HVP contribution to \((g - 2)_{\mu}\) from lattice QCD including electromagnetic corrections

* motivations

* application of the RM123 approach to the evaluation of the e.m. and isospin breaking corrections to \(a_{\mu}^{\text{HVP}}\)

* results for the strange and charm contributions (connected diagrams only) and preliminary results for the light contribution

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**motivations**

* the muon magnetic anomaly $a_\mu = (g - 2)/2$ is measured to 0.5 ppm [Muon G-2 Coll. ’06]

* within the SM $a_\mu$ is known to 0.4 ppm [PDG ’16]

* tension with SM prediction at $\sim 3.5$ standard deviations:

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (27.6 \pm 8.0) \cdot 10^{-10} \quad [\text{Muon G-2 Coll. ’06}]$$

$$= (28.8 \pm 8.0) \cdot 10^{-10} \quad [\text{PDG ’16}]$$

* future experiments at FermiLab [E989] and J-PARC (E34) aim at a target precision of $\sim 2 \cdot 10^{-10}$

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storage ring (FNAL) | ultra cold muon beam (JPARC)
* largest uncertainties from Hadronic Vacuum Polarization (HVP) and Hadronic Light by Light (HLbL)

**HVP:**

\[ a^\text{HVP}_\mu (\alpha^2_e) = (692.3 \pm 4.2) \times 10^{-10} \]  
[Davier et al. '11]

\[ = (694.9 \pm 4.3) \times 10^{-10} \]  
[Hagiwara et al. '11]

**HLbL:**

\[ a^\text{HLbL}_\mu = (10.5 \pm 2.6) \times 10^{-10} \]  
[Hagiwara et al. '11, PDG '16]

* dispersion formalism much more involved for HLbL
\[ a_{\mu}^{HVP} \left( \alpha_{em}^3 \right) \quad \text{[Jegerlehner\&Nyffeler '09]} \]

\[ a_{\mu}^{HVP} \left( \alpha_{em}^3 \right) = \left( -9.84 \pm 0.07 \right) \cdot 10^{-10} \]

\[ a_{\mu}^{HLbL} = (10.5 \pm 2.6) \cdot 10^{-10} \quad \text{[Hagiwara et al. '11, PDG '16]} \]

\[ \delta a_{\mu}^{HVP} \sim (3.9 \pm 0.1) \cdot 10^{-10} \]

usually not included in \( a_{\mu}^{HVP} \left( \alpha_{em}^3 \right) \), but taken into account in the experimental determination of \( a_{\mu}^{HVP} \left( \alpha_{em}^2 \right) \)

[mainly \( e^+ e^- \rightarrow \pi^+ \pi^- \gamma \)]
***** lattice QCD calculations of $a_{\mu}^{HVP}(\alpha_{em}^2)$ and $\delta a_{\mu}^{HVP}$ are mandatory *****

* several lattice QCD calculations of $a_{\mu}^{HVP}(\alpha_{em}^2)$ are available: RBC/UKQCD, HPQCD, ETMC, ...

* no lattice results available for the e.m. corrections to the HVP $\delta a_{\mu}^{HVP}$

* during the last few years the issue of the electromagnetic corrections to hadron observables has been addressed on the lattice:

  - hadron spectrum [BMW, RM123, RBC/UKQCD]: no IR divergencies (various techniques)
  - leptonic hadron decays [RM123 + Soton PRD ’15]: presence of IR divergencies

***** first application of the RM123 approach to the IB corrections for $a_{\mu}^{HVP}$ *****

the RM123 approach is based on a double expansion in the “small” parameters $\alpha_{em}$ and $(m_d - m_u) \sim 1\%$ [JHEP ’12, PRD ’13]

only isospin-symmetric QCD gauge configurations are required
\[
a_{\mu}^{HVP} = 4\alpha_{em}^2 \int_0^\infty dQ^2 \frac{1}{m_\mu^2} f \left( \frac{Q^2}{m_\mu^2} \right) \left[ \Pi(Q^2) - \Pi(0) \right]
\]

\( Q = \) Euclidean 4-momentum

kinematical kernel: \( f(s) = \frac{1}{s} \sqrt{\frac{s}{4+s}} \left( \frac{\sqrt{4+s} - \sqrt{s}}{\sqrt{4+s} + \sqrt{s}} \right)^2 \) peaked at \( s = \sqrt{5} - 2 = 0.24 \)

\( \Pi(Q^2) \) = HVP form factor appearing in the covariant decomposition of the HVP tensor:

\[
\Pi_{\mu\nu}(Q) = \int d^4x \, e^{iQ\cdot x} \langle J_\mu(x)J_\nu(0) \rangle = \left[ \delta_{\mu\nu}Q^2 - Q_\mu Q_\nu \right] \Pi(Q^2)
\]

\[
J_\mu(x) = \sum_{f=u,d,s,c,...} q_f \bar{\psi}_f(x) \gamma_\mu \psi_f(x)
\]

[Bernecker&Meyer EPJA '11]

\[
\Pi(Q^2) - \Pi(0) = 2 \int_0^\infty dt \, V(t) \left[ \frac{\cos(Qt) - 1}{Q^2} + \frac{1}{2} t^2 \right]
\]

\[
V(t) \equiv \frac{1}{3} \sum_{i=1,2,3} \int d\vec{x} \, \langle J_i(\vec{x},t)J_i(0) \rangle = \text{vector correlator}
\]

calculable on the lattice
**hybrid method**: lattice evaluation of the HVP form factor $\Pi(Q^2)$ as FT (periodic momenta)

- low $Q^2$: parameterization using lattice data (Padé approximants, conformal polynomials, VMD, ...)

