The MSR Mass and the $\mathcal{O}(\Lambda_{\rm QCD})$ Renormalon Sum Rule

Christopher Lepenik

in collaboration with
André H. Hoang, Ambar Jain, Vicent Mateu, Moritz Preisser, Ignazio Scimemi, Iain W. Stewart

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Outline

Motivation and Introduction - Renormalons

- When interactions in QFT are "weak" $\rightarrow$ perturbation theory allows to express observable $\mathcal{O}$ as series in the (renormalized) interaction strength α

$$\mathcal{O} = \sum_n c_n \alpha^n \,.$$

- Almost always divergent for any α with behavior

$$c_n \sim a^n n!\, n^b \qquad (n \rightarrow \infty)\,.$$

- Particular source of divergence: Renormalons.
  - Related to sensitivity of $\mathcal{O}$ to small and large momenta (long and short distance).

Motivation and Introduction - Renormalons

- Relation of observable $\mathcal{O}$ to its perturbative series?
- QFTs of phenomenological relevance:
  - Not possible to construct non-perturbatively from perturbative expansions and analyticity properties of Green's function.
  - Non-trivial, non-perturbative structure of vacuum and its excitations.
  - Non-perturbative power corrections.

Motivation and Introduction - Renormalons

- Important in QCD: Infrared Renormalons related to large distances/small momenta.
  - α_s grows with distance $\rightarrow$ sensitivity to regions where QCD is non-perturbative.
  - Leads to irreducible error and bad perturbative behavior.
  - Examples: Pole mass in QCD, soft function in effective field theories.

Motivation and Introduction - Renormalons

Let's consider the relation between the pole and $\overline{\rm MS}$ mass in more detail:

- Pole mass:
  - Absorbs all contributions from on-shell self energy diagrams, including contributions from energies < 1 GeV - clearly IR sensitive
    $$\not{p} - m_Q + \Sigma(m_Q^2) = \not{p} - m_Q^{\rm pole}\,.$$
  - Often appropriate scheme when dealing with on-shell particles.
- $\overline{\rm MS}$ mass:
  - Absorb only UV $1/\varepsilon$ divergences from on-shell self energy diagrams - by construction only sensitive to short distance aspects of QCD - "short distance mass".

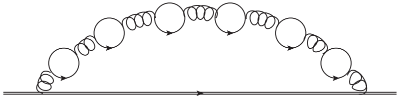
Motivation and Introduction - Renormalons

- Relation between pole and $\overline{\rm MS}$ $(\overline{m}_Q(\overline{m}_Q) \equiv \overline{m}_Q)$:

$$m_Q^{\rm pole} = \overline{m}_Q + \overline{m}_Q \sum_{n=1}^{\infty} a_n^{\overline{\rm MS}} \left(\frac{\alpha_s^{(n_l+1)}(\overline{m}_Q)}{4\pi} \right)^n .$$

- Observation: Intrinsic scale of the $\overline{\rm MS}$ mass is $\overline{m}_Q$ itself.
  - Perturbative corrections of order $\overline{m}_Q$.
  - No logs if $\mu = \overline{m}_Q$.

Motivation and Introduction - Renormalons



- Explore contributions to $a_n^{\overline{\rm MS}}$ from class of diagrams with massless quark bubble insertions to all orders.
- Leads to ($\mu = \overline{m}_Q$, $\alpha_s \equiv \alpha_s(\overline{m}_Q)$, $\hat{q}^2 \equiv q^2/\overline{m}_Q^2$)

$$A(\alpha_s) \sim \sum_{n=0}^{\infty} \frac{\alpha_s^n n_\ell^n}{(6\pi)^n} \int_0^\infty {\rm d}\hat{q}^2 \underbrace{F\left(\hat{q}^2\right)}_{\text{“Gluon momentum distribtion”}} \log^n\left(\hat{q}^2\,{\rm e}^{-5/3}\right) .$$

- Large logarithmic enhancement for $\hat{q}^2 \ll 1$!

