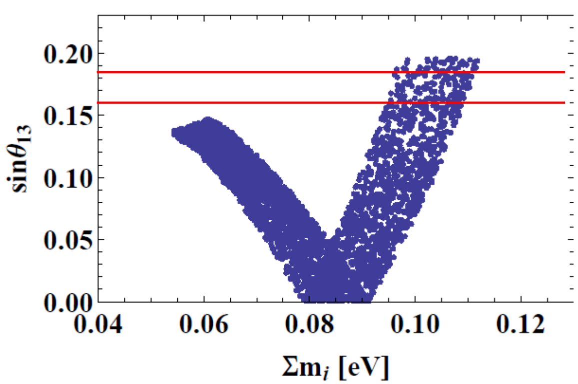
Both normal and inverted mass hierarchies are possible.

$$M_\nu = V_{\text{tri-bi}}\begin{pmatrix}a+c-\frac{d}{2} & 0 & \frac{\sqrt{3}}{2}d\\ 0 & a+3b+c+d & 0\\ \frac{\sqrt{3}}{2}d & 0 & a-c+\frac{d}{2}\end{pmatrix}V^T_{\text{tri-bi}}$$

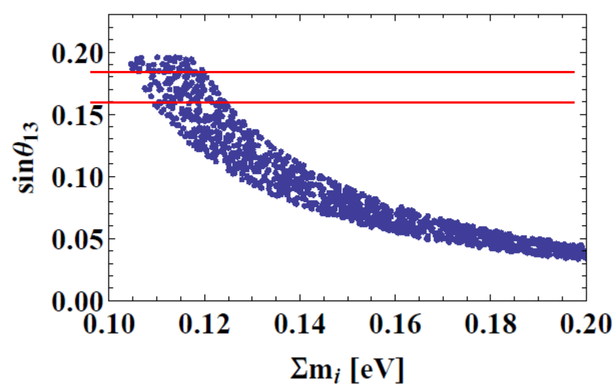
Tri-maximal mixing: TM2

$$\Delta m^2_{31} = -4a\sqrt{c^2+d^2-cd}\,, \qquad \Delta m^2_{21} = (a+3b+c+d)^2 - (a+\sqrt{c^2+d^2-cd})^2$$

Normal hierarchy



Inverted hierarchy



4 Minimal seesaw model with flavor symmetry

We search for a simple scheme to examine the flavor structure of quark/lepton mass matrices because the number of available data is much less than unknown parameters.

For neutrinos, 2 mass square differences,
3 mixing angles in experimental data
however, 9 parameters in neutrino mass matrix

Remove a certain of parameters
in neutrino mass matrix by assuming

- 2 Right-handed Majorana Neutrinos m_1 or m_3 vanishes
- Flavor Symmetry S_4

Yusuke Simizu, Kenta Takagi, M.T, arXiv:1709.02136

S_4 : irreducible representations 1, 1', 2, 3, 3'

Assign: Lepton doublets L: 3'   Right-handed neutrinos ν_R: 1

Introduce: two flavons (gauge singlet scalars) 3' in S_4 Φ_{atm} , Φ_{sol}

Consider specific vacuum alignments for 3'

$$\langle\phi_{\rm atm}\rangle \sim \begin{pmatrix} \frac{b+c}{2} \\ c \\ b \end{pmatrix}, \qquad \langle\phi_{\rm sol}\rangle \sim \begin{pmatrix} \frac{e+f}{2} \\ f \\ e \end{pmatrix}$$

preserves Z_2 {1, SU} symmetry for 3'.

S_4 generators : S, T, U

$$SU\ (US) = \mp\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{pmatrix}$$ for 3 and 3' . $$SU\begin{pmatrix} \frac{b+c}{2} \\ b \\ c \end{pmatrix} = \begin{pmatrix} \frac{b+c}{2} \\ b \\ c \end{pmatrix}$$

S_4 invariant Yukawa Couplings

$$\frac{y_{\rm atm}}{\Lambda}\phi_{\rm atm}LH_u\nu^c_{R1} + \frac{y_{\rm sol}}{\Lambda}\phi_{\rm sol}LH_u\nu^c_{R2}$$

3' × 3' × 1 3' × 3' × 1

Since $L(3')\phi(3') = L_1\phi_1 + L_2\phi_3 + L_3\phi_2$,

we obtain a simple Dirac neutrino mass matrix.

$$M_D = \begin{pmatrix} \frac{b+c}{2} & \frac{e+f}{2} \\ b & e \\ c & f \end{pmatrix}$$

$$M_D = \begin{pmatrix} \frac{b+c}{2} & \frac{e+f}{2} \\ b & e \\ c & f \end{pmatrix} \qquad M_R = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} = M_0 \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

After seesaw, Mv is rotated by V_{TBM}

$$V_{\rm TBM} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$M_\nu = -M_D M_R^{-1} M_D^T \qquad \hat{M}_\nu \equiv V_{\rm TBM}^T M_\nu V_{\rm TBM}$$

$$\hat{M}_\nu = \frac{1}{M_0} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{4}\left((b+c)^2 p + (e+f)^2\right) & \frac{1}{2}\sqrt{\frac{3}{2}}\left((c^2-b^2)p - e^2 + f^2\right) \\ 0 & \frac{1}{2}\sqrt{\frac{3}{2}}\left((c^2-b^2)p - e^2 + f^2\right) & \frac{1}{2}\left((b-c)^2 p + (e-f)^2\right) \end{pmatrix}$$

m_1=0: Normal Hierarchy of Neutrino Masses

Trimaximal mixing TM_1

$$U_{\rm PMNS} = V_{\rm TBM} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & e^{-i\sigma}\sin\phi \\ 0 & -e^{i\sigma}\sin\phi & \cos\phi \end{pmatrix}$$

$m_1=0$: Normal Hierarchy of Neutrino Masses

Prediction of CP violation

Input Data (Global Analyses) 2 masses, 3 mixing angles

Output: CP violating phase δ_{CP}

$$M_D = \begin{pmatrix} \frac{b+c}{2} & \frac{e+f}{2} \\ b & e \\ c & f \end{pmatrix}$$