- mid $Q^2$: direct integration of the lattice data in $Q^2$

- high $Q^2$: matching with pQCD

* alternatively, the sine-cardinal method: direct FT at arbitrary $Q$ [exp. suppressed finite-T effects]

* time moments [HPQCD '14]

- HVP form factor $\Pi(Q^2)$ reconstructed from the time behavior of the vector correlator $V(t)$

\[
\Pi(Q^2) - \Pi(0) = \sum_{j=1}^{\infty} \Pi_j Q^{2j}
\]

\[
\Pi_j = (-)^{j+1} \frac{V_{2j+2}}{(2j+2)!}
\]

\[
V_{2j+2} = a^4 \sum_t t^{2j+2} V(t)
\]

- few moments ($j \leq 4$) and Padé approximants

ETMC '14, RBC/UKQCD '16, HPQCD '14 and '16, CLS '17...
\[ a_{\mu}^{HVP} = 4 \alpha_{em}^2 \int_0^\infty dt \ \tilde{f}(t) V(t) \]

\[ V(t) = \frac{1}{3} \sum_{i=1,2,3} \int d\vec{x} \left\langle \sum_f \bar{\psi}_f(\vec{x},t) \gamma_i \psi_f(\vec{x},t) \sum_{f'} \bar{\psi}_{f'}(0) \gamma_i \psi_{f'}(0) \right\rangle \]

\[ \tilde{f}(t) \equiv 2 \int_0^\infty dQ^2 \frac{1}{m^2_\mu} f(Q^2 \cos(Qt) - 1 + \frac{1}{2} t^2) \]

\[ f(s) = \frac{1}{s} \sqrt{\frac{s}{4+s}} \left( \frac{\sqrt{4+s} - \sqrt{s}}{\sqrt{4+s} + \sqrt{s}} \right)^2 = \text{kinematical kernel} \]

\[ \tilde{f}(t) \] is proportional to \( t^4 \) at small \( t \) and to \( t^2 \) at large \( t \)

\[ \tilde{f}(t) = \frac{1}{36} m^2_\mu t^4 + O(t^4), \quad \tilde{f}(t) \xrightarrow{t \to \infty} \frac{1}{2} t^2 \]

* enhancement of the large time distance behavior of the vector correlator \( V(t) \)
- we will limit ourselves to **connected diagrams** only (each quark flavor \( f \) contributes separately)

- the vector correlator \( V(t) \) can be calculated at discretized values of \( t \) between \( t = 0 \) and \( t = T / 2 \)

\[
a_{\mu}^{HVP} = a_{\mu}^{HVP}(<) + a_{\mu}^{HVP}(>)
\]

\[
a_{\mu}^{HVP}(<) = 4\alpha_{em}^2 q_f^2 \sum_{\bar{t} = 0}^{T_{data}} w(\bar{t}) \bar{f}(\bar{t}) \bar{V}(\bar{t})
\]

\[
a_{\mu}^{HVP}(>) = 4\alpha_{em}^2 q_f^2 \sum_{\bar{t} = T_{data} + 1}^{\infty} w(\bar{t}) \bar{f}(\bar{t}) \frac{\bar{G}_V}{2\bar{M}_V} e^{-\bar{M}_V \bar{t}}
\]

\[
\bar{f}(\bar{t}) \equiv \frac{4}{\bar{m}_{\mu}^2} \int d\omega \sqrt{\frac{1}{4 + \omega^2}} \left( \frac{\sqrt{4 + \omega^2} - \omega}{\sqrt{4 + \omega^2} + \omega} \right)^2 \left[ \frac{\cos(\omega \bar{m}_{\mu} \bar{t}) - 1}{\omega^2} + \frac{1}{2} \bar{m}_{\mu}^2 \bar{t}^{-2} \right]
\]

\[
\bar{G}_V \equiv \frac{1}{3} \sum_{i=1,2,3} |\langle 0|J_i(0)|V \rangle|^2 = \text{coupling constant with the ground-state}
\]

\[
w(\bar{t}) = \text{weights of the (cubic) Simpson formula}
\]

\( t \leq T_{data} < T/2 \) (to avoid backward signals)

\( t > T_{data} > t_{\text{min}} \) (ground-state dominance)

Overlined quantities are in lattice units

\( \bar{m}_{\mu} = a m_{\mu}, \bar{t} = t / a, \bar{M}_V = a M_V, \ldots \)
the kernel function $f(t)$

\[ f(t) = \frac{1}{36} \left( \frac{m_\mu^2}{t^2} + O\left( \frac{1}{t^4} \right) \right), \quad \lim_{t \to \infty} \frac{f(t)}{t^2} = \frac{1}{2} \]

* some sensitivity to the lattice spacing

* sensitivity to the mass of the lepton:

enhancement of the large time distances for light leptons

$T/2 = 48$

$\bar{m}_\mu = 105.7$ MeV

$a \approx 0.089$ fm

$a \approx 0.082$ fm

$a \approx 0.062$ fm

$m_\mu = 105.7$ MeV

$T/2 = 48$

$a \approx 0.062$ fm

$m_\mu = 105.7$ MeV

Saturday, May 20, 17
avoid the mixing of strange and charm quarks in the valence sector we adopted a non-unitary set up in which the valence strange and charm quarks are regularized as Osterwalder-Seiler fermions, while the valence up and down quarks have the same action of the sea. Working at maximal twist such a setup guarantees an automatic $O(a)$-improvement. We considered three values of the inverse bare lattice coupling and different lattice volumes, as shown in Table 1, where the number of configurations analyzed ($N_{cfg}$) correspond to 20 trajectories. A different lattice spacing, different values of the light sea quark masses have been considered. The light valence and sea quark masses are always taken to be degenerate. The bare masses of both the strange ($a_{µs}$) and the charm ($a_{µc}$) valence quarks are obtained, at each, using the physical strange and charm masses and the mass renormalization constants (method M1) determined in Ref. [3].

\[
a = \{0.0885, 0.0815, 0.0619\} \text{ fm at } \beta = \{1.90, 1.95, 2.10\},
\]

pion masses in the range 210 - 450 MeV

<table>
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<th>ensemble</th>
<th>$\beta$</th>
<th>$V/a^4$</th>
<th>$a_{µs} = a_{µ\ell}$</th>
<th>$a_{µ\sigma}$</th>
<th>$a_{µ\delta}$</th>
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Table 1: Values of the simulated sea and valence quark bare masses for the 16 ETMC gauge ensembles with $N_f = 2+1+1$ dynamical quarks adopted in this work (see Ref. [3]). The values of the strange and charm quark bare masses $a_{µs}$ and $a_{µc}$, given for each gauge ensemble, correspond to the physical strange and charm quark masses determined in Ref. [3].