Motivation and Introduction - Renormalons

- Evaluate integral for small momenta:

$$F(\hat{q}^2) = \frac{2}{\sqrt{\hat{q}^2}} + \mathcal{O}\left(\sqrt{\hat{q}^2}\right) ,$$

$$\Rightarrow \quad A(\alpha_s) \sim \sum_{n=0}^{\infty} \alpha_s^n \left(\frac{-2n_\ell}{6\pi}\right)^n n! + \ldots$$

- Infrared renormalon behavior!
- How can we deal with the result?
- Can we assign a number to the series?

Motivation and Introduction - Renormalons

- How to deal with divergent series?
  - Generalize summation process - has to give right answer for convergent series.
  - Particularly useful for summing divergent asymptotic series is Borel summation:

A series $s(x) = \sum_{n=0}^{\infty} c_n x^n$ is called Borel-summable if the Borel transform

$$B[s](t) = \sum_{n=0}^{\infty} \frac{c_n t^n}{n!}$$

is convergent for $t > 0$ and if the integral

$$S(x) = \int_0^{\infty} \mathrm{d}t \; \mathrm{e}^{-t} B[s](tx)$$

exists. $S(x)$ is the value of the series.

Motivation and Introduction - Renormalons

- Assumption: Perturbative series is asymptotic in the sense

$$\left|\mathcal{O}(\alpha) - \sum_{i=0}^{n} c_i \alpha^i\right| < K_{n+1}\alpha^{n+1}\,.$$

  - Can not be proven but is reasonable since phenomenology using perturbation theory works very well.
- Ordinary summation: Best approximation typically given when truncating at smallest term $\rightarrow$ irreducible error.
- Note: While a divergent perturbative series implies non-analyticity at $\alpha = 0$, non-analyticity does not imply divergence. A convergent series can still differ from $\mathcal{O}$ by exponentially small terms $\exp(-1/\alpha)$.

Motivation and Introduction - Renormalons

$$A(\alpha_s) \sim \sum_{n=0}^{\infty} \alpha_s^n \left(\frac{\beta_0}{2\pi}\right)^n n! + \ldots$$

- Before applying tools to our example: Use a “dirty trick” - naive non-Abelianization $n_\ell \to -3/2(11 - 2/3 n_\ell) = -3/2\beta_0$.
  - Can be justified diagrammatically - includes some non-Abelian corrections. Profound consequences!
- Apply Borel summation technique ($u \equiv t\beta_0/4\pi$)

$$B[A](u) \sim \frac{1}{1 - \frac{t\beta_0}{2\pi}} + \cdots = \frac{1}{2}\,\frac{1}{1/2 - u} + \ldots .$$

- Pole of the Borel transform at $u = 1/2$.

Motivation and Introduction - Renormalons

$$B[A](u) \sim \frac{1}{1/2 - u} + \ldots$$

- Meaning of the pole? Let's Borel sum the series:

$$A(\alpha_s) = \int_0^\infty \mathrm{d}u \; \mathrm{e}^{-\frac{4\pi u}{\beta_0 \alpha_s}} B[A](u)\,.$$

- Integral exists, but one has to choose path in complex plain to avoid singularity $\rightarrow$ Ambiguity of the Borel summation!

Motivation and Introduction - Renormalons

$$B[A](u) \sim \frac{1}{1/2 - u} + \dots$$

- Size of ambiguity

$$\Delta\left[\int_0^\infty \mathrm{d}u\ \mathrm{e}^{-\frac{4\pi u}{\beta_0 \alpha_s(\overline{m}_Q)}} \frac{1}{u-k}\right] \sim \left(\frac{\Lambda_{\mathrm{QCD}}^2}{\overline{m}_Q^2}\right)^k .$$

- Gives rise to non-perturbative power corrections.
- Pole-$\overline{\mathrm{MS}}$ relation: $k = 1/2$, multiplied with $\overline{m}_Q \Rightarrow \mathcal{O}(\Lambda_{\mathrm{QCD}})$ ambiguity!
- More informations about renormalons: [Beneke, hep-ph/9807443]

Motivation and Introduction - Revisiting Mass Schemes

Why is this important?

- Quark masses:
  - Important parameters for SM predictions.
  - Quark masses are no physical observables (confinement) and are renormalization scheme dependent!
  - Can choose appropriate mass scheme depending on the application.
  - Pole mass:
    - Bad perturbative behavior even at low orders $\rightarrow$ much larger errors in extractions!
    - Irreducible error of order $\Lambda_{\rm QCD}$ unacceptable for future precision measurements (ILC: $\Delta m_t \lesssim 100$ MeV).