4 real parameters
2 phases

Toward
minimal seesaw

putting one zero
is allowed

$$M_D = \begin{pmatrix} 0 & \frac{e+f}{2} \\ b & e \\ -b & f \end{pmatrix}$$

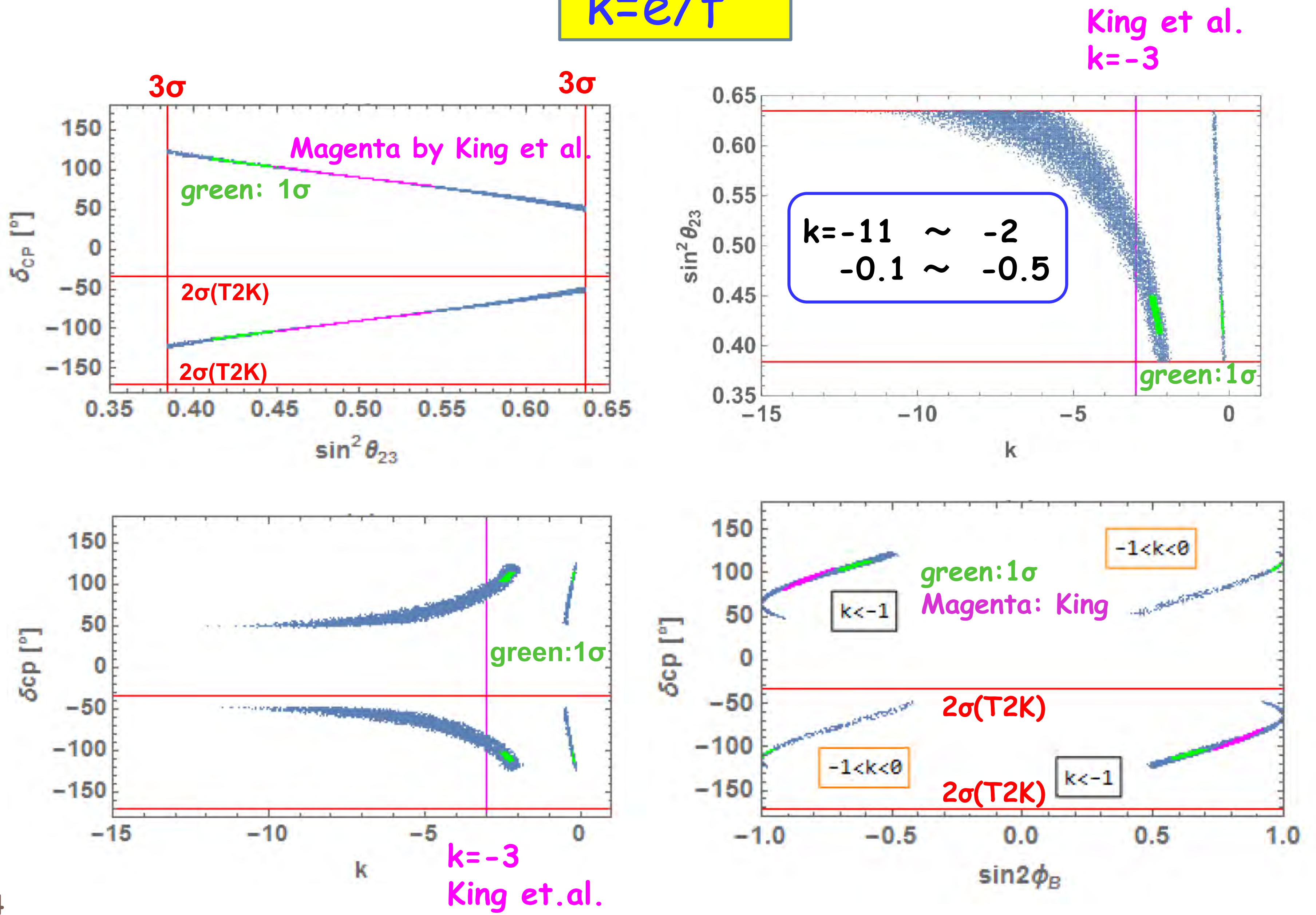
3 real parameters + 1 phase

Arg [b/f]=Φ_B

King et al. $$M_D = \begin{pmatrix} 0 & f \\ b & 3f \\ -b & -f \end{pmatrix}$$

2 real parameters + 1 phase

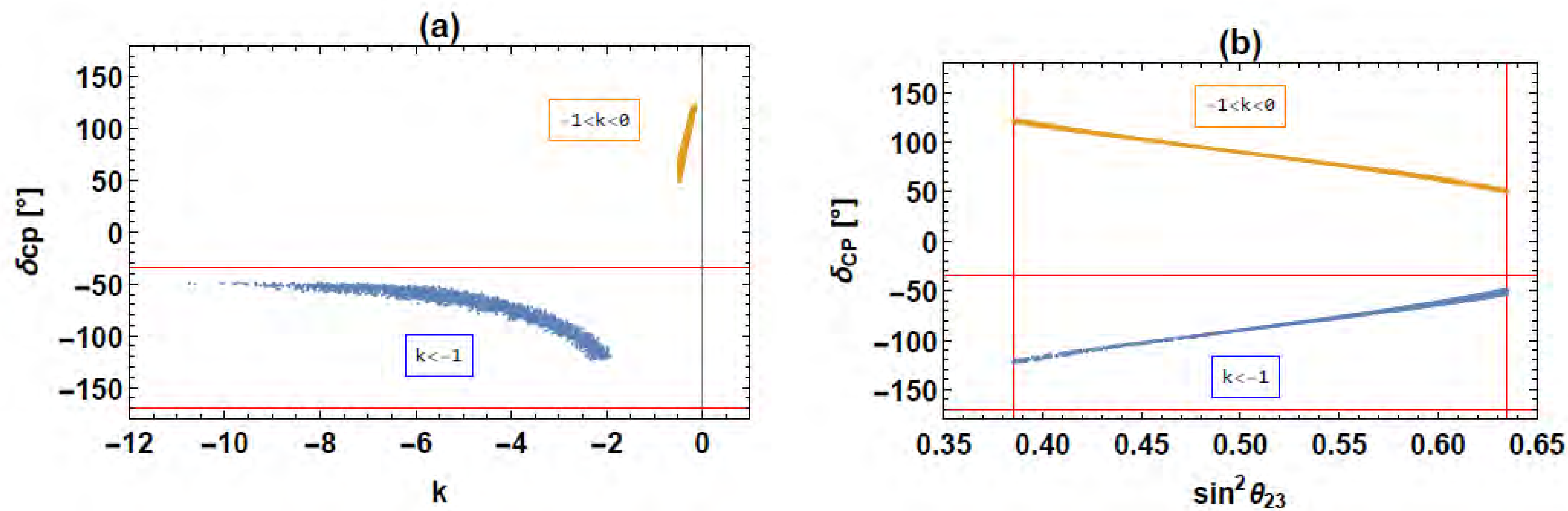
k=e/f



King et al.
k=-3
3σ
3σ
Magenta by King et al.
green: 1σ
2σ(T2K)
2σ(T2K)
δCP [°]
sin²θ23
k=-11 ~ -2
-0.1 ~ -0.5
green:1σ
k
green:1σ
δcp [°]
k
k=-3
King et.al.
-1<k<0
k<-1
green:1σ
Magenta: King
2σ(T2K)
-1<k<0
k<-1
2σ(T2K)
sin2φB

Input of cosmological baryon asymmetry

by leptogenesis with $M_1 << M_2$



Y.Shimzu, K, Takagi and M.T (2017)

4 Prospect

Quark Sector ?

☆How can Quarks and Leptons become reconciled ?

T', S_4, A_5 and $\Delta(96)$ can be embeded in $SU(5)$
S_3, S_4, $\Delta(27)$ and $\Delta(96)$ can be embeded in $SO(10)$ GUT.
A_4 and S_4 can be embeded in PS

For example: See references S.F. King, 1701.0441

quark sector (2, 1) for SU(5) 10

lepton sector (3) for SU(5) 5

Different flavor structures of quarks and leptons appear !

Cooper, King, Luhn (2010,2012), Callen,Volkas (2012), Meroni, Petcov, Spinrath (2012)
Antusch, King, Spinrath (2013), Gehrlein, Oppermann, Schaefer, Spinrath (2014)
Gehrlein, Petcov,Spinrath (2015), Bjoreroth, Anda, Medeiros Varzielas, King (2015) …

Origin of Cabibbo angle ?

☆ Flavour symmetry in Higgs sector ?

Does a Finite group control Higgs sector ?
2HDM, 3HDM …
an interesting question since Pakvasa and Sugawara 1978

☆ How is Flavor Symmetry tested ?

∗ Mixing angle sum rules

Example: TM1

$$\sin^2\theta_{23} = \frac{1}{2}\frac{1}{\cos^2\theta_{13}} \geq \frac{1}{2}\,, \qquad \sin^2\theta_{12} \simeq \frac{1}{\sqrt{3}} - \frac{2\sqrt{2}}{3}\sin\theta_{13}\cos\delta_{CP} + \frac{1}{3}\sin^2\theta_{13}\cos 2\delta_{CP}$$

TM2

$$\sin^2\theta_{12} = \frac{1}{3}\frac{1}{\cos^2\theta_{13}} \geq \frac{1}{3}\,, \qquad \cos\delta_{CP}\tan 2\theta_{23} \simeq \frac{1}{\sqrt{2}\sin\theta_{13}}\left(1 - \frac{5}{4}\sin^2\theta_{13}\right)$$

∗ Neutrino mass sum rules in FLASY ⇔ neutrinoless double beta decays

∗ Prediction of CP violating phase.

We obtained the predictable minimal seesaw mass matrices, which is based on

- Two right-handed Majorana neutrinos M_1 and M_2

- Trimaxial mixing

  This is reproduced by the S_4 flavor symmetry.

$$M_D = vY_\nu = v\begin{pmatrix} 0 & \frac{e+f}{2} \\ b & e \\ -b & f \end{pmatrix}$$

Three real parameters and one phase

Normal Hierarchy of masses

will be tested by δ_{CP} and $\sin^2\theta_{23}$.

The cosmological baryon asymmetry can determine the sign of δ_{CP} by leptogenesis !

Backup slides

A lager group

is constructed from more than two groups *by a certain product.*

A simple one is the **direct product**.

Consider e.g. two groups G_1 and G_2. Their direct product is denoted as $G_1 \times G_2$.

Multiplication rule $(a_1,a_2)\ (b_1,b_2) = (a_1b_1,\ a_2b_2)$ for $a_1,b_1 \in G_1$ and $a_2,b_2 \in G_2$

(outer) semi-direct product

It is defined such as

$$(a_1,a_2)\ (b_1,b_2) = (a_1\ f_{a2}\ (b_1),\ a_2b_2) \quad \text{for } a_1,b_1 \in G_1 \text{ and } a_2,b_2 \in G_2$$

where $f_{a2}\ (b_1)$ *denotes a homomorphic map from* G_2 *to* G_1.

This semi-direct product is denoted as $G_1 \rtimes_f G_2$.

We consider the group G and its **subgroup H** and **normal subgroup N**, whose elements are h_i and n_j , respectively.

When $G = NH = HN$ and $N \cap H = \{e\}$, *the semi-direct* product $N \rtimes_f H$ *is isomorphic to G, where* we use the map f *as* $f_{hi}\ (n_j) = h_i\ n_j\ (h_i)^{-1}$.

Since $(\chi_{1'}(C_2))^3 = 1, \quad (\chi_{1'}(C_3))^2 = 1$ are satisfied,

Orthogonarity conditions determine the Character Table

	h	χ_1	$\chi_{1'}$	χ_2
C_1	1	1	1	2
C_2	3	1	1	-1
C_3	2	1	-1	0

C_1 : {e}, C_2 : {ab, ba}, C_3 : {a, b, bab}.

By using this table, we can construct the representation matrix for 2.

Because of $\chi_2(C_3) = 0$, we choose $a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **C_3 : {a, b, bab}**

Recalling $b^2 = e$, we can write $b = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \quad bab = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

C_2 : {ab, ba} $ab = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad ba = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$

Consider the case of A_4 flavor symmetry:

A_4 has subgroups:

three Z_2, four Z_3, one $Z_2 \times Z_2$ (klein four-group)

$S^2=T^3=(ST)^3=1$

Z_2: {1,S}, {1,T^2ST}, {1,TST^2}
Z_3: {1,T,T^2}, {1,ST,T^2S}, {1,TS, ST^2}, {1,STS,ST^2S}
K_4: {1,S,T^2ST,TST^2}

Suppose A_4 is spontaneously broken to one of subgroups:
Neutrino sector preserves Z_2: {1,S}
Charged lepton sector preserves Z_3: {1,T,T^2}

$$S^T m_{LL}^\nu S = m_{LL}^\nu, \quad T^\dagger Y_e Y_e^\dagger T = Y_e Y_e^\dagger$$

⬇

$$[S, m_{LL}^\nu] = 0, \quad [T, Y_e Y_e^\dagger] = 0$$

Mixing matrices diagonalise m_{LL}^ν, $Y_e Y_e^\dagger$ also diagonalize S and T, respectively !

For the triplet, the representations are given as

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}; \quad \omega = e^{2\pi i/3}$$

$$V_\nu^T S V_\nu = \mathrm{diag}(-1, 1, -1)$$

$$V_\nu = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -c/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$$

Independent of mass eigenvalues !

Freedom of the rotation between 1st and 3rd column because a column corresponds to a mass eigenvalue.

Then, we obtain PMNS matrix.

$$V_\nu = \begin{pmatrix} 2c/\sqrt{6} & 1/\sqrt{3} & 2s/\sqrt{6} \\ -c/\sqrt{6} + s/\sqrt{2} & 1/\sqrt{3} & -s/\sqrt{6} - c/\sqrt{2} \\ -c/\sqrt{6} - s/\sqrt{2} & 1/\sqrt{3} & -s/\sqrt{6} + c/\sqrt{2} \end{pmatrix}$$

$$c = \cos\theta, \quad s = \sin\theta$$

Tri-maximal mixing : so called TM_2

θ is not fixed.

Semi-direct model

In general, s is complex.
CP symmetry can predict this phase as seen later.

another Mixing sum rules

$$\sin^2\theta_{12} = \frac{1}{3}\frac{1}{\cos^2\theta_{13}} \geq \frac{1}{3}, \qquad \cos\delta_{CP}\tan 2\theta_{23} \simeq \frac{1}{\sqrt{2}\sin\theta_{13}}\left(1 - \frac{5}{4}\sin^2\theta_{13}\right)$$

A_4 model easily realizes non-vanishing θ_{13} .

Modify G. Altarelli, F. Feruglio, Nucl.Phys. B720 (2005) 64

	(l_e, l_μ, l_τ)	e^c	μ^c	τ^c	$h_{u,d}$	ϕ_l	ϕ_ν	ξ	ξ'
$SU(2)$	2	1	1	1	2	1	1	1	1
A_4	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1''}$	$\mathbf{1'}$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1'}$
Z_3	ω	ω^2	ω^2	ω^2	1	1	ω	ω	ω

Y. Simizu, M. Tanimoto, A. Watanabe, PTP 126, 81(2011)

$$\mathbf{3} \times \mathbf{3} \Rightarrow \mathbf{1} \quad = a_1 * b_1 + a_2 * b_3 + a_3 * b_2$$
$$\mathbf{3} \times \mathbf{3} \Rightarrow \mathbf{1'} \quad = a_1 * b_2 + a_2 * b_1 + a_3 * b_3$$
$$\mathbf{3} \times \mathbf{3} \Rightarrow \mathbf{1''} \quad = a_1 * b_3 + a_2 * b_2 + a_3 * b_1$$

ξ $\qquad\qquad\qquad$ ξ'

$$\mathbf{1} \times \mathbf{1} \Rightarrow \mathbf{1} \quad , \qquad \mathbf{1''} \times \mathbf{1'} \Rightarrow \mathbf{1}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

TM_1 with NH

$$M_D = \begin{pmatrix} \frac{b+c}{2} & \frac{e+f}{2} \\ b & e \\ c & f \end{pmatrix}$$

After rotating Mv by V_{TBM} $\hat{M}_\nu \equiv V_{\mathrm{TBM}}^T M_\nu V_{\mathrm{TBM}}$, we obtain

$$V_{\mathrm{TBM}} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\hat{M}_\nu = \frac{1}{M_0} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{4}\left((b+c)^2 p + (e+f)^2\right) & \frac{1}{2}\sqrt{\frac{3}{2}}\left((c^2-b^2)p - e^2 + f^2\right) \\ 0 & \frac{1}{2}\sqrt{\frac{3}{2}}\left((c^2-b^2)p - e^2 + f^2\right) & \frac{1}{2}\left((b-c)^2 p + (e-f)^2\right) \end{pmatrix}$$