* correlators calculated in the PRACE project on “QED corrections to meson decay rates in LQCD”

* number of stochastic sources per gauge configuration not optimal for the vector correlator
at maximal twist: \( J^L_\mu (x) = q_f Z_V \bar{\psi}_f(x) \gamma_\mu \psi_f(x) \) \( Z_V = \) vector RC

thanks to the findings of Burger et al. [ETMC] JHEP '15

\[
\int d^4x \, e^{iQx} \left\langle J^L_\mu(x) J^L_\nu(0) \right\rangle = \Pi_{\mu\nu}(Q) + \delta_{\mu\nu}Z_1 \left( \frac{1}{a^2} - S_6 + \frac{S_5^2}{2} \right) + \delta_{\mu\nu}Z_m m^2 \\
+ \delta_{\mu\nu}Z_L Q^2 + \delta_{\mu\nu}Z_T \left( \delta_{\mu\nu}Q^2 - Q_\mu Q_\nu \right) + O(a^2)
\]

\( \Pi_{\mu\nu}(Q) \) = transverse polarization tensor

\( S_{5(6)} = \) v.e.v. of dim-5(6) terms of Symanzik expansion of twisted-mass action

\( Z_{1,m,L,T} \) = non-perturbative mixing coefficients

\[
\int d^4x \left( e^{iQx} - 1 \right) \left\langle J^L_\mu(x) J^L_\nu(0) \right\rangle = \Pi_{\mu\nu}(Q) + \delta_{\mu\nu}Z_L Q^2 + \delta_{\mu\nu}Z_T \left( \delta_{\mu\nu}Q^2 - Q_\mu Q_\nu \right) + O(a^2)
\]

\[
\Pi_{\mu\nu}(Q) = \left[ \delta_{\mu\nu}Q^2 - Q_\mu Q_\nu \right] \Pi(Q^2)
\]

\( Q = \{ Q, \bar{Q} \} \)

\[
\int dt \left( \frac{\cos(Qt) - 1}{Q^2} \right) \int d\tilde{x} \frac{1}{3} \sum_{i=1,2,3} \left\langle J^L_i(\tilde{x}, t) J^L_i(0) \right\rangle = \Pi(Q^2) + (Z_L + Z_T) + O(a^2)
\]

\[
2 \int_0^\infty dt \left( \frac{\cos(Qt) - 1}{Q^2} + \frac{1}{2} t^2 \right) V(t) = \Pi(Q^2) - \Pi(0) + O(a^2)
\]
matching with pQCD

once-subtracted dispersion relation: \[
\Pi_R (Q^2) = \Pi (Q^2) - \Pi (0) = \frac{1}{12 \pi^2} \int_0^\infty ds \frac{Q^2}{s(s+Q^2)} R_{\text{had}}^0 (s)
\]

\[
\sigma^{e^+e^-\rightarrow \text{hadrons}} (s) = \frac{4 \pi \alpha_{\text{em}}^2}{s} R_{\text{had}} (s)
\]

\[
V (t) = \int_{-\infty}^{\infty} dQ \ e^{-iQt} Q^2 \Pi_R (Q^2) = \frac{1}{24 \pi^2} \int_{4 M_\pi^2}^\infty ds \ \sqrt{s} \ e^{-\sqrt{s} t} R_{\text{had}}^0 (s) + \left[ \ldots \infty \delta (t), \ \delta'' (t) \ldots \right]
\]

pQCD behavior @ LO: \[
R_{\text{pQCD}} (s) = q_f^2 N_c \sqrt{1 - \frac{4 m^2}{s}} \left(1 + \frac{2 m^2}{s}\right) \Theta \left[ s - 4 m^2 \right] + O (\alpha_s)
\]
m = on-shell quark mass

\[
V_{\text{pQCD}} (t) \xrightarrow{t \to 0} \frac{q_f^2 N_c}{6 \pi^2} \left\{ \frac{1}{t^3} e^{-2 m t} \left(1 + 2 m t + 2 m^2 t^2\right) + 4 m^3 \int_{1}^{\infty} dy y^2 \left\{ \sqrt{1 - \frac{1}{y^2} \left(1 + \frac{1}{2 y^2}\right)} - 1 \right\} e^{-2 m y t} \right\}
\]

\[
\xrightarrow{m t \ll 1} \frac{q_f^2}{2 \pi^2} \frac{1}{t^3} + O (m^3)
\]

massless limit
A30.32, B25.32, D20.48 share a common value of the light-quark mass ($m_l \sim 12$ MeV) and differ in the value of the lattice spacing ($a \sim 0.089, 0.082, 0.062$ fm)

**dashed green lines:** massless pQCD behavior ($\sim 1 / t^3$)

**solid green line:** massive pQCD behavior for the charm case

**quark-hadron duality @ work:** up to $t \sim 1$ fm $>> 1 / \Lambda_{QCD}$

* possible issue: contributions from $t < a$ (or equivalently $Q^2 > 1 / a^2$)

thanks to our values of the lattice spacing, which correspond to $Q^2 \geq 5$ GeV$^2$, the contribution from $t < a$ turns out always to be well within the uncertainties
FIG. 4: The vector correlator $V(t)/q_f^2$ (in physical units) in the case of the light (left panel) and strange (right panel) quarks for the ETMC gauge ensembles specified in the inset, which share an approximate common value of the light-quark mass $m_\ell \approx 12$ MeV and differ in the values of the lattice spacing.