Motivation and Introduction - Revisiting Mass Schemes

$$m_Q^{\rm pole} - \overline{m}_Q = \overline{m}_Q \sum_{n=1}^{\infty} a_n^{\overline{\rm MS}} \left(\frac{\alpha_s^{(n_\ell+1)}(\overline{m}_Q)}{4\pi} \right)^n$$

- Reconsider the $\overline{\rm MS}$ mass:
  - Intrinsic physical scale: $\overline{m}_Q$.
  - Problem: Only physically relevant for $\mu > m_Q$
    - Perturbative corrections of order $\overline{m}_Q$.
    - Scales $\ll m_Q$: Virtual heavy quark effects should be integrated out.
  - Standard scheme for most high energy applications, but not right choice for low energy experiments (e.g. at threshold).

Motivation and Introduction - Revisiting Mass Schemes

$$m_Q^{\rm 1S} - m_Q^{\rm pole} = M_B \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} c_{n,k} \left(\frac{\alpha_s^{(n_\ell)}(\mu)}{4\pi}\right)^n \log^k\left(\frac{\mu}{M_B}\right),$$

$\left(M_B \equiv C_F \alpha_s(\mu) m_Q^{\rm pole}\right)$

- Low scale short distance mass: 1S mass [Hoang et al., hep-ph/9809423]:
  - Define as half of the mass of heavy quarkonium spin triplet ground state.
  - Intrinsic physical scale: Inverse Bohr radius $M_B \equiv C_F \alpha_s m_Q^{\rm pole}$.
  - (By definition) well suited for threshold experiments.
- Still: Conversion to $\overline{\rm MS}$ involves large scale hierarchy...
- Other short distance mass schemes: PS, Kinetic, RS, jet, static, ...

MSR Mass - Basic Idea and Definitions

- Need concept of a short distance mass with freely adjustable, universal scale R.
- Makes it possible to relate heavy quark mass values extracted at widely separated scales by using an IR-renormalization group equation.
  - Resums large logarithms,
  - Should be free of $\mathcal{O}(\Lambda_{\text{QCD}})$ renormalon.
- Mass can then be used in arbitrary low energy processes and evolved to high energy scales without any troubles.

MSR Mass - Basic Idea and Definitions

$$m_Q^{\rm pole} - m_Q^{\rm MSR}(R) = R\sum_{n=1}^{\infty} a_n \left(\frac{\alpha_s^{(n_\ell)}(R)}{4\pi}\right)^n$$
$$a_n = a_n^{\overline{\rm MS}}(n_h = 0)$$

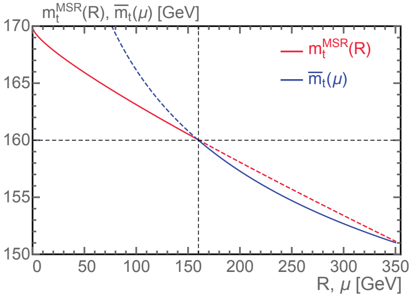
- Define the MSR mass [Hoang et al., 0803.4214]:
- Idea:
  - Asymptotic behavior of perturbative pole-$\overline{\rm MS}$ relation depends on number of massless quarks n_ℓ, but not on m_Q.
  - Can replace $\overline{m}_Q$ by arbitrary scale R and use relation as definition of a new mass scheme.

MSR Mass - Basic Idea and Definitions

- Properties:
  - Low scale short distance mass with direct relation to on-shell self energy diagrams.
  - Very easy and direct relation to $\overline{\rm MS}$ mass.
    - MSR mass automatically inherits conceptual cleanness and good infrared properties.
    - (Almost) no additional computational effort.

MSR Mass - Basic Idea and Definitions

- MSR mass expected to have applications primarily for $R < m_Q$ $\rightarrow$ integrate out virtual heavy quark effects.
  - Change flavor scheme $\alpha_s^{(n_\ell+1)} \rightarrow \alpha_s^{(n_\ell)}$ - matching to $\overline{\rm MS}$.
  - Note: This is called the "natural" MSR scheme.
- MSR mass is the natural generalization of $\overline{\rm MS}$ mass for renormalization scales $\ll m_Q$!