$m_1=0$

$$m_2^2 + m_3^2 = \frac{f^4}{16}\left[B^4(5j^2+2j+5)^2 + 2B^2(5jk+j+k+5)^2 \cos 2\phi_B + (5k^2+2k+5)^2\right]$$

$$m_2^2 m_3^2 = \frac{9}{4}(j-k)^4 B^4 f^8 \qquad \frac{e}{f} = k\,, \qquad \frac{b}{c} = j\,, \qquad \frac{c}{f} = Be^{i\phi_B}$$

Leptogenesis

CP lepton asymmetry

at the decay of the lighter right-handed Majorana neutrino N_1

$$\epsilon_{N_1} \simeq -\frac{3}{16\pi}\sum_j \frac{\mathrm{Im}[\{(Y_\nu^\dagger Y_\nu)_{j1}\}^2]}{(Y_\nu^\dagger Y_\nu)_{11}}\frac{1}{p}$$

$P = M_{R2}/M_{R1}$

↑ assumption

SM with two right-handed neutrinos

$$\frac{\mathrm{Im}[\{(Y_\nu^\dagger Y_\nu)_{21}\}^2]}{(Y_\nu^\dagger Y_\nu)_{11}} = \frac{1}{2}f^2(k-1)^2 \sin 2\phi_B \ ,$$

$$Y_{B-L} = -\epsilon_{N_1}\kappa Y_{N_1}^{eq}(T \gg M_1) \qquad \eta_B \equiv \frac{n_B}{n_\gamma} = 7.04 \times \frac{28}{79} Y_{B-L}$$

η_B is proportional to $(k-1)^2 \sin 2\Phi_B$

J_{CP} is proportional to $(k-1)(k-1)^5 \sin 2\Phi_B$

Correlation between δ_{CP} and cosmological baryon asymmetry

$$M_D = vY_\nu = v\begin{pmatrix} 0 & \frac{e+f}{2} \\ b & e \\ -b & f \end{pmatrix} \qquad \frac{e}{f} = k\ , \qquad \arg[b] = \phi_B\ , \qquad \frac{b}{f} = Be^{i\phi_B}$$

One phase !

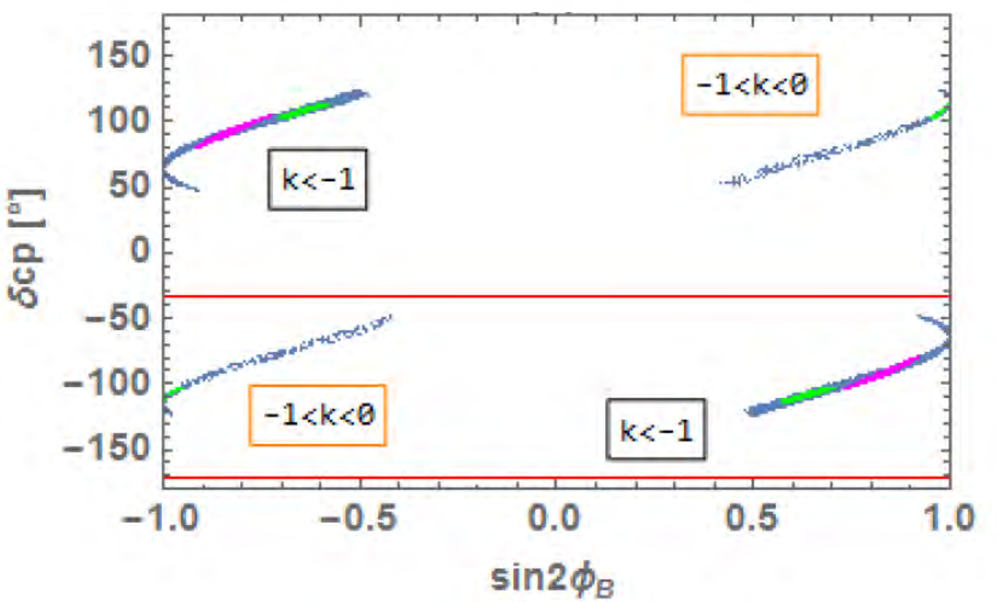
$$J_{CP} = -\frac{3}{8}\frac{f^{12}}{M_0^6}(B\sqrt{p})^6(k-1)(k+1)^5\sin 2\phi_B\frac{v^{12}}{(\Delta m_{13}^2 - \Delta m_{12}^2)\Delta m_{13}^2\Delta m_{12}^2}$$

J_{CP} is proportional to

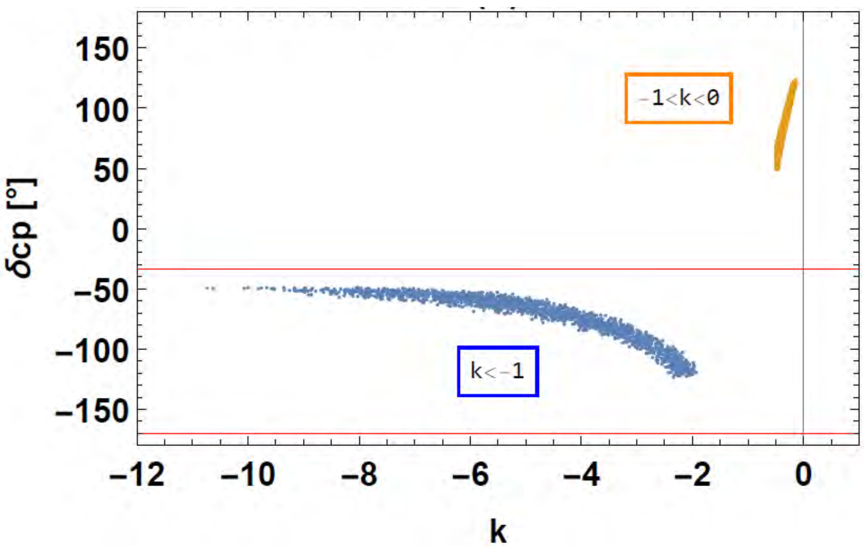
$\sin 2\phi_B$ for $-1 \le k \le 1$;

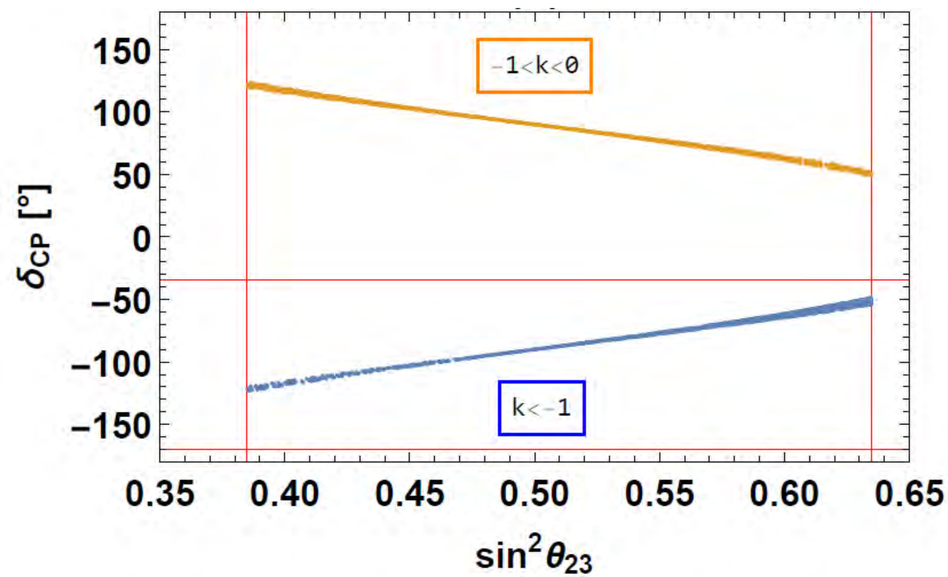
$-\sin 2\phi_B$ for $k \le -1,\ k \ge 1$

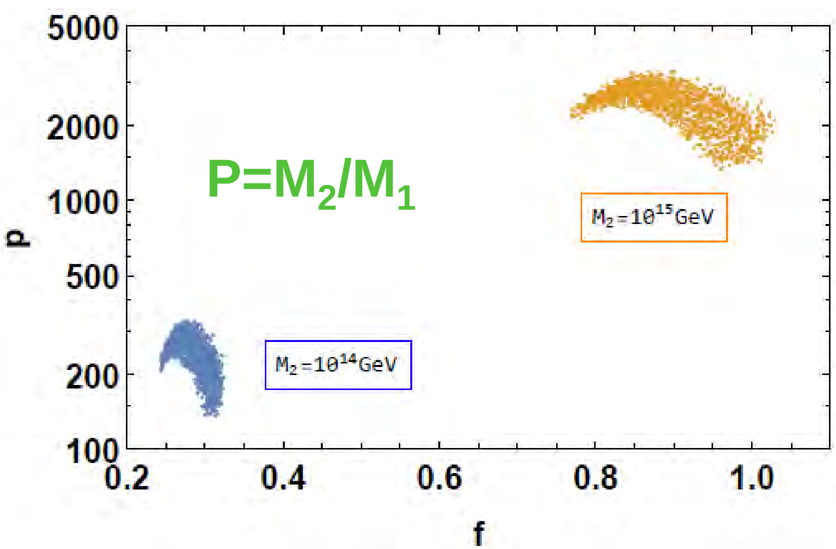
k= -11 ~ -2, -0.1~ -0.5



Inputting $\eta_B = (5.8 - 6.6) \times 10^{-10}$ 95% C.L.







$$M_D = vY_\nu = v\begin{pmatrix} 0 & \frac{e+f}{2} \\ b & e \\ -b & f \end{pmatrix}$$

$$\frac{e}{f} = k\ , \qquad \arg[b] = \phi_B\ , \qquad \frac{b}{f} = Be^{i\phi_B}$$

Our Dirac neutrino mass matrix predicts
both the signs of δ_{CP} and cosmological baryon asymmetry

$$M_D = \begin{pmatrix} 0 & 2f \\ b & 5f \\ -b & -f \end{pmatrix}, \quad \begin{pmatrix} 0 & f \\ b & 4f \\ -b & -2f \end{pmatrix}, \quad \begin{pmatrix} 0 & f \\ b & 3f \\ -b & -f \end{pmatrix} \quad \ldots\ldots$$

K= -5 K= -2 King, et al.

is preferred by T2K and Nova data if M2> M1.

$\delta_{CP} < 0$ $\eta_B > 0$

3.2 Origin of Flavor symmetry

Is it possible to realize such discrete symmetres in string theory?
Answer is yes !

Superstring theory on a certain type of six dimensional compact space leads to stringy selection rules for allowed couplings among matter fields in four-dimensional effective field theory.

Such stringy selection rules and geometrical symmetries result in discrete flavor symmetries in superstring theory.