FIG. 5: Effective mass of the vector correlator $\overline{V}(t)$ in the case of the strange (left panel) and charm (right panel) contributions for the ETMC gauge ensembles specified in the insets.

ground-state identification

number of stochastic sources per gauge configuration not optimal for the vector correlator

not OK for the light contribution

~ OK for the strange contribution

OK for the charm contribution

$$\tilde{M}_{\text{eff}} = \log \frac{\overline{V}(t)}{\overline{V}(t-1)}$$
strange contribution

\[ a_\mu^{HVP} = a_\mu^{HVP}(<) + a_\mu^{HVP}(>) \]

\[ a_\mu^{HVP}(<) = 4\alpha_{em} q_f^2 \sum_{\bar{t} = 0}^{\bar{T}_{data}} f(\bar{t}) \bar{V}(\bar{t}) \]

\[ a_\mu^{HVP}(>) = 4\alpha_{em} q_f^2 \sum_{\bar{t} = \bar{T}_{data} + 1}^{\infty} f(\bar{t}) \frac{G_{V}}{2M_{V}} e^{-M_{V}\bar{t}} \]

\((t_{\text{min}}, t_{\text{max}}) = \text{ground-state dominance}\)

**four choices:**

\[ \bar{T}_{data} = \left\{ \left(\bar{t}_{\text{min}} + 2\right), \left(\bar{t}_{\text{min}} + \bar{t}_{\text{max}}\right)/2, \left(\bar{t}_{\text{max}} - 2\right), \left(\bar{T}/2 - 4\right) \right\} \]

* the sum is independent on \( T_{data} \)

in what follows \( \bar{T}_{data} = \bar{T}/2 - 4 \)

\[ a_\mu^{HVP}(>) / a_\mu^{HVP} < 1\% \]

and within the statistical errors
instead of $am^\text{phys}_\mu$:

$$am^{ELM}_\mu = aM_V M^{\text{phys}}_V$$

[ETMC JHEP '14]

- no need of the value of the lattice spacing (no sensitivity to the lattice scale setting)

- sensitivity to the precision of the vector meson mass $aM_V$

much better precision with the ELM procedure
strange contribution

charm contribution

\[ a_{\mu}^s = A_0^s \left[ 1 + A_1^s \xi + D^s \alpha^2 + F^{s,c} \frac{e^{-M_\pi L}}{M_\pi L} \right] \]

\[ \xi = \frac{M_\pi^2}{(4\pi f_0)^2} \]
fitting functions: \[ a_{\mu}^{s,c} = A_0^{s,c} \left[ 1 + A_1^{s,c} \xi + D^{s,c} a^2 + F^{s,c} \frac{\xi e^{-M_\pi L}}{M_\pi L} \right] \]
\[ \xi = \frac{M_\pi^2}{(4\pi f_0)^2} \]

error budget: \[ a^2: \text{with/without the ELM procedure} \]
\[ FSE: F = 0 \text{ or } F \neq 0 \]
\[ chir: A_1 = 0 \text{ or } A_1 \neq 0 \]

input: uncertainties due to the scale setting, to the physical quark masses, ...

stat+fit: statistical + fitting procedure errors

strange contribution:
\[ a_{\mu}^s (\text{phys}) = \left( 53.1 \pm 1.6_{\text{stat+fit}} \pm 1.5_{\text{input}} \pm 1.3_{a^2} \pm 0.2_{FSE} \pm 0.1_{\text{chiral}} \right) \cdot 10^{-10} \]
\[ = (53.1 \pm 2.5) \cdot 10^{-10} \]

\[ a_{\mu}^s (\text{phys}) = (53.41 \pm 0.59) \cdot 10^{-10} \quad \text{[HPQCD '14, } N_f = 2 + 1 + 1 \text{]} \]
\[ = (53.1 \pm 0.9^{+0.1}_{-0.3}) \cdot 10^{-10} \quad \text{[RBC/UKQCD '16, } N_f = 2 + 1 \text{]} \]
\[ = (51.1 \pm 1.7 \pm 0.4) \cdot 10^{-10} \quad \text{[CLS/Mainz '17, } N_f = 2 \text{]} \]

charm contribution:
\[ a_{\mu}^c (\text{phys}) = \left( 14.75 \pm 0.42_{\text{stat+fit}} \pm 0.36_{\text{input}} \pm 0.10_{a^2} \pm 0.03_{FSE} \pm 0.01_{\text{chiral}} \right) \cdot 10^{-10} \]
\[ = (14.75 \pm 0.56) \cdot 10^{-10} \]

\[ a_{\mu}^c (\text{phys}) = (14.42 \pm 0.39) \cdot 10^{-10} \quad \text{[HPQCD '14, } N_f = 2 + 1 + 1 \text{]} \]
\[ = (14.3 \pm 0.2 \pm 0.1) \cdot 10^{-10} \quad \text{[CLS/Mainz '17, } N_f = 2 \text{]} \]
For each quark flavor $f$ one has:

$$\delta V(t) = \delta V^{self}(t) + \delta V^{exch}(t) + \delta V^{tad}(t) + \delta V^{PS}(t) + \delta V^{S}(t)$$