MSR Mass - Basic Idea and Definitions

- Possible interpretation of R: MSR mass contains self-energy corrections of the pole mass only for scales larger than R.
  - Note: Handy property for analyzing mass parameter in Monte Carlo generators ($R \leftrightarrow$ Shower cut-off).
- MSR mass formally agrees with pole mass for $R \to 0$.
  - However: Limit ambiguous (involves evolving through Landau pole) $\to$ manifestation of renormalon ambiguity.

MSR Mass - R-evolution

$$R\frac{\mathrm{d}}{\mathrm{d}R}m_Q^{\rm MSR}(R) = R\frac{\mathrm{d}}{\mathrm{d}R}\left[m_Q^{\rm pole} - R\sum_{n=1}^{\infty} a_n\left(\frac{\alpha_s^{(n_\ell)}(R)}{4\pi}\right)^n\right]$$

$$= -R\sum_{n=0}^{\infty}\gamma_n^R\left(\frac{\alpha_s(R)}{4\pi}\right)^{n+1}$$

- R-evolution equation:
- RGE for the IR scale R, relating MSR masses at different scales.
- Easily solved numerically - good numerical behavior even for R close to Landau pole.

MSR Mass - R-evolution

$$R\frac{\mathrm{d}}{\mathrm{d}R}m_Q^{\mathrm{MSR}}(R) = -R\sum_{n=0}^{\infty}\gamma_n^R\left(\frac{\alpha_s(R)}{4\pi}\right)^{n+1}$$

- Properties:
  - Sums systematically asymptotic renormalon series and large logarithms to all orders.
  - Free of the $\mathcal{O}(\Lambda_{\mathrm{QCD}})$ renormalon (renormalon behavior independent of R).
  - R-evolution equation does not only have logarithmic (like most RGEs) but also linear dependence on R.

MSR Mass - R-evolution

- Compare R-evolution and fixed order perturbation theory (FOPT):
  - Error estimation in R-evolution: Expand $\alpha_s(R)$ in terms of $\alpha_s(\lambda R)$ and vary λ. Leads to common logarithmic scale variation.
  - FOPT: Renormalon behavior cancels only if same scale for all α_s is used - possibly large logarithms for widely separated scales.

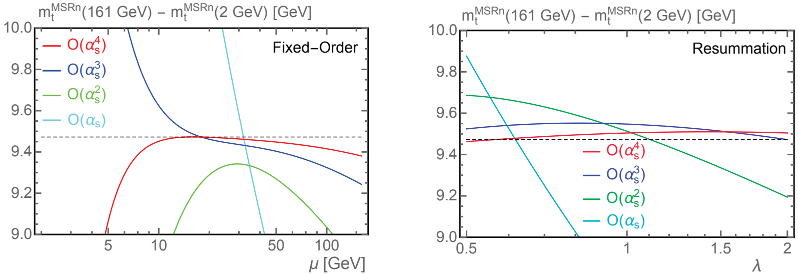
MSR Mass - R-evolution

- Renormalon cancellation for equal μ choice at large β_0:

$$\left[m_Q^{\rm pole} - m_Q^{\rm MSR}(R)\right]_{\beta_0} \sim R\sum_{n=0}^{\infty}\left(\frac{\beta_0\alpha_s(R)}{2\pi}\right)^{n+1} n!$$
$$= R\sum_{n=0}^{\infty}\left(\frac{\beta_0\alpha_s(\mu)}{2\pi}\right)^{n+1} n!\sum_{k=0}^{n}\frac{1}{k!}\log^k\frac{\mu}{R}\,,$$