- Heterotic orbifold models (Kobayashi, Nilles, Ploger, Raby, Ratz, 07)

- Magnetized/Intersecting D-brane Model
  (Kitazawa, Higaki, Kobayashi,Takahashi, 06)
  (Abe, Choi, Kobayashi, HO, 09, 10)

Stringy origin of non-Abelian discrete flavor symmetries

T. Kobayashi, H. Niles, F. PloegerS, S. Raby, M. Ratz, hep-ph/0611020

D_4, $\Delta(54)$

Non-Abelian Discrete Flavor Symmetries from Magnetized/Intersecting Brane Models

H. Abe, K-S. Choi, T. Kobayashi, H. Ohki, 0904.2631

D_4, $\Delta(27)$, $\Delta(54)$

Non-Abelian Discrete Flavor Symmetry from T^2/Z_N Orbifolds

A.Adulpravitchai, A. Blum, M. Lindner, 0906.0468

A_4, S_4, D_3, D_4, D_6

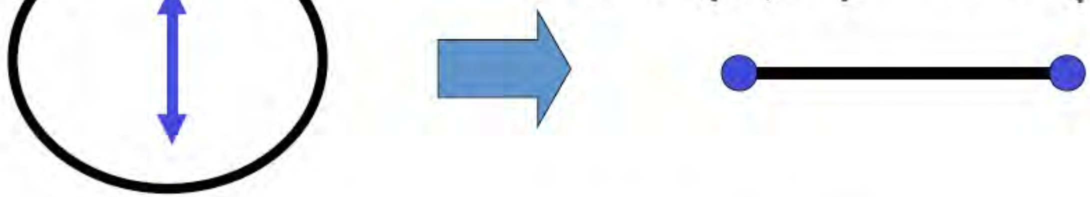
Non-Abelian Discrete Flavor Symmetries of 10D SYM theory with Magnetized extra dimensions

H. Abe, T. Kobayashi, H. Ohki, K.Sumita, Y. Tatsuta 1404.0137

S_3, $\Delta(27)$, $\Delta(54)$

$S^1/\mathbf{Z}_2$ **orbifold** (Kobayashi, Nilles, Ploger, Raby, Ratz, 07)

$$1:(\theta,0) \quad 2:(\theta,e_1)$$



There are two fixed point under the orbifold twist

These two fixed points can be represented by space group elements which act (θ, v)

$$(\theta, v)\alpha = \theta\alpha + v$$

e_1 : shift vector in one torus $(y \sim y + e_1)$

charge assignment of $\mathbf{Z}_2$: $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(stringy selection rule: Coupling is only allowed in matching of the string boundary conditions)

Discrete flavor symmetry from orbifold $S^1/\mathbf{Z}_2$

This effective Lagrangian also have permutation symmetry of these two fixed point (orbifold geometry).

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Closed algebra of these transformations $\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

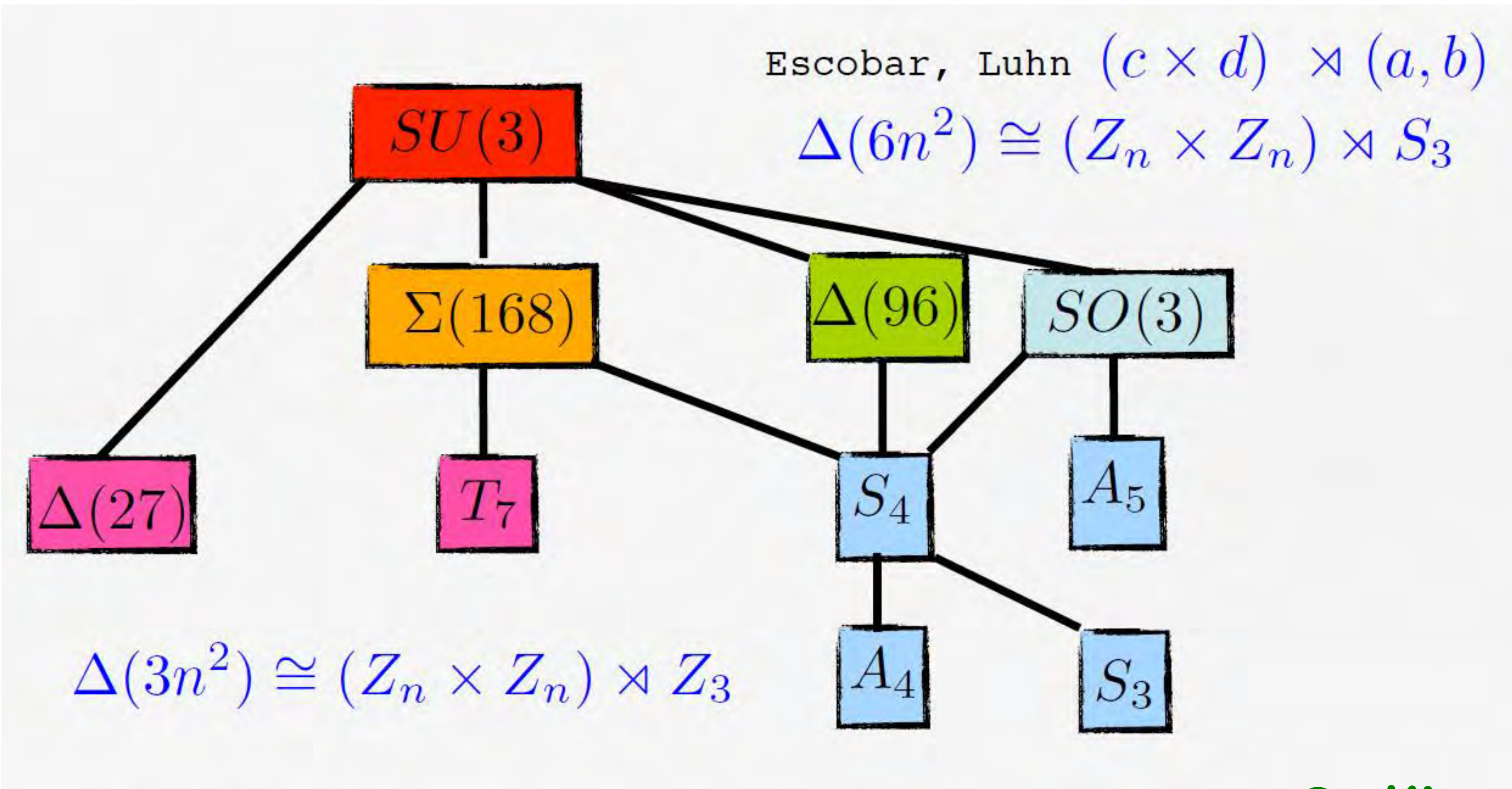
➡ $D_4 \sim S^2 \cup (\mathbf{Z}_2 \times \mathbf{Z}_2)$

Two field localized at two fixed points : doublet of D4 $\mathbf{2}$

Bulk mode (untwisted mode) : singlet of D4 $\mathbf{1}$

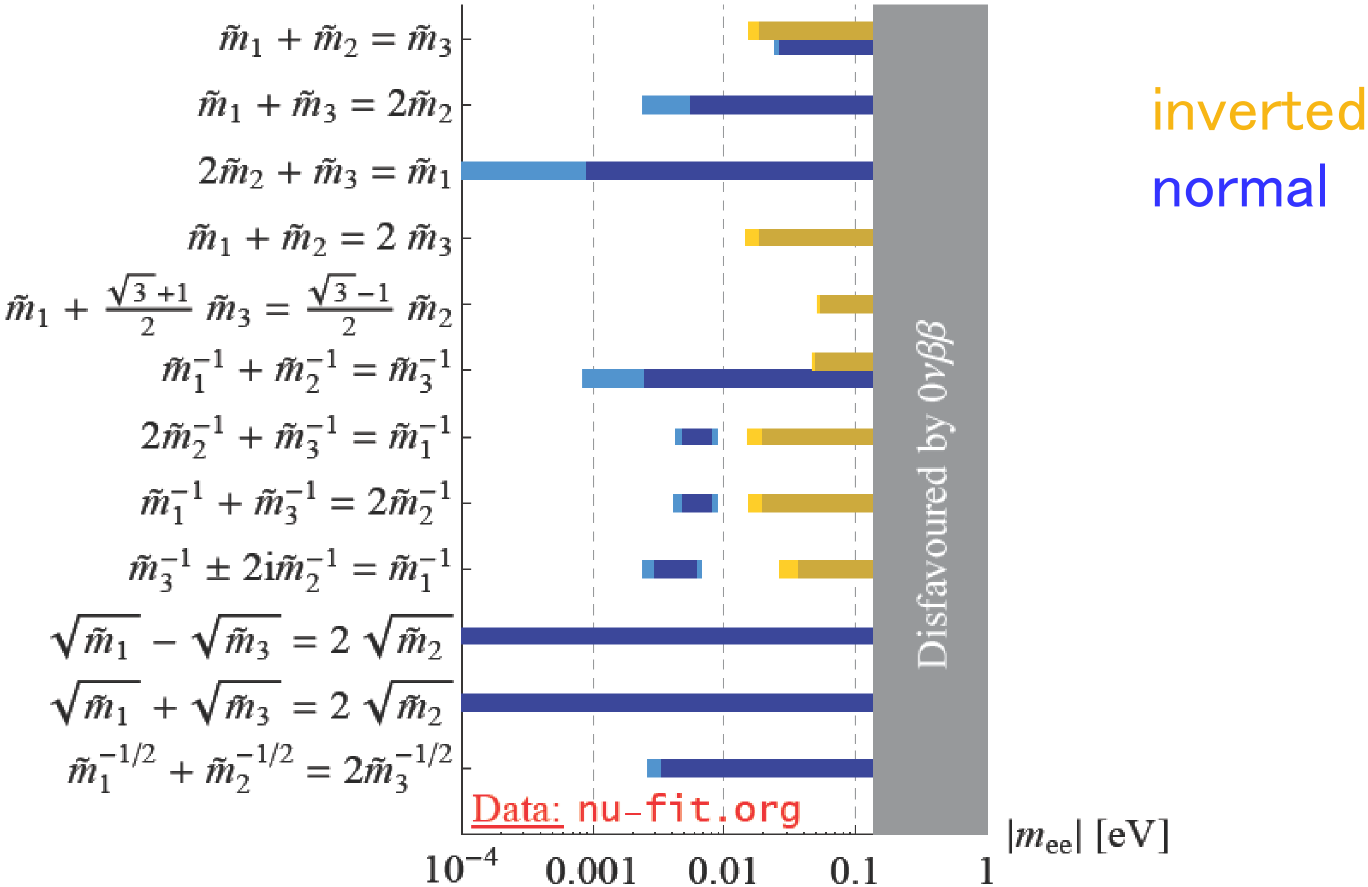
Thus full symmetry is larger than geometric symmetry

Alternatively, discrete flavor symmetries may be originated from continuous symmetries



S. King

Restrictions by mass sum rules on $|m_{ee}|$



King, Merle, Stuart, JHEP 2013, arXiv:1307.2901

Mass sum rules in A_4, T', S_4, A_5, $\Delta(96)$ …

(Talk of Spinrath) Barry, Rodejohann, NPB842(2011) arXiv:1007.5217

Different types of neutrino mass spectra correspond to the neutrino mass generation mechanism.

$$\chi \tilde{m}_2 + \xi \tilde{m}_3 = \tilde{m}_1$$ (Χ=2, ξ=1) (Χ=-1, ξ=1)

$$\frac{\chi}{\tilde{m}_2} + \frac{\xi}{\tilde{m}_3} = \frac{1}{\tilde{m}_1}$$ **M_R structre in See-saw**

$$\chi \sqrt{\tilde{m}_2} + \xi \sqrt{\tilde{m}_3} = \sqrt{\tilde{m}_1}$$ **M_D structre in See-saw**

$$\frac{\chi}{\sqrt{\tilde{m}_2}} + \frac{\xi}{\sqrt{\tilde{m}_3}} = \frac{1}{\sqrt{\tilde{m}_1}}$$ **M_R in inverse See-saw**

Χ and ξ are model specific complex parameters

King, Merle, Stuart, JHEP 2013, arXiv:1307.2901
King, Merle, Morisi, Simizu, M.T, arXiv: 1402.4271

Let us study irreducible representations of S_3.
The number of irreducible representations must be equal to three, because there are three conjugacy classes.