\begin{align*}
\delta V^{self}(t) + \delta V^{exch}(t) &= \frac{4\pi \alpha_{em}}{3} \sum_{i=1,2,3} \sum_{x,y_1,y_2} \langle 0 | T \left\{ J^\dagger_i(\vec{x}, t) \sum_{\mu} J^C_i(y_1) J^C_\mu(y_2) J_i(0) \right\} | 0 \rangle , \\
\delta V^{tad}(t) &= \frac{4\pi \alpha_{em}}{3} \sum_{i=1,2,3} \sum_{x,y} \langle 0 | T \left\{ J^\dagger_i(\vec{x}, t) \sum_{\nu} T_\nu(y) J_i(0) \right\} | 0 \rangle , \\
\delta V^{PS}(t) &= \frac{\delta m^{crit}_f}{3} \sum_{i=1,2,3} \sum_{x,y} \langle 0 | T \left\{ J_i^\dagger(\vec{x}, t) i\bar{\psi}_f(y)\gamma_5\psi_f(y) J_i(0) \right\} | 0 \rangle , \\
\delta V^{S}(t) &= -\frac{m_f}{3Z_f Z_m} \sum_{i=1,2,3} \sum_{\vec{x},y} \langle 0 | T \left\{ J_i^\dagger(\vec{x}, t) \bar{\psi}_f(y)\psi_f(y) J_i(0) \right\} | 0 \rangle ,
\end{align*}

lattice conserved current

$$J^C_\mu(x) = q_f \frac{1}{2} \left[ \bar{\psi}_f(x)(\gamma_\mu - i\tau^3\gamma_5)U_\mu(x)\psi_f(x + a\hat{\mu}) + \bar{\psi}_f(x + a\hat{\mu})(\gamma_\mu + i\tau^3\gamma_5)U_\mu^\dagger(x)\psi_f(x) \right].$$

\begin{align*}
\delta m^{crit}_f \propto \alpha_{em} q_f^2 &= \text{e.m. shift of the critical mass (breaking of chiral symmetry)} \\
\frac{1}{Z_m} = Z_p(\overline{MS}, \mu) \rightarrow \text{mass RC (maximal twist LQCD)} \\
\frac{1}{Z_f}(\overline{MS}, \mu) = \frac{\alpha_{em} q_f^2}{4\pi} \left[ 6 \log(a\mu) - 22.596 \right] \rightarrow \text{mass RC (LO in QED)}
\end{align*}
* separation of QCD and QED effects is prescription dependent [see Gasser et al. EPJC ’03]

- mass anomalous dimensions in QCD and QCD+QED are different
- matching possible only at a given renormalization scale $\mu^*$

\[ \hat{m}_f(MS, \mu^*) = m_f(MS, \mu^*) \quad \text{our choice: } \mu^* = 2 \text{ GeV} \]

renormalized mass in QCD+QED $\quad \longrightarrow$ renormalized mass in QCD only

- bare mass difference: $\hat{\mu}_f - \mu_f = \frac{\hat{m}_f}{\hat{Z}_{m,f}} - \frac{m_f}{Z_m}$

\[ \frac{1}{\hat{Z}_{m,f}} = \frac{1}{Z_m} \left( 1 + \frac{1}{Z_f} \right) \implies \hat{\mu}_f - \mu_f \approx \frac{1}{Z_m} \left( \hat{m}_f - m_f \right) + \frac{1}{Z_m Z_f} m_f \]

$O(\alpha_{em})$

\[ (\hat{\mu}_f - \mu_f) \times \text{ (insertion of scalar density) } \xrightarrow{\mu = \mu^*} \frac{m_f}{Z_m Z_f Z_{fact}^f} \]

for details see arXiv:1704.06561 (to appear in PRD)

LO in QED: \[ \frac{1}{Z_f}(MS, \mu^*) = \frac{\alpha_{em} q_f^2}{4\pi} \left[ 6 \log(a\mu^*) - 22.596 \right] \]

[Aoki et al. PRD ’98]

\[ Z_m^{fact} = 1 \pm 0.2 \quad \longrightarrow \quad \text{“factorization approximation” between QED and QCD vertex corrections} \]
* lattice formulations of QCD which break chiral symmetry \( \Rightarrow \) additive mass renormalization

(twisted) Wilson term \( \Rightarrow \) power-divergent (1/a) mass counterterm \( \Rightarrow \) critical mass \( m_{\text{crit}} \)

QED contribution: \( \delta m_f^{\text{crit}} = \) e.m. shift of the critical mass

vector WT identity: \( \delta m_f^{\text{crit}} = -\frac{\nabla_0 \left[ \delta V_f^J(t) + \delta V_f^T(t) \right]}{\nabla_0 \delta V_f^{P_f}(t)} \)

\[
\begin{align*}
\delta V_f^J(t) &= \frac{1}{L^5} \sum_{\bar{x}, \gamma_1, \gamma_2} \langle 0 | T \{ \bar{\psi}_{f}(\bar{x}, t) \gamma_0 \psi_f(\bar{x}, t) J^C_{\mu}(y_1) J^C_{\mu}(y_2) i \bar{\psi}_f(0) \gamma_5 \psi_f(0) \} | 0 \rangle \\
\delta V_f^T(t) &= \frac{1}{L^5} \sum_{\bar{x}, y} \langle 0 | T \{ \bar{\psi}_{f}(\bar{x}, t) \gamma_0 \psi_f(\bar{x}, t) T(y) i \bar{\psi}_f(0) \gamma_5 \psi_f(0) \} | 0 \rangle \\
\delta V_f^{P_f}(t) &= \frac{1}{L^3} \sum_{\bar{x}, y} \langle 0 | T \{ \bar{\psi}_{f}(\bar{x}, t) \gamma_0 \psi_f(\bar{x}, t) i \bar{\psi}_f(y) \gamma_5 \psi_f(y) i \bar{\psi}_f(0) \gamma_5 \psi_f(0) \} | 0 \rangle
\end{align*}
\]

for details see arXiv:1704.06561 (to appear in PRD)
* e.m. corrections to the renormalization of the (local) e.m. current:

we have adopted a maximally twisted-mass setup with quarks and anti-quarks regularized with opposite values of the Wilson r-parameter: the vector current renormalizes multiplicatively with \( Z_A \)

\[
Z_A = Z_A^{(0)} + \alpha_{em} Z_A^{(1)} + O(\alpha_{em}^2) = Z_A^{(0)}\left(1 - 2.51406 \alpha_{em} q_f^2 Z_A^{\text{fact}}\right) + O(\alpha_{em}^2)
\]