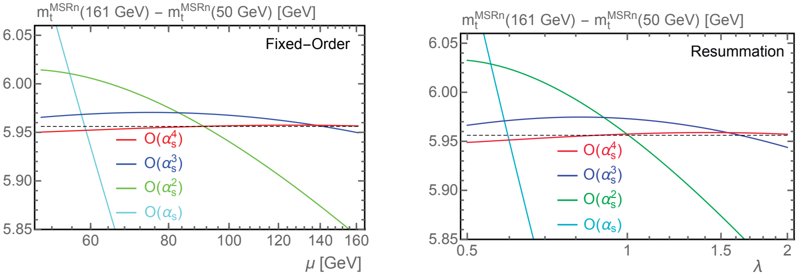
$$\left[m_Q^{\rm MSR}(R_0) - m_Q^{\rm MSR}(R_1)\right]_{\beta_0} \sim$$
$$\sim \sum_{n=0}^{\infty}\left(\frac{\beta_0\alpha_s(\mu)}{2\pi}\right)^{n+1} n!\left(R_1\sum_{k=0}^{n}\frac{1}{k!}\log^k\frac{\mu}{R_1} - R_0\sum_{k=0}^{n}\frac{1}{k!}\log^k\frac{\mu}{R_0}\right)$$

MSR Mass - R-evolution



- R-evolution vs. FOPT for widely separated scales:
  - Significant μ dependence in FOPT - large logs spoil convergence!

MSR Mass - R-evolution



- R-evolution vs. FOPT for similar scales:
  - Very similar in behavior and size.
  - Shows equivalence of λ variation in R-evolution and μ variation in FOPT.

$\mathcal{O}(\Lambda_{\rm QCD})$ Renormalon Sum Rule - Overview

- Analytic solution of R-evolution equation provides useful expression for Borel transform of the pole-MSR series.
- Conceptional feature: Systematic reordering of terms in asymptotic series associated to renormalon ambiguity in leading and subleading.
- Possible to derive analytically the Borel transform of any given perturbative series from perspective of carrying a $\mathcal{O}(\Lambda_{\rm QCD})$ renormalon.
- Yields analytic expression for normalization of singular term in Borel transform.
- Will show some applications.

$\mathcal{O}(\Lambda_{\rm QCD})$ Renormalon Sum Rule - Derivation

$$R\frac{\mathrm{d}}{\mathrm{d}R}m_Q^{\rm MSR}(R) = -R\sum_{n=0}^{\infty}\gamma_n^R\left(\frac{\alpha_s(R)}{4\pi}\right)^{n+1}$$

$$m_Q^{\rm MSR}(R) - m_Q^{\rm pole} = -\int_0^R \mathrm{d}\bar{R}\,\gamma^R(\alpha_s(\bar{R}))$$

$$= \Lambda_{\rm QCD}\sum_{k=0}^{\infty}\mathrm{e}^{i\pi(\hat{b}_1+k)}\,S_k\,\Gamma(-\hat{b}_1-k,t_R)$$

$$t_R = -2\pi/(\beta_0\alpha_s(R)), \qquad \hat{b}_1 = \beta_1/(2\beta_0), \qquad S_k = S_k(\{a_n\},\{\beta_n\})$$

- Solve R-evolution equation describing the pole-MSR difference analytically:
  - All-order representation of the original series.
  - Can use solution to study the encoded $\mathcal{O}(\Lambda_{\rm QCD})$ renormalon.
  - Sum over k: Reordering of series provides information about leading and subleading contributions to renormalon.
    - $k=0$: Resums leading large β_0 terms.

$\mathcal{O}(\Lambda_{\rm QCD})$ Renormalon Sum Rule - Derivation

$$m_Q^{\rm MSR}(R) - m_Q^{\rm pole} = \Lambda_{\rm QCD} \sum_{k=0}^{\infty} {\rm e}^{i\pi(\hat{b}_1+k)}\, S_k\, \Gamma(-\hat{b}_1 - k, t_R)$$

- Ambiguity visible through multivaluedness of incomplete gamma function ($t_R < 0$).
  - Arises in integration over Landau pole at $t = 0$.
- Compare:

$$m_Q^{\rm MSR}(R_0) - m_Q^{\rm MSR}(R_1) =$$
$$= \Lambda_{\rm QCD} \sum_{k=0}^{\infty} {\rm e}^{i\pi(\hat{b}_1+k)}\, S_k\, \Big[\Gamma(-\hat{b}_1 - k, t_0) - \Gamma(-\hat{b}_1 - k, t_1)\Big]$$

  - Ambiguity cancels in the difference.