These elements are classified to three conjugacy classes,

$C_1 : \{e\}, \quad C_2 : \{ab, ba\}, \quad C_3 : \{a, b, bab\}.$

The subscript of C_n, n, denotes the number of elements in the conjugacy class C_n.
Their orders are found as

$(ab)^3 = (ba)^3 = e, \quad a^2 = b^2 = (bab)^2 = e$

Due to the orthogonal relation

$$\sum_\alpha [\chi_\alpha(C_1)]^2 = \sum_n m_n n^2 = m_1 + 4m_2 + 9m_3 + \cdots = 6$$
$$\sum_n m_n = 3 \qquad m_n \geq 0$$

We obtain a solution: $(m_1, m_2) = (2, 1)$

Irreducible representations of S_3 are **two singlets 1 and 1′ , one doublet 2**.

All permutations of S_3 are represented on the reducible triplet (x_1, x_2, x_3) as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

$e : (x_1, x_2, x_3) \rightarrow (x_1, x_2, x_3)$
$a_1 : (x_1, x_2, x_3) \rightarrow (x_2, x_1, x_3)$
$a_2 : (x_1, x_2, x_3) \rightarrow (x_3, x_2, x_1)$
$a_3 : (x_1, x_2, x_3) \rightarrow (x_1, x_3, x_2)$
$a_4 : (x_1, x_2, x_3) \rightarrow (x_3, x_1, x_2)$
$a_5 : (x_1, x_2, x_3) \rightarrow (x_2, x_3, x_1)$

We change the representation through the unitary transformation, $U^\dagger g U$, e.g. by using the unitary matrix U_{tribi},

$$U_{\mathrm{tribi}} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}.$$

Then, the six elements of S_3 are written as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

These are completely reducible and that the (2×2) submatrices are exactly the same as those for the doublet representation . The unitary matrix U_{tribi} is called tri-bimaximal matrix.

T' group

Double covering group of A_4, 24 elements

24 elements are generated by S, T and R:

$S^2 = R, \quad T^3 = R^2 = 1,$

$(ST)^3 = 1, \quad RT = TR$

	h	χ_1	$\chi_{1'}$	$\chi_{1''}$	χ_2	$\chi_{2'}$	$\chi_{2''}$	χ_3
C_1	1	1	1	1	2	2	2	3
C_1'	2	1	1	1	-2	-2	-2	3
C_4	3	1	ω	ω^2	-1	$-\omega$	$-\omega^2$	0
C_4'	3	1	ω^2	ω	-1	$-\omega^2$	$-\omega$	0
C_4''	6	1	ω	ω^2	1	ω	ω^2	0
C_4'''	6	1	ω^2	ω	1	ω^2	ω	0
C_6	4	1	1	1	0	0	0	-1

Ireducible representations

1, 1', 1", 2, 2', 2", 3

For triplet $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega^2 & -1 & 2\omega \\ 2\omega & 2\omega^2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}; \quad \omega = e^{2\pi i/3}$$

TM_1 with IH

$m_3=0$

After taking
$$M_D = \begin{pmatrix} -2b & \frac{e+f}{2} \\ b & e \\ b & f \end{pmatrix}$$
, we get

$$\hat{M}_\nu = \frac{1}{M_0}\begin{pmatrix} 6b^2 & 0 & 0 \\ 0 & \frac{3}{4}(e+f)^2 & -\frac{1}{2}\sqrt{\frac{3}{2}}(e-f)(e+f) \\ 0 & -\frac{1}{2}\sqrt{\frac{3}{2}}(e-f)(e+f) & \frac{1}{2}(e-f)^2 \end{pmatrix}$$

$$\frac{e}{f} = ke^{i\phi_k}$$

$$= \frac{6b^2}{M_0}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{f^2}{M_0}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{4}(ke^{i\phi_k}+1)^2 & -\frac{1}{2}\sqrt{\frac{3}{2}}(k^2e^{2i\phi_k}-1) \\ 0 & -\frac{1}{2}\sqrt{\frac{3}{2}}(k^2e^{2i\phi_k}-1) & \frac{1}{2}(ke^{i\phi_k}-1)^2 \end{pmatrix}$$

Mixing angles and CP phase are given only by k and Φ_k

TM_2 with NH or IH

$m_1=0$ or $m_3=0$

After taking
$$M_D = \begin{pmatrix} b & -e-f \\ b & e \\ b & f \end{pmatrix}$$
, we get

$$\frac{e}{f} = k e^{i\phi_k}$$

$$\hat{M}_\nu = \frac{1}{M_0} \begin{pmatrix} \frac{3}{2}(e+f)^2 & 0 & \frac{\sqrt{3}}{2}(e^2-f^2) \\ 0 & 3b^2 & 0 \\ \frac{\sqrt{3}}{2}(e^2-f^2) & 0 & \frac{1}{2}(e-f)^2 \end{pmatrix}$$

$$= \frac{3b^2}{M_0} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{f^2}{M_0} \begin{pmatrix} \frac{3}{2}(ke^{i\phi_k}+1)^2 & 0 & \frac{\sqrt{3}}{2}(k^2e^{2i\phi_k}-1) \\ 0 & 0 & 0 \\ \frac{\sqrt{3}}{2}(k^2e^{2i\phi_k}-1) & 0 & \frac{1}{2}(ke^{i\phi_k}-1)^2 \end{pmatrix}$$

TM_2 with NH or IH

$m_1=0$ or $m_3=0$

After taking

$$M_D = \begin{pmatrix} b & -e-f \\ b & e \\ b & f \end{pmatrix}$$

, we get

$$\frac{e}{f} = k e^{i\phi_k}$$

$$\hat{M}_\nu = \frac{1}{M_0}\begin{pmatrix} \frac{3}{2}(e+f)^2 & 0 & \frac{\sqrt{3}}{2}(e^2-f^2) \\ 0 & 3b^2 & 0 \\ \frac{\sqrt{3}}{2}(e^2-f^2) & 0 & \frac{1}{2}(e-f)^2 \end{pmatrix}$$

$$= \frac{3b^2}{M_0}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{f^2}{M_0}\begin{pmatrix} \frac{3}{2}(ke^{i\phi_k}+1)^2 & 0 & \frac{\sqrt{3}}{2}(k^2e^{2i\phi_k}-1) \\ 0 & 0 & 0 \\ \frac{\sqrt{3}}{2}(k^2e^{2i\phi_k}-1) & 0 & \frac{1}{2}(ke^{i\phi_k}-1)^2 \end{pmatrix}$$

★ TM$_1$ with IH in S_4 flavor symmetry

3 × 3 × 1   **3' × 3 × 1'**

$$\frac{y_{\rm atm}}{\Lambda}\phi_{\rm atm} L H_u \nu^c_{R1} + \frac{y_{\rm sol}}{\Lambda}\phi_{\rm sol} L H_u \nu^c_{R2}$$

$$\langle \phi_{\rm atm} \rangle \sim \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \qquad \langle \phi_{\rm sol} \rangle \sim \begin{pmatrix} \frac{e+f}{2} \\ f \\ e \end{pmatrix}$$

3   3'

$$M_D = \begin{pmatrix} -2b & \frac{e+f}{2} \\ b & e \\ b & f \end{pmatrix}$$

$\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ preserves SU symmetry for 3.

★ TM$_2$ with NH or IH in A_4 or S_4 flavor symmetry

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ preserves S symmetry for 3.

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$ for 3 and 3'.

S is a generator of A_4 and S_4 generator

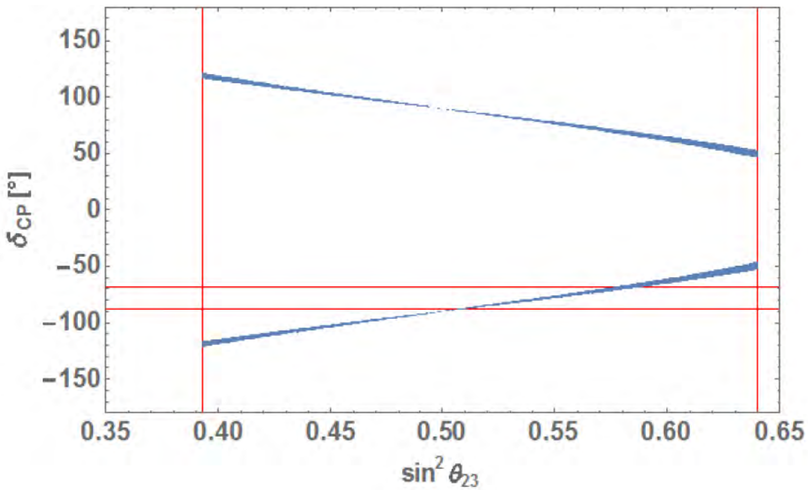
$\begin{pmatrix} -e-f \\ e \\ f \end{pmatrix}$ breaks S, T , U, SU unless e=f .

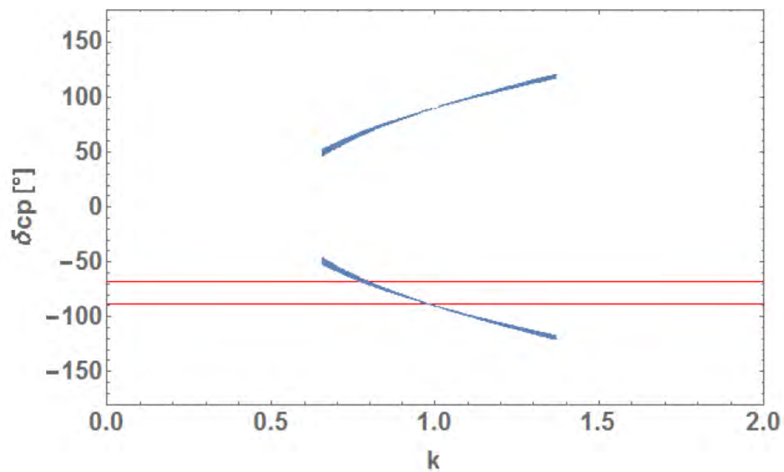
We need auxiliary Z_2 symmetry to obtain

$$M_D = \begin{pmatrix} b & -e-f \\ b & e \\ b & f \end{pmatrix}$$

TM_1 with IH $m_3=0$

$$M_D = \begin{pmatrix} -2b & \frac{e+f}{2} \\ b & e \\ b & f \end{pmatrix}$$





$$\frac{e}{f} = k e^{i\phi_k}$$

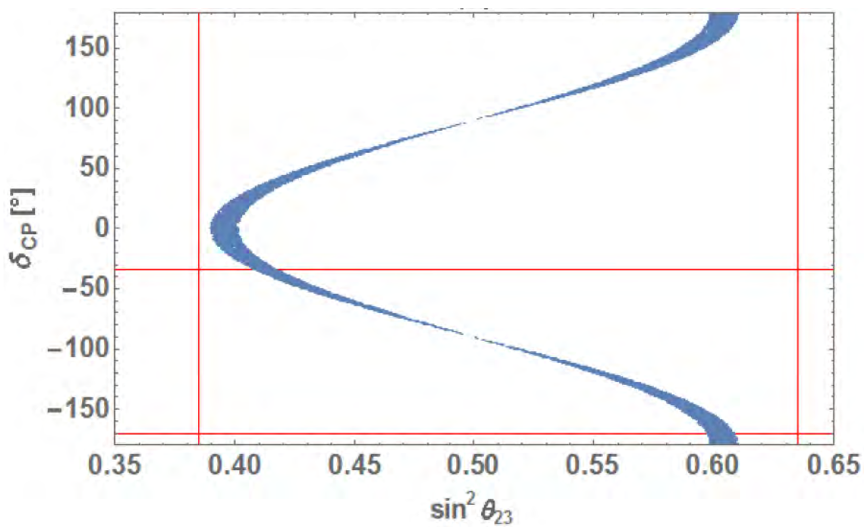
k=|e/f|=0.65～1.37 $\Phi_k=\pm(25^\circ \sim 38^\circ)$