\( Z_A^{\text{fact}} = 1 \pm 0.2 \quad \text{“factorization approximation” between QED and QCD vertex corrections with violations at 20% level (preliminary estimate)} \)

* addition of a further contribution:

\[
\delta V(t) = \delta V^{\text{self}}(t) + \delta V^{\text{exch}}(t) + \delta V^{\text{tad}}(t) + \delta V^{PS}(t) + \delta V^S(t) + \delta V^{Z_A}(t)
\]

\[
\delta V^{Z_A}(t) = -2.51406 \alpha_{em} q_f^2 Z_A^{\text{fact}} V(t)
\]
* neglect of the electric sea-quark charges: \( q_{f}^{\text{sea}} = 0 \)

- in the RM123 approach \( \Rightarrow \) neglect of (noisy) fermionic disconnected diagrams

\[
q_{f} \sum_{f'} q_{f'}^{\text{sea}} \quad \sum_{f'} q_{f'}^{\text{sea}} q_{f'}^{\text{sea}} \quad \sum_{f_{1}, f_{2}} q_{f_{1}}^{\text{sea}} q_{f_{2}}^{\text{sea}}
\]
\[
\delta a_{\mu}^{HVP} = \delta a_{\mu}^{HVP} (<) + \delta a_{\mu}^{HVP} (>)
\]

\[
\delta a_{\mu}^{HVP} (<) = 4\alpha_{em}^3 q_f^4 \sum_{\bar{t}=0}^{N_{data}} f(\bar{t}) \delta \bar{V}(\bar{t})
\]

directly from lattice data

\[
\delta a_{\mu}^{HVP} (>) = 4\alpha_{em}^3 q_f^4 \sum_{\bar{t}=N_{data}+1}^{\infty} f(\bar{t}) \delta \left[ \frac{\bar{G}_V}{2\bar{M}_V} e^{-\bar{M}_V\bar{t}} \right]
\]

analytic representation

\[
= 4\alpha_{em}^3 q_f^4 \sum_{\bar{t}=N_{data}+1}^{\infty} f(\bar{t}) \frac{\bar{G}_V}{2\bar{M}_V} \left[ \frac{\delta \bar{G}_V}{\bar{G}_V} - \frac{\delta \bar{M}_V}{\bar{M}_V} (1 + \bar{M}_V\bar{t}) \right] e^{-\bar{M}_V\bar{t}}
\]

RM123 approach

\[ [\text{JHEP '12, PRD '13}] \]

\[
\frac{\delta \bar{V}(\bar{t})}{\bar{V}(\bar{t})} \to \frac{\delta \bar{G}_V}{\bar{G}_V} - \frac{\delta \bar{M}_V}{\bar{M}_V} (1 + \bar{M}_V\bar{t})
\]
\[
\delta a_{\mu}^{HVP} (<) = 4\alpha_{em}^3 q_f^4 \sum_{\bar{t}=0}^{N_{data}} f(\bar{t}) \delta \bar{V} (\bar{t})
\]
directly from lattice data

\[
\delta \bar{V} (\bar{t}) = \delta \bar{V}^{self}(\bar{t}) + \delta \bar{V}^{tad}(\bar{t}) + \delta \bar{V}^{PS}(\bar{t}) + \delta \bar{V}^{exch}(\bar{t}) + \delta \bar{V}(\bar{t})^{scalar} + \delta \bar{V}(\bar{t})^{ZA}
\]

- contributions with different signs
- partial cancellations among the various terms

\[
Z_m^{fact} = Z_A^{fact} = 1
\]

Saturday, May 20, 17
strange contribution

\( \delta a_{\mu}^{HVP} = \delta a_{\mu}^{HVP}(<) + \delta a_{\mu}^{HVP}(>) \)

\[
\delta a_{\mu}^{HVP}(<) = 4\alpha_{em} q_{f}^{2} \sum_{\bar{t}=0}^{T_{\text{data}}} \bar{f} (\bar{t}) \delta \bar{V}(\bar{t})
\]

\[
\delta a_{\mu}^{HVP}(>) = 4\alpha_{em} q_{f}^{2} \sum_{\bar{t}=T_{\text{data}}+1}^{\infty} \bar{f} (\bar{t}) \delta \left[ \frac{G_{V}}{2M_{V}} e^{-\bar{M}_{V} \bar{t}} \right]
\]

\((t_{\text{min}}, t_{\text{max}}) = \) ground-state dominance

four choices:

\[
\bar{T}_{\text{data}} = \left\{ (\bar{t}_{\text{min}} + 2), \frac{\bar{t}_{\text{min}} + \bar{t}_{\text{max}}}{2}, \right. \\
\left. \frac{\bar{t}_{\text{max}} - 2}{2}, \frac{\bar{T} / 2 - 4}{2} \right\}
\]

* the sum is independent on \( T_{\text{data}} \)

\[
\delta a_{\mu}^{HVP}(>) / \delta a_{\mu}^{HVP} < 2 \%
\]