$\mathcal{O}(\Lambda_{\rm QCD})$ Renormalon Sum Rule - Derivation

- Goal: Useful Borel space expression. Next steps:
  - Asymptotic expansion of incomplete Gamma-function,
  - Borel transform,
  - Hypergeometric function identities.
- Result:

$$B\left[m_Q^{\rm MSR}(R) - m_Q^{\rm pole}\right](u) =$$

$$= -N_{1/2}\left[R\,\frac{4\pi}{\beta_0}\sum_{\ell=0}^{\infty} g_\ell\,\frac{\Gamma(1+\hat{b}_1-\ell)}{\Gamma(1+\hat{b}_1)}(1-2u)^{-1-\hat{b}_1+\ell}\right] + 2R\sum_{\ell=0}^{\infty} g_\ell\, Q_\ell(u)$$

$$N_{1/2} = \frac{\beta_0\,\Gamma(1+\hat{b}_1)}{2\pi}\sum_{k=0}^{\infty}\frac{S_k}{\Gamma(1+\hat{b}_1+k)}$$

$$Q_\ell(u) = \sum_{k=0}^{\infty} S_k \sum_{i=0}^{k+\ell-1}\frac{2^i\,\Gamma(1+\hat{b}_1+i-\ell)}{\Gamma(1+\hat{b}_1+k)\,\Gamma(i+1)}\,u^i$$

$\mathcal{O}(\Lambda_{\rm QCD})$ Renormalon Sum Rule - Derivation

$$-N_{1/2}\left[R\,\frac{4\pi}{\beta_0}\sum_{\ell=0}^{\infty} g_\ell\,\frac{\Gamma(1+\hat{b}_1-\ell)}{\Gamma(1+\hat{b}_1)}(1-2u)^{-1-\hat{b}_1+\ell}\right]+2R\sum_{\ell=0}^{\infty} g_\ell\,Q_\ell(u)$$

$$N_{1/2}=\frac{\beta_0\,\Gamma(1+\hat{b}_1)}{2\pi}\,\sum_{k=0}^{\infty}\frac{S_k}{\Gamma(1+\hat{b}_1+k)}$$

- First term:
  - Non analytic - leads to renormalon ambiguity.
  - Provides analytic expression of normalization $N_{1/2}$ of non-analytic term → $\mathcal{O}(\Lambda_{\rm QCD})$ Renormalon Sum Rule.
- Second term:
  - Purely polynomial in u. Contributions from original series that go beyond pure renormalon corrections.

$\mathcal{O}(\Lambda_{\rm QCD})$ Renormalon Sum Rule - Properties

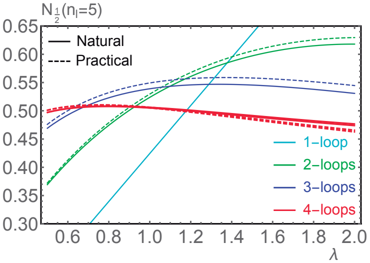
$$N_{1/2} = \frac{\beta_0\,\Gamma(1+\hat{b}_1)}{2\pi}\sum_{k=0}^{\infty}\frac{S_k}{\Gamma(1+\hat{b}_1+k)}$$

- Sum rule can be applied to any perturbative series to probe for an $\mathcal{O}(\Lambda_{\rm QCD})$ renormalon.
- Note: No rigorous proof of (non-)existence of renormalon - in practice applied to truncated series.
  - Projection of known terms onto renormalon behavior.
  - Rigorous proof would need all-order studies.

Applications - Pole Mass Renormalon

- First application: Estimation of $N_{1/2}$ for pole mass.
  - Use recent multi-loop results for terms up to $k = 3$:
  - 4-loop $\overline{\rm MS}$ coefficients,
  - 5-loop β-function.
- Note on error estimation: Implement renormalization scale variations by rewriting $R \to R\lambda$ in original series and expand in $\alpha_s(R)$.
  - Sum rule invariant under variations of λ in the asymptotic limit (R-independence of asymptotic behavior).