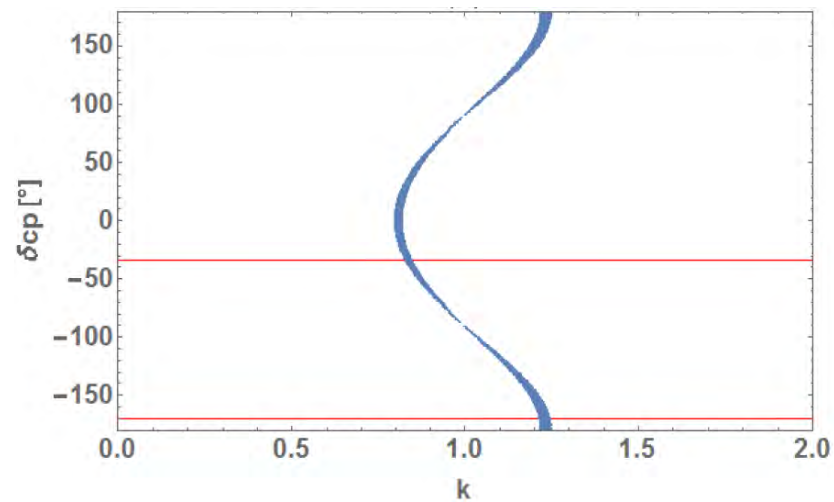
$|m_{ee}|$～50 meV

TM_2 with NH $m_1=0$

$$M_D = \begin{pmatrix} -2b & \frac{e+f}{2} \\ b & e \\ b & f \end{pmatrix}$$

$$\cos\delta_{CP}\tan 2\theta_{23} \simeq \frac{1}{\sqrt{2}\sin\theta_{13}}\left(1-\frac{5}{4}\sin^2\theta_{13}\right)$$





Predicted δ_{CP} is sensitive to k

$$\frac{e}{f} = k e^{i\phi_k}$$

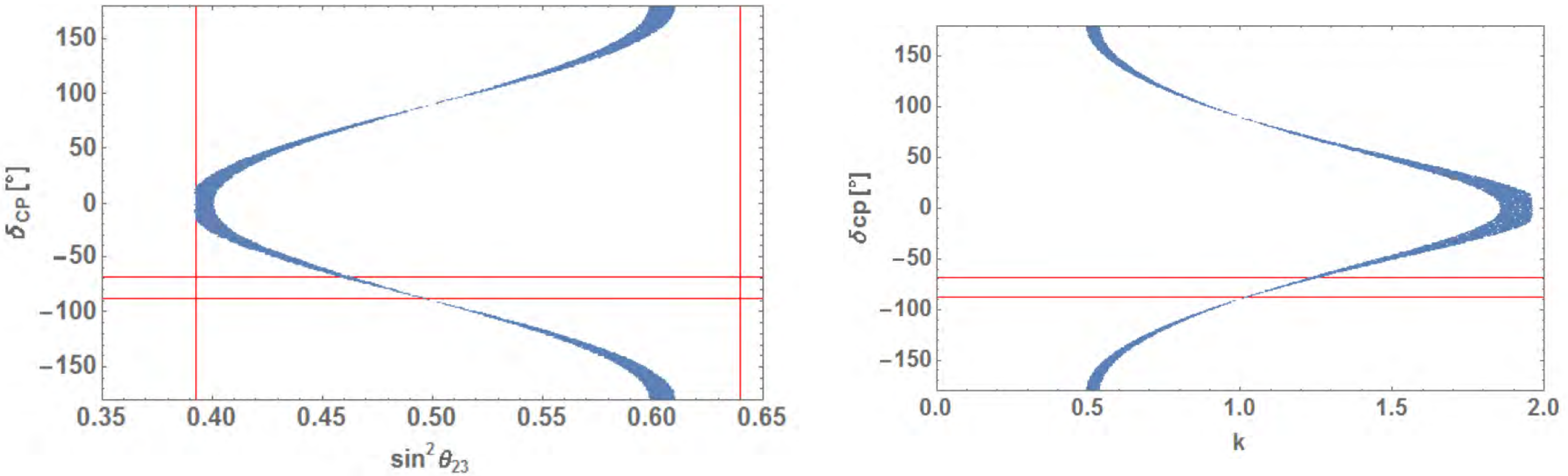
k=|e/f|=0.78∼1.24 $\Phi_k=\pm(165^\circ \sim 180^\circ)$

$|m_{ee}|$=(2∼4) meV

TM_2 with IH $m_3=0$

$$M_D = \begin{pmatrix} -2b & \frac{e+f}{2} \\ b & e \\ b & f \end{pmatrix}$$

$$\cos\delta_{CP}\tan 2\theta_{23} \simeq \frac{1}{\sqrt{2}\sin\theta_{13}}\left(1-\frac{5}{4}\sin^2\theta_{13}\right)$$

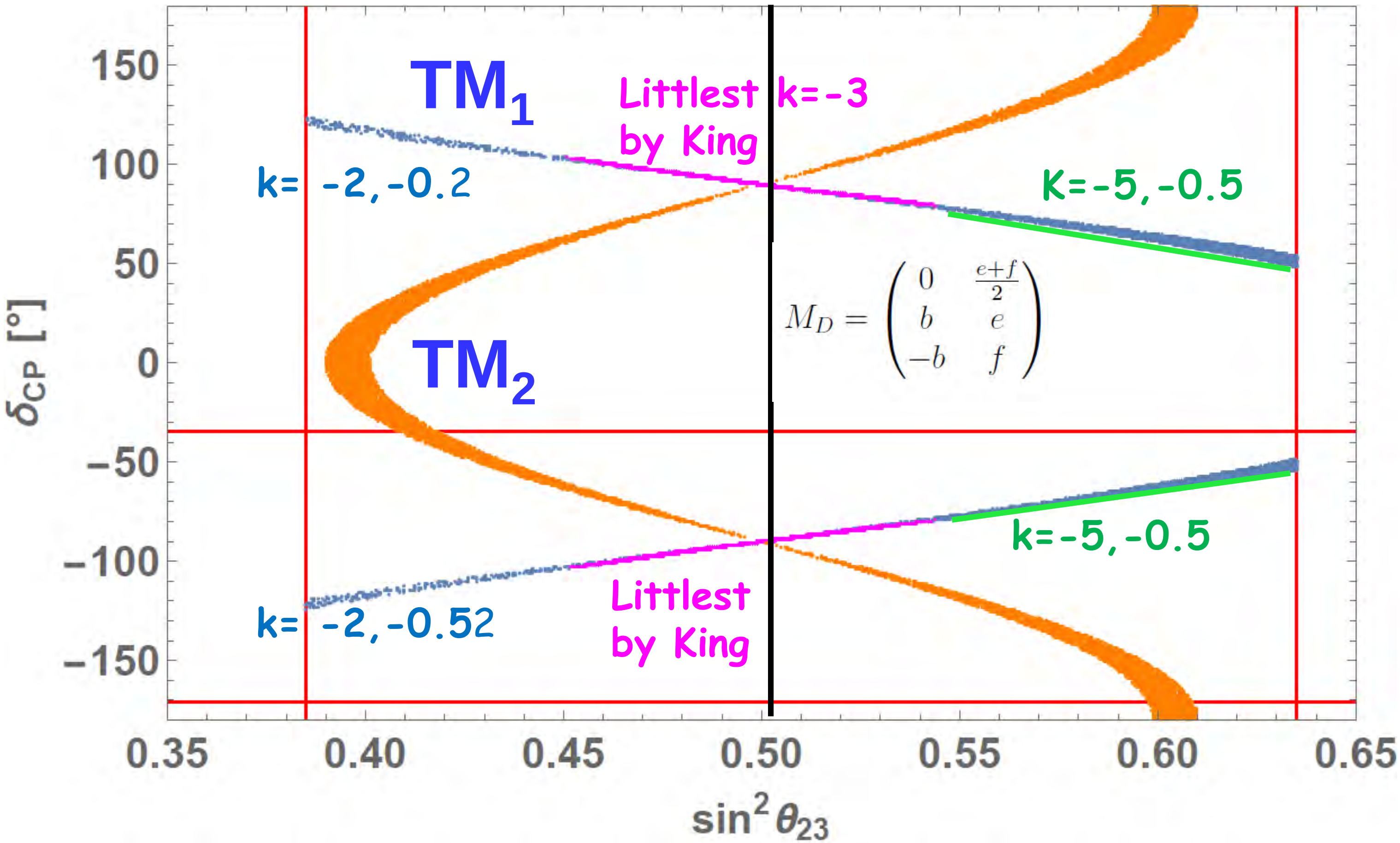


$$\frac{e}{f} = k e^{i\phi_k}$$

k=|e/f|=0.49∼1.95 Φ_k= -40° ∼40°

$|m_{ee}|$∼50 meV

Combined result



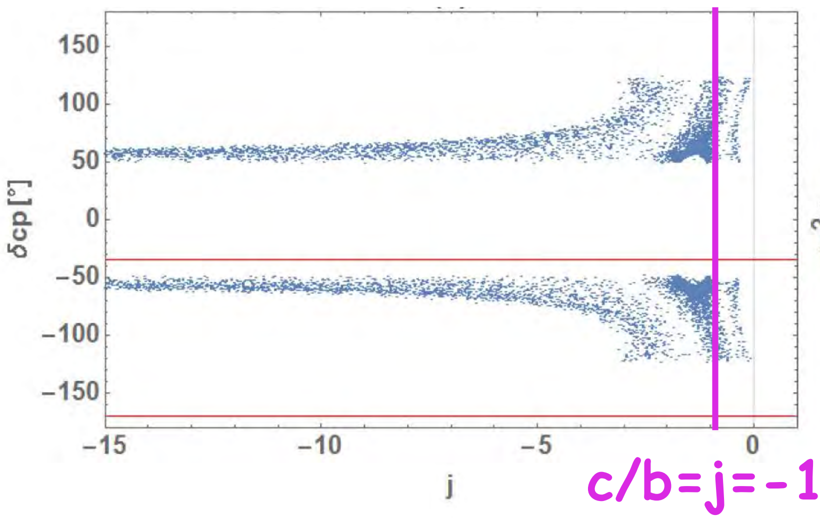
e/f will be fixed by the observation of δ_{cp} .