and well within the statistical errors

\begin{array}{|c|c|c|c|c|}
\hline
s & \bar{s} & (\bar{t}_{\text{min}} + 2) & (\bar{t}_{\text{min}} + \bar{t}_{\text{max}})/2 & (\bar{t}_{\text{max}} - 2) & (\bar{T} / 2 - 4) \\
\hline
\hline
\text{ensemble A40.24} & & & & & \\
\hline
\delta a_{\mu}^{had}(<) & -2.05 (22) & -2.13 (23) & -2.17 (24) & -2.24 (27) & \\
\delta a_{\mu}^{had}(>) & -0.25 (14) & -0.17 (11) & -0.13 (10) & -0.05 (6) & \\
\delta a_{\mu}^{had} & -2.30 (31) & -2.30 (30) & -2.29 (30) & -2.29 (30) & \\
\hline
\text{ensemble A30.32} & & & & & \\
\hline
\delta a_{\mu}^{had}(<) & -2.13 (15) & -2.45 (20) & -2.52 (25) & -2.52 (26) & \\
\delta a_{\mu}^{had}(>) & -0.36 (18) & -0.07 (8) & -0.01 (2) & -0.01 (1) & \\
\delta a_{\mu}^{had} & -2.49 (30) & -2.52 (27) & -2.53 (26) & -2.53 (26) & \\
\hline
\text{ensemble B25.32} & & & & & \\
\hline
\delta a_{\mu}^{had}(<) & -2.78 (18) & -3.26 (20) & -3.54 (24) & -3.55 (25) & \\
\delta a_{\mu}^{had}(>) & -0.76 (20) & -0.28 (12) & -0.05 (4) & -0.03 (2) & \\
\delta a_{\mu}^{had} & -3.54 (29) & -3.54 (27) & -3.59 (26) & -3.58 (26) & \\
\hline
\text{ensemble D15.48} & & & & & \\
\hline
\delta a_{\mu}^{had}(<) & -2.44 (24) & -3.17 (41) & -3.31 (44) & -3.30 (43) & \\
\delta a_{\mu}^{had}(>) & -0.82 (21) & -0.14 (7) & -0.02 (2) & -0.01 (1) & \\
\delta a_{\mu}^{had} & -3.26 (41) & -3.30 (44) & -3.33 (44) & -3.31 (43) & \\
\hline
\end{array}
- in the ratios $\delta a_{\mu}^{HVP}/a_{\mu}^{HVP}$ various systematics cancel out

- errors dominated by the uncertainties in $Z_m^{fact}$ and $Z_A^{fact}$

- the ELM procedure does not improve the precision

- no FSEs are visible

fitting functions:

$$\delta a_{\mu}^{s,c} = \delta A_0^{s,c} \left[ 1 + \delta A_1^{s,c} M_\pi^2 + \delta D^{s,c} a^2 \right]$$

need of a non-perturbative determination of $Z_m^{fact}$ and $Z_A^{fact}$
at the physical pion mass and in the continuum limit:

\[ \frac{\delta a^s}{a^s} = -0.07 \ (3) \% \quad \frac{\delta a^c}{a^c} = -0.30 \ (7) \% \]

using our lowest order results

\[ a^s = (53.1 \pm 2.5) \cdot 10^{-10} \quad a^c = (14.75 \pm 0.56) \cdot 10^{-10} \]

we have

\[ \delta a^s = -(3.9 \pm 1.4) \cdot 10^{-12} \quad \delta a^c = -(4.4 \pm 1.0) \cdot 10^{-12} \]

for the strange and charm contributions: \textbf{e.m. corrections} \ll \textbf{uncertainties of the lowest order}
* number of stochastic sources / gauge configuration

\[
\text{variance: } \propto e^{-2M_\pi t}
\]

\[
\text{noise / signal: } \propto e^{(M_\rho - M_\pi)t}
\]

similar to the nucleon case \[\text{[Parisi, Lepage]}\]

- thanks to improvements in the Dirac inverter (DD\(\alpha\)AMG versus CG) we plan to reach 160 sources / gauge conf. for all ETMC ensembles (presently only 4 out of 16)
- FSEs are clearly visible

- the ELM procedure makes the pion mass dependence milder, but increases the statistical uncertainty
Finite Size Effects

* infinite volume limit of two-pion states [Meyer '11]

\[
V_{2\pi}(t) \xrightarrow{L \to \infty} \frac{1}{24\pi^2} \int ds \sqrt{s} e^{-\sqrt{s}t} R_{2\pi}(s) = \frac{1}{24\pi^2} \int ds \sqrt{s} \left(1 - \frac{4M_{\pi}^2}{s}\right)^{\frac{3}{2}} e^{-\sqrt{s}t} |F_{\pi}(s)|^2
\]

time-like pion form factor

* the case of finite volume [Meyer '11]

\[
V_{2\pi}(t; L) = \sum_n |A_n|^2 e^{-\omega_n t}
\]
\[\omega_n \equiv 2\sqrt{M_{\pi}^2 + k_n^2}\]

- interacting pions [Luscher '91]

\[
k_n : \quad \delta_{11}(k_n) + \phi\left(\frac{k_n L}{2\pi}\right) = n\pi
\]
\[\delta_{11} = \text{scattering phase shift (p-wave, T=1)}\]
\[\phi = \text{known kinematic function}\]

\[
|A_n|^2 : \quad |F_{\pi}(\omega_n^2)|^2 = \left\{k_n \frac{\partial \delta_{11}(k_n)}{\partial k_n} + \frac{k_n L}{2\pi} \phi'\left(\frac{k_n L}{2\pi}\right)\right\} \frac{3\pi\omega_n^2}{2k_n^2} |A_n|^2
\]

- non-interacting pions [Francis et al '13]:

\[
k_n = \frac{2n\pi}{L}
\]

\[
V_{2\pi}(t; L) - V_{2\pi}(t; \infty) = \frac{M_{\pi}^4}{3\pi^2} t \sum_{\bar{n} \neq 0} \left\{K_2 \left[\frac{M_{\pi} \sqrt{L^2 \bar{n}^2 + 4t^2}}{M_{\pi}^2 (L^2 \bar{n}^2 + 4t^2)}\right] - \frac{1}{M_{\pi} L |\bar{n}|} \int dy K_0 \left[M_{\pi} y \sqrt{L^2 \bar{n}^2 + 4t^2}\right] \sinh\left[M_{\pi} L |\bar{n}| (y-1)\right]\right\}
\]

\[
a_{\mu}^{(2\pi)}(L) - a_{\mu}^{(2\pi)}(\infty) \xrightarrow{K_1(z) \gg 1} \sqrt{\frac{\pi}{2z}} e^{-z} \to \propto \left(M_{\pi} L\right)^2 e^{-M_{\pi} L}
\]
sizable effects expected at the physical pion for current lattice volumes [CLS/Mainz '17]
\[ a^{(u,d)}_\mu = \left( 572 \pm 11 \right) \cdot 10^{-10} \quad \text{[ETMC '14]} \]
\[ = \left( 598 \pm 11 \right) \cdot 10^{-10} \quad \text{[HPQCD '16]} \]
\[ = \left( 588 \pm 36 \right) \cdot 10^{-10} \quad \text{[CLS/Mainz '17]} \]
*light (u, d) contribution:*