Applications - Pole Mass Renormalon



$$N_{1/2}(n_\ell = 3) = 0.526 \pm 0.012\,,$$
$$N_{1/2}(n_\ell = 4) = 0.492 \pm 0.016\,,$$
$$N_{1/2}(n_\ell = 5) = 0.446 \pm 0.024\,.$$

[Beneke et al., 1605.03609], [Ayala et al., 1407.2128]:

$$N_{1/2}(n_\ell = 5) = 0.4616^{+0.027}_{-0.070} \pm 0.002.$$

- Good convergence.
- Result fully compatible with previous ones.
  - Method: Compare explicit loop calculation with pure asymptotic behavior numerically.

Applications - Estimating Higher Order Coefficients

- Invert line of arguments: Use sum rule to estimate coefficients.
  - Estimated 4-loop coefficient (based on consistancy with lower orders) compatible with recently computed result:

$$a_4^{\overline{\rm MS}}(n_\ell = 4, 1) = 230192 \pm 14747 \qquad \text{[Our estimate, 1704.01580]},$$
$$a_4^{\overline{\rm MS}}(n_\ell = 4, 1) = 211807 \pm 5504 \qquad \text{[Marquard et al., 1502.01030]},$$
$$a_4^{\overline{\rm MS}}(n_\ell = 4, 1) = 214828 \pm 422 \qquad \text{[Marquard et al., 1606.06754]}.$$

Applications - Estimating Higher Order Coefficients

n_ℓ	$a_5^{\overline{\rm MS}} \times 10^{-7}$	$a_6^{\overline{\rm MS}} \times 10^{-9}$	$a_7^{\overline{\rm MS}} \times 10^{-11}$	$a_8^{\overline{\rm MS}} \times 10^{-13}$
3	3.400 ± 0.077	3.315 ± 0.075	3.824 ± 0.087	5.099 ± 0.116
4	2.254 ± 0.075	2.023 ± 0.067	2.151 ± 0.072	2.644 ± 0.088
5	1.382 ± 0.074	1.130 ± 0.060	1.097 ± 0.059	1.233 ± 0.066

- Possible to estimate higher order coefficients:
  - Manipulate solution of R-evolution equation to get

$$a_n = (2\beta_0)^n \sum_{k=0}^{n-1} S_k \sum_{\ell=0}^{n-1-k} g_\ell \left(1 + \hat{b}_1 + k\right)_{n-1-\ell-k} .$$

$$(b)_n = \Gamma(b+n)/\Gamma(b)$$

  - Separation of coefficients of the original series into leading and subleading contribution to asymptotic high order behavior.
  - Truncation in k, l (limited knowledge of $\beta_{n>4}$, $a_{n>4}$) still provides correct asymptotic behavior (Pochhammer symbol suppresses higher order terms).

Other Applications of the Sum Rule

- Can also test sum rule by applying it to series known not to have an $\mathcal{O}(\Lambda_{\rm QCD})$ renormalon. Examples:
  - β-function:

$$N^{\beta}_{1/2} = (0.829 \pm 0.497, -0.004 \pm 0.272, 0.065 \pm 0.092, 0.038 \pm 0.032)\,.$$

  - Hadronic R-ratio:

$$N^{R}_{1/2} = (0.398 \pm 0.239, -0.003 \pm 0.1311, -0.071 \pm 0.105, -0.009 \pm 0.029)\,.$$

- All orders (beyond first) compatible with zero as expected.
- Remember: Sum rule not sensitive to higher order renormalons.

Summary and Conclusions

- Aim of introduction: Build awareness of the renormalon problem in QCD.
  - Relevant in determinations of heavy quark masses.
- Detailed presentation of the MSR mass:
  - Low scale short distance mass with freely adjustable scale R.
  - Natural generalization of $\overline{\mathrm{MS}}$ for $\mu \ll m_Q$.
  - IR-RGE (R-evolution equation) to relate MSR masses at arbitrary scales - resums logs and asymptotic renormalon series.
- Analytic solution of R-evolution equation provides reordering of terms regarding leading and subleading contributions to asymptotic behavior.
  - Can be used to analyze ambiguity - analytic expression for normalization of $\mathcal{O}(\Lambda_{\mathrm{QCD}})$ renormalon ambiguity.
  - Can be applied to any perturbative series - many applications.
- Thank you for your attention.