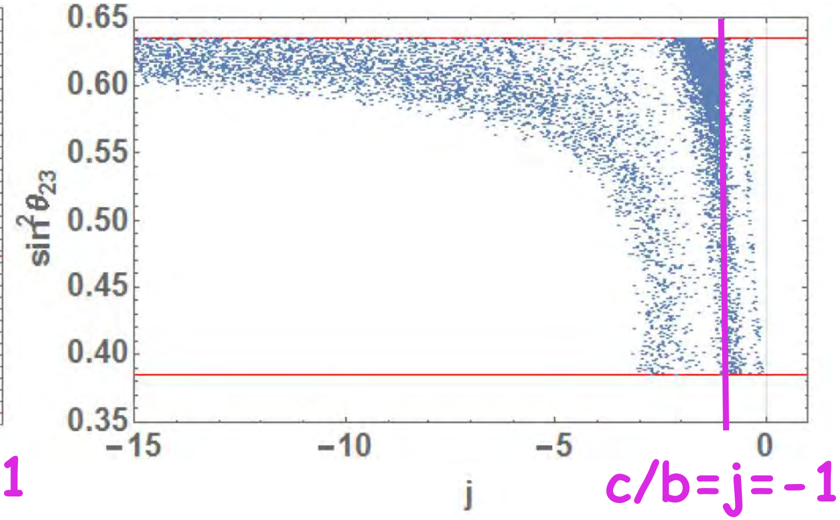
Predictions at arbitrary c/b=j

$$\frac{e}{f} = k\,, \qquad \frac{b}{c} = j\,, \qquad \frac{c}{f} = Be^{i\phi_B}$$

5 parameters by supposing j to be real

$$\hat{M}_\nu = \frac{f^2}{M_0}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{4}\left[B^2e^{2i\phi_B}(1+j)^2+(k+1)^2\right] & \frac{1}{2}\sqrt{\frac{3}{2}}\left[B^2e^{2i\phi_B}(1-j^2)-k^2+1\right] \\ 0 & \frac{1}{2}\sqrt{\frac{3}{2}}\left[B^2e^{2i\phi_B}(1-j^2)-k^2+1\right] & \frac{1}{2}\left[B^2e^{2i\phi_B}(1-j)^2+(k-1)^2\right] \end{pmatrix}$$





case 1: c/b=-1, case 2: c/b=-∞, case 3: c/b=0

Subgroups and decompositions of multiplets

S_4 group is isomorphic to $\Delta(24) = (Z_2 \times Z_2) \rtimes S_3$.

A_4 group is isomorphic to $\Delta(12) = (Z_2 \times Z_2) \rtimes Z_3$.

$S_4 \to S_3$

$$\begin{array}{cccccc} S_4 & \mathbf{1} & \mathbf{1}' & \mathbf{2} & \mathbf{3} & \mathbf{3}' \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ S_3 & \mathbf{1} & \mathbf{1}' & \mathbf{2} & \mathbf{1}+\mathbf{2} & \mathbf{1}'+\mathbf{2} \end{array}$$

$S_4 \to A_4$

$$\begin{array}{cccccc} S_4 & \mathbf{1} & \mathbf{1}' & \mathbf{2} & \mathbf{3} & \mathbf{3}' \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ A_4 & \mathbf{1} & \mathbf{1} & \mathbf{1}'+\mathbf{1}'' & \mathbf{3} & \mathbf{3} \end{array}$$

$S_4 \to (Z_2 \times Z_2) \rtimes Z_2$

Subgroups and decompositions of multiplets

A_4 group is isomorphic to $\Delta(12) = (Z_2 \times Z_2) \rtimes Z_3$.

$A_4 \rightarrow Z_3$

$$\begin{array}{ccc} A_4 \simeq \Delta(12) & \mathbf{1}_k & \mathbf{3} \\ & \downarrow & \downarrow \\ Z_3 & \mathbf{1}_k & \mathbf{1}_0 + \mathbf{1}_1 + \mathbf{1}_2 \end{array}$$

(k = 0, 1, 2)

$A_4 \rightarrow Z_2 \times Z_2$

$$\begin{array}{ccc} A_4 \simeq \Delta(12) & \mathbf{1}_k & \mathbf{3} \\ & \downarrow & \downarrow \\ Z_2 \times Z_2 & \mathbf{1}_{0,0} & \mathbf{1}_{1,1} + \mathbf{1}_{0,1} + \mathbf{1}_{1,0} \end{array}$$

Subgroups and decompositions of multiplets

$A_5 \to A_4$

$$\begin{array}{cccccc} A_5 & \mathbf{1} & \mathbf{3} & \mathbf{3'} & \mathbf{4} & \mathbf{5} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ A_4 & \mathbf{1} & \mathbf{3} & \mathbf{3} & \mathbf{1}+\mathbf{3} & \mathbf{1'}+\mathbf{1''}+\mathbf{3} \end{array}$$

$A_5 \to D_5$

$$\begin{array}{cccccc} A_5 & \mathbf{1} & \mathbf{3} & \mathbf{3'} & \mathbf{4} & \mathbf{5} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ D_5 & \mathbf{1}_+ & \mathbf{1}_- + \mathbf{2}_1 & \mathbf{1}_- + \mathbf{2}_2 & \mathbf{2}_1 + \mathbf{2}_2 & \mathbf{1}_+ + \mathbf{2}_1 + \mathbf{2}_2 \end{array}$$

$A_5 \to S_3 \simeq D_3$

$$\begin{array}{cccccc} A_5 & \mathbf{1} & \mathbf{3} & \mathbf{3'} & \mathbf{4} & \mathbf{5} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ D_3 & \mathbf{1}_+ & \mathbf{1}_- + \mathbf{2} & \mathbf{1}_- + \mathbf{2} & \mathbf{1}_+ + \mathbf{1}_- + \mathbf{2} & \mathbf{1}_+ + \mathbf{2} + \mathbf{2} \end{array}$$

$A_5 \to Z_2 \times Z_2$

5 Klein four groups

$$K_1 = \{v_1, v_2, v_3, e\} \ , \quad K_2 = \{v_4, v_5, v_6, e\}$$
$$K_3 = \{v_7, v_8, v_9, e\} \ , \quad K_4 = \{v_{10}, v_{11}, v_{12}, e\} \quad \text{and} \quad K_5 = \{v_{13}, v_{14}, v_{15}, e\}$$

$$\begin{array}{lllll} v_1 = s \ , & v_2 = st^2st^3st^2 \ , & v_3 = t^2st^3st^2 \ , & v_4 = t^4st \ , & v_5 = st^3st^2s \ , \\ v_6 = t^2st^3sts \ , & v_7 = tst^4 \ , & v_8 = st^2st^3s \ , & v_9 = stst^3st^2 \ , & v_{10} = st^2st \ , \\ v_{11} = t^2st^3 \ , & v_{12} = tst^3st^2s \ , & v_{13} = tst^2s \ , & v_{14} = t^3st^2 \ , & v_{15} = st^2st^3st \ . \end{array}$$

Monstrous moonshine

Modular J function

$$J(q) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + 333202640600q^5 + \cdots$$

$$q = e^{2\pi i\tau},\ Im(\tau) > 0,\ J(\tau) = J(\frac{a\tau + b}{c\tau + d}),\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z)$$

It turns out q-expansion coefficients of J-function are decomposed into a sum of dimensions of some irreducible representations of the monster group M

$$196884 = 1 + 196883,\ \ 21493760 = 1 + 196883 + 21296876,$$

$$864299970 = 2 \times 1 + 2 \times 196883 + 21296876 + 842609326,$$

$$20245856256 = 1 \times 1 + 3 \times 196883 + 2 \times 21296876$$

$$+842609326 + 19360062527, \cdots$$

Dimensions of some irreducible representations of monster :

$$\{1,\ 196883,\ 21296876,\ 842609326,$$

$$18538750076,\ 19360062527 \cdots\}$$

☆CP is conserved in HE theory before FLASY is broken.
☆CP is a dicrete symmetry.

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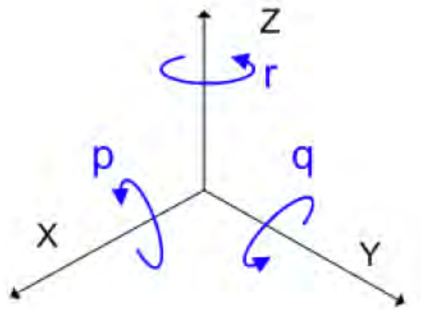
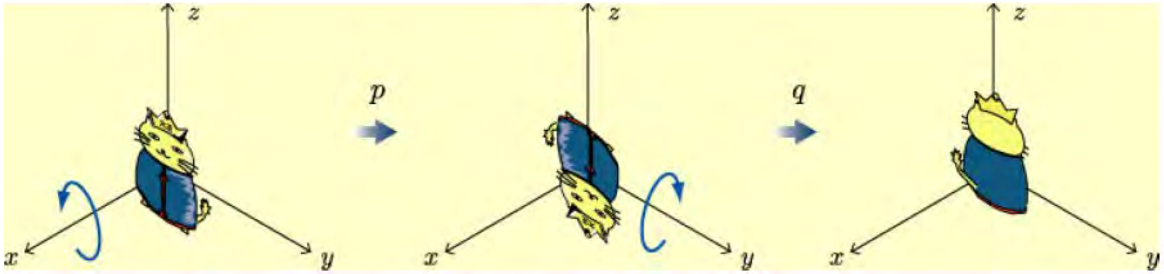
Klein four group


Felix Klein





Multiplication table

	e	p	q	r
e	e	p	q	r
p	p	e	r	q
q	q	r	e	p
r	r	q	p	e

With four elements, the Klein four group is the smallest non-cyclic group, and the cyclic group of order 4 and the Klein four-group are, up to isomorphism, the only groups of order 4. Both are abelian groups.

Normal subgroup of A_4

$Z_2 \times Z_2$ $\quad V$ = < identity, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3) >

Taking both the charged lepton mass matrix and the right-handed Majorana neutrino one to be real diagonal:

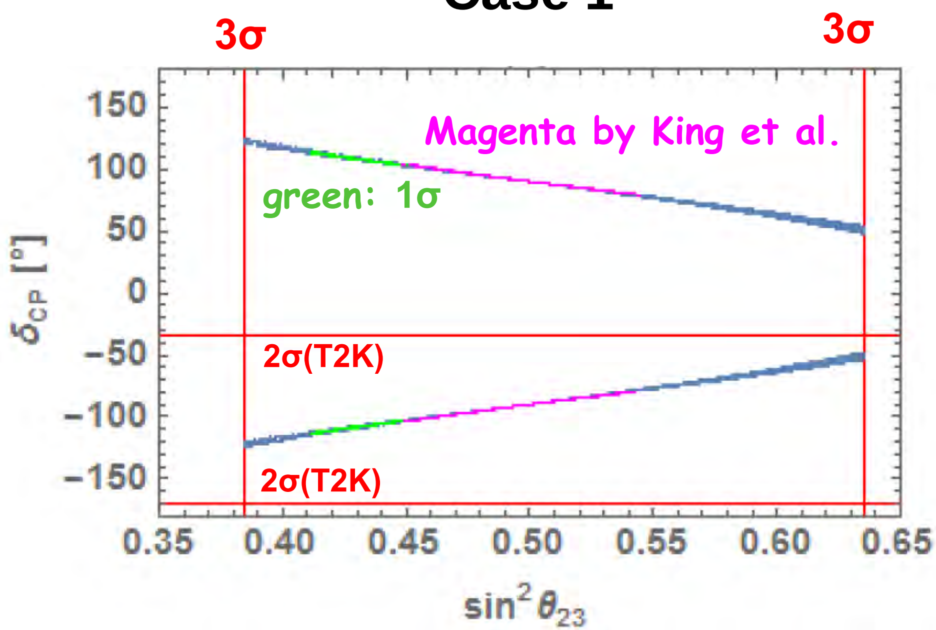
$$M_R = M_0 \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \qquad M_D = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}_{LR}$$

$p = M_{R2} / M_{R1}$

Let us consider the condition in M_D to realize the case of TM_1 .

$$U_{PMNS} = V_{TBM} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & e^{-i\sigma}\sin\phi \\ 0 & -e^{i\sigma}\sin\phi & \cos\phi \end{pmatrix} \qquad V_{TBM} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

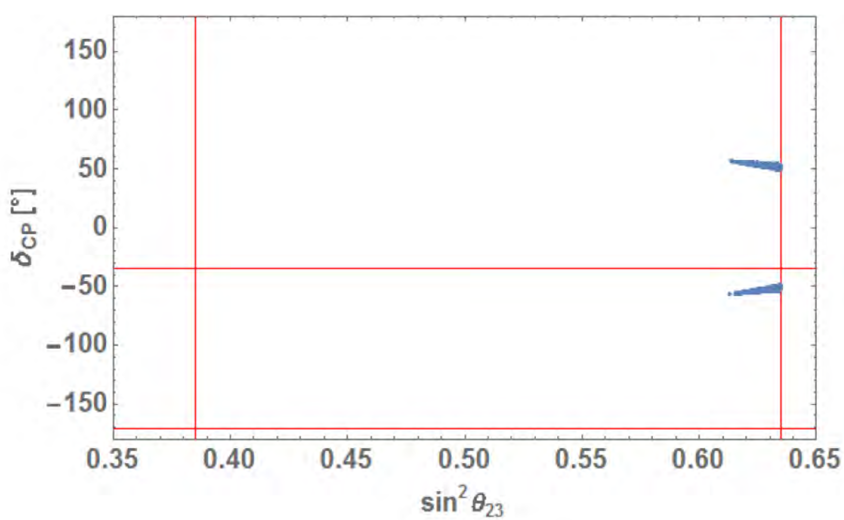
Case 1



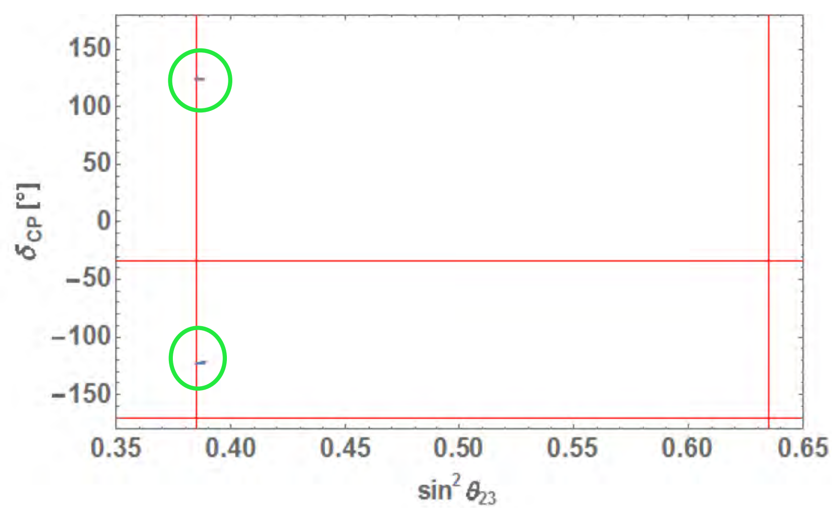
TM$_1$ sum rule

$$\cos\delta_{CP}\tan 2\theta_{23} \simeq -\frac{1}{2\sqrt{2}\sin\theta_{13}}\left(1-\frac{7}{2}\sin^2\theta_{13}\right)$$

Case 2



Case 3



TM_1 with NH $m_1=0$

Consider specific three cases
(Remove 2 parameters by adding one zero in M_D)

Case I $b+c=0$

$$M_D = \begin{pmatrix} 0 & \frac{e+f}{2} \\ b & e \\ -b & f \end{pmatrix}$$

Case 2 $c=0$

$$M_D = \begin{pmatrix} \frac{b}{2} & \frac{e+f}{2} \\ b & e \\ 0 & f \end{pmatrix}$$

Case 3 $b=0$

$$M_D = \begin{pmatrix} \frac{c}{2} & \frac{e+f}{2} \\ 0 & e \\ c & f \end{pmatrix}$$

3 real parameters + 1 phase e, f are real : b is complex

$$M_D = \begin{pmatrix} 0 & f \\ b & 3f \\ -b & -f \end{pmatrix}$$

$e/f=-3$, 2 real parameters + 1 phase

Littlest seesaw model by King et al.

New simple Dirac neutrino mass matrices with different k=e/f

$$M_D = \begin{pmatrix} 0 & 2f \\ b & 5f \\ -b & -f \end{pmatrix}, \quad \begin{pmatrix} 0 & 2f \\ b & -f \\ -b & 5f \end{pmatrix}, \quad \begin{pmatrix} 0 & f \\ b & 4f \\ -b & -2f \end{pmatrix}, \quad \begin{pmatrix} 0 & f \\ b & -2f \\ -b & 4f \end{pmatrix}$$

k= -5	k= -1/5	k= -2	k= -1/2
$\delta_{CP}=\pm(50\text{-}70)^\circ$	$\delta_{CP}=\pm 120^\circ$	$\delta_{CP}\sim\pm 120^\circ$	$\delta_{CP}=\pm(50\text{-}70)^\circ$
$\sin^2\theta_{23}\geqq 0.55$	$\sin^2\theta_{23}\sim 0.4$	$\sin^2\theta_{23}\sim 0.4$	$\sin^2\theta_{23}\geqq 0.55$

Littlest seesaw model by King et al.

$$M_D = \begin{pmatrix} 0 & f \\ b & 3f \\ -b & -f \end{pmatrix}$$

k=-3

$\delta_{CP}=\pm(80\text{-}105)^\circ$

$\sin^2\theta_{23}=0.45\sim 0.55$

Since simple patterns predict vanishing θ_{13}, larger groups may be used to obtain non-vanishing θ_{13}.

R.de Adelhart Toorop, F.Feruglio, C.Hagedorn, Phys. Lett 703} (2011) 447
G.J.Ding, Nucl. Phys.B 862 (2012) 1
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G.J.Ding and S.F.King, Phys.Rev.D89 (2014) 093020
C.Hagedorn, A.Meroni and E.Molinaro, Nucl.Phys. B 891 (2015) 499

Δ(96) group

Generator S, T and U : $S^2=(ST)^3=T^8=1$, $(ST^{-1}ST)^3=1$

Irreducible representations: 1, 1', 2, 3_1 - 3_6, 6

Subgroup : fifteen Z_2 , sixteen Z_3 , seven K_4 , twelve Z_4 , six Z_8

For triplet 3,

$$S=\frac{1}{2}\begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1 & 1 \\ \sqrt{2} & 1 & -1 \end{pmatrix} \qquad T=\begin{pmatrix} e^{\frac{6\pi i}{4}} & 0 & 0 \\ 0 & e^{\frac{7\pi i}{4}} & 0 \\ 0 & 0 & e^{\frac{3\pi i}{4}} \end{pmatrix}$$

If neutrino sector preserves {S, ST^4ST^4} ($Z_2\times Z_2$)
charged lepton sector preserves
ST (Z_3)

$$U_{TFH1}=\begin{pmatrix} \frac{1}{6}(3+\sqrt{3}) & \frac{1}{\sqrt{3}} & \frac{1}{6}(-3+\sqrt{3}) \\ \frac{1}{6}(-3+\sqrt{3}) & \frac{1}{\sqrt{3}} & \frac{1}{6}(3+\sqrt{3}) \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

$\Theta_{13}\sim 12^\circ$ rather large

If A_5 is broken to other subgroups: for example,

Neutrino sector preserves S or $T^2ST^3ST^2$ (both are K_4 generator)
Charged lepton sector preserves T (Z_5)

$$\begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ \frac{\sin\theta_{12}}{\sqrt{2}} & -\frac{\cos\theta_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{\sin\theta_{12}}{\sqrt{2}} & -\frac{\cos\theta_{12}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \times \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\tan\theta_{12} = 1/\phi \qquad \phi = \frac{1+\sqrt{5}}{2}$$

Θ is not fixed, however, there appear testable sum rules:

$$\sin^2\theta_{12} = \frac{\sin^2\varphi}{1-\sin^2\theta_{13}} \approx \frac{0.276}{1-\sin^2\theta_{13}} \qquad \sin^2\theta_{23} \approx \frac{1}{2}\left(1 \pm (1-\sqrt{5})\sin\theta_{13}\right)$$

A.Di Iura, C.Hagedorn and D.Meloni, JHEP1508 (2015) 037

Monster group is maximal one in sporadic finite group, which is related to the string theory.

Vertex Operator Algebra

Moonshine phenomena

On the other hand,
A_5 is the minimal simple finite group except for cyclic groups.
This group is succesfully used to reproduce the lepton flavor structure.

There appears a flavor mixing angle with Golden ratio.

Platonic solids (tetrahedron, cube, octahedron, regular dodecahedron, regular icosahedron) have symmetries of A_4, S_4 and A_5 ,
which may be related with flavor structure of leptons.

Moonshine phenomena was discovered in Monster group.

Monster group: largest sporadic finite group, of order 8×10^{53} .

808 017 424 794 512 875 886 459 904 961 710 757 005 754 368 000 000 000

McKay, Tompson, Conway, Norton (1978) observed :
strange relationship between modular form and an isolated discrete group.

q-expansion coefficients of Modular *J*-function are decomposed into a sum of dimensions of some irreducible representations of the monster group.

Moonshine phenomena

Phenomenon of monstrous moonshine has been solved mathematically in early 1990's using the technology of vertex operator algebra in string theory.
However, we still do not have a 'simple' explanation of this phenomenon.

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$$J(\tau) = J\left(\frac{a\tau + b}{c\tau + d}\right)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z)$$

$$196884 = 1 + 196883,\ 21493760 = 1 + 196883 + 21296876,$$
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$$20245856256 = 1 \times 1 + 3 \times 196883 + 2 \times 21296876 + 842609326 + 19360062527, \cdots$$

$$\{1,\ 196883,\ 21296876,\ 842609326,\ 18538750076,\ 19360062527 \cdots\}$$

Dimensions of irreducible representations

Phenomenon of monstrous moonshine has been solved mathematically in early 1990's using the technology of vertex operator algebra in string theory

However, we still do not have a 'simple' explanation of this phenomenon.