Figure 1: Fermionic connected diagrams contributing at $O(\epsilon^2)$ and $O(m_d - m_u)$ to the IB corrections to meson masses: exchange (a), self energy (b), tadpole (c), pseudoscalar insertion (d) and scalar insertion (e).

\[
\frac{\delta a^{(u,d)}_\mu}{a^{(u,d)}_\mu}(\text{phys}) \sim 0.5 \pm 0.5 \%
\]
\[
\delta a^{(u,d)}_\mu(\text{phys}) \sim (3 \pm 3) \cdot 10^{-10}
\]

*more statistics is needed*
CONCLUSIONS

* the HVP contribution to the muon (g-2), $a_\mu^{HVP}$, is presently one of the major sources of the theoretical uncertainty

* in the past few years several lattice results have been obtained and many more are expected in the next future

* no lattice results are currently available for the e.m. and strong isospin-breaking corrections to $a_\mu^{HVP}$

* the RM123 approach is based on a double expansion in the “small” parameters $\alpha_{em}$ and $(m_d - m_u)$, and has been already applied successfully to the calculation of the charged meson masses and to the leptonic decay rates of pseudoscalar mesons

* using the time-momentum representation of the HVP form factor both the lowest-order $a_\mu(\alpha_{em}^2)$ and the e.m. and strong isospin-breaking corrections $a_\mu(\alpha_{em}^3)$ have been determined with good precision in the case of the strange and charm contributions:

\[
\begin{align*}
    a_\mu^s(\alpha_{em}^2) &= (53.1 \pm 2.5) \cdot 10^{-10} \\
    a_\mu^c(\alpha_{em}^2) &= (14.75 \pm 0.56) \cdot 10^{-10} \\
    a_\mu^s(\alpha_{em}^3) &= -(3.9 \pm 1.4) \cdot 10^{-12} \\
    a_\mu^c(\alpha_{em}^3) &= -(4.4 \pm 1.0) \cdot 10^{-12}
\end{align*}
\]
* in the case of u- and d-quarks more statistics is required and we have reached till now only preliminary results:

\[ a_{\mu}^{(u,d)}(\alpha_{em}^2) = (589 \pm 21_{\text{stat}}) \cdot 10^{-10} \]

\[ a_{\mu}^{(u,d)}(\alpha_{em}^3) = (3 \pm 3) \cdot 10^{-10} \]

---

**open issues**

* removal of the quenched QED approximation (effects of the sea-quark electric charges)

* non-diagonal flavor contributions to \( a_{\mu}^{\text{HVP}} \)

\[ J_\mu(x)J_\nu(y) \subseteq q_f \bar{\psi}_f(x) \gamma_\mu \psi_f(x) q_{f'} \bar{\psi}_{f'}(y) \gamma_\nu \psi_{f'}(y) \]

- preliminary lattice estimates [HPQCD, RBC/UKQCD, CLS/Mainz] are in the range - (1 – 2 %)
- new lattice results expected in the near future

---

evaluation of fermionic disconnected diagrams

work is in progress ...
- plateaux for $\delta m_{f}^{\text{crit}}$:

\begin{align*}
\beta &= 1.95, \ L/a = 32 \\
\beta &= 2.10, \ L/a = 48
\end{align*}

\[ a\mu_{\text{light}} = 0.0025 \]

\[ a\mu_{\text{light}} = 0.0015 \]
The sea quark propagators have been drawn in blue (and with a different line) and the isosymmetric vacuum polarization diagrams have not been displayed explicitly. By combining the previous expressions we find the elegant formula

\[ \delta C(t) / C^0(t) = \frac{M_\pi}{C_0} + \frac{M_\kappa}{C_1} + \ldots \]

where heavier quark masses do not contribute to observables that are shown in Eqs. (1a)-(1b) and to the variation of the strong coupling constant \( \alpha_s \). The use of the conserved e.m. current guarantees the absence of contact terms in the product \( \delta m_m \delta m_T / C^0 \) for all these reasons, the expression for the pion mass \( M_\pi \sim 300 \text{ MeV} \) and of the neutral pion \( M_\pi \sim 550 \text{ MeV} \) to within the discretization errors on \( \delta \alpha \) and masses.

Furthermore, as already stressed, the electric charge does not need to be renormalized at this order and, for all these reasons, the expression for the pion mass \( M_\pi \sim 300 \text{ MeV} \) and the flavor splitting can be considered a ''clean'' theoretical prediction.

\[ \delta C(t) / C^0(t) = \frac{M_\pi}{C_0} + \frac{M_\kappa}{C_1} + \ldots \]

On the other hand, the lattice calculation of the disconnected sea quark loop contributions explicitly.

Figure 1:

- (a) diagrams (1a)+(1b)
- (b) diagram (1e)
- (c) diagram (1e)
- (d) diagram (1e)
- (e) diagram (1e)
* local vector current with maximally twisted-mass setup

\[ V_\mu = Z \bar{\psi}_{f'}(\vec{x}, t) \gamma_\mu \psi_f(\vec{x}, t) \]

* two choices of the Wilson r-parameters:

\[ r_{f'} = r_f \quad Z = Z_V \]

\[ r_{f'} = -r_f \quad Z = Z_A \]

* the two currents differ by \( O(a^2) \